2015

Constructing the Quantum Steering Ellipsoid Using State Measurement of Biphotons

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A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

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Annandale-on-Hudson, New York
May, 2015

Abstract

This paper investigates a recently discovered geometric representation of two-qubit states, the Quantum Steering Ellipsoid, by exploring quantum state measurements on mixtures of polarization-entangled photons produced using spontaneous parametric down conversion (SPDC). Measurements of the second photon of a mixed biphoton system that are conditioned on observations of the first photon demonstrate all possible Bloch Vectors that the second photon can be collapsed to by way of "steering," which is only possible due to the nonlocal nature of entangled systems. This paper examines methods to experimentally verify the Quantum Steering Ellipsoid as a geometric classification of quantum states.

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Dedication

To all the Physicists who helped craft the lens through which I view the universe.

Acknowledgments

Physics—working on it, discussing it, playing with it—has to me always been a social activity, and thus this work is indebted to all the people who aided, directly or indirectly, in bringing it to fruition. I owe the most to my advisors Paul, Matt, Chris, and Hal, for serving as my guides through the thick of the quantum forest and the many technical difficulties of lab design. I am especially grateful for Hal not only steering me to this subject, but for helping me keep to a consistent work schedule this past semester, checking to ensure I was up and in the lab at 8:30am every morning as promised. I'd also like to thank the Bard Physics Department as a whole for giving me the financial resources to build the optics lab on which this paper is modeled; and though I was in the end unable to perform the experiment I set out to do, going through the motions was invaluable to writing this project, and reenlivened my love of playing in the laboratory. Finally, I'd like to thank my mathematician mother and physicist grandfather, not only for helping review this paper the night before I handed it in, but for whatever peculiar genes that made this such an enjoyable process—in spite of the many obstacles that emerge in trying to explore the mysteries of the infinite universe.

1

Introduction

In the physics lab, it is tricky to repeatedly produce quantum particles (electrons, protons, photons, etc.) that are in the same state. For example, let's say we wanted to play with light in its simplest form: a single photon. Photons can be viewed as electromagnetic waves oscillating through space. In other words, the eletric field oscillates—it moves side to side, up and down, or in a circular pattern—and the direction of that oscillation is what we call the state of a photon's **polarization**. If we could prepare a beam of photons perfectly, the polarization state of all the photons in that beam would be the same each time our laser produced one. That means if we could pull a single photon from a laser beam, regardless of which one we pulled out we know that our single photon would be oscillating in the same direction every time. Our ability to know exactly which polarization state our individual photons of this beam are in, with 100% certainty, means the photons of that beam are all in the same **pure state**. In other words, we do not need to measure the photons to figure out what state they are in—there was never any ambiguity as to their polarization. Mathematically then, we can describe what we know about these pure state photons with a column vector, even more simply represented in the handy bra-ket notation by the general symbol $|\gamma\rangle$. What is peculiar about pure state particles, however, is that when they are measured in a way that transforms or rotates the particle there is an inherent probability that we will get certain outcomes. For example, if we had a photon in a horizontally polarized state $|H\rangle$ and then tried to measure it such that we found it in the diagonal $|D\rangle$ or antidiagonal $|A\rangle$ polarization state, there is a 50-50 chance we would get$|A\rangle$ or $|D\rangle$—and there is no way to know which it will be until we measure it. In quantum mechanics, this is an ontological ignorance; it is an uncertainty inherent in the state of the particle itself.

However, most of the time our lab equipment will prepare photons imperfectly, meaning some photons will come out in any number of different polarization states. Because of this, we do not know which pure states our individual photons are in until we measure them. In this case, if we wanted to pull a single photon out to play with, all we know about the photon before we observe it is that there is some probability p_α that it came out of the laser prepared in state $|\alpha\rangle$, some probability p_β that it was prepared in state $|\beta\rangle$, some probability p_ϵ that it was prepared in state $|\epsilon\rangle$, and so on. The description of this beam of photons is now a mixture of possible pure states, or a **mixed state**, meaning we can only describe it as a combination of those possible states and their respective probabilities. Take for example a laser which prepared photons in two possible polariztion states: horizontally $|H\rangle$ or vertically polarized $|V\rangle$. In this case, a simple example of a mixed state could be that 50% of the photons come out horizontally polarized and 50% come out vertically polarized. If we wanted to express this as an equation, the state of this mixed state beam of photons (which we'll call the Greek letter *rho*) could be written using bra-ket notation as such:

$$\rho = \frac{1}{2} |V\rangle \langle V| + \frac{1}{2} |H\rangle \langle H| .$$

Looking at this we see the probabilities of measuring a photon in one of the two possible states are written as fractions multiplied times that state. The components in this equation tell us that it's essentially a coin toss whether we'll measure a vertically or horizontally polarized photon from our beam of photons. However, unlike a real coin toss where you could ostensibly calculate the coin's angular velocity and trajectory to predict whether it will turn up heads or tails, there is no way to physically predict which state each photon will be in once the coin has been flipped. We can't open up our laser and look at the materials generating the photons in hopes that it will tell us exactly what pure state the next photon is about to be prepared in—we could only figure out what the probability is that it will be prepared in that state. In quantum mechanics, this is an epistemological ignorance; the photon is in a pure state, but we don't know which pure state it is in before measurement.

These sorts of curiosities in the world of the very-tiny is part of what makes quantum mechanics so fascinating—and also so perplexing. The overarching goal of this paper is to explore some of the bizarre phenomena of quantum mechanics in a way that makes them easier to understand. The focus in particular is on a quantum phenomenon known as *steering*, and how we can represent it through the gemoetric shapes of spheres and 3D ellipses (aka ellipsoids). This by itself is beautiful, but by doing this we can make it easier to visualize the phenomenon and what it can mean on the quantum level. It is my hope that armed with a rudimentary understanding of the language we use to describe quantum mechanics, this type of geometric representation can make some of the most fascinating parts of physics, at least conceptually, less esoteric. In that spirit, I did my best to write this paper with the goal of accessibility, taking time to explain concepts that undoubtedly appear basic to those already versed in the language of quantum physics and translating more complicated ideas and notations to simpler versions wherever possible. It shall not be taken for granted that readers know what I mean when I say this paper deals with finite-dimensional non-relativistic theory—instead I'll explain that we're dealing with physics with a defined number of dimensions (2 to 4 at most), and that we don't have to worry about the other weird stuff that happens when you start moving close to the speed of light. Likewise, I will find ways to communicate ideas without using higher level mathematical jargon, which in the end isn't necessary to the aims of this paper. I intentionally avoid concepts such as a basis and Hilbert space when I can just as effectively explain the same ideas in simpler, more real-world oriented terms (as the physicist Asher Peres put it: Quantum phenomena do not occur in a

Hilbert space. They occur in a laboratory." [12]). I do this in part because I remember wondering when I first set out to study physics if it was necessary that the mysteries of the quantum world be guarded behind the gates of advanced mathematics. Part of me still, stubbornly, believes we can even explain the mysteries of a field in physics known for its complexity and oddity in a way that far more people than currently do can understand. In that sense this paper is, too, an experiment.

In such, I will walk through the theory step by step to build the foundation upon which the experiment I modeled is built. The section on Theory will detail all the knowledge necessary for this exploration of the study of light on the particle level—a field we call quantum optics. I first discuss terms and ideas that will be needed to conceptualize what we are physically doing when running the experiment, as well as the mathematical background for the calculations we will need for modeling it. After demonstrating what we'll be doing with some examples, I then demonstrate the Experimental Model by looking at the Apparatus that would be used to run this experiment, as well as a detailed explanation as to how it would be run and the results one should anticipate. Examples of computationally modeled Quantum Steering Ellipsoids conclude the paper, drawing a well illustrated roadmap for anyone who should want to verify this experimentally.

2

Theory

2.1 Photon Polarization and Superposition

Before going forward it is important to expand a bit more on our knowledge of polarization. First, a clarification: polarization does not tell you everything about the state of a photon. An example of another part or facet of the photon's whole state (what's known as the global state), which is important to distinguish from polarization, is the direction of a photon's propogation—in other words, which way the wave is headed. Remember polarization is the direction in which its electric field is oscillating back and forth.

The available polarization states for photons come in a variety of flavors[4]:

- <u>linear polarization</u>: the direction of the electric field of the photon oscillates in the horizontal or vertical plane, in the diagonal or antidiagonal plane, and so on. In other words, the oscillations of a photon's electric field are confined to a single plane.

- circular polarization: the direction of the electric field rotates as the photon moves, circling in the counterclockwise (left) or clockwise (right) direction with respect to the direction of propogation.

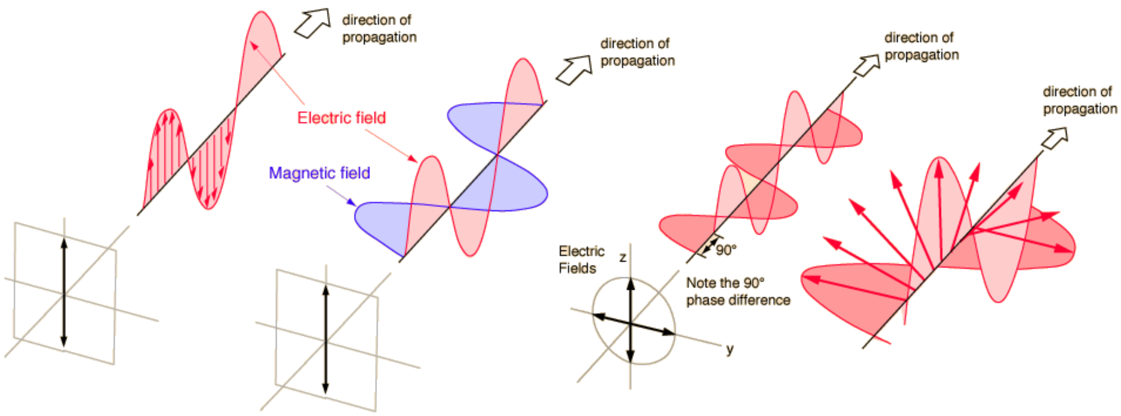

Figure 2.1.1: Linear and Circular Polarization
[See electronic version for figure in color.]

- elliptical polarization: the direction of the electric field rotates as the photon moves, drawing an ellipse in the counterclockwise or clockwise direction with respect to the direction of propogation.

- unpolarized: photons are randomly polarized, which is how normal light from a desk lamp or the sun is typically prepared (and has to be transformed to become a particular state).

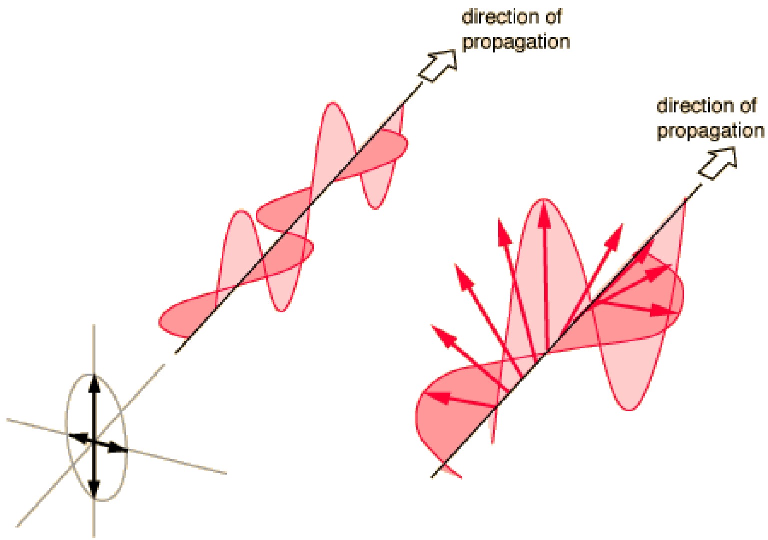

Figure 2.1.2: Elliptical Polarization
[See electronic version for figure in color.]

Having defined these types of polarizations, this is where we start using *bra-ket notation* to help abbreviate the different polarization states. Bra-ket notation is a way of writing the information we know about a state as simple symbols, and those symbols are shorthand for the information written in the form of a vector. In that case, $\langle bra's|$ represent row vectors such as $\begin{pmatrix}1 & 0\end{pmatrix}$ and $|ket's\rangle$ for column vectors such as $\begin{pmatrix}1\\0\end{pmatrix}$, and the same will be true for this paper.

It is important to note that when we write equations that multiply bra's and ket's that we know the difference between an *inner product* and an *outer product*. An inner product is when we have a bra and a ket facing away from one another ($\langle bra|\,|ket\rangle$, usually abridged to be $\langle bra|ket\rangle$), and their product always equals a number (or scalar). A special case is when the bra and ket represent the same state, and in that case the result is alwas equal to 1:

$$\langle\gamma|\gamma\rangle = 1 \tag{2.1.1}$$

Likewise the outer product is when a ket and a bra are facing towards one another ($|ket\rangle \langle bra|$), and they equal a matrix. We will see examples of both of these products later in the paper.

With this introduction to the notation, particular cases of the polarization types we just described can be represented by a ket and its associated column vector [4]:

Horizontal Polarization	$\|H\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}$
Vertical Polarization	$\|V\rangle$	$\begin{pmatrix}0\\1\end{pmatrix}$
Diagonal Polarization	$\|D\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\frac{1}{\sqrt{2}}\|H\rangle + \frac{1}{\sqrt{2}}\|V\rangle$
Antidiagonal Polarization	$\|A\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $\frac{1}{\sqrt{2}}\|H\rangle - \frac{1}{\sqrt{2}}\|V\rangle$
Left-Circularly Polarized	$\|L\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ $\frac{1}{\sqrt{2}}\|H\rangle + \frac{i}{\sqrt{2}}\|V\rangle$
Right-Circularly Polarized	$\|R\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ $\frac{1}{\sqrt{2}}\|H\rangle - \frac{i}{\sqrt{2}}\|V\rangle.$

Note that when we talk about a photon being in a particular polarization state we're saying we know exactly what its state is, meaning that those states we just listed are all pure states. Notice, too, that the diagonal/antidiagonal and left/right circular polarization states we wrote above can also be described as combinations (known as linear combinations) of the horizontal and vertical polarization states. This **principle of superposition** is true for all pure states: they can be rewritten as a linear combination of other pure states. The reason for this is because a light wave can also be described as an overlaying (hence superposition) of two or more other light waves [15]. In the same way that two sound waves interfere to either create a bigger (and to us, louder) sound wave or cancel one another out, polarization states interfere to create different wave shapes as well, in effect forming a different polarization. That does not mean that the diagonal polarization is not its own definite polarization state; the horizontal polarization state can be written as a linear combination of diagonal and antidiagonal polarizations as well. But thinking of a state also being a combination of two other states can be useful in ways we will see in the next section.

It is important to distinguish how superposition is different from a mixed state, in that superposed states are pure states and not a mix of different photons in one state or another. Recall our simplest-case mixed state beam from the Introduction: we saw that half the photons of our laser beam were horizontal and the other half were vertical. Compare that to the individual photons of a diagonally polarized beam, which are always a single state that is also *both* horizontal and vertical at the same time. What this tells us is that in the quantum world, superposition means photons can be thought of as moving in two (or more) directions at once—in essence, they can occupy multiple states at once.

2.2 Qubits

Related to superposition is another important concept for this paper that comes from quantum information theory. Quantum information is a way of looking at the states of quantum systems (like the polarization state of our single photon) as information—in essence applying the field of data science (with binary bits and computation and all that) to the weird world of quantum physics, producing some interesting ways of viewing these concepts.

The simplest quantum systems are those that exist in two-dimensional space. This is where our single photon's polarization lives: a space defined by two axes that are perpendicular to one another. For example, the vertical and horizontal polarizations form two perpendicular axes that create a 2D space; the same could go for our diagonal/antidiagonal and left/right circular polarizations. The polarization of our single photon is thus an example of the simplest quantum system. These simplest 2D systems are known in quantum information as a **qubit**. A qubit is simply taking the binary bit to the quantum level, where systems can exist simultaneously as both a 1 and a 0—making good use of the phenomenon of superposition [15]. Since one qubit represents two dimensions, like our horizontal-vertical photon, a two-qubit system represents four dimensions, such as two entangled photons both with their own horizontal-vertical space. The notion of qubits will be helpful when we discuss the construction of the Bloch Sphere, which we'll adress in a few sections.

2.3 Density Matrices and the Trace

It is also important to understand how mixed states can be most handily represented. Keeping with our vector and bra-ket notation, the best way to mathematically represent a mixed state is to describe it as what we call a **density matrix**. A density matrix is simply what you get when you add up the mixture of pure states that the quantum system could possibly be in, writing all that we know about the beam of photons in the form of a matrix [1].

For example, let's say we take our mixed state from the Introduction again: the mixed state in which 50% of the photons come out of the laser horizontally polarized and 50% are prepared vertically polarized. Each polarization state has a probability of $\frac{1}{2}$. Thus our representation of its state, a density matrix ρ, is a sum of the two possible states and their respective probabilities made into a matrix:

$$\rho = \frac{1}{2}\left|V\right\rangle\left\langle V\right| + \frac{1}{2}\left|H\right\rangle\left\langle H\right|,$$

which using vector multiplication becomes

$$\rho = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$
$$\Rightarrow \rho = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\Rightarrow \rho = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

So far we've only seen density matrices that are *diagonalized*—meaning all elements not on the left-to-right diagonal of the matrix are zero. Any density matrix can be diagonalized if rewritten in a way that uses the states that form the density matrix as the $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ vectors. For example, had we defined the $|D\rangle$ polarization state to be equal to $\begin{pmatrix}1\\0\end{pmatrix}$ and the $|A\rangle$ polarization state to be equal to $\begin{pmatrix}0\\1\end{pmatrix}$, their respective density matrices would be diagonalized.

If we think of the density matrix as the sum of the possible pure states and the probabilities of their preparation, we get a something that more generally looks like [15]:

$$\rho = p_\alpha |\alpha\rangle\langle\alpha| + p_\beta |\beta\rangle\langle\beta| + p_\epsilon |\epsilon\rangle\langle\epsilon| + ...,$$

which is commonly written in the form:

$$\rho = \sum_\gamma p_\gamma |\gamma\rangle\langle\gamma| . \tag{2.3.1}$$

The simplest examples of density matrices are those that are formed by pure states. In the pure state case, we know the probability of our single photon being in a particular polarization state is 1, meaning our equation for a density matrix is simplified to just one element of the sum:

$$\rho = |\alpha\rangle\langle\alpha| . \tag{2.3.2}$$

Hence the density matrices for all possible polarization states we listed in Section 2.1 are

$$\rho_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho_D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho_A = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$\rho_L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad \rho_R = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

It is important to note that it is a condition of density matrices that the sum of their diagonal elements, known as the **trace** of the density matrix, is always equal to 1:

$$\mathrm{Tr}[\rho] = 1. \tag{2.3.3}$$

The other condition of density matrices is that they are **Hermitian**. That means if we flipped the elements across the left-to-right diagonal of the matrix, and then turned any imaginary elements negative (so i becomes -i and vice versa), it would be the same matrix. These concepts will be useful going forward.

We can do all sorts of neat things with the density matrix, but the function that matters to us here is how we can represent density matrices (and thus the states of quantum systems) in a way that's easier to visualize, like a point on a graph. This is where the Bloch Sphere comes in.

2.4 Constructing the Bloch Sphere

Let's say we want to plot polarization states on a graph. As it turns out, there's a way to turn any density matrix, which as we've shown can represent both pure and mixed states, into a unique 3D vector. With the three components of that vector, we can plot a point on a 3D graph that represents a given photon's (or photons') density matrix, and thus its polarization state.

To do this we use a set of matrices that when written as a linear combination can describe any possible matrix. This set is known as the **Pauli Matrices**, which is made up of three matrices, plus the identity matrix:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that each Pauli Matrix is associated with a pair of polarizations, and this pair forms an axis. One polarization represents the positive direction on the axis, and the other polarization represents the negative direction on the axis [3]:

$$\sigma_x = |D\rangle\langle D| - |A\rangle\langle A|$$
$$\Rightarrow \sigma_x = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$\Rightarrow \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = |L\rangle\langle L| - |R\rangle\langle R|$$
$$\Rightarrow \sigma_y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$
$$\Rightarrow \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = |H\rangle \langle H| - |V\rangle \langle V|$$
$$\Rightarrow \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To rewrite any density matrix for a qubit as a linear combination of these Pauli Matrices, we use the equation[15]:

$$\rho = \frac{1}{2}(\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z) \tag{2.4.1}$$

which can be re-written again as

$$\rho = \frac{1}{2}(\mathbb{1} + \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z \end{pmatrix}\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}) = \frac{1}{2}(\mathbb{1} + \vec{\sigma} \cdot \vec{a}). \tag{2.4.2}$$

You'll notice that there are three unspecified elements in Equation (2.4.1) and (2.4.2) that can be represented as a column vector: a_x, a_y, and a_z. This column vector $\vec{a}$, which is specifically associated with the three Pauli Matrices, we call the **Bloch Vector**, and it is what we'll be plotting on our 3D graph. The Bloch Vector will always be a set of three real numbers that we can show as a point in 3D space.

The way to acquire the values of components of a_x, a_y, and a_z is by taking the associated trace of the density matrix times each Pauli Matrix. We know this because, by taking Equation (2.4.2) and multiplying it times the Pauli Matrix written as a vector, we have:

$$\rho \cdot \vec{\sigma} = \frac{1}{2}(\mathbb{1} \cdot \vec{\sigma} + \vec{\sigma}^2 \cdot \vec{a}),$$

which we then take the trace of both sides we get:

$$\mathrm{Tr}[\rho \cdot \vec{\sigma}] = \frac{1}{2}(\mathrm{Tr}[\mathbb{1} \cdot \vec{\sigma}] + \mathrm{Tr}[\vec{\sigma}^2 \cdot \vec{a}]).$$

The first term on the right hand side simply becomes the trace of $\vec{\sigma}$, since the Identity Matrix does not transform the vector, and looking at all the Pauli Matrices we see the $\mathrm{Tr}[\vec{\sigma}]$ is equal to 0. Likewise the square of any Pauli Matrix is the Identity Matrix, meaning the second component reduces to $\vec{a}\,\mathrm{Tr}[\mathbb{1}]$ which gives us $2\vec{a}$:

$$\mathrm{Tr}[\rho \cdot \vec{\sigma}] = \vec{a}.$$

Here's an example of how we would solve for a Bloch Vector. Let's say we wanted to figure out what the Bloch Vector is for the pure state horizontally polarized density matrix. We've shown that the density matrix for the horizontally polarized photon is

$$\rho_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we know we need to solve for the a-components of the Bloch Vector in the following equation:

$$\rho_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_y \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} + a_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).$$

To solve for the three unknowns, we can utilize the trace function. The trace of the density matrix for a horizontally polarized photon would thus be 1 (which is good because as we noted above the trace of any density matrix should be equal to 1). To solve for a_x in particular, we take the trace of the density matrix of the horizontally polarized photon multiplied times the Pauli Matrix associated with that component of $\vec{a}$, which in this case is σ_x. Thus, our equation for a_x would be

$$a_x = \text{Tr}[\rho_H \cdot \sigma_x]$$

$$\Rightarrow a_x = \text{Tr}[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}].$$

We apply the same to the other two components of $\vec{a}$. Thus, for the horizontally-polarized photon, our Bloch Vector is this column vector:

$$\vec{a}_H = \begin{pmatrix} Tr[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] \\ \text{Tr}[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}] \\ \text{Tr}[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}] \end{pmatrix}$$

$$\Rightarrow \vec{a}_H = \begin{pmatrix} \mathrm{Tr}[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] \\ \mathrm{Tr}[\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}] \\ \mathrm{Tr}[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] \end{pmatrix}$$

$$\Rightarrow \vec{a}_H = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

And to double check that this Bloch Vector fits into our original equation, we simply plug the components of $\vec{a}_H$ in:

$$\rho = \frac{1}{2}(\mathbb{1} + (0)\sigma_x + (0)\sigma y + (1)\sigma z)$$

$$\Rightarrow \rho_H = \frac{1}{2}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (0)\begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} + (1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$$

$$\Rightarrow \rho_H = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

Thus the density matrix ρ_H can also be described by the vector $\vec{a}_H = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$.

This case shows that the Bloch Vector for our horizontally polarized photon has a length of 1. As it turns out, *all pure states when turned into Bloch Vectors* *have a length of 1.* We can prove this mathematically based on everything we know about pure states and density matrices:

$$\rho_{\text{pure}} = |\gamma\rangle \langle\gamma| \,,$$

$$\rho_{\text{pure}}^2 = |\gamma\rangle \langle\gamma|\gamma\rangle \langle\gamma| \,,$$

and since $\langle\gamma|\gamma\rangle = 1$,

$$\Rightarrow \rho_{\text{pure}}^2 = |\gamma\rangle \langle\gamma| \,,$$

which means that the trace of ρ_{pure}^2 is also equal to 1. All this to say that we know a density matrix represents a pure state if $\text{Tr}[\rho_{\text{pure}}]$=$\text{Tr}[\rho_{pure}^2] = 1$.

Now if we look at the trace of the density matrix equation using the Bloch Vector, then the result would be:

$$\text{Tr}[\rho^2] = \text{Tr}[\frac{1}{4}(\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)(\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)]$$

Since the trace is commutative, it operates indivudally on each component of the equation once we've factored it all out, with the a-components being unaffected by the trace and thus moved outside the function

$$\begin{aligned}\text{Tr}[\rho^2] =& \frac{1}{4}(\mathbb{1} + a_x \text{Tr}[\sigma_x] + a_y \text{Tr}[\sigma_y] + a_z \text{Tr}[\sigma_z] + a_x^2 \text{Tr}[\sigma_x^2] + a_x a_y \text{Tr}[\sigma_x\sigma_y] \\ &+ a_x a_z \text{Tr}[\sigma_x\sigma_z] + a_y a_x \text{Tr}[\sigma_y\sigma_x] + a_y^2 \text{Tr}[\sigma_y^2] + a_y a_z \text{Tr}[\sigma_y\sigma_z] \\ &+ a_z a_x \text{Tr}[\sigma_z\sigma_x] + a_z a_y \text{Tr}[\sigma_z\sigma_y] + a_z^2 \text{Tr}[\sigma_z^2])\end{aligned}$$

With that in mind, we get rid of all the terms of different Pauli Matrices being multiplied by one another as they are all equal to zero:

$$\mathrm{Tr}[\sigma_x\sigma_y] = \mathrm{Tr}[\sigma_x\sigma_z] = \mathrm{Tr}[\sigma_y\sigma_x] = \mathrm{Tr}[\sigma_y\sigma_z] = \mathrm{Tr}[\sigma_z\sigma_x] = \mathrm{Tr}[\sigma_z\sigma_y] = 0,$$

as well as get rid of all terms that are the trace of a Pauli Matrix (or a Pauli Matrix times any number):

$$a_x\,\mathrm{Tr}[\sigma_x] = a_y\,\mathrm{Tr}[\sigma_y] = a_z\,\mathrm{Tr}[\sigma_z] = 0.$$

Finally, because a Pauli Matrix times itself will give you the Identity Matrix, and the trace of the Identity Matrix is 2, we have:

$$\mathrm{Tr}[\rho^2] = \frac{1}{4}(2 + 2a_x^2 + 2a_y^2 + 2a_z^2)$$
$$\Rightarrow \mathrm{Tr}[\rho^2] = \frac{1}{2}(1 + a_x^2 + a_y^2 + a_z^2)$$
$$\Rightarrow \mathrm{Tr}[\rho^2] = \frac{1}{2}(1 + |\vec{a}|^2),$$

which means anytime that $\mathrm{Tr}[\rho^2]$=1, and thus is a pure state, $|\vec{a}|^2$ must also be equal to one, meaning the length $|\vec{a}|$ (the length of the Bloch Vector) is always 1.

Since the maximum length any Bloch Vector can have is 1, if we took all possible Bloch Vectors and plotted them out they would form what is known as the **Bloch Sphere**, where all pure states of this qubit are on the surface of the sphere and all mixed states of the qubit are on the inside of the sphere.

In the Figure 2.4.1 below, you'll see what this looks like. Our axes for the Bloch Sphere are chosen to be pairs of the usual culprits for pure polarization states: diagonal-antidiagonal is the x-axis, and the left-right ciruclar is the y-axis, and horizontal-vertical is the z-axis.

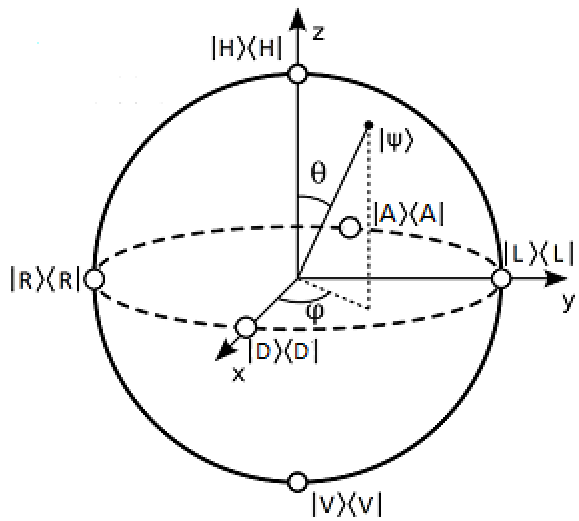

Figure 2.4.1: The Bloch Sphere for our single photon

What's helpful about the Bloch Sphere is that this represents all the states available to a qubit, including the quantum system of our beam of photons. We have a graphical representation of all its possible states described in these three polarization axes—the horizontal-vertical, the diagonal-antidiagonal, and the left-right circularly polarized axis. The beam of photons can thus be represented as a point anywhere both on or inside the Bloch Sphere.

2.5 Entanglement

Having explored different representations of simple polarization states, we can explore some more complicated phenomena. As we've seen in quantum mechanics, simple quantum systems like the polarization of photons are described as being in a given state, whether it be a mixed state or a pure one. But where some of the quantum weirdness really starts is when we have a quantum system that is composed of two (or more) particles' states that are said to be **entangled**, meaning that the states of these two particles form one composite system rather than two one-particle states [8]. In other words, we cannot think of an entangled system of two photons as separate entities—this is why some refer to two entangled phtons as a single **biphoton** [10]. To express the system as two separate photons would mean they were not entangled.

Entanglement can be represented mathematically by taking what is known as the *Kronecker product* between two states. If we have two horizontally or vertically polarized photons whose polarizations are entangled, they could be written in one of three ways:

$$|H_1\rangle \otimes |H_2\rangle = |H_1\rangle\,|H_2\rangle = |H_1, H_2\rangle\,,$$

$$|V_1\rangle \otimes |V_2\rangle = |V_1\rangle\,|V_2\rangle = |V_1, V_2\rangle\,.$$

The Kronecker product is simply taking two vectors/matrices and matching each element of the first vector/matrix with the entire second vector/matrix. A quick sketch of how this vector multiplication works for the horizontally polarized case would be:

$$|H_1, H_2\rangle = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}$$

$$|H_1, H_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

As you can see, the entanglement of two qubits moves us up into four dimensions—in this case being the pair of axes formed by each photon's individual horizontal and vertical polarization states. The simplest examples of entangled pure states are known as the four Bell States, which are written here using the horizontal and vertical polarization states:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|H_1, H_2\rangle + |V_1, V_2\rangle),$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|H_1, H_2\rangle - |V_1, V_2\rangle),$$
$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|H_1, V_2\rangle + |V_1, H_2\rangle),$$
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|H_1, V_2\rangle - |V_1, H_2\rangle).$$

We will be looking at an example of steering photons using the third of those four Bell States.

2.6 Down-Conversion

We can produce entangled photons in a number of different ways. The earliest optical experiments on entangled photons involved sending ultraviolet light at calcium atoms, which causes electrons to jump into a more excited state. When the electrons fell back to their natural resting orbits (their ground state), the spontaneous decay would emit a pair of correlated pairs of photons emanating in opposite directions[10]. A technologically simpler method (and the way this experiment would produce entangled photons) is by sending a beam of photons through down-conversion crystals.

Spontaneous Parametric Down-Conversion (SPDC) is the splitting of individual photons into entangled photons by sending them through specially prepared crystals. The process was first described in 1970 [2], improving on the efficiency of atomic emission of two photons to having one out of every 10^{12} photons sent into the crystal turned into a pair [11]. The process is "parametric" because the down-conversion is dependent on the electric fields of the photons, not just the intensity of the light we send in to the crystals. Specifically, in our experiment one of the blue-wavelength photons we send into the crystals will split into two, entangled, red-wavelength photons. The reason for the color change is because in the process of being down-converted, energy and momentum of the soon-to-be-down-converted-photon are conserved by being split evenly between the two entangled photons, meaning that they are less energetic and take longer to oscillate, producing longer wavelengths (hence "down-conversion"). How down-conversion in the crystals works is still unknown (hence "spontaneous"), but since the focus of this paper is on the entanglement we will have to leave that as an open question [7].

2.7 Quantum Steering

The phenomenon of entanglement is what allows for a quantum system as a whole to react to actions performed on its individual parts. This is how the concept of **quantum steering** works. First discovered by Schrödinger, steering can be thought of as taking measurements on one particle of an entangled system (like our biphoton), causing the other particle to collapse (i.e. to be ”steered”) to another state [14]. In effect, by using optical elements to rotate one of the entangled photons (called the idler photon) into different polarization states and then measuring that photon, we can steer the other photon (called the signal photon) into a range of other states.

Steering is not as active a process as it may sound. By taking measurements on the idler photon, we are not causing the signal photon to become another state. Instead, steering is simply finding a correlation that we defined: if we purposefully measure diagonal polarization on the idler side for the state $|D_1, D_2\rangle$ by projecting it on to the diagonal polarization state, then we're going to get diagonal polarization on the signal side as well; and that stays true even if we try to measure the idler photon by projecting it on $|H\rangle$ or $|R\rangle$ or any pure polarization state. The correlation always holds, and is what allows steering to take place. One way to think of this is by taking trials of rolling a pair of dice. Let's say we labeled each of the 6 numbers on our dice with the common polarization states, where the opposite side of the die is the perpendicular polarization: 1 $\rightarrow |H\rangle$, 2 $\rightarrow |L\rangle$, 3 $\rightarrow |D\rangle$, 4 $\rightarrow |A\rangle$, 5 $\rightarrow |R\rangle$, 6 $\rightarrow |V\rangle$. Now say these dice were entangled, so that each time we got one state on the first die, we got its perpendicular polarization on the second die. The trials would look something like this:

Trial	1st Die	1st State	2nd State
1	3	$\|D\rangle$	$\|A\rangle$
2	5	$\|R\rangle$	$\|L\rangle$
3	1	$\|H\rangle$	$\|V\rangle$
4	1	$\|H\rangle$	$\|V\rangle$
5	4	$\|A\rangle$	$\|D\rangle$

As you can see, each time we find $|H\rangle$ for the 1st die, we will always find $|V\rangle$ for the 2nd die, and vice versa. Thus steering, in this example, would be if we did something to the first die mid-roll that made it land with a 1 facing up. By projecting it onto the state 1, we have in effect steered the second die to come up with a 6 because of that correlation. This is a subtle difference between causation and correlation in quantum mechanics, but an important one to make.

2.7.1 Rotating States with Wave Plates

In our experimental model, we rotate our down-coverted idler and signal photons using wave plates and then measure the states using Rochon Polarizers and detectors, which transmit horizontally and redirect vertically polarized light (see Figure

3.1.1). Depending on how we angle the wave plates, we can rotate our photons into a modified polarization state. Thus, if we were to place a wave plate in front of the beam of the idler photon, the signal photon would be rotated such that when we measure it as a result of their entanglement it will be projected onto a pure or mixed state.

The way we mathematically represent what wave plates and polarizers do to the polarization state of photons is by using the **Jones Matrices**. Each Jones Matrix mathematically represents the operation a given wave plate performs on a beam of photons. In our experiment, we are using common wave plates known as half-wave plates and quarter-wave plates. Rotating these wave plates to different angles transforms the states. The general Jones Matrices for half-wave plates and quarter-wave plates are listed below, as are the two matrix operations that the Rochon Polarizer performs on the state (since it acts as both a horizontal and vertical polarizer) [1]:

Horizontal Polarizer	$\mathbf{J}_H$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Vertical Polarizer	$\mathbf{J}_V$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Half-Wave Plate	$\mathbf{J}_{\frac{\lambda}{2}}(\phi)$	$\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$
Quarter-Wave Plate	$\mathbf{J}_{\frac{\lambda}{4}}(\chi)$	$\frac{\sqrt{2}}{2}\begin{pmatrix} \cos^2(\chi) + i\sin^2(\chi) & (1-i)\sin(\chi)\cos(\chi) \\ (1-i)\sin(\chi)\cos(\chi) & \sin^2(\chi) + i\cos^2(\chi) \end{pmatrix}$

When we talk about orienting or rotating the wave plates, we mean that we are rotating it like a wheel about its center. The Figure 2.7.1 depicts this.

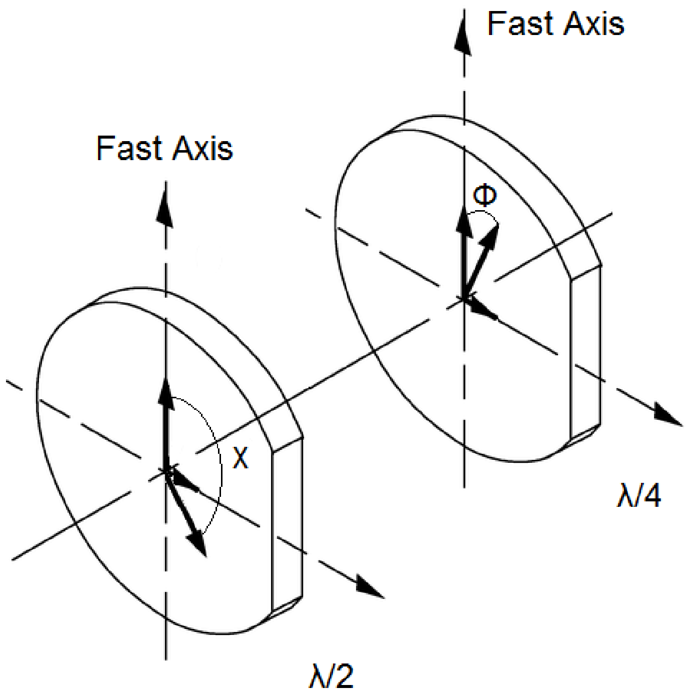

Figure 2.7.1: Wave Plate Orientations

There are particular orientations of both wave plates that will be important for taking measurements. On the signal side of the biphoton, we'll need to rotate the signal state in three different ways, each correlated with one of the three Pauli Matrices and their associated polarization axis for the Bloch Sphere that we mentioned earlier. In essence, we're rotating the state of the photon beam from wherever it is in the Bloch Sphere to one of these axes.

The wave plate settings for our signal photon will thus be

Polarization Axis	QWP Angle χ	HWP Angle ϕ
Horizontal-Vertical	0^o	0^o
Diagonal-Antidiagonal	0^o	22.5^o
Left-Right Circular	45^o	0^o

with the fast axis of the wave plates being rotated relative to the horizontal (i.e. 0^o means perpendicular with the lab table).

It is important to understand that the wave plates after down-conversion only act on the individual photons in the biphoton system. However, we mathematically have to operate on the entire biphoton state due to their entanglement. We do this by taking the Kronecker product of the Identity Matrix with the operators we listed above, so that it only acts on one of the components of the Kronecker product that makes up the entanglement.

For example, this operation $(\mathbb{1}\otimes\mathbf{J}_{\frac{\lambda}{4}}(45^o))\,|H_s, H_i\rangle$ only acts on the idler photon, leaving the horizontal polarization state of the signal photon unchanged, which gives us with $|H_s, R_i\rangle$.

2.7.2 An Example with a Bell State Biphoton

Let's flesh all this out with an example using our third Bell State biphoton:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|H_s, V_i\rangle + |V_s, H_i\rangle).$$

The idler passes through a quarter- and half-wave plate and the signal through either a quarter- or half-wave plate (as they will in the experiment we model, see Figure 3.1.1). We take the Kronecker product of a general Jones Matrix and the Identity Matrix, the Kronecker product of the Identity Matrix and the Jones Matrix for the half-wave plate, and then the Kronecker product of the Jones Matrix for the quarter-wave plate and the Identity Matrix:

$$(\mathbf{J}_{\frac{\lambda}{2,4}}(\theta) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{2}}(\phi)(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{4}}(\chi)) |\psi^+\rangle$$

where the angles θ, ϕ, and χ are all dependent on our rotations of the wave plates. Note we must order the Jones Matrices right to left according to the order that they rotate the state.

As this is a pure state case, and later we will be dealing with mixed state cases, we're going to convert our pure state from a ket to a density matrix. Using Equation (2.3.2), our density matrix is simply the product $|\psi^+\rangle \langle\psi^+|$:

$$\rho_{\psi^+} = \frac{1}{\sqrt{2}}(|H_s, V_i\rangle + |V_s, H_i\rangle)\frac{1}{\sqrt{2}}(\langle H_s, V_i| + \langle V_s, H_i|$$

$$\Rightarrow \rho_{\psi^+} = \frac{1}{2}(|H_s, V_2 i\rangle \langle H_s, V_i| + |H_s, V_i\rangle \langle V_s, H_i| + |V_s, H_i\rangle \langle H_s, V_i| + |V_s, H_i\rangle \langle V_s, H_i|).$$

This may look clunky in bra-ket notation, but what this tells us is the value of each component in the 4x4 density matrix that represents our third Bell State biphoton $|\psi^+\rangle$. We're going to use bra-ket notation for the next part of our calculation, but just to see the density matrix at this juncture we convert it to vector notation and find:

$$\rho_{\psi^+} = \frac{1}{2}(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \rho_{\psi^+} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Much neater. Now back to bra-ket notation. Having prepared our biphoton, let's make up some settings for our wave plates. We can do a simple demonstration of how the operations work by rotating the half-wave plate to 22.5^o and leaving the quarter-wave plate oriented at 0^o. Our equation is thus:

$$(\mathbf{J}_{\frac{\lambda}{2,4}}(\theta) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{2}}(22.5^o))(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{4}}(0^o)) \left|\psi^+\right\rangle,$$

which we'll introduce a shorthand for in order to make our equations less messy:

$$\hat{W} \left|\psi^+\right\rangle \equiv (\mathbf{J}_{\frac{\lambda}{2,4}}(\theta) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{2}}(22.5^o))(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{4}}(0^o)) \left|\psi^+\right\rangle.$$

Now looking at the density matrix representation of this we'd have:

$$\hat{W} \left|\psi^+\right\rangle \left\langle\psi^+\right| \hat{W}^\dagger$$

where the dagger next to the Jones Matrix shorthand stands for the Hermitian conjugate, which we saw in our discussion of density matrices. Expanding that, we get:

$$= \hat{W}(\frac{1}{2}(|H_s, V_i\rangle \langle H_s, V_i| + |H_s, V_i\rangle \langle V_s, H_i| + |V_s, H_i\rangle \langle H_s, V_i| + |V_s, H_i\rangle \langle V_s, H_i|))\hat{W}.$$

To save us the headache of multiplying all these 4-element vectors with 4x4 matrices, we can simply create a key that shows us:

$$\mathbf{J}_{\frac{\lambda}{2}}(22.5^o)\mathbf{J}_{\frac{\lambda}{4}}(0^o)|H\rangle = |D\rangle$$

$$\langle H| \mathbf{J}_{\frac{\lambda}{4}}(0^o)^\dagger \mathbf{J}_{\frac{\lambda}{2}}(22.5^o)^\dagger = \langle D|$$

$$\mathbf{J}_{\frac{\lambda}{2}}(22.5^o)\mathbf{J}_{\frac{\lambda}{4}}(0^o)|V\rangle = i|A\rangle$$

$$\langle V| \mathbf{J}_{\frac{\lambda}{4}}(0^o)\mathbf{J}_{\frac{\lambda}{2}}(22.5^o)^\dagger = -i\langle A|,$$

which we can use to more quickly convert our pure state density matrix in bra-ket notation:

$$= (\mathbf{J}_{\frac{\lambda}{2,4}}(\theta) \otimes \mathbb{1})\cdot$$

$$\frac{1}{2}(|H_s\rangle\, i|A_i\rangle \langle H_s|(-i)\langle A_i| + |H_s\rangle\, i|A_i\rangle \langle V_s| \langle D_i| +$$

$$|V_s\rangle |D_i\rangle \langle H_s|(-i)\langle A_i| + |V_s\rangle |D_i\rangle \langle V_s| \langle D_i|)$$

$$\cdot(\mathbf{J}_{\frac{\lambda}{2,4}}(\theta)^\dagger \otimes \mathbb{1}).$$

Notice how all of the idler states have been rotated to become diagonal or antidiagonal polarization, sometimes producing an imaginary number i, which means that the wave of the polarization was shifted backward or forward relative to its original oscillation. Now if we wanted to find the tranformed density matrices for each of the three possible wave plate orientations, so that we could the find the three components of the Bloch Vector from those density matrices, we need to do three separate transformations for measuring the signal state using no wave plate, a quarter-wave plate, and then a half-wave plate .

No Wave Plate

$$\rho_{\text{nwp}} = \frac{1}{2}(|H_s\rangle\,(-i)\,|A_i\rangle\,\langle H_s|\,i\,\langle A_i| + |H_s\rangle\,i\,|A_i\rangle\,\langle V_s|\,\langle D_i|$$

$$+\,|V_s\rangle\,|D_i\rangle\,\langle H_s|\,(-i)\,\langle A_i| + |V_s\rangle\,|D_i\rangle\,\langle V_s|\,\langle D_i|)$$

$$= \frac{1}{2}(|H_s\rangle\,((\frac{1}{\sqrt{2}})(|H_i\rangle - |V_i\rangle))\,\langle H_s|\,((\frac{1}{\sqrt{2}})(\langle H_i| - \langle V_i|)))$$

$$+i\,|H_s\rangle\,((\frac{1}{\sqrt{2}})(|H_i\rangle - |V_i\rangle))\,\langle V_s|\,((\frac{1}{\sqrt{2}})(\langle H_i| + \langle V_i|))$$

$$-i\,|V_s\rangle\,((\frac{1}{\sqrt{2}})(|H_i\rangle + |V_i\rangle))\,\langle H_s|\,i((\frac{1}{\sqrt{2}})(|H_i\rangle - |V_i\rangle))$$

$$+\,|V_s\rangle\,((\frac{1}{\sqrt{2}})(|H_i\rangle + |V_i\rangle))\,\langle V_s|\,((\frac{1}{\sqrt{2}})(\langle H_i| + \langle V_i|)))$$

$$\Rightarrow = \frac{1}{2}(\frac{1}{2}(|H_s\rangle\,|H_i\rangle - |H_s\rangle\,|V_i\rangle)(\langle H_s|\,\langle H_i| - \langle H_s|\,\langle V_i|)$$

$$+\,\frac{i}{2}(|H_s\rangle\,|H_i\rangle - |H_s\rangle\,|V_i\rangle)(\langle V_s|\,\langle H_i| + \langle V_s|\,\langle V_i|)$$

$$-\,\frac{i}{2}(|V_s\rangle\,|H_i\rangle + |V_s\rangle\,|V_i\rangle)(\langle H_s|\,\langle H_i| - \langle H_s|\,\langle V_i|)$$

$$+\,\frac{1}{2}(|V_s\rangle\,|H_i\rangle + |V_s\rangle\,|V_i\rangle)(\langle V_s|\,\langle H_i| + \langle V_s|\,\langle V_i|))$$

$$\Rightarrow = \frac{1}{4}(|H_s\rangle |H_i\rangle \langle H_s| \langle H_i| - |H_s\rangle |H_i\rangle \langle H_s| \langle V_i| - |H_s\rangle |V_i\rangle \langle H_s| \langle H_i| + |H_s\rangle |V_i\rangle \langle H_s| \langle V_i|)$$

$$+\frac{i}{4}(|H_s\rangle |H_i\rangle \langle V_s| \langle H_i| + |H_s\rangle |H_i\rangle \langle V_s| \langle V_i| - |H_s\rangle |V_i\rangle \langle V_s| \langle H_i| - |H_s\rangle |V_i\rangle \langle V_s| \langle V_i|)$$

$$-\frac{i}{4}(|V_s\rangle |H_i\rangle \langle H_s| \langle H_i| - |V_s\rangle |H_i\rangle \langle H_s| \langle V_i| + |V_s\rangle |V_i\rangle \langle H_s| \langle H_i| - |V_s\rangle |V_i\rangle \langle H_s| \langle V_i|)$$

$$+\frac{1}{4}(|V_s\rangle |H_i\rangle \langle V_s| \langle H_i| + |V_s\rangle |H_i\rangle \langle V_s| \langle V_i| + |V_s\rangle |V_i\rangle \langle V_s| \langle H_i| + |V_s\rangle |V_i\rangle \langle V_s| \langle V_i|)$$

$$\Rightarrow = \frac{1}{4}\begin{pmatrix} 1 & -1 & i & i \\ -1 & 1 & -i & -i \\ -i & i & 1 & 1 \\ -i & i & 1 & 1 \end{pmatrix}.$$

This is our transformed biphoton for this case just before it hits the Rochon Polarizers, untransformed by a wave plate on the signal side of the apparatus. When we take the trace of the biphoton state's density matrix times the associated Pauli Matrix for this wave plate arrangement (in this case σ_z), we will get the a_z component of the Bloch Vector. Now for the next component a_y, we transform the biphoton density matrix using a half-wave plate.

Half-Wave Plate (Oriented at 22.5^o)

Doing the same steps as above get us the following state (abbreviated here):

$$\rho_{\text{hwp}} = (\mathbf{J}_{\frac{\lambda}{2}}(22.5^o) \otimes \mathbb{1})\cdot$$

$$(\frac{1}{2})(|H_s\rangle\, i\,|A_i\rangle \langle H_s|\,(-i)\,\langle A_i| + |H_s\rangle\, i\,|A_i\rangle \langle V_s| \langle D_i|$$

$$+ |V_s\rangle |D_i\rangle \langle H_s|\,(-i)\,\langle A_i| + |V_s\rangle |D_i\rangle \langle V_s| \langle D_i|)\cdot$$

$$(\mathbf{J}_{\frac{\lambda}{2}}(22.5^o)^\dagger \otimes \mathbb{1})$$

$$\Rightarrow \rho_{\text{hwp}} = (\frac{1}{2})(|D_s\rangle |A_i\rangle \langle D_s| \langle A_i| + i\,|D_s\rangle |A_i\rangle \langle A_s| \langle D_i| +$$

$$i\,|A_s\rangle |D_i\rangle \langle D_s| \langle A_i| + |A_s\rangle |D_i\rangle \langle A_s| \langle D_i|)$$

$$\Rightarrow \rho_{\text{hwp}} = \frac{1}{4}\begin{pmatrix} 1 & i & -i & -1 \\ -i & 1 & -1 & i \\ i & -1 & 1 & -i \\ -1 & -i & i & 1 \end{pmatrix}.$$

Quarter-Wave Plate (Oriented at 45^o)

$$\rho_{\text{qwp}} = (\mathbf{J}_{\frac{\lambda}{4}}(45^o) \otimes \mathbb{1})\cdot$$

$$(\frac{1}{2})(|H_s\rangle\,(-i)\,|A_i\rangle \langle H_s|\, i\,\langle A_i| + |H_s\rangle\, i\,|A_i\rangle \langle V_s| \langle D_i| +$$

$$|V_s\rangle |D_i\rangle \langle H_s|\,(-i)\,\langle A_i| + |V_s\rangle |D_i\rangle \langle V_s| \langle D_i|)\cdot$$

$$(\mathbf{J}_{\frac{\lambda}{4}}(45^o)^\dagger \otimes \mathbb{1})$$

The key for this being

$$\mathbf{J}_{\frac{\lambda}{4}}(45^o)\,|H\rangle = |R\rangle$$

$$\langle H| \mathbf{J}_{\frac{\lambda}{4}}(45^o)^\dagger = \langle R|$$

$$\mathbf{J}_{\frac{\lambda}{4}}(22.5^o) |V\rangle = i|L\rangle$$

$$\langle V| \mathbf{J}_{\frac{\lambda}{4}}(45^o)^\dagger = -i\langle L|$$

$$\begin{aligned}\rho_{\text{qwp}} =& (\frac{1}{2})(|R_s\rangle |A_i\rangle \langle R_s| \langle A_i| + i|R_s\rangle |A_i\rangle (-i) \langle L_s| \langle D_i| \\ &+ i|L_s\rangle |D_i\rangle \langle R_s| \langle A_i| + (i) |L_s\rangle |D_i\rangle (i) \langle L_s| \langle D_i|).\end{aligned}$$

We'd then rewrite the diagonally and antidiagonally polarized photons in terms of the right and left circular polarizations. Our resultant matrix would be:

$$\rho_{\text{qwp}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.7.3 Converting Coincidence Counts

Now we have these three matrices representing three different transformations of the signal state after we rotate the idler photon.

In order to finally determine the signal state, we send both the idler and the signal photons through Rochon Polarizers, which transmit horizontally polarized photons and redirect vertically polarized photons to separate detectors. Our single-photon counting modules receive those beams, telling us how many coincidence counts occur between them. In other words, the entangled photons will arrive at two separate detectors at the same time—hence their coincidence—and the proportion of those counts give us information about the signal state.

For example, in our apparatus the idler side has two detectors (A and A'), and the same goes for the signal side (B and B'). Detector A and B receive horizontally polarized light and A' and B' receive vertically polarized light. Let's say we sent the signal beam through the quarter-wave plate oriented at 45^o: if the proportion of coincidences between Detectors A and B is 100% of all coincidences, then we know the incoming state was $|H, H\rangle$. If we then work backwards to see what the quarter-wave plate on the signal side rotated the biphoton state from to achieve a pure $|H, H\rangle$ state, we know what our signal photon was before we rotated it to horizontal or vertical polarization—in other words, what state we had steered the signal photon to. In the case where we get 100% of the coincidence counts at A and B after transforming the signal state with a quarter-wave plate, we know that $\mathbf{J}_{\frac{\lambda}{4}}(45^o)\,|L\rangle = |H\rangle$, meaning our signal state had been steered to $|L\rangle$.

Knowing which setting of wave plates we used, we can conclude using the coincidence counts the value of each component of our 3D Bloch Vector, associated with the polarization axis that we rotated to using our signal wave plate setting. In the example above, we just showed a proportion of 100% (or 1) came up for our measurement using the quarter-wave plate on the signal side, meaning we projected it from the left-right circular polarization axis onto the horizontal-vertical polarization axis. Were we running the experiment, we would have achieved this proportion using the formula [3]:

$$\frac{N_{AB} - N_{AB'}}{N_{AB} + N_{AB'}}. \tag{2.7.1}$$

This allows for negative proportions as well, which means they lie on the "negative" side of the axes that make up the Bloch Sphere. If we had gotten a proportion of -1, that would tell us that we had steered our state to $|R\rangle$.

For the purpose of modeling, we can achieve this same coincidence proportion by taking the trace of the three density matrices after we operate on them with the Rochon Polarizers. Since the Rochon Polarizers function as both horizontal and vertical polarizers acting on the idler and signal part of the biphoton, we take the Kronecker Product of $(\mathbf{J}_H - \mathbf{J}_V)$ with itself, giving us

$$(\mathbf{J}_H - \mathbf{J}_V) \otimes (\mathbf{J}_H - \mathbf{J}_V) = (\mathbf{J}_H \otimes \mathbf{J}_H - \mathbf{J}_H \otimes \mathbf{J}_V - \mathbf{J}_V \otimes \mathbf{J}_H + \mathbf{J}_V \otimes \mathbf{J}_V), \tag{2.7.2}$$

which is the 4x4 matrix equal to the Kronecker Product of the Pauli Matrix σ_z with itself:

$$\begin{aligned} \hat{R} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Rightarrow \hat{R} &= \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \\ \Rightarrow \hat{R} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Meaning for our third Bell State biphoton case, the values of our Bloch Vector would be

$$a_x = \mathrm{Tr}[\rho_{\mathrm{nwp}} \cdot \hat{R}]$$
$$\Rightarrow= a_x\ \mathrm{Tr}[\frac{1}{4}\begin{pmatrix} 1 & -1 & i & i \\ -1 & 1 & -i & -i \\ -i & i & 1 & 1 \\ -i & i & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}]$$
$$\Rightarrow a_x = 0,$$

$$a_y = \mathrm{Tr}[\rho_{\mathrm{qwp}} \cdot \hat{R}]$$
$$\Rightarrow a_y = \mathrm{Tr}[\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}]$$
$$\Rightarrow a_y = -1,$$

$$a_z = \mathrm{Tr}[\rho_{nwp} \cdot \hat{R}]$$
$$\Rightarrow a_z = \ Tr[\frac{1}{4}\begin{pmatrix} 1 & -1 & i & i \\ -1 & 1 & -i & -i \\ -i & i & 1 & 1 \\ -i & i & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}]$$
$$\Rightarrow a_z = 0.$$

Thus our Bloch Vector for the signal state is $\vec{a}_{\psi^+} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ or simply the right circularly polarized state $|R\rangle$.

We can do this for any combination of angles rotating the idler state. Here are the answers to a few more ϕ and χ angle combinations inserted into the equation

$$(\mathbf{J}_{\frac{\lambda}{2,4}}(\theta) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{2}}(\phi))(\mathbb{1} \otimes \mathbf{J}_{\frac{\lambda}{4}}(\chi))\,|\psi^+\rangle\,,$$

$\phi = 10^o$ and $\chi = 60^o$:

$$\vec{a}_{\psi^+} = \begin{pmatrix} .150384 \\ .984808 \\ .0868241 \end{pmatrix}$$

$\phi = 150^o$ and $\chi = 35^o$:

$$\vec{a}_{\psi^+} = \begin{pmatrix} -.925417 \\ -.173648 \\ .336824 \end{pmatrix}$$

$\phi = 292^o$ and $\chi = 108^o$:

$$\vec{a}_{\psi^+} = \begin{pmatrix} 0.361877 \\ 0.788011 \\ -0.498081 \end{pmatrix}$$

It's not obvious at first glance, but all of these Bloch Vectors have a length of 1, meaning that we have only been steering our third Bell State biphoton to pure states. As it turns out, we can only steer this biphoton to any pure state on the Bloch Sphere. Having plotted a number of the angled pairs, we can see how the possible Bloch Vectors for this case outline the Bloch Sphere in Figure 2.7.2.

Now that we've shown this for the pure state biphoton case, we can use the same process both computationally for and experimental measurement of mixed state biphotons. In the case of mixed state biphotons, the states that the signal photon can be steered to would produce an interesting shape: a 3D ellipse, otherwise known as an ellipsoid [9]. The experiment we model will then produce mixed state biphotons that we then steer to different ellipsoids.

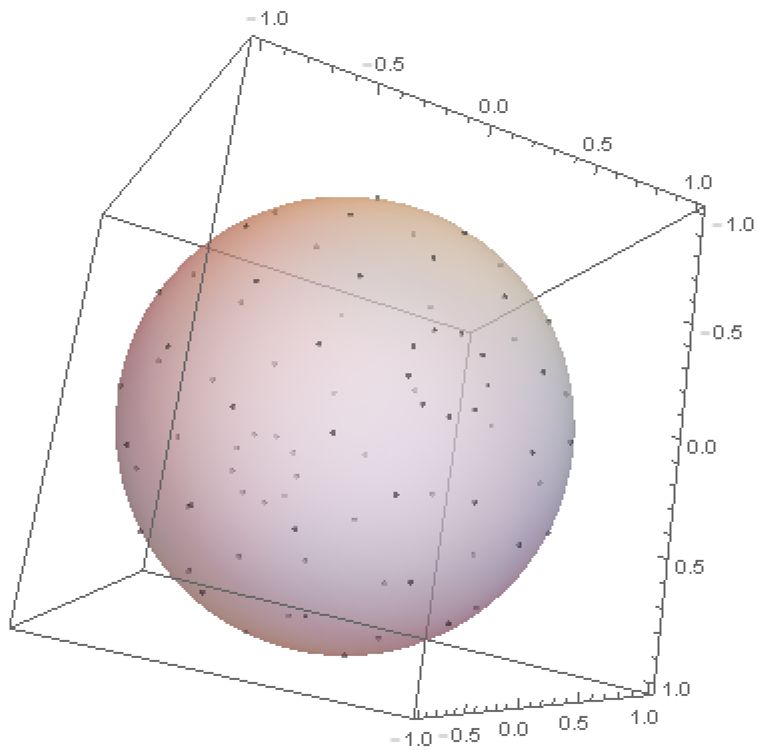

Figure 2.7.2: Signal Steering from the Bell State $|\psi^+\rangle$. These are the results for a list of different angle pairs ((0^o, 0^o), (10^o, 10^o), and so on) that we oriented our quarter-wave plate and half-wave plate to on the idler side of the apparatus. We then calculated the Bloch Vectors through the equations above and plotted as points on the graph, specifically for the state of the biphoton $|\psi^+\rangle$, which are all pure states on the surface of the Bloch Sphere as seen above. [See electronic version for figure in color.]

3

Experimental Model

3.1 Apparatus

With all this information in our tool belt, let's look at a model of running this experiment in full. The experimental apparatus is depicted in the Figure 3.1.1 below, and builds off the correlated-photon experiments run by Mark Beck [1] and Enrique Galvez [5]. In the model, a 40-mW Gallium-Nitride 405-nm blue laser diode sends a beam into beta-barium borate (BBO) down-conversion crystals. Down converted photons emerge at angles of 3^o with respect to the pump beam. Each down-converted photon, due to conservation of momentum, has a wavelength of approximately 810nm.

We use two types of down-conversion crystals in this apparatus: Type I and Type II. Type I down conversion produces entangled photons that are in the same polarization state, while Type II crystals produce entangled photons that are in perpendicular polarization states [1]. What states they are produced in depend on how we prepare the pump beam (using the half-wave plate, the birefringent crystal, or polarizer) as well as the angle of the crystal itself.

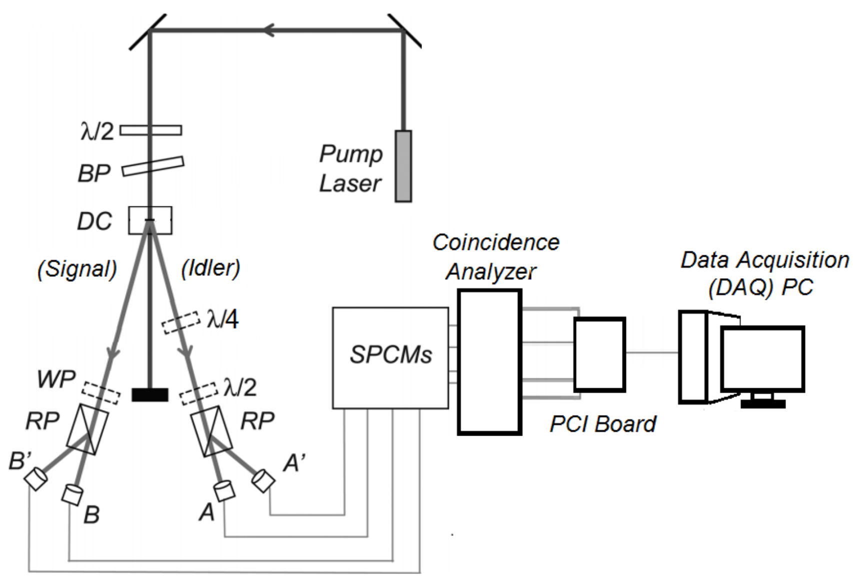

Figure 3.1.1: The laser (called the pump for historical reasons) passes through a half-wave plate ($\lambda/2$) and then through a birefringent plate (BP) before hitting the down-conversion crystal (DC). The idler beam passes through two wave plates, first the quarter-wave plate and then the half-wave plate. The signal beam passes through one of the three wave plate settings (WP). Both the signal and the idler then hit Rochon Polarizers (RP), which transmit horizontally-polarized photons and displace vertically-polarized photons. Each beam then hits a detector, which we use to count coincidences.

After passing through the wave plates, whose orientation we vary by rotating them, and then through the Rochon Polarizers the photon hits the single-photon counting modules (SPCMs) that then send a 5V pulse to a coincidence analyzer, picking up when two signals strike at exactly the same time (or rather, within a 30 ns range). The pulses from the coincidence analyzer then go through a PCI board that transmits them to the computer programmed to acquire the data.

3.2 Computational Results

Though I was unable to run the experiment myself, with some computer coding of the physical operations discussed in the Theory section, I was able to accurately model the outcome of running these experiments for various mixed state biphotons. The code I wrote that generated these models is included in the Appendix. A few examples of Quantum Steering Ellipsoids below are ones that could be created in a laboratory setting, where mixed states are prepared in the way that Galvez mimicks a mixed state in one of his experiments [6], which is by putting a polarizer, which only allows photons in a particular polarization state through, before the down conversion crystal, and then using the half-wave plate or a quarter-wave plate to rotate that state. For example, if we place a polarizer and then a properly rotated half-wave plate in front of the pump beam, then from Type I down conversion we get $|D_s, D_i\rangle$ entangled photons, and from Type II down conversion we could get $|D_s, A_i\rangle$ entangled states.

If we ran an experiment in which we prepared biphotons in the state $|H_s, H_i\rangle$ for half the time and biphotons in the state $|V_s, V_i\rangle$ for half the time, and then added the two data sets together, we would be mixing the state manually to create the state

$$\rho_1 = \frac{1}{2} |H_s, H_i\rangle \langle H_s, H_i| + \frac{1}{2} |V_s, V_i\rangle \langle V_s, V_i| \,.$$

When we put that mixed state into the computation model, we get a graph of all the points inside the Bloch Sphere representing all the states the signal photon can be steered to, which is an ellipsoid of sorts (see Figure 3.2.1).

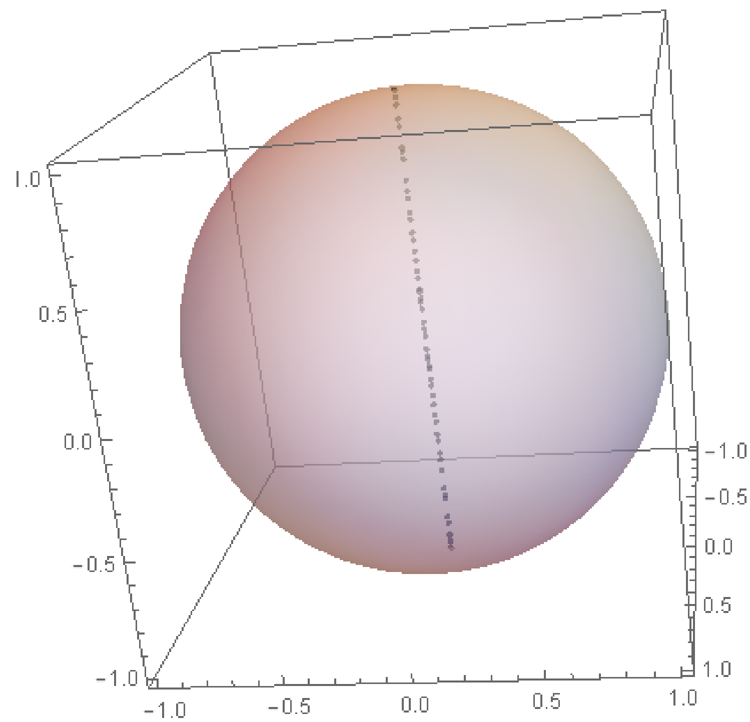

Figure 3.2.1: Steering Ellipsoid for the $\rho_1 = \frac{1}{2} |H_s, H_i\rangle \langle H_s, H_i| + \frac{1}{2} |V_s, V_i\rangle \langle V_s, V_i|$ biphoton. The signal state is only collapsed to other mixtures of $|H\rangle$ and $|V\rangle$ states, forming an "ellipsoid" that is a straight line along the axis formed by the horizontal-vertical polarization states. [See electronic version for figure in color]

The signal photon states that we can steer an evenly mixed biphoton of $|H_s, H_i\rangle$ and $|V_s, V_i\rangle$ states to are mixtures of $|H\rangle$ and $|V\rangle$ states. This makes sense if we consider that our mixed state was prepared such that the idler and signal photon were correlated on one another both being horizontally or vertically polarized. Regardless of how we rotated and then measured the idler photon by sending it through wave plates, we could never steer the signal photon to anything other than mixtures of its original entangled polarization states.

If we wanted to steer our signal photon to a set of states that looked more like an ellipsoid than a straight line, then we would need to create a mixed state including polarization states that are not just horizontal and vertical. In that case, say we used a a combination of polarizers and wave plates in the pump beam that were sent through a Type II down conversion crystal to produce the state $|H, V\rangle$. Then say we changed the polarizers and wave plates and a Type I down conversion crystal to produce the state $|R, R\rangle$, as well as $|D, D\rangle$. We then measured those states for periods of time that, when we added their data together, produced the state:

$$\rho_2 = \frac{1}{4} |H, V\rangle \langle H, V| + \frac{1}{2} |R, R\rangle \langle R, R| + \frac{1}{4} |D, D\rangle \langle D, D|,$$

producing the density matrix

$$\begin{pmatrix} \frac{3}{16} & \frac{1}{16} + \frac{i}{8} & \frac{1}{16} + \frac{i}{8} & -\frac{1}{16} \\ \frac{1}{16} - \frac{i}{8} & \frac{7}{16} & \frac{3}{16} & \frac{1}{16} + \frac{i}{8} \\ \frac{1}{16} - \frac{i}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} + \frac{i}{8} \\ -\frac{1}{16} & \frac{1}{16} - \frac{i}{8} & \frac{1}{16} - \frac{i}{8} & \frac{3}{16} \end{pmatrix}$$

This would give us a Steering Ellipsoid in which the signal state is now collapsed to a number mixtures of $|H\rangle$, $|V\rangle$, $|D\rangle$, $|A\rangle$,$|L\rangle$, and $|R\rangle$ states. Had we steered the signal using all possible angles for our two wave plates on the idler side of the apparatus, the ellipsoid would have a closed surface area, representing all possible mixed states of the steered signal photon.

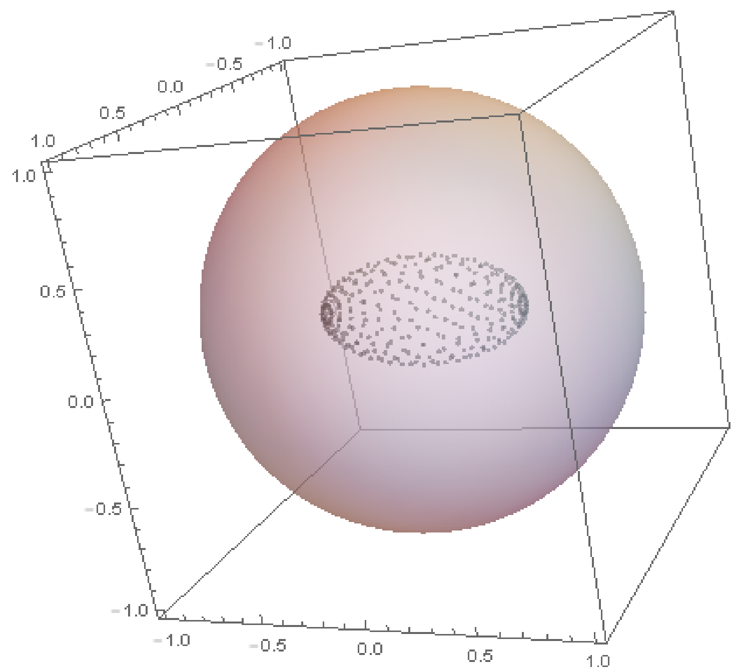

Figure 3.2.2: Steering Ellipsoid for the $\rho_2 = \frac{1}{4}|H,V\rangle\langle H,V| + \frac{1}{2}|R,R\rangle\langle R,R| + \frac{1}{4}|D,D\rangle\langle D,D|$ biphoton. The signal state is now collapsed to a number mixtures of $|H\rangle$, $|V\rangle$, $|D\rangle$, $|A\rangle$,$|L\rangle$, and $|R\rangle$ states. Had we collapsed all possible angles for our two wave plates, the ellipsoid would have a closed surface area, representing all possible mixed states of the steered signal photon. [See electronic version for figure in color.]

Beautiful: a Steering Ellipsoid we can create right here in the lab utilizing the method described above. It is important to note that we could, using the method described above in which we add data sets of pure states entanglements together, create mixed states that could not physically exist. To ensure this did not happen, I used an equation that converts the density matrices of biphotons (or any two-qubit systems) into a block matrix that includes the Bloch Vectors of the idler (represented below by $\vec{\mathbf{b}}$, which in the matrix is turned into a row vector by taking its tranpose) and signal photon (represented below by $\vec{\mathbf{a}}$), were they disentangled from one another and then measured as separate states [9]:

$$\Theta = \begin{pmatrix} 1 & \vec{\mathbf{b}}^T \\ \vec{\mathbf{a}} & T \end{pmatrix}.$$

The T in this block matrix is its own 3x3 matrix that contains information about the correlation between the two photons. We solve for this block matrix in the same way we solved for the elements of the Bloch Vector: by taking the trace of the original density matrix times the Pauli Matrices—though this time, it is all possible Kronecker products between each the Pauli Matrices, creating 16 traces that represent each element in the block matrix.

Through this equation we know that if the Bloch Vectors had a length greater than 1, then we know that this density matrix cannot be accurate, and we mimicked a state that does not exist in reality: the beam of photons would have probabilities of being in a particular pure state that were greater than 1.

We can, however, mimick mixed states that are weighted such that the ellipsoid takes on very different shapes. For example, I modeled the preparation of the mixed state:

$$\rho_3 = \frac{1}{14} |H, V\rangle \langle H, V| + \frac{6}{7} |R, L\rangle \langle R, L| + \frac{1}{14} |D, A\rangle \langle D, A|,$$

which because the values were so heavily weighted towards the entangled $|R, L\rangle$ state, stretched the Steering Ellipsoid along the axis formed by the right/left circularly polarized states.

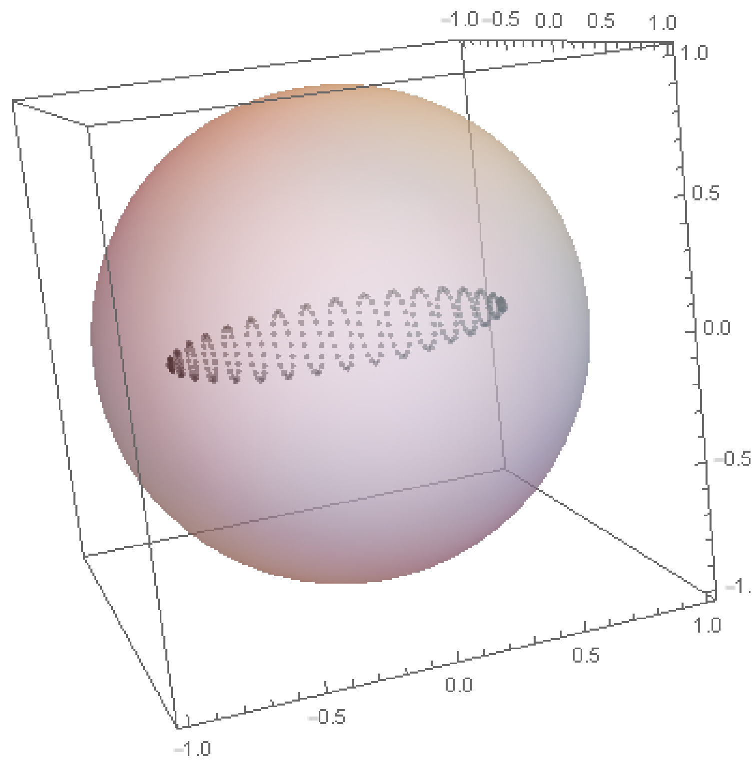

Figure 3.2.3: Steering Ellipsoid for the $\rho_3 = \frac{1}{14}|H, V\rangle\langle H, V| + \frac{6}{7}|R, L\rangle\langle R, L| + \frac{1}{14}|D, A\rangle\langle D, A|$ biphoton. [See electronic version for figure in color.]

The method we used above only allows for Quantum Steering Ellipsoids that are centered in the Bloch Sphere, as these biphotons carry only one type of correlation. There are possible mixed states that would create Steering Ellipsoids off the center, however, which relates to other types of correlation between two particles. Readers interested in more about these other types of Steering Ellipsoids should see the paper that proved the Quantum Steering Ellipsoid theoretically [9].

4

Conclusion

To date I know of no experimental test of the claim to the concept of the Quantum Steering Ellipsoid since its theoretical proof less than a year ago [9]. I've shown examples of the Steering Ellipsoid using models based on real quantum mixed states, as welll as demonstrated the steps to experimentally verify their existence. This experiment can and should be run, because the Quantum Steering Ellipsoid has potential to make a concept that initially sounds complicated (a two-qubit mixed state) and yield clear and intuitive understandings about the concepts of entanglement and steering. The Steering Ellipsoid could also have uses for visualizing the results of two-qubit state tomography, which is the reconstruction of a system of source states based on measurements taken on parts of that system—not too dissimilar from the experimental procedure we described here. It ought to be the goal of physics to continue searching for these intuitive and salient representations of quantum mechanical phenomena, not only so further discoveries can be made in the field, but so more people can access the strange realities of our physical world. I hope this roadmap for experimentally verifying the Quantum Steering Ellipsoid serves of use to those pursuing that end.

5

Appendix

The code below was written in Wolfram Mathematica. It contains abbreviated functions that allow for variable inputs; all I had to do was insert some new [$\rho mixed$] and then run all of the functions to return a Bloch Sphere with the ellipsoid mapped inside of it. I left out the returned values for the lists below, but they can easily be recalculated by reinserting the code into Mathematica.

```
HWP[θ_] := {{Cos[2 * θ], Sin[2 * θ]}, {Sin[2 * θ], -Cos[2 * θ]}}

QWP[θ_] := {{Cos[θ] ^ 2 + i * Sin[θ] ^ 2,  (1 - i) * Sin[θ] * Cos[θ]},
  {(1 - i) * Sin[θ] * Cos[θ], Sin[θ] ^ 2 + i * Cos[θ] ^ 2}}

QWP45 := (1 / Sqrt[2]) * {{1, -i}, {-i, 1}}

HP = {{1, 0}, {0, 0}}

{{1, 0}, {0, 0}}

VP = {{0, 0}, {0, 1}}

{{0, 0}, {0, 1}}

IdentityMatrix[2]

{{1, 0}, {0, 1}}

ρBHV[{HWPD_, QWPD_}] :=
 Tr[
  (KroneckerProduct[IdentityMatrix[2], HWP[HWPD Degree]].
     KroneckerProduct[IdentityMatrix[2], QWP[QWPD Degree]].
     ρmixed.ConjugateTranspose[
      KroneckerProduct[IdentityMatrix[2], QWP[QWPD Degree]]].
     ConjugateTranspose[KroneckerProduct[IdentityMatrix[2],
       HWP[HWPD Degree]]]).
   (KroneckerProduct[HP, HP] - KroneckerProduct[VP, HP] +
     KroneckerProduct[VP, VP] - KroneckerProduct[HP, VP])]
```

```
ρBLR[{HWPD_, QWPD_}] :=
 Tr[
  (KroneckerProduct[IdentityMatrix[2], HWP[HWPD Degree]].
     KroneckerProduct[IdentityMatrix[2], QWP[QWPD Degree]].
     KroneckerProduct[QWP45, IdentityMatrix[2]].ρmixed.
     ConjugateTranspose[KroneckerProduct[QWP45, IdentityMatrix[2]]].
     ConjugateTranspose[KroneckerProduct[IdentityMatrix[2],
       QWP[QWPD Degree]]].
     ConjugateTranspose[KroneckerProduct[IdentityMatrix[2],
       HWP[HWPD Degree]]]).
   (KroneckerProduct[HP, HP] - KroneckerProduct[VP, HP] +
     KroneckerProduct[VP, VP] - KroneckerProduct[HP, VP])]

ρBDA[{HWPD_, QWPD_}] :=
 Tr[
  (KroneckerProduct[IdentityMatrix[2], HWP[HWPD Degree]].
     KroneckerProduct[IdentityMatrix[2], QWP[QWPD Degree]].
     KroneckerProduct[HWP[22.5 Degree], IdentityMatrix[2]].
     ρmixed.ConjugateTranspose[
      KroneckerProduct[HWP[22.5 Degree], IdentityMatrix[2]]].
     ConjugateTranspose[KroneckerProduct[IdentityMatrix[2],
       QWP[QWPD Degree]]].
     ConjugateTranspose[KroneckerProduct[IdentityMatrix[2],
       HWP[HWPD Degree]]]).
   (KroneckerProduct[HP, HP] - KroneckerProduct[VP, HP] +
     KroneckerProduct[VP, VP] - KroneckerProduct[HP, VP])]

Listaz = Map[ρBHV, AnglePairs] // N // Chop

Listay = Map[ρBLR, AnglePairs] // N // Chop

Listax = Map[ρBDA,  AnglePairs] // N // Chop
```

```
BlochSphere = Transpose[{Listc1, Listc2, Listc3}]

Graphics3D[{Opacity[.5], Sphere[{0, 0, 0}, 1], Point[BlochSphere]},
 Axes → True]

AnglePairs = Outer[List, Range[0, 360, 5], Range[0, 360, 5]] //
  Flatten[#, 1] &
```

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