Basis Criteria for n-cycle Integer Splines

Ester Gjoni
Bard College, eg3085@bard.edu

Basis Criteria for *n-cycle* Integer Splines

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Ester Gjoni

Annandale-on-Hudson, New York
May, 2015

Abstract

In this project we work with integer splines on graphs with positive integer edge labels. We focus on graphs that are n-cycles for some natural number n. We find an explicit condition for when a set of splines can form a module basis for *n-cycle* splines. In general, a set of splines forms a $\mathbb{Z}$-module basis if and only if their determinant is equal to the product of the edge labels divided by the greatest common divisor of those edge labels.

Contents

List of Figures

Dedication

To my father. Words can't describe how influential you have been in my life. To my mother- for her strength and devotion to our family. To my sister - for all the times she has made me smile and laugh.

Acknowledgments

My life would have been on a different and misguided path if I did not attend BHSEC Queens. Hence, I would like to thank all the faculty members who played a major role in nurturing and inspiring my love for knowledge. A special thanks goes to Arup Mukherjee, my 9^{th} grade advisor, professor, and friend. In honor of all the students who were in our 9^{th} grade advisory, we thank you for accidentally writing *forced* free writes instead of *focused* free writes. In a way, you subconsciously knew how some of us felt about coming back to school. Nonetheless, that was the moment we all bonded as an advisory and have all remained friends to this day. So thanks Arup, for your kind words and your dedication to making sure we all succeeded, even after high school.

I would like to thank my advisor, Lauren Rose. Without her guidance and persistent feedback, this project would have been inconclusive and a disaster. I would like to thank her for pushing me out of my comfort zone and constantly reassuring me that we were on the right path. Most importantly, I thank her for encouraging me to find my own voice throughout my senior project. It was truly a pleasure to work with you side by side.

1

Introduction

Splines appear in many branches of mathematics and have many applications in several fields. Originally, the word spline referred to a thin strand of wooden beam or thin metal used in the construction of ships and aircraft. A metal weight was placed at specific control points to keep the spline steady. Once the spline was secured in place, it was then shaped according to the desired form.

Mathematically, a spline is a special type of curve, formed by a collection of piecewise polynomial functions joined together at key points to achieve a certain degree of smoothness. In particular, we say F is a spline over an internal $I = (-\infty, a] \cup [a, \infty)$, if $f_1(a) = f_2(a)$. We denote a spline as $F = (f_1, f_2)$.

Example 1.0.1. Consider the following piecewise function:

$$F(x) = \begin{cases} f_1 = x^2 : & x \leq 0 \\ f_2 = x : & x \geq 0 \end{cases} .$$

Clearly, $F = (x^2, x)$ is a spline over $I = (-\infty, 0] \cup [0, \infty)$ since, $f_1(0) = f_2(0)$. Note, Figures 1.0.1 and 1.0.2 depict two ways we can visually represent F. ◊

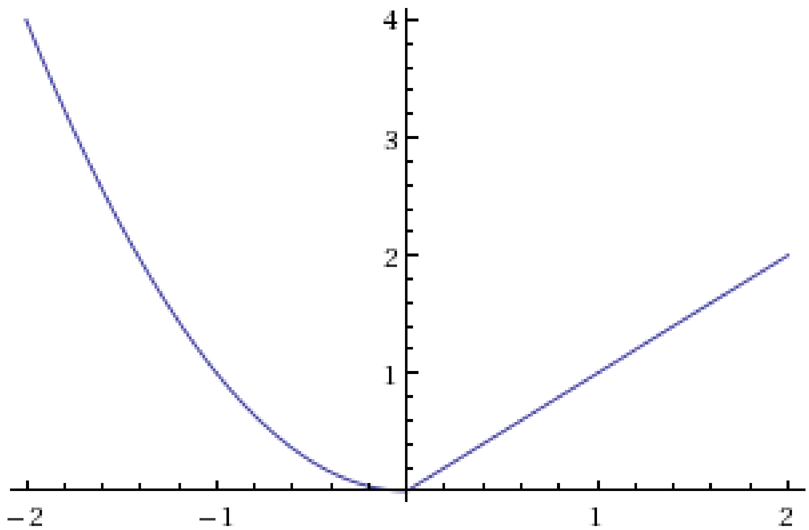

Figure 1.0.1. The polynomial spline $F = (x^2, x)$.

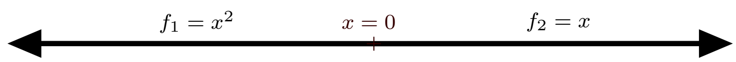

Figure 1.0.2. A second way to represent a polynomial spline.

In addition, we are interested in the degree of smoothness of a spline. In general, the degree of smoothness is equal to the number of continuous derivatives of a spline. These splines are referred to as C^r *splines*.

Definition 1.0.2. [3, Definition 3.0.15] A C^r *spline* over I is a spline with r continuous derivatives. △

The following theorem tells us when a spline is C^r.

Theorem 1.0.3. [3, Theorem 3.0.18] *Let $F = (f_1, f_2)$ be a spline defined over the interval $I = I_1 \cup I_2 = (-\infty, a] \cup [a, \infty)$. Then,*

$$F \text{ is } C^r \iff f_1 \equiv f_2 \mod k^{r+1},$$

where $k(x) = x - a$ is the linear polynomial defining the boundary $x - a = 0$ between I_1 and I_2.

In our example, $f_1 = x^2, f_2 = x$ and $k^{r+1} = (x-0)^{r+1}$. Theorem 1.0.3 tells us that F is C^0 since $x^2 \equiv x \mod (x-0)^{0+1}$, but F is not C^1 since $x^2 \not\equiv x \mod (x-0)^{1+1}$.

Aside from Figures 1.0.1 and 1.0.2, we can represent a C^r spline as a graph, where f_1 and f_2 are the vertices and $(x-a)^{r+1}$ is the edge. This is shown in Figure 1.0.3.

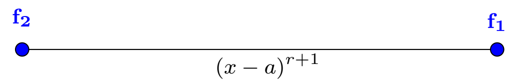

Figure 1.0.3. A graph representation of a C^r spline.

Furthermore, we can have the following C^r spline, $F = (f_1, f_2, f_3)$, on an interval I. Similarly, we illustrate this spline as a graph in Figure 1.0.4 (a).

As it turns out, we can *generalize* the theory of polynomial splines to what are called integer splines. Hence, instead of working with polynomials over $\mathbb{R}$, we work over the ring of integers. This transition is shown in Figure 1.0.4 (b), where we label the vertices with integers, whereas before, they were labeled with polynomials. Similarly, we label the edges with natural numbers. If the following is satisfied:

$$f_1 \equiv f_2 \mod a_1,\ f_2 \equiv f_3 \mod a_2,\ \text{and}\ f_3 \equiv f_1 \mod a_3,$$

then $F = (f_1, f_2, f_3)$ is called a *generalized integer spline*.

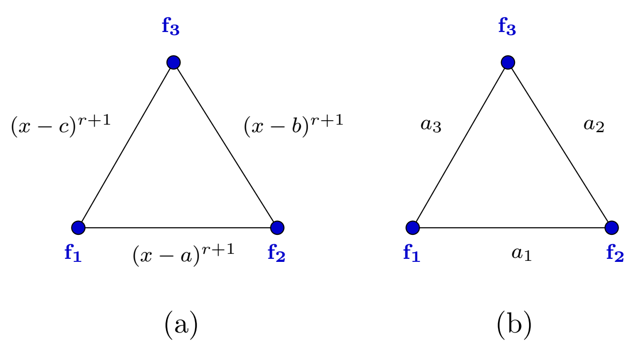

Figure 1.0.4. (a) illustrates a *polynomial spline* and (b) illustrates a *generalized integer spline*.

In this project, we focus on generalized integer splines called *n-cycle splines*. These are splines on an *n-cycle* graph and are illustrated in Figure 1.0.5.

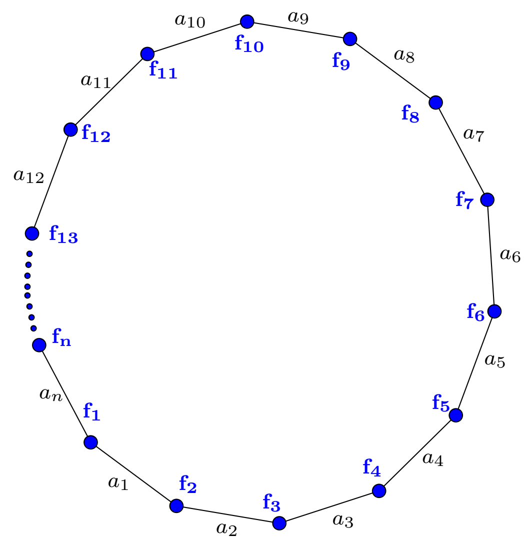

Figure 1.0.5. $F = (f_1, \ldots, f_n)$ is an *n-cycle* spline.

Our work builds on the study done by Handschy et al. [1]. They construct a special type of *n-cycle* splines called flow-up classes. They identify the smallest flow-up class as having the smallest elements in a flow-up class and show that the smallest flow-up classes form a module basis for *n-cycle* splines over the integers.

While Handschy et al. show that a group of splines form a basis for *n-cycle* splines, our main finding shows that there is a criterion for when a set of splines form a $\mathbb{Z}$-module basis.

We list a brief outline of our chapters:

In Chapter 2, we introduce the reader to some basic number theory.

In Chapter 3, we give a conventional definition for generalized integer splines as well as *n-cycle* splines. Then, we introduce flow-up classes and smallest flow-up classes as defined by Handschy et al.

In Chapter 4, we present the majority of our findings. We focus on *3-cycle* splines and show that the determinant of the set of the smallest flow-up classes is equal to the product of the edge labels divided by the greatest common divisor of those edges. Then, in Section 4.3 we present our main result and show that a set of splines form a module basis for *3-cycle* splines if and only if their determinant is equal to plus or minus the product of the edge labels divided by the greatest common divisor of those edges.

In Chapter 5, we generalize our findings from Section 4.3 for *n-cycle* splines.

2
Preliminaries

The purpose of this chapter is to introduce the reader to some basic number theory. We define basic terminology as well as present proofs that will be useful in the forthcoming chapters.

2.1 Elementary Number Theory

Definition 2.1.1. [2, Chapter 1, Section 1.5] Let $a, b \in \mathbb{Z}$. We say b *divides* a, if there exists an $x \in \mathbb{Z}$ such that $a = bx$. △

Notation: We denote b divides a by $b|a$ and we call b the *divisor/factor* of a.

Note that $b|a$ does not say the same thing as $\frac{b}{a}$. The former is a statement, while the latter is an expression. When we say b divides a, we are implicitly stating that the remainder is zero. When we say b over a, we are not assuming that the remainder is zero.

Definition 2.1.2. [2, Chapter 3, Section 1.1] Let a and b be integers and m be a positive integer. We say *a is congruent to b modulo m*, denoted $a \equiv b \mod m$, if $m|a - b$. △

It follows from Definition 2.1.2 that two integers are congruent each other modulo m if there exists an $x \in \mathbb{Z}$ such that $mx = a - b$ or $a = b + mx$.

The following are some important properties of congruences.

Theorem 2.1.3. [2, Theorem 4.2] *Let m be a positive integer. Congruences modulo m satisfy the following properties:*

$$(i)\ \ \textit{Reflexive Property}: \ \textit{If } a \textit{ is an integer, then } a \equiv a \mod m.$$

$$(ii)\ \textit{Symmetric Property}: \textit{If } a \textit{ and } b \textit{ are integers such that } a \equiv b \mod m,$$
$$\textit{then } b \equiv a \mod m.$$

$$(iii)\ \ \textit{Transitivity Property}: \textit{If } a,\ b,\ \textit{and } c \textit{ are integers with } a \equiv b \mod m$$
$$\textit{and } b \equiv c \mod m,\ \ \textit{then } a \equiv c \mod m.$$

We now define the *greatest common divisor* of two integers.

Definition 2.1.4. [2, Chapter 2, Section 2.1] The *greatest common divisor* (*gcd*) of two integers a and b, where both integers are not zero, is the largest integer that divides both a and b. △

Notation: We denote the *gcd* of two integers as (a, b).

We can also define the *gcd* of more than two integers.

Definition 2.1.5. [2, Chapter 3, Section 3.2] Let $a_1, a_2, \ldots, a_n$ be integers not all 0. The *gcd* of these integers is the largest integer that is a divisor of all the integers in the set. The *gcd* of $a_1, a_2, \ldots, a_n$ is denoted by $(a_1, a_2, \ldots, a_n)$. △

Example 2.1.6. We easily see that $(2, 10, 12) = 2$ and $(4, 8, 12) = 4$. ♢

Next, we state some useful properties of the *gcd*.

Theorem 2.1.7. [2, Theorem 3.8] *The greatest common divisor of integers a and b, not both 0, is the least positive integer that is a linear combination of a and b i.e. $(a, b) = ax+by$ for some $x, y \in \mathbb{Z}$.*

It is an easy exercise to see that Theorem 2.1.7 can be generalized for more that two integers. We omit the proof.

Theorem 2.1.8. *Let a, b, and g be positive integers. If $d = (a, b)$, $g|a$ and $g|b$, then $g|d$.*

Proof. From Theorem 2.1.7, we know that for some $x, y \in \mathbb{Z}$, $ax+by = d$. From Definition 2.1.1, we know that there exists some $s, t \in \mathbb{Z}$ such that $gs = a$ and $gt = b$. Substituting this in the first equation we see that, $d = ax + by = (gs)x + (gt)y = g(sx + ty)$. Hence, $g|d$. □

We can extend Theorem 2.1.8 for more than two integers. The proof is omitted but follows the same method as above.

Theorem 2.1.9. [2, Theorem 3.6] *If $d = (a, b)$, then $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.*

Now, we define the *least common multiple* of two integers.

Definition 2.1.10. [2, Chapter 2, Section 2.3] The *least common multiple* (*lcm*) of two integers a and b is the smallest positive integer that is divided by both a and b. △

Notation: We denote the *lcm* of two integers as $[a, b]$.

Example 2.1.11. We have the following least common multiples: $[5, 10] = 10, [12, 20] = 60$, and $[2, 13] = 26$. ♢

We can also define the *lcm* of more than two integers.

Definition 2.1.12. [2, Chapter 3, Section 3.4] The *lcm* of the integers $a_1, a_2, \ldots, a_n$ which are not all zero, is the smallest positive integer that is divisible by all the integers $a_1, a_2, \ldots, a_n$; it is denoted by $[a_1, a_2, \ldots, a_n]$. △

In general, we can write every natural number uniquely as a product of one or more primes. This is known as *The Fundamental Theorem of Arithmetic.*

Theorem 2.1.13. [2, Theorem 2.3] *Every positive integer greater than 1 can be written uniquely as a product of primes i.e.* $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$ *for some distinct primes* $p_1, p_2, \dots, p_m$ *and* $a_1, a_2, \dots, a_m \geq 0$.

The reader should keep Theorem 2.1.13 in mind for it is used in some of our proofs. The next example summarizes what we have discussed so far.

Example 2.1.14. We compute $[234, 552]$ and $(234, 552)$. First, we find the prime factorization of each integer: $234 = 2 \cdot 3^2 \cdot 13$ and $552 = 2^3 \cdot 3 \cdot 23$. By definition of *gcd* and *lcm*, $(234, 552) = 2 \cdot 3$ and $[234, 552] = 2^3 \cdot 3^2 \cdot 13 \cdot 23$. Note that $(234, 552) \cdot [234, 552] = 2^4 \cdot 3^3 \cdot 13 \cdot 23 = 234 \cdot 552$. ♢

Example 2.1.14 illustrates an interesting relationship between the *gcd* and *lcm*. As it turns out, the product of two positive integers is equal to the product of their *gcd* and *lcm*.

Theorem 2.1.15. [2, Theorem 2.8] *If* a *and* b *are positive integers, then* $(a, b)[a, b] = ab$.

We now expand Theorem 2.1.15 for three positive integers. This fact will be useful in Chapter 4. Before we continue, we provide an example.

Example 2.1.16. Let's compute $(6, 10, 12)$ and $[6, 10, 12]$. We see that $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, and $12 = 2^2 \cdot 3$. By definition, $(6, 10, 12) = 2$ and $[6, 10, 12] = 2^2 \cdot 3 \cdot 5 = 60$. However, note that $(6, 10, 12) \cdot [6, 10, 12] = 2^3 \cdot 3 \cdot 5 \neq 2^4 \cdot 3^2 \cdot 5 = 6 \cdot 10 \cdot 12$. ♢

While we would expect the relationship for three positive integers to be similar to Theorem 2.1.15, as Example 2.1.16 illustrates, the same method does not hold for three integers. Example 2.1.16 motivates the following theorem.

Theorem 2.1.17. *If* $a, b,$ *and* c *are positive integers, then* $(a, b, c)[ab, bc, ca] = abc$.

Before we prove Theorem 2.1.17, we will need some lemmas.

Lemma 2.1.18. *If $a, b,$ and c are positive integers, then*

$$\max(a,b,c) = a+b+c-\min(a,b)-\min(b,c)-\min(a,c)+\min(a,b,c).$$

Proof. Without loss of generality, let $c \leq b \leq a$. Then, $\max(a,b,c) = a+b+c-b-c-c+c = a+b+c-\min(a,b)-\min(b,c)-\min(a,c)+\min(a,b,c)$. □

Lemma 2.1.19. *If $a, b,$ and c are positive integers, then* $\max(a+b, b+c, c+a) = a+b+c-\min(a,b,c)$.

Proof. Let $A = \max(a+b, b+c, c+a)$ and $B = a+b+c-\min(a,b,c)$. From Lemma 2.1.18, we know $A = (a+b)+(b+c)+(c+a)-\min(a+b,b+c)-\min(b+c,c+a)-\min(a+b,c+a)+\min(a+b,b+c,c+a)$. Without loss of generality, let $c \leq b \leq a$. Then, $b \leq a$ implies $b+c \leq a+c$ and $c \leq b$ implies $a+c \leq a+b$. Hence, $b+c \leq a+c \leq a+b$. Now,

$$\begin{aligned}
A &= (a+b)+(b+c)+(c+a)-\min(a+b,b+c)-\min(b+c,c+a) \\
&\quad -\min(a+b,c+a)+\min(a+b,b+c,c+a) \\
&= (a+b)+\cancel{(b+c)}+\cancel{(c+a)}+\cancel{(-b-c)}+\cancel{(-b-c)}+\cancel{(-c-a)}+\cancel{(b+c)} \\
&= (a+b+c)-c \\
&= (a+b+c)-\min(a,b,c) \\
&= B.
\end{aligned}$$

□

Proof of Theorem 2.1.17. By the *Fundamental Theorem of Arithmetic*, we can express $a, b,$ and c uniquely as a product of primes. Hence, $a = \prod_{i=1}^{n} p_i^{a_i}$, $b = \prod_{i=1}^{n} p_i^{b_i}$, and $c = \prod_{i=1}^{n} p_i^{c_i}$, where $a_i, b_i, c_i \geq 0$. For simplicity, let $x_i = a_i + b_i, y_i = b_i + c_i$, and $z_i = a_i + c_i$, where $1 \leq i \leq n$. By definition of lcm, we know $[ab, bc, ca] = \left[\prod_{i=1}^{n} p_i^{x_i}, \prod_{i=1}^{n} p_i^{y_i}, \prod_{i=1}^{n} p_i^{z_i}\right] = \prod_{i=1}^{n} p_i^{\max(x_i,y_i,z_i)}$. From Lemma 2.1.19, we know that $\max(x_i, y_i, z_i) = a_i + b_i + c_i - \min(a_i, b_i, c_i)$. In other words,

$$[ab, bc, ca] = \prod_{i=1}^{n} p_i^{\max(x_i,y_i,z_i)} = \prod_{i=1}^{n} p_i^{a_i+b_i+c_i-\min(a_i,b_i,c_i)}.$$

By definition of gcd , we know that $(a, b, c) = \prod_{i=1}^{n} p_i^{\min(a_i,b_i,c_i)}$. This implies that

$$\prod_{i=1}^{n} p_i^{\min(a_i,b_i,c_i)} \Big| \prod_{i=1}^{n} p_i^{a_i+b_i+c_i}$$

and so,

$$\frac{abc}{(a, b, c)} = \frac{\prod_{i=1}^{n} p_i^{a_i+b_i+c_i}}{\prod_{i=1}^{n} p_i^{\min(a_i,b_i,c_i)}} = \prod_{i=1}^{n} p_i^{a_i+b_i+c_i-\min(a_i,b_i,c_i)} = [ab, bc, ac].$$

Hence, $(a, b, c)[ab, bc, ca] = abc$. □

The following example illustrates Theorem 2.1.17.

Example 2.1.20. Refer back to Example 2.1.16, where it was shown that $[6, 10, 12](6, 10, 12) \neq 6 \cdot 10 \cdot 12$. Now from Theorem 2.1.17, we know that $[60, 120, 72](6, 10, 12) = 6 \cdot 10 \cdot 12$. To see this, lets compute the multiples of $60, 120$, and 72, respectively:

multiples of 60 : $60, 120, 180, 240, 300, \mathbf{360}, 420, \ldots$

multiples of 120 : $120, 240, \mathbf{360}, 480, 600, \ldots$

multiples of 72 : $72, 144, 216, 288, \mathbf{360}, 432, 504, \ldots$

Hence, $[60, 120, 72] = 360 = 2^3 \cdot 3^2 \cdot 5$. Now, $[60, 120, 72](6, 10, 12) = 2^4 \cdot 3^2 \cdot 5 = 6 \cdot 10 \cdot 12$. ♢

The next lemma shows that an associative property for the gcd holds.

Lemma 2.1.21. *Let a, b, and c be positive integers. Then, $(a, b, c) = (a, (b, c))$.*

Proof. Let $K = (a, b, c)$, $M = (a, m)$, where $m = (b, c)$. This means that $K|a, K|b$ and $K|c$. Since, $M|a$ and $M|m$ implies that $M|a, M|b$ and $M|c$, then by an extended version of Theorem 2.1.8, $M|K$. Now, we know that $K|a, K|b$ and $K|c$. Since $K|b$ and $K|c$, then by Theorem 2.1.8, $K|m$. Then, since $K|m$ and $K|a$, from Theorem 2.1.8, $K|M$. Hence, $M|K$ and $K|M$ implies $K = M$. □

Similarly, it can be shown that Lemma 2.1.21 holds for $a_1, a_2, \ldots, a_n \in \mathbb{N}$ such that $(a_1, a_2, \ldots, a_n) = (a_1, (a_2, \ldots, a_n))$. Although the proof is omitted, the reader is encouraged to verify this.

The following theorem is known as *The Chinese Remainder Theorem* (CRT).

Theorem 2.1.22. [2, Theorem 3.12]
Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime positive integers. Then, the system of

$$\begin{aligned} x &\equiv a_1 \mod m_1 \\ x &\equiv a_2 \mod m_2 \\ &\vdots \\ x &\equiv a_r \mod m_r, \end{aligned}$$

has a unique solution module $M = m_1 m_2 \ldots m_r$.

As it turns out, we can generalize Theorem 2.1.22.

Theorem 2.1.23. *The system of congruences*

$$\begin{aligned} x &\equiv a_1 \mod m_1 \\ x &\equiv a_2 \mod m_2 \\ &\vdots \\ x &\equiv a_n \mod m_n, \end{aligned}$$

has a solution if and only if $(m_i, m_j)|a_i - a_j$ *for all pairs of integers* (i, j)*, where* $1 \leq i < j \leq n$*. If a solution exists, then it is unique modulo* $[m_1, m_2, \ldots, m_n]$.

Before we prove Theorem 2.1.23, we will show that it holds for two system of congruences. The resulting lemma will guide the proof for Theorem 2.1.23.

Lemma 2.1.24. *The system of congruences*

$$x \equiv a_1 \mod m_1$$
$$x \equiv a_2 \mod m_2,$$

has a solution if and only if $(m_1, m_2)|(a_1 - a_2)$*. The solution is unique modulo* $[m_1, m_2]$.

Proof. $\Leftarrow$ Suppose that $(m_1, m_2)|a_1 - a_2$. From Theorem 2.1.7, we know that for some $n_1, n_2 \in \mathbb{Z}$, $m_2n_2 - m_1n_1 = a_1 - a_2$ or equivalently, $a_1 + m_1n_1 = a_2 + m_2n_2$. Now, let $x = a_1 + n_1m_1$. Then,

$$a_1 + m_1n_1 \equiv a_1 \mod m_1 \qquad (1)$$
$$a_1 + m_1n_1 \equiv a_2 \mod m_2 \qquad (2).$$

Equation (1) can be written as $a_1 + m_1n_1 \equiv a_1 \mod \frac{m_1}{(m_1,m_2)}$. From Theorem 2.1.9, this means $(\frac{m_1}{(m_1,m_2)}, m_2) = 1$. Then, according to Theorem 2.1.22, a unique solution exists and is congruent modulo $\frac{m_1}{(m_1,m_2)}m_2 = [m_1, m_2]$.

$\Rightarrow$ Suppose that x is a solution. This means that $x \equiv a_1 \mod m_1$ and $x \equiv a_2 \mod m_2$. Then, for some $n_1, n_2 \in \mathbb{Z}$, $m_1n_1 = x - a_1$ and $m_2n_2 = x - a_2$. In other words, $m_1n_1 + a_1 = m_2n_2 + a_2$ or $m_2n_2 - m_1n_1 = a_1 - a_2$. Multiplying the latter equation by $\frac{(m_1,m_2)}{(m_1,m_2)}$, we get:

$$(m_1, m_2)\Big(\frac{m_2n_2}{(m_1, m_2)} - \frac{m_1n_1}{(m_1, m_2)}\Big) = a_1 - a_2.$$

Note that (m_1, m_2) divides m_1 and m_2, hence the equation on the left is still an integer. If x is a solution then $(m_1, m_2)|(a_1 - a_2)$. □

Proof of Theorem 2.1.23. From Lemma 2.1.24, we know that x is congruent to some $Q \mod [m_1, m_2]$. Then, applying Lemma 2.1.24 to $x \equiv Q \mod [m_1, m_2]$ and $x \equiv a_3 \mod m_3$, we know a solution exists and that it is congruent to some Q' mod $[[m_1, m_2], m_3] = [m_1, m_2, m_3]$. This step can be repeated multiple times until we note that x is congruent to mod $[m_1, m_2, \ldots, m_n]$. Hence, there is a solution as long as $(m_i, m_j)|(a_i - a_j)$ for 1≤i and $j \geq n$. $\square$

3

Generalized Integer Splines

In Section 3.1 we give a formal definition of generalized integer splines and introduce *n-cycle* splines. In Section 3.2 we show that *n-cycle* splines form a $\mathbb{Z}$-module. Finally, in Section 3.3 we discuss flow-up classes and define the smallest flow-up class.

3.1 Definitions and Examples of Integer Splines

We begin by defining an edge-labeled graph, which is simply assigning a label to an edge.

Definition 3.1.1. [1, Defintion 2.1] Let G be a graph with k edges $e_1, e_2, \ldots, e_k$ and n vertices $v_1, v_2, \ldots, v_n$. For $1 \leq i \leq k$, let $a_i \in \mathbb{N}$ be the label on edge e_i and let $\mathbf{A} = \{a_1, \ldots, a_k\}$ be the set of edge labels. Then,$(G, \mathbf{A})$ is called an **edge-labeled graph**. △

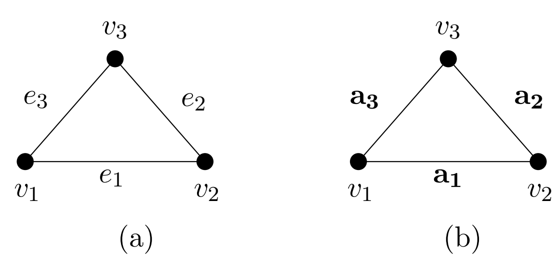

Figure 3.1.1. (a) is a graph while (b) is an *edge-labeled graph* .

Figure 3.1.1 illustrates Definition 3.1.1.

Definition 3.1.2. [1, Defintion 2.2] Let $(G, \mathbf{A})$ be an edge-labeled graph. A **generalized integer spline** is a vertex labeling $(f_1, \ldots, f_n) \in \mathbb{Z}^n$ such that if v_i and v_j are connected by an edge e_k, then $f_i \equiv f_j \mod a_k$. We denote the set of all splines on $(G, \mathbf{A})$ by $\mathcal{S}_{(G,\mathbf{A})}$. △

The following figure illustrates Definition 3.1.2.

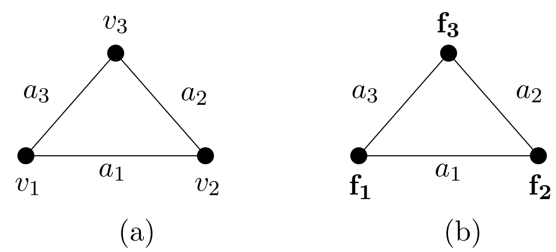

Figure 3.1.2. (a) is an *edge-labeled graph*, while (b) is a graphical representation of a *generalized integer spline*.

Note: From now on we refer to generalized integer splines as splines.

In this project, we label the edges with natural numbers and the vertices with integers.

Example 3.1.3. In Figure 3.1.3 (a), we see that $F = (12, 25, 9, 2, 13)$ is a spline since

$$13 \equiv 12 \mod 1, \;\; 25 \equiv 12 \mod 13, \;\; 25 \equiv 9 \mod 2,$$

$$9 \equiv 12 \mod 3, \;\; 2 \equiv 12 \mod 5, \;\text{ and } 13 \equiv 2 \mod 11.$$

Similarly in graph (b), $G = (34, 65, 11, 29)$ is a spline. However, in graph (c), $H = (11, 23, 63, 89)$ is not a spline since $11 \not\equiv 23 \mod 8$. ◊

We now introduce a particular type of spline called an *n-cycle* spline. As the name indicates, these are splines on $(G, \mathbf{A})$, where G is an *n-cycle* graph. Our project focuses on *n-cycle* splines.

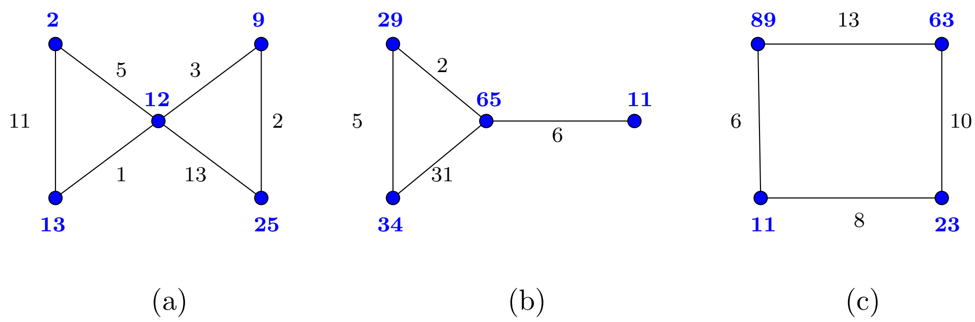

Figure 3.1.3. (a) and (b) are examples of splines, where $F = (12, 25, 9, 2, 13)$ and $G = (34, 65, 11, 29)$, respectively. Graph (c) is not a spline.

Let $\mathbf{A} = \{a_1, a_2, \ldots, a_n\}$ be the ordered set of edge-labels on an *n-cycle* graph with ordered vertices $\{v_1, v_2, \ldots, v_n\}$. If the following conditions are satisfied

$$\begin{aligned} f_1 &\equiv f_2 \mod a_1 \\ f_2 &\equiv f_3 \mod a_2 \\ &\vdots \\ f_{n-1} &\equiv f_n \mod a_{n-1} \\ f_n &\equiv f_1 \mod a_n, \end{aligned}$$

then $F = (f_1, f_2, \ldots, f_n)$ is an ***n-cycle* spline**.

Notation: We denote the set of all *n-cycle* splines by $\mathcal{S}_n(\mathbf{A})$. In other words, $\mathcal{S}_n(\mathbf{A}) = \mathcal{S}_{(G,\mathbf{A})}$, where G is an *n-cycle*.

Note: From now on a graph G refers to an *n-cycle* graph.

Before we continue, we use a conventional way to number the vertices and edges on an *n-cycle* graph. In general, $e_i = \{v_i, v_{i+1}\}$ for $1 \leq i \leq n-1$ and $e_n = \{v_n, v_1\}$. This is shown in Figure 3.1.4.

The following is an example of *n-cycle* splines.

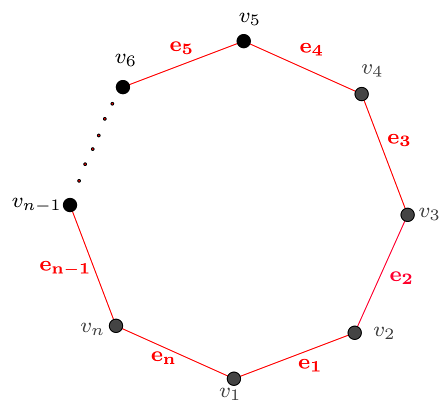

Figure 3.1.4. An *n-cycle* Graph.

Example 3.1.4. Figure 3.1.5 (a) represents a *3-cycle* spline, where $F = (5, 8, 30)$. Similarly, in Figure 3.1.5 (b), $G = (24, 16, 26, 5)$ is a *4-cycle* spline. In graph (c), $H = (12, 18, 30, 56, 23)$ is a *5-cycle* spline. ◊

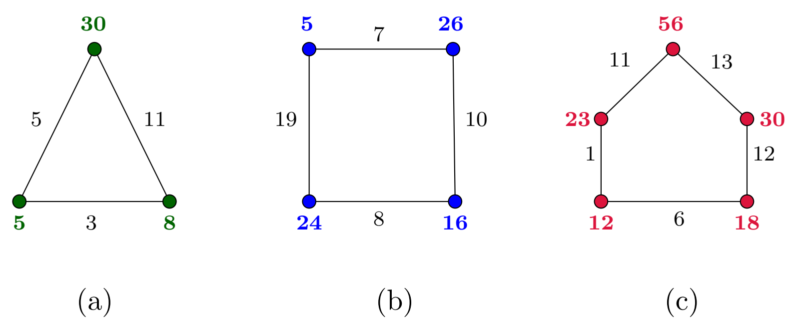

Figure 3.1.5. (a), (b), and (c) illustrate *3-cycle*, *4-cycle*, and *5-cycle* splines, respectively.

3.2 *n-cycle* Splines form a $\mathbb{Z}$-module

Before we go any further, we show that the set of all integer splines on $(G, \mathbf{A})$ form a $\mathbb{Z}$-module. While modules can be thought of as vector spaces, where the *scalars* are from a ring R instead of a field F, there is no guarantee that a module defined over R will have a basis. However, before we can state whether or not $\mathcal{S}_n(\mathbf{A})$ has a basis, we will show that it forms a $\mathbb{Z}$-module. This is equivalent to showing that $\mathcal{S}_n(\mathbf{A}) \subseteq \mathbb{Z}^n$ [4]. In other words, we must show that *n-cycle* splines are a subgroup of $\mathbb{Z}^n$.

Below we recall the definition of a subgroup.

Definition 3.2.1. A subset H of a group G is a **subgroup** if

1. e $\in$ H;
2. if $x, y \in H$, then $x \star y \in H$;
3. if $x \in H$, then $x^{-1} \in H$.

△

Theorem 3.2.2. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, \dots, a_n\}$*. Then,* $\mathcal{S}_n(\boldsymbol{A})$ *is a subgroup of* $\mathbb{Z}^n$*.*

Proof. According to Definition 3.2.1, we must show the identity element of $\mathbb{Z}^n$, $E = (0, \dots, 0)$, is in $\mathcal{S}_n(\mathbf{A})$. As well as show that, $\mathcal{S}_n(\mathbf{A})$ is closed under addition and the inverse of $\mathbb{Z}^n$, $F^{-1} = (-f_1, -f_2, \dots, -f_n)$, is in $\mathcal{S}_n(\mathbf{A})$.

Clearly, $E = (0, \dots, 0) \in \mathcal{S}_n(\mathbf{A})$ since $a_i | 0 - 0$, for $1 \leq i \leq n$.

Now, let $F = (f_1, f_2, \dots, f_n), G = (g_1, g_2, \dots, g_n) \in \mathcal{S}_n(\mathbf{A})$. This means that for $1 \leq i \leq n - 1$:

$$x_i a_i = (f_{i+1} - f_i) \text{ and } x_n a_n = (f_n - f_1), \quad \text{where} \quad x_i, x_n \in \mathbb{Z} \tag{3.2.1}$$

$$y_i a_i = (g_{i+1} - g_i) \text{ and } y_n a_n = (g_n - g_1), \quad \text{where} \quad y_i, y_n \in \mathbb{Z}. \tag{3.2.2}$$

Adding the equations in 3.2.1 with the ones in 3.2.2, we get:

$$a_i(x_i + y_i) = a_i x_i + a_i y_i = (f_{i+1} - f_i) + (g_{i+1} - g_i) = (f_{i+1} + g_{i+1}) - (f_i + g_i) \tag{3.2.3}$$

and

$$a_n(x_n + y_n) = a_n x_n + a_n y_n = (f_n - f_1) + (g_n - g_1) = (f_n + g_n) - (f_1 + g_1). \tag{3.2.4}$$

Hence, $a_i|(f_{i+1}+g_{i+1})-(f_i+g_i)$ and $a_n|(f_n+g_n)-(f_1+g_1)$. Therefore, $F+G \in \mathcal{S}_n(\mathbf{A})$ which implies that $\mathcal{S}_n(\mathbf{A})$ is closed under addition.

Now, suppose $F=(f_1,f_2,\ldots,f_n) \in \mathcal{S}_n(\mathbf{A})$. If we multiply the equations in 3.2.1 by -1, we get:

$$-x_ia_i = -(f_{i+1}-f_i) = -f_{i+1}-(-f_i) \quad \text{and} \quad -x_na_n = -(f_n-f_1) = -f_n-(-f_1).$$

In other words, $a_i|-f_{i+1}-(-f_i)$ and $a_n|-f_n-(-f_1)$. Hence, $F^{-1}=(-f_1,-f_2,\ldots,-f_n) \in \mathcal{S}_n(\mathbf{A})$.

Therefore, $\mathcal{S}_n(\mathbf{A})$ is a *subgroup* of $\mathbb{Z}^n$. □

How do we know if $\mathcal{S}_n(\mathbf{A})$ has a basis? We defer to the following theorem, which we include without a proof.

Theorem 3.2.3. [6, Theorem 6.1] *Let F be a free module over a principal ideal domain R and G a submodule of F. Then, G is a free R-module and rank G $\leq$ rank F.*

Since $\mathbb{Z}$ is a principal ideal domain and finitely generated modules over a principal ideal domain are free, then according to Theorem 3.2.3, $\mathcal{S}_n(\mathbf{A})$, which is a $\mathbb{Z}-$submodule of $\mathbb{Z}^n$, is also free. Hence, we know a basis for $\mathcal{S}_n(\mathbf{A})$ exists.

3.3 Flow-Up Classes

A flow-up class is an *n-cycle* spline with i, where $0 \leq i < n$, leading zeros. We provide a formal definition below.

Definition 3.3.1. [1, Defintion 2.3] Fix the edge labels on $(G,\mathbf{A})$. Fix i, where $0 \leq i < n$. A **flow-up class** is any spline in $\mathcal{S}_n(\mathbf{A})$ with i leading zeros, i.e. $F_i = (0,\ldots,0,f_{i+1},\ldots,f_n)$. △

Notation: We denote the set of all flow-up classes with i fixed leading zeros as $\mathcal{F}_i(\mathbf{A})$.

The following theorem tells us that flow-up classes exist in $\mathcal{S}_n(\mathbf{A})$.

Theorem 3.3.2. [1, Theorem 4.3] *Fix the edge labels on* $(G, \boldsymbol{A})$. *Let* $0 \leq i < n$. *There exists a flow-up class* $F_i \in \mathcal{S}_n(\boldsymbol{A})$.

In the following example, we illustrate Theorem 3.3.2 for $\mathcal{S}_3(\mathbf{A})$.

Example 3.3.3. Fix the edges on $(G, \mathbf{A})$, where $\mathbf{A} = \{5, 12, 13\}$. Let $F_0 = (1, 1, 1), F_1 = (0, 5, 65)$, and $F_2 = (0, 0, 156)$. Clearly, F_0, F_1, and F_2 are flow-up classes in $\mathcal{S}_3(\mathbf{A})$. F_0, F_1 and F_2 are visually represented in Figure 3.3.1. $\diamondsuit$

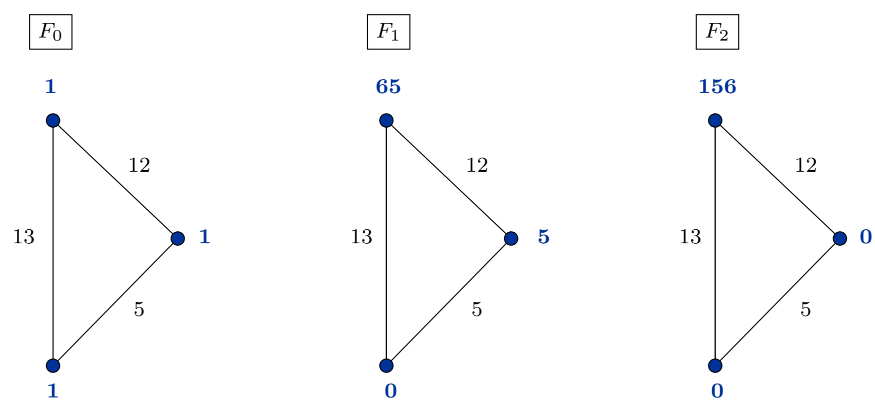

Figure 3.3.1. Flow-up classes F_0, F_1, and F_2.

Next, we define the smallest flow-up class. We say a spline is the smallest flow-up class if its entries are the smallest relative to another flow-up class.

Definition 3.3.4. [1, Defintion 2.4] Fix the edge labels on $(G, \mathbf{A})$. Let $F_i \in \mathcal{F}_i(\mathbf{A})$, then F_i is the **smallest flow-up class** if for every $H_i \in \mathcal{F}_i(\mathbf{A})$, $f_{j+1} \leq h_{j+1}$, where $i \leq j < n$. $\triangle$

The following theorem states that we can algebraically identify the smallest leading element of a flow-up class.

Theorem 3.3.5. [1, Theorem 4.5] *Fix the edge labels on* $(G, \boldsymbol{A})$, *where* $\boldsymbol{A}= \{a_1, a_2, \ldots, a_n\}$. *Let* $1 \leq i \leq n-1$ *and let* $F_i = (0, \ldots, 0, f_{i+1}, f_{i+2}, \ldots, f_n)$ *be a flow-up class in* $\mathcal{S}_n(\boldsymbol{A})$. *Then, the leading element,* f_{i+1}, *is a multiple of* $[a_i, (a_{i+1}, \ldots, a_n)]$ *and* $f_{i+1} =$

$[a_i,(a_{i+1},\dots,a_n)]$ is the smallest positive value satisfying the a_i and a_{i+1} edge labeled conditions.

The following proposition states that the smallest flow-up class exists in $\mathcal{F}_0(\mathbf{A})$.

Proposition 3.3.6. [1, Proposition 2.5] *Fix the edge labels on $(G,\boldsymbol{A})$. Then, $F_0 = (1,\dots,1)$ is the smallest flow-up class in $\mathcal{F}_0(\boldsymbol{A})$ and is in $\mathcal{S}_n(\boldsymbol{A})$.*

Note: Any flow-up class in $\mathcal{F}_0(\mathbf{A})$ is a multiple of $F_0 = (1,\dots,1)$.

The next theorem tells us that the smallest flow-up classes exist in $\mathcal{S}_n(\mathbf{A})$.

Theorem 3.3.7. [1, Theorem 4.6] *Fix the edge labels on $(G,\boldsymbol{A})$. Let $1 \le i < n$. There exists a smallest flow-up class $F_i = (0,\dots,0,f_{i+1},\dots,f_n) \in \mathcal{S}_n(\boldsymbol{A})$.*

Example 3.3.8. Refer back to Example 3.3.3. Not only are F_0, F_1, and F_2 flow-up classes, but they are the smallest flow-up classes in $\mathcal{F}_0(\mathbf{A})$, $\mathcal{F}_1(\mathbf{A})$, and $\mathcal{F}_2(\mathbf{A})$, respectively. To see this, we know from Proposition 3.3.6, that the smallest flow-up class in $\mathcal{F}_0(\mathbf{A})$ is $F_0 = (1,1,1)$. By Theorem 3.3.5, we know that the smallest leading element of F_1 is equal to $[5,(12,13)] = 5$. Now, the third element in F_1, f_3, must be such that:

$$f_3 \equiv 5 \mod 12 \quad \text{and} \quad f_3 \equiv 0 \mod 13.$$

The smallest number that satisfies this condition is 65. Hence, $F_1 = (0,5,65)$ is the smallest flow-up class in $\mathcal{F}_1(\mathbf{A})$. A similar application of Theorem 3.3.5 tells us that $[12,13] = 156$ is the smallest leading element of F_2. Hence, $F_2 = (0,0,156)$ is the smallest flow-up class in $\mathcal{F}_2(\mathbf{A})$. ◊

The following theorem tells us that any *n-cycle* spline on $(G,\mathbf{A})$ can be written as a linear combination of the smallest flow-up classes, $F_0,\dots,F_{n-1}$. In other words, if $F \in \mathcal{S}_n(\mathbf{A})$, then there exist $x_0,\dots,x_{n-1} \in \mathbb{Z}$ such that, $F = x_0F_0 + x_1F_1 + \cdots + x_{n-1}F_{n-1}$.

Theorem 3.3.9. [1, Theorem 4.7] *The smallest flow-up classes, $F_0, F_1, \dots, F_{n-1}$ form a basis over the integers for the module of splines $\mathcal{S}_n(\boldsymbol{A})$.*

We illustrate Theorem 3.3.9 with an example.

Example 3.3.10. We showed in Example 3.3.8 that $F_0 = (1, 1, 1), F_1 = (0, 5, 65)$, and $F_2 = (0, 0, 156)$ were the smallest flow-up classes in $\mathcal{S}_3(\mathbf{A})$, where $\mathbf{A} = \{5, 12, 13\}$. According to Theorem 3.3.9, to show that $F = (3, 28, 328) \in span\{F_0, F_1, F_2\}$, we must find an $x_0, x_1, x_2 \in \mathbb{Z}$ such that

$$x_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 5 \\ 65 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 156 \end{bmatrix} = \begin{bmatrix} 3 \\ 28 \\ 328 \end{bmatrix}.$$

In other words we must solve,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 65 & 156 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 28 \\ 328 \end{bmatrix}.$$

Using elementary row reduction we get:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 1 & 5 & 0 & 28 \\ 1 & 65 & 156 & 328 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 25 \\ 0 & 65 & 156 & 325 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 25 \\ 0 & 0 & 156 & 0 \end{bmatrix}.$$

Hence, $x_0 = 3$, $x_1 = \frac{25}{5} = 5$, and $x_2 = 0$. Thus, F is a linear combination of F_0, F_1, and F_2. $\Diamond$

4
Bases for *3-cycle* splines

In this chapter we present the majority of our results. In Section 4.3, we prove that we do not need to find the smallest flow-up classes in order to span $\mathcal{S}_3(\mathbf{A})$. In fact, all we need are a set of splines in $\mathcal{S}_3(\mathbf{A})$ that fulfill a certain criteria. Sections 4.1 and 4.2 are the building blocks needed to arrive at this result.

4.1 Flow-Up Classes on *3-cycle* Splines

We omit the proof showing that the smallest flow-up class in $\mathcal{F}_0(\mathbf{A})$ is $F_0 = (1, 1, 1)$ and that $F_0 \in \mathcal{S}_3(\mathbf{A})$ since this is easily verifiable. Hence, we start by determining when a flow-up class, $F_1 \in \mathcal{S}_3(\mathbf{A})$. The following theorems are from Handschy et al.[1]. We simply present alternative proofs.

Theorem 4.1.1. *Fix the edge labels on* $(G, \boldsymbol{A})$*. Then, a flow-up class,* $F_1 = (0, f_2, f_3)$*, exists in* $\mathcal{S}_3(\boldsymbol{A})$ *if and only if* f_2 *is a multiple of* $[a_1, (a_2, a_3)]$.

Proof. In order for $F_1 \in \mathcal{S}_3(\mathbf{A})$ we must show that f_2 and f_3 exist. In other words, f_2 and f_3 must satisfy:

$$0 \equiv f_2 \mod a_1 \tag{4.1.1}$$

$$f_3 \equiv f_2 \mod a_2 \tag{4.1.2}$$

$$0 \equiv f_3 \mod a_3. \tag{4.1.3}$$

From Theorem 2.1.22, we know that f_3 exists if and only if $(a_2, a_3)|f_2$. In other words, $f_2 \equiv 0 \mod (a_2, a_3)$. In addition, Equation 4.1.1 tells us that $f_2 \equiv 0 \mod a_1$. Now, from Theorem 2.1.22, we know f_2 exists if and only if f_2 is congruent $0 \mod [a_1, (a_2, a_3)]$. In other words, f_2 is a multiple of $[a_1, (a_2, a_3)]$. Hence, $F_1 \in \mathcal{S}_3(\mathbf{A})$ as long as f_2 is a multiple of $[a_1, (a_2, a_3)]$. □

Note: Theorem 4.1.1 implies that f_3 exists since $(a_2, a_3)|f_2$.

The next theorem guarantees that a flow-up class F_2 exists in $\mathcal{S}_3(\mathbf{A})$.

Theorem 4.1.2. *Fix the edge labels on* $(G, \boldsymbol{A})$*. Then, the flow-up class* $F_2 = (0, 0, f_3)$ *exists in* $\mathcal{S}_3(\boldsymbol{A})$ *iff* f_3 *is a multiple of* $[a_2, a_3]$.

Proof. From Theorem 2.1.22, we know that a solution to $f_3 \equiv 0 \mod a_2$ and $f_3 \equiv 0 \mod a_3$ exists and this solution is unique congruent $0 \mod [a_2, a_3]$. In other words, f_3 exists iff it is a multiple of $[a_2, a_3]$. Hence, $F_2 \in \mathcal{S}_3(\mathbf{A})$. □

The following theorems show that the smallest flow-up classes exist in $\mathcal{S}_3(\mathbf{A})$.

Theorem 4.1.3. *Fix the edge labels on* $(G, \boldsymbol{A})$*. Then, the smallest flow-up class* F_1 *exists in* $\mathcal{S}_3(\boldsymbol{A})$.

Proof. From Theorem 4.1.1, we know that the leading element of a flow-up class in $\mathcal{F}_1(\mathbf{A})$ is a multiple of $[a_1, (a_2, a_3)]$. The smallest such element is when $f_2 = [a_1, (a_2, a_3)]$. Theorem 4.1.1 also tells us that f_3 exists as long as $(a_2, a_3)|f_2$. Rewriting f_2 we see that, for some

$x \in \mathbb{N}$, $f_2 = \dfrac{x \cdot a_1 \cdot (a_2, a_3)}{(a_1, (a_2, a_3))} = \dfrac{x \cdot a_1}{(a_1, (a_2, a_3))} \cdot (a_2, a_3)$. Hence, f_3 exists and by the Well Ordering Property, we know that in the set of all possible f_3, the smallest element exists. Now, let the smallest possible value of f_3 be f_3'. Then, $F_1 = (0, [a_1, (a_2, a_3)], f_3')$ is the smallest flow-up class in $\mathcal{S}_3(\mathbf{A})$. □

Theorem 4.1.4. *Fix the edge labels on* $(G, \boldsymbol{A})$. *Then, the smallest flow-up class* F_2 *exists in* $\mathcal{S}_3(\boldsymbol{A})$.

Proof. From Theorem 4.1.2, we know that the leading element of a flow-up class in $\mathcal{F}_2(\mathbf{A})$ is a multiple of $[a_2, a_3]$. In other words $f_3 = x \cdot [a_2, a_3]$, where $x \in \mathbb{N}$. The smallest number in $\mathbb{N}$ is 1. Hence, $F_2 = (0, 0, [a_2, a_3])$ is the smallest flow-up class in $\mathcal{S}_3(\mathbf{A})$. □

For now, we omit the proof showing that the smallest flow-up classes form a module basis for $\mathcal{S}_3(\mathbf{A})$. We give a formal proof in Section 4.3.

We provide additional examples, similar to Example 3.3.8 and Example 3.3.10, to demonstrate the theorems once again.

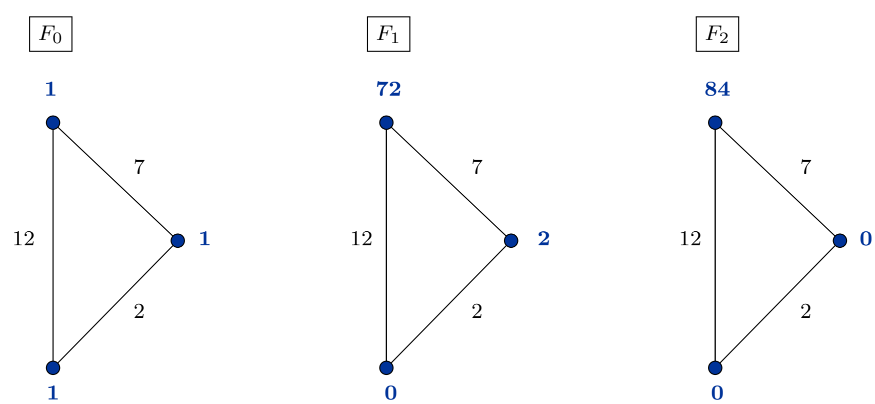

Figure 4.1.1. The smallest flow-up classes on $(G, \mathbf{A})$, where $\mathbf{A} = \{2, 7, 12\}$, are $F_0 = (1, 1, 1), F_1 = (0, 2, 72)$ and $F_2 = (0, 0, 84)$.

Example 4.1.5. Fix the edge labels on $(G, \mathbf{A})$, where $\mathbf{A} = \{2, 7, 12\}$. From Theorem 4.1.3 and Theorem 4.1.4, we know that the smallest flow-up classes exist in $\mathcal{S}_3(\mathbf{A})$. In addition, Theorem 4.1.3 tells us that the smallest leading element of F_1 is $[2,(7,12)] = 2$. With some additional computation, we find that if $f_3 = 72$, then $F_1 = (0, 2, 72)$ is the smallest flow-up class. From Theorem 4.1.4, we know that F_2 is the smallest flow-up class when the leading element is $[7, 12] = 84$. Hence, $F_2 = (0, 0, 84)$. Additionally, we know that the smallest flow-up class for F_0 is $(1, 1, 1)$. Figure 4.1.1 illustrate the smallest flow-up classes, F_0, F_1 and F_2. ♢

Example 4.1.6. From Example 4.1.5, we know that $F_0 = (1, 1, 1), F_1 = (0, 2, 72)$, and $F_2 = (0, 0, 74)$ are the smallest flow-up classes in $\mathcal{S}_3(\mathbf{A})$, where $\mathbf{A}=\{2, 7, 12\}$. To find out if $F = (12, 6, 39) \in span\{F_0, F_1, F_2\}$ we solve for,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 72 & 84 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 39 \end{bmatrix}.$$

Using elementary row reduction we get:

$$\begin{bmatrix} 1 & 0 & 0 & 12 \\ 1 & 2 & 0 & 6 \\ 1 & 72 & 84 & 39 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 2 & 0 & -6 \\ 0 & 72 & 84 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 84 & 243 \end{bmatrix}.$$

Since $x_2 = \dfrac{243}{84} \notin \mathbb{Z}$, then $(12, 6, 39) \notin span\{F_0, F_1, F_2\}$. Note, it is really easy to eyeball that $F \notin \mathcal{S}_3(\mathbf{A})$ since, $39 \not\equiv 6 \mod 7$. ♢

4.2 Determinant of Flow-Up Classes

Before we continue, consider the following example.

Example 4.2.1. Recall Example 4.1.5, where the edge labels on $(G, \mathbf{A})$ were $\mathbf{A} = \{2, 7, 12\}$ and $F_0 = (1, 1, 1), F_1 = (0, 2, 72),$ and $F_2 = (0, 0, 84)$ were the smallest flow-up classes. Now, let

$$M = [F_0, F_1, F_2] = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 1 & \mathbf{2} & 0 \\ 1 & 74 & \mathbf{84} \end{bmatrix}.$$

Since M is a lower triangular matrix, it follows that $|M| = 1 \cdot 2 \cdot 84 = 2 \cdot 7 \cdot 12$. Note, $|M|$ is equal to the product of the edge labels. ♢

As Example 4.2.1 demonstrates, the determinant of the smallest flow-up classes in $\mathcal{S}_3(\mathbf{A})$ follows a particular pattern. In fact, the following theorem was motivated by this pattern.

Theorem 4.2.2. *Fix the edge labels on* $(G, \boldsymbol{A})$, *where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$. *Let* F_0, F_1 *and* F_2 *be flow-up classes in* $\mathcal{S}_3(\boldsymbol{A})$. *Then* $|F_0, F_1, F_2| = c \cdot \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$, *where* $c \in \mathbb{N}$.

Proof. Let F_0, F_1 and F_2 be flow-up classes in $\mathcal{S}_3(\mathbf{A})$. Then, from Theorem 4.1.1, we know that the leading element of F_1 is a multiple of $[a_1, (a_2, a_3)]$ and from Theorem 4.1.2, we know that the leading element of F_2 is a multiple of $[a_2, a_3]$. We also know that the flow-up class F_0 is a multiple of the trivial case, namely $(1, 1, 1)$. Hence, for some $x_0, x_1, x_2 \in \mathbb{N}$, we have:

$$F_0 = x_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ x_1 \cdot [a_1, (a_2, a_3)] \\ f_3 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 0 \\ 0 \\ x_2[a_2, a_3] \end{bmatrix},$$

where f_3 is an integer that satisfies the necessary conditions for $F_1 \in \mathcal{S}_3(\mathbf{A})$. Now, let

$$M = \begin{bmatrix} \underline{F_0} & \underline{F_1} & \underline{F_2} \\ \mathbf{x_0} & 0 & 0 \\ x_0 & \mathbf{x_1[a_1, (a_2, a_3)]} & 0 \\ x_0 & f_3 & \mathbf{x_2[a_2, a_3]} \end{bmatrix}.$$

Since M is a lower triangular matrix, it follows that $|M| = x_0x_1x_2[a_1, (a_2, a_3)][a_2, a_3]$.

Let $c = x_0x_1x_2$, then

$$\begin{aligned} |M| &= c \cdot [a_1, (a_2, a_3)] \cdot [a_2, a_3] && (4.2.1) \\ &= c \cdot \frac{a_1 \cdot (a_2, a_3)}{(a_1, (a_2, a_3))} \cdot \frac{a_2 \cdot a_3}{(a_2, a_3)} && \text{By Theorem 2.1.15} \quad (4.2.2) \\ &= c \cdot \frac{a_1 \cdot (a_2, a_3)}{(a_1, a_2, a_3)} \cdot \frac{a_2 \cdot a_3}{(a_2, a_3)} && \text{By Lemma 2.1.21} \quad (4.2.3) \\ &= c \cdot \frac{a_1a_2a_3}{(a_1, a_2, a_3)}. && (4.2.4) \end{aligned}$$

Hence, $|M|$ is a multiple of $\dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$. □

We now generalize Theorem 4.2.2.

Theorem 4.2.3. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$*. Let* $F_0, F_1, \ldots, F_{n-1}$ *be flow-up classes in* $\mathcal{S}_n(\boldsymbol{A})$*. Then,* $|F_0, F_1, \ldots, F_{n-1}| = c \cdot \dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$*, where* $c \in \mathbb{N}$.

Proof. Let $F_0, F_1, \ldots, F_{n-1}$ be flow-up classes in $\mathcal{S}_n(\mathbf{A})$ defined by,

$$F_i = \begin{cases} (0, \ldots, 0, f^i_{i+1}, f^i_{i+2}, \ldots, f^i_n) & \text{for } 1 \le i \le n-1 \\ x_0(1, 1, \ldots, 1, 1) & \text{where } x_0 \in \mathbb{N}, i = 0 \end{cases}.$$

From Theorem 3.3.5, we know that the leading entry of any flow-up class, excluding F_0, is a multiple of $[a_i, (a_{i+1}, \ldots, a_n)]$. In other words, for $1 \le i \le n-1$ and $x_i \in \mathbb{N}$, $f^i_{i+1} = x_i[a_i, (a_{i+1}, \ldots, a_n)]$. Now, let

$$M = \begin{array}{c} \begin{array}{ccccc} \underline{F_0} & \underline{F_1} & \underline{F_2} & \cdots & \underline{F_{n-1}} \end{array} \\ \begin{bmatrix} \mathbf{x_0} & 0 & 0 & \cdots & 0 \\ x_0 & \mathbf{x_1}[\mathbf{a_1}, (\mathbf{a_2}, \ldots, \mathbf{a_n})] & 0 & \cdots & 0 \\ x_0 & f^1_3 & \mathbf{x_2}[\mathbf{a_2}, (\mathbf{a_3}, \ldots, \mathbf{a_n})] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0 & f^1_n & f^2_n & \cdots & \mathbf{x_{n-1}}[\mathbf{a_{n-1}}, \mathbf{a_n}] \end{bmatrix} \end{array}.$$

Since M is a lower triangular matrix we have,

$$|M| = x_0 x_1 \cdots x_{n-2} x_{n-1} [a_1, (a_2, \ldots, a_n)][a_2, (a_3, \ldots, a_n)] \cdots [a_{n-2}, (a_{n-1}, a_n)][a_{n-1}, a_n].$$

Let $c = x_0 x_1 \cdots x_{n-2} x_{n-1}$. Then,

$$|M| = c \cdot \frac{a_1(a_2, \ldots, a_n)}{(a_1, (a_2, \ldots, a_n))} \cdot \frac{a_2(a_3, \ldots, a_n)}{(a_2, (a_3, \ldots, a_n))} \cdots \frac{a_{n-2}(a_{n-1}, a_n)}{(a_{n-2}, (a_{n-1}, a_n))} \cdot \frac{a_{n-1} a_n}{(a_{n-1}, a_n)} \tag{4.2.5}$$

$$= c \cdot \frac{a_1(a_2, \ldots, a_n)}{(a_1, a_2, \ldots, a_n)} \cdot \frac{a_2(a_3, \ldots, a_n)}{(a_2, \ldots, a_n)} \cdots \frac{a_{n-2}(a_{n-1}, a_n)}{(a_{n-2}, a_{n-1}, a_n)} \cdot \frac{a_{n-1} a_n}{(a_{n-1}, a_n)} \tag{4.2.6}$$

$$= c \cdot \frac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)} \cdot \frac{(a_2, \ldots, a_n)}{(a_2, \ldots, a_n)} \cdots \frac{(a_{n-2}, a_{n-1}, a_n)}{(a_{n-2}, a_{n-1}, a_n)} \cdot \frac{(a_{n-1}, a_n)}{(a_{n-1}, a_n)} \tag{4.2.7}$$

$$= c \cdot \frac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}. \tag{4.2.8}$$

Hence, $|M|$ is a multiple of $\dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$. □

Note: To get from Equation 4.2.5 to 4.2.6, we use an extended version of Lemma 2.1.21. To get from Equation 4.2.7 to 4.2.8, we see that every numerator of the form $(a_i, \ldots, a_n)$

for $2 \leq i \leq n-1$ has an equivalent and accompanying denominator, hence the terms cancel each other out.

It follows from Thereom 4.2.3, that the determinant for the smallest flow-up classes in $\mathcal{S}_n(\mathbf{A})$ follow a similar pattern. We present this below.

Corollary 4.2.4. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A}$=$\{a_1, a_2, \ldots, a_n\}$. Let $F_0, F_1 \ldots, F_{n-1}$ be the smallest flow-up classes in $\mathcal{S}_n(\boldsymbol{A})$. Then, $|F_0, F_1, \ldots, F_{n-1}| = \dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$.*

Proof. Let $F_0', F_1', \ldots, F_{n-1}'$ be flow-up classes in $\mathcal{S}_n(\mathbf{A})$. From Theorem 4.2.3, we know $|F_0', F_1', \ldots, F_{n-1}'| = c \cdot \dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$, where $c = x_0 x_1 \cdots x_{n-1}$. Now, c is equal to the product of the multiple of each leading entry in a flow-up class, i.e x_0 is the multiple of the leading entry in F_0', x_1 is the multiple of the leading entry of F_1', x_2 is the multiple of the leading entry of F_2' and etc. Since $F_0, F_1, F_2, \ldots, F_{n-1}$ are the smallest flow-up classes, then $x_0 = x_1 = x_2 = \cdots = x_{n-1} = 1 \Rightarrow c = 1$. Therefore, $|F_0, F_1, \ldots, F_{n-1}| = \dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$. □

4.3 Basis Criteria for *3-cycle* splines

As we showed in Section 4.2, the determinant of a basis matrix follows a certain pattern. In this section we prove that any set of splines form a module basis for $\mathcal{S}_3(\mathbf{A})$ if and only if their determinant is **equal** to $\pm \dfrac{a_1 a_2 a_3}{(a_1, a_2, a_3)}$.

Before we prove our big theorem, we need a couple of lemmas.

Lemma 4.3.1. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2, a_3\}$. Let $Q = \dfrac{a_1 a_2 a_3}{(a_1, a_2, a_3)}$ and $F, G, H, D \in \mathcal{S}_3(\boldsymbol{A})$. Suppose $\big|F, G, H\big| = \pm Q$. Then, QD is in the span of $\{F, G, H\}$.*

Proof. Let $F = (f_1, f_2, f_3), G = (g_1, g_2, g_3), H = (h_1, h_2, h_3)$ and $D = (d_1, d_2, d_3)$. Let $M = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{bmatrix}$ and suppose $|M| = \pm Q$. In order to show that $QD \in span\{F, G, H\}$, we must show that QD is a linear combination of F, G, and H. In other words, show that there exists some $x_1, x_2, x_3 \in \mathbb{Z}$, such that the following can be solved:

$$\begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} Qd_1 \\ Qd_2 \\ Qd_3 \end{bmatrix}.$$

Using Cramer's Rule over $\mathbb{Q}$, we get:

$$x_1 = \frac{\begin{vmatrix} Qd_1 & g_1 & h_1 \\ Qd_2 & g_2 & h_2 \\ Qd_3 & g_3 & h_3 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} d_1 & g_1 & h_1 \\ d_2 & g_2 & h_2 \\ d_3 & g_3 & h_3 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} d_1 & g_1 & h_1 \\ d_2 & g_2 & h_2 \\ d_3 & g_3 & h_3 \end{vmatrix}.$$

Similarly,

$$x_2 = \frac{\begin{vmatrix} f_1 & Qd_1 & h_1 \\ f_2 & Qd_2 & h_2 \\ f_3 & Qd_3 & h_3 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} f_1 & d_1 & h_1 \\ f_2 & d_2 & h_2 \\ f_3 & d_3 & h_3 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} f_1 & d_1 & h_1 \\ f_2 & d_2 & h_2 \\ f_3 & d_3 & h_3 \end{vmatrix}$$

and

$$x_3 = \frac{\begin{vmatrix} f_1 & g_1 & Qd_1 \\ f_2 & g_2 & Qd_2 \\ f_3 & g_3 & Qd_3 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} f_1 & g_1 & d_1 \\ f_2 & g_2 & d_2 \\ f_3 & g_3 & d_3 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} f_1 & g_1 & d_1 \\ f_2 & g_2 & d_2 \\ f_3 & g_3 & d_3 \end{vmatrix}.$$

Since all the entries in these matrices are in $\mathbb{Z}$, we have that $x_1, x_2, x_3 \in \mathbb{Z}$. Hence, $QD \in span_{\mathbb{Z}}\{F, G, H\}$. Note this implies $QD \in \mathcal{S}_3(\mathbf{A})$. □

We illustrate Lemma 4.3.1 with an example.

Example 4.3.2. Fix the edge labels on $(G, \mathbf{A})$, where $\mathbf{A} = \{2, 3, 4\}$. Let $F = (1, 3, 9), G = (0, 6, 0)$, and $H = (0, 10, 4)$. Clearly, $F \in \mathcal{S}_3(\mathbf{A})$ since $3 \equiv 1 \mod 2$, $9 \equiv 3 \mod 3$, and $9 \equiv 1 \mod 4$. Similarly, $G, H \in \mathcal{S}_3(\mathbf{A})$. Let $Q = \dfrac{2 \cdot 3 \cdot 4}{(2, 3, 4)} = 24$ and $M = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 6 & 10 \\ 9 & 0 & 4 \end{bmatrix}$. Then,

$$|M| = 1 \cdot (6 \cdot 4 - 10 \cdot 0) = 24 = Q.$$

Let $D = (10, 6, 18)$. We see that $D \in \mathcal{S}_3(\mathbf{A})$ since $6 \equiv 10 \mod 2$, $18 \equiv 6 \mod 3$, and $10 \equiv 18 \mod 4$. To show that $QD = (240, 144, 432) \in span\{F, G, H\}$, we need to find

$x_1, x_2, x_3 \in \mathbb{Z}$ such that

$$\begin{bmatrix}1 & 0 & 0\\ 3 & 6 & 10\\ 9 & 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = 24\begin{bmatrix}10\\ 6\\ 18\end{bmatrix}.$$

We see that $x_1 = 240, x_2 = 624$, and $x_3 = -432$ is a solution. Hence, $QD \in span\{F, G, H\}$.

$\Diamond$

Next we show that each edge label divides the determinant of $F, G, H \in \mathcal{S}_3(\mathbf{A})$.

Lemma 4.3.3. *Fix the edge labels on* $(G, \boldsymbol{A})$, *where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$. *Let* $F, G, H \in \mathcal{S}_3(\boldsymbol{A})$. *Then,* $a_1 \big| |F, G, H|$, $a_2 \big| |F, G, H|$, *and* $a_3 \big| |F, G, H|$.

Proof. Since $F \in \mathcal{S}_3(\mathbf{A})$, we know that $a_1|(f_1 - f_2)$, $a_2|(f_2 - f_3)$, and $a_3|(f_3 - f_1)$. Similarly for $G, H \in \mathcal{S}_3(\mathbf{A})$. Now, let $M = \big|F, G, H\big|$. Then,

$$M = \begin{vmatrix} f_1 & g_1 & h_1\\ f_2 & g_2 & h_2\\ f_3 & g_3 & h_3\end{vmatrix} = \begin{vmatrix} f_1 - f_2 & g_1 - g_2 & h_1 - h_2\\ f_2 & g_2 & h_2\\ f_3 & g_3 & h_3\end{vmatrix} = a_1\begin{vmatrix} x_1 & x_2 & x_3\\ f_2 & g_2 & h_2\\ f_3 & g_3 & h_3\end{vmatrix},$$

for some $x_1, x_2, x_3 \in \mathbb{Z}$. Similarly,

$$M = \begin{vmatrix} f_1 & g_1 & h_1\\ f_2 & g_2 & h_2\\ f_3 & g_3 & h_3\end{vmatrix} = \begin{vmatrix} f_1 & g_1 & h_1\\ f_2 - f_3 & g_2 - g_3 & h_2 - h_3\\ f_3 & g_3 & h_3\end{vmatrix} = a_2\begin{vmatrix} f_1 & g_1 & g_1\\ y_1 & y_2 & y_3\\ f_3 & g_3 & h_3\end{vmatrix}$$

and

$$M = \begin{vmatrix} f_1 & g_1 & h_1\\ f_2 & g_2 & h_2\\ f_3 & g_3 & h_3\end{vmatrix} = \begin{vmatrix} f_1 & g_1 & h_1\\ f_2 & g_2 & h_2\\ f_3 - f_1 & g_3 - g_1 & h_3 - h_1\end{vmatrix} = a_3\begin{vmatrix} f_1 & g_1 & h_1\\ f_2 & g_2 & h_2\\ z_1 & z_2 & z_3\end{vmatrix}$$

where $y_1, y_2, y_3, z_1, z_2, z_3 \in \mathbb{Z}$. Hence, $a_1|M, a_2|M$ and $a_3|M$. $\square$

We can strengthen the statement of Lemma 4.3.3. This fact will be useful in the following theorem.

Lemma 4.3.4. *Fix the edge labels on* $(G, \boldsymbol{A})$, *where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$. *Let* $F, G, H \in \mathcal{S}_3(\boldsymbol{A})$, *then* $a_1a_2 \big| |F, G, H|$, $a_2a_3 \big| |F, G, H|$, *and* $a_3a_1 \big| |F, G, H|$.

Proof. Let $M = \big|F, G, H\big|$. From Lemma 4.3.3, we know that

$$M = a_1 \begin{vmatrix} x_1 & x_2 & x_3 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} = a_1 \begin{vmatrix} x_1 & x_2 & x_3 \\ f_2 - f_3 & g_2 - g_3 & h_2 - h_3 \\ f_3 & g_3 & h_3 \end{vmatrix} = a_1 a_2 \begin{vmatrix} f_1 & g_1 & h_1 \\ k_1 & k_2 & k_3 \\ f_3 & g_3 & h_3 \end{vmatrix},$$

for some $k_1, k_2, k_3 \in \mathbb{Z}$. Hence, $a_1 a_2 \big| |F, G, H|$. By a similar argument, $a_2 a_3 \big| |F, G, H|$ and $a_3 a_1 \big| |F, G, H|$. □

The following theorem is an adaptation of Proposition 2.2 from Rose [5].

Theorem 4.3.5. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$*. Let* $Q = \dfrac{a_1 a_2 a_3}{(a_1, a_2, a_3)}$*. If* $F, G, H \in \mathcal{S}_3(\boldsymbol{A})$*, then* $Q \big| |F, G, H|$*.*

Proof. For simplicity, let $M = \big|F, G, H\big|$. From Lemma 4.3.4, we know that $a_1 a_2 | M$, $a_2 a_3 | M$, and $a_3 a_1 | M$. This implies that $[a_1 a_2, a_2 a_3, a_3 a_1] \big| M$. From Theorem 2.1.17, we know that we can rewrite this as $\dfrac{a_1 a_2 a_3}{(a_1, a_2, a_3)} \Big| M$. Hence, $Q | M$. □

The next example highlights Theorem 4.3.5.

Example 4.3.6. Fix the edge labels on $(G, \mathbf{A})$, where $\mathbf{A} = \{2, 3, 4\}$. It is easily verifiable that $F = (6, 4, 22), G = (5, 3, 9)$, and $H = (11, 5, 23)$ are in $\mathcal{S}_3(\mathbf{A})$. Now,

$$\begin{aligned} |F, G, H| = \begin{vmatrix} 6 & 5 & 11 \\ 4 & 3 & 5 \\ 22 & 9 & 23 \end{vmatrix} \\ &= 6(3 \cdot 23 - 9 \cdot 5) - 5(4 \cdot 23 - 22 \cdot 5) + 11(4 \cdot 9 - 22 \cdot 3) \\ &= 144 + 90 - 330 \\ &= -96. \end{aligned}$$

Since, $Q = \dfrac{2 \cdot 3 \cdot 4}{(2, 3, 4)} = 24$ and $24 | -96$, this verifies Theorem 4.3.5. ◊

Lemma 4.3.7. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$*. If* F, G*, and* H *form a basis for* $\mathcal{S}_3(\boldsymbol{A})$*, and* J, K*, and* L *are linear combinations of* F, G*, and* H*, then* $|F, G, H| \Big| |J, K, L|$*.*

Proof. Since, F, G, and H form a basis for $\mathcal{S}_3(\mathbf{A})$ and J, K, and L are linear combinations of F, G, and H, then

$$J = aF + bG + cH \qquad \text{for some a,b,c} \in \mathbb{Z},$$

$$K = dF + eG + fH \qquad \text{for some d,e,f} \in \mathbb{Z},$$

$$\text{and } L = gF + hG + iH \qquad \text{for some g,h,i} \in \mathbb{Z}.$$

Now,

$$\begin{aligned}\begin{bmatrix} J, & K, & L \end{bmatrix} &= \begin{bmatrix} af_1 + bg_1 + ch_1 & df_1 + eg_1 + fh_1 & gf_1 + hg_1 + ih_1 \\ af_2 + bg_2 + ch_2 & df_2 + eg_2 + fh_2 & gf_2 + hg_2 + ih_2 \\ af_3 + bg_3 + ch_3 & df_3 + eg_3 + fh_3 & gf_3 + hg_3 + ih_3 \end{bmatrix} \\ &= \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{bmatrix} \cdot \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.\end{aligned}$$

By the properties of determinants, we know that $|AB| = |A||B|$. Hence,

$$\begin{aligned}|J, K, L| &= \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} \cdot \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \\ &= \begin{vmatrix} F, & G, & H \end{vmatrix} \cdot \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}.\end{aligned}$$

Therefore, $|F, G, H| \Big| |J, K, L|$, since $a, b, \ldots, i \in \mathbb{Z}$ implies $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \in \mathbb{Z}$. □

Lemma 4.3.8. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2, a_3\}$. If $\{F, G, H\}$ is a basis for $\mathcal{S}_3(\boldsymbol{A})$ and $\{J, K, L\}$ is another basis for $\mathcal{S}_3(\boldsymbol{A})$, then $|F, G, H| = \pm|J, K, L|$.*

Proof. Let $|F, G, H| = D$. From Lemma 4.3.7, we know that $D\Big||J, K, L|$. Hence, for some $x \in \mathbb{Z}, Dx = |J, K, L|$. Now, since $\{J, K, L\}$ is another basis, then from Lemma 4.3.7, $|J, K, L|\Big||F, G, H|$. Hence, for some $y \in \mathbb{Z}$, we have:

$$|J, K, L| \cdot y = |F, G, H| \Rightarrow Dxy = D \Rightarrow xy = 1 \Rightarrow y = \pm 1.$$

Hence, $|F, G, H| = \pm|J, K, L|$. □

The following theorem and techniques were inspired by Theorem 2.3 from Rose [5].

Theorem 4.3.9. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$*. Let* $Q = \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$ *and let* $F, G, H \in \mathcal{S}_3(\boldsymbol{A})$*. Then,* $\{F, G, H\}$ *form a module basis for* $\mathcal{S}_3(\boldsymbol{A})$ *if and only if* $\big|F, G, H\big| = \pm Q$*.*

Proof. $\Rightarrow$ From Theorem 3.3.9, we know that the smallest flow-up classes, $\{F_0, F_1, F_2\}$, form a module basis for $\mathcal{S}_3(\mathbf{A})$. From Corollary 4.2.4, we know that $|F_0, F_1, F_2| = \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$. Now, from Lemma 4.3.8, we know that $|F_0, F_1, F_2| = \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)} = \pm|F, G, H|$, where F, G, H is another module basis for $\mathcal{S}_3(\mathbf{A})$. Hence, $\pm|F, G, H| = \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$, or $|F, G, H| = \pm Q$.

$\Leftarrow$ To show that $\{F, G, H\}$ form a basis for $\mathcal{S}_3(\mathbf{A})$ we must show that $\{F, G, H\}$ is linearly independent and spans $\mathcal{S}_3(\mathbf{A})$.

We see that F, G and H are linearly independent since $\pm Q \neq 0$. Now, let $D \in \mathcal{S}_3(\mathbf{A})$. To show that $\{F, G, H\}$ span $\mathcal{S}_3(\mathbf{A})$, we need to show that D is a linear combination of $\{F, G, H\}$. From Lemma 4.3.1, we know that

$$QD = x_1F + x_2G + x_3H$$

for some $x_1, x_2, x_3 \in \mathbb{Z}$. Now,

$$\begin{aligned}
\pm x_1 Q =& x_1|F, G, H| \\
=& |x_1F, G, H| \\
=& |(x_1F + x_2G + x_3H), G, H| \\
=& |QD, G, H| \\
=& Q|D, G, H|.
\end{aligned}$$

This implies $x_1 = \pm|D, G, H|$. Now, from Theorem 4.3.5, we know that $Q\big||D, G, H|$. Hence, for some $s_1 \in \mathbb{Z}$, $s_1Q = |D, G, H| \Rightarrow x_1 = \pm s_1Q$. Similarly, $x_2 = \pm s_2Q$ and $x_3 = \pm s_3Q$, where $s_2, s_3 \in \mathbb{Z}$. Now,

$$\begin{aligned}
&QD = x_1F + x_2G + x_3H \\
\Rightarrow &QD = \pm(s_1Q)F \pm (s_2Q)G \pm (s_3Q)H \\
\Rightarrow &QD = Q(\pm s_1F \pm s_2G \pm s_3H) \\
\Rightarrow &D = \pm s_1F \pm s_2G \pm s_3H.
\end{aligned}$$

Hence, D is a linear combination of $F, G,$ and H, and $\{F, G, H\}$ span $\mathcal{S}_3(\mathbf{A})$. □

We now show that the smallest flow-up classes form a module basis for $\mathcal{S}_3(\mathbf{A})$.

Corollary 4.3.10. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, a_3\}$*. Let* F_0, F_1 *and* F_2 *be the smallest flow-up classes in* $\mathcal{S}_3(\boldsymbol{A})$*. Then,* $\{F_0, F_1, F_2\}$ *form a module basis for* $\mathcal{S}_3(\boldsymbol{A})$*.*

Proof. From Theorem 4.3.9, we know that F_0, F_1 and F_2 form a basis for $\mathcal{S}_3(\mathbf{A})$ if and only if $\left|F_0, F_1, F_2\right| = \pm\dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$. From Corollary 4.2.4, we know that $\left|F_0, F_1, F_2\right| = \dfrac{a_1a_2a_3}{(a_1, a_2, a_3)}$. Hence, the smallest flow-up classes, $F_0, F_1,$ and F_2, form a module basis for $\mathcal{S}_3(\mathbf{A})$. □

The next example incorporates Theorem 4.3.9 and Corollary 4.3.10.

Example 4.3.11. Lets find the smallest flow-up classes for Example 4.3.2, where the edge labels of the *3-cycle* graph were $\mathbf{A}= \{2, 3, 4\}$. We know that , $F_0 = (1, 1, 1)$ is the smallest flow-up class in $\mathcal{F}_0(\mathbf{A})$. Through several applications of Theorem 4.1.3 and Theorem 4.1.4, we find that the smallest flow-up classes in $\mathcal{F}_1(\mathbf{A})$ and $\mathcal{F}_2(\mathbf{A})$ are $F_1 = (0, 2, 8)$ and $F_2 = (0, 0, 12)$, respectively. Hence, we verify $D = (3, 11, 47) \in span\{F_0, F_1, F_2\}$ by finding $x_0, x_1, x_2 \in \mathbb{Z}$ such that $x_0F_0 + x_1F_1 + x_2F_2 = D$. Now, if we let $x_0 = 3, x_1 = 4$ and $x_2 = 1$, then

$$3\begin{bmatrix}1\\1\\1\end{bmatrix} + 4\begin{bmatrix}0\\2\\8\end{bmatrix} + 1\begin{bmatrix}0\\0\\12\end{bmatrix} = \begin{bmatrix}3+0+0\\3+8+0\\3+32+12\end{bmatrix} = \begin{bmatrix}3\\11\\47\end{bmatrix}.$$

Hence, we see that $D = (3, 11, 47) \in span\{F_0, F_1, F_2\}$. Now, Theorem 4.3.9 tells us that we do not need to find the smallest flow-up classes in order to span $\mathcal{S}_3(\mathbf{A})$. All we need is a set of three arbitrary splines, whose determinant is equal to $+24$ or -24. Now, from Example 4.3.2, we saw that if $F = (1, 3, 9), G = (0, 6, 0)$, and $H = (0, 10, 4)$ then, $|F, G, H| = 24$. The next step is to show that D is a linear combination of F, G, and H, i.e. show that there exists $y_1, y_2, y_3 \in \mathbb{Z}$, such that, $D = y_1 F + y_2 G + y_3 H$. We see that,

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 6 & 10 & 11 \\ 9 & 0 & 4 & 47 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 6 & 10 & 2 \\ 0 & 0 & 4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 6 & 0 & -48 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Hence, when $y_1 = 3, y_2 = -8$ and $y_3 = 5$, $D = 3 \cdot F - 8 \cdot G + 5 \cdot H$ thus, $D \in span\{F, G, H\}$.

$\Diamond$

5

Bases for *n-cycle* splines

In this chapter, we generalize our findings from Section 4.3 for *n-cycle* splines.

5.1 Basis Criteria for *n-cycle* Splines

Before we expand Theorem 4.3.9, we extend the lemmas from Section 4.3. We start off by expanding Lemma 4.3.4.

Lemma 5.1.1. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$*. Let* $F^1, \ldots, F^n \in \mathcal{S}_n(\boldsymbol{A})$*. Define* $\widehat{a}_j = a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n$*, where* $1 \leq j \leq n$*. Then,* $\widehat{a}_j \mid |F^1, \ldots, F^n|$*.*

Proof. Without loss of generality, choose $\widehat{a}_j$ and let $M = |F^1, \ldots, F^n|$. Let $F^i = (f_1^i, f_2^i, \cdots, f_n^i)$, where $1 \leq i \leq n$. Let

$$M = \begin{vmatrix} f_1^1 & f_1^2 & f_1^3 & \cdots & f_1^n \\ f_2^1 & f_2^2 & f_2^3 & \cdots & f_2^n \\ f_3^1 & f_3^2 & f_3^3 & \cdots & f_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}^1 & f_{n-1}^2 & f_{n-1}^3 & \cdots & f_{n-1}^n \\ f_n^1 & f_n^2 & f_n^3 & \cdots & f_n^n \end{vmatrix}. \tag{5.1.1}$$

Applying the rules of determinants to Equation 5.1.1, we get:

$$M = \begin{vmatrix} f_1^1 - f_2^1 & f_1^2 - f_2^2 & f_1^3 - f_2^3 & \cdots & f_1^n - f_2^n \\ f_2^1 - f_3^1 & f_2^2 - f_3^2 & f_2^3 - f_3^3 & \cdots & f_2^n - f_3^n \\ f_3^1 - f_4^1 & f_3^2 - f_4^2 & f_3^3 - f_4^3 & \cdots & f_3^n - f_4^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{j-1}^1 - f_j^1 & f_{j-1}^2 - f_j^2 & f_{j-1}^3 - f_j^3 & \cdots & f_{j-1}^n - f_j^n \\ \boxed{f_j^1} & \boxed{f_j^2} & \boxed{f_j^3} & \cdots & \boxed{f_j^n} \\ f_{j+1}^1 - f_{j+2}^1 & f_{j+1}^2 - f_{j+2}^2 & f_{j+1}^3 - f_{j+2}^3 & \cdots & f_{j+1}^n - f_{j+2}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}^1 - f_n^1 & f_{n-1}^2 - f_n^2 & f_{n-1}^3 - f_n^3 & \cdots & f_{n-1}^n - f_n^n \\ f_n^1 - f_1^1 & f_n^2 - f_1^2 & f_n^3 - f_1^3 & \cdots & f_n^n - f_1^n \end{vmatrix}. \tag{5.1.2}$$

Note, for clarity we box the entries that are not subtracted by anything. Now, since $F^1, \dots, F^n \in \mathcal{S}_n(\mathbf{A})$, this means that for all $1 \le k \le n-1$, $a_k | f_k^i - f_{k+1}^i$ and $a_n | f_n^i - f_1^i$. Let s denote the row number of an entry and t denote the column number of an entry, then there exists $x_{s,t} \in \mathbb{Z}$ such that Equation 5.1.2 can be rewritten as:

$$M = \begin{vmatrix} a_1 x_{1,1} & a_1 x_{1,2} & a_1 x_{1,3} & \cdots & a_1 x_{1,n} \\ a_2 x_{2,1} & a_2 x_{2,2} & a_2 x_{2,3} & \cdots & a_2 x_{2,n} \\ a_3 x_{3,1} & a_3 x_{3,2} & a_3 x_{3,3} & \cdots & a_3 x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1} x_{j-1,1} & a_{j-1} x_{j-1,2} & a_{j-1} x_{j-1,3} & \cdots & a_{j-1} x_{j-1,n} \\ \boxed{f_j^1} & \boxed{f_j^2} & \boxed{f_j^3} & \cdots & \boxed{f_j^n} \\ a_{j+1} x_{j+1,1} & a_{j+1} x_{j+1,2} & a_{j+1} x_{j+1,3} & \cdots & a_{j+1} x_{j+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} x_{n-1,1} & a_{n-1} x_{n-1,2} & a_{n-1} x_{n-1,3} & \cdots & a_{n-1} x_{n-1,n} \\ a_n x_{n,1} & a_n x_{n,2} & a_n x_{n,3} & \cdots & a_n x_{n,n} \end{vmatrix}. \tag{5.1.3}$$

Once we factor out the common multiples, we see that Equation 5.1.3 can be written as,

$$M = \widehat{a}_j \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{f_j^1} & \boxed{f_j^2} & \boxed{f_j^3} & \cdots & \boxed{f_j^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{vmatrix}. \tag{5.1.4}$$

Hence, $\widehat{a}_j | M$, since the determinant from Equation 5.1.4 is also an integer. □

The following lemma is an extension of Theorem 2.1.17.

Lemma 5.1.2. *Let $a_1, a_2, \ldots, a_n$ be integers not all zero. Define $\widehat{a_i} = a_1 \cdots a_{i-1}a_{i+1} \cdots a_n$, for $1 \leq i \leq n$. Then $[\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_n}] = \dfrac{a_1 a_2 \ldots a_n}{(a_1, a_2, \ldots, a_n)}$.*

Proof. Let p be a prime occurring in $a_1, \ldots, a_n$. Let x_i be the power of p in a_i. Then in $\widehat{a_i}$, the power of p is $\sum_{j=1}^{n} x_j - x_i$. Then, in $[\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_n}]$, p occurs with power

$$\max_i \left[\left(\sum_{j=1}^{n} x_j \right) - x_i \right] = \sum_{i=1}^{n} x_i - \min_{j=1}^{n} x_j,$$

Denote

$$[\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_n}] = \prod_{k=1}^{r} p_{(k)}^{\sum_{i=1}^{n} x_i^{(k)} - \min_{j=1}^{n} x_j^{(k)}}.$$

Then,

$$\begin{aligned}
[\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_n}] &= \prod_{k=1}^{r} p_{(k)}^{\sum_{i=1}^{n} x_i^{(k)} - \min_{j=1}^{n} x_j^{(k)}} \\
&= \frac{\prod_{k=1}^{r} p_{(k)}^{\sum_{i=1}^{n} x_i^{(k)}}}{\prod_{k=1}^{r} p_{(k)}^{\min_{j=1}^{n} x_j^{(k)}}} = \frac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}.
\end{aligned}$$

□

From Lemma 5.1.2, we prove the following.

Lemma 5.1.3. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2 \ldots, a_n\}$. Let $Q = \dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)}$. If $F^1, F^2, \ldots, F^n \in \mathcal{S}_n(\boldsymbol{A})$, then $Q \Big| |F^1, F^2, \ldots, F^n|$.*

Proof. From Lemma 5.1.1, we know that $\widehat{a_j} \Big| |F^1, \ldots, F^n|$, where $\widehat{a_j}$ is defined as $\widehat{a_j} = a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n$, for $1 \leq j \leq n$. This implies, $[\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_n}] \Big| |F^1, F^2, \ldots, F^n|$. Then, from Lemma 5.1.2, this means that $\dfrac{a_1 a_2 \cdots a_n}{(a_1, a_2, \ldots, a_n)} \Big| |F^1, \ldots, F^n|$. □

We now generalize Lemma 4.3.7, but we will not go in depth with the proof since it follows the same logic.

Lemma 5.1.4. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$. Let $F^1, \ldots, F^n$ form a basis for $\mathcal{S}_n(\boldsymbol{A})$ and let $J^1, \ldots, J^n$ be linear combinations of $F^1, \ldots, F^n$. Then, $|F^1, \ldots, F^n| \Big| |J^1, \ldots, J^n|$.*

Proof. Since $F^1, \ldots, F^n$ form a basis for $\mathcal{S}_n(\mathbf{A})$, then $J^1, \ldots, J^n$ can be written as a linear combination of $F^1, \ldots, F^n$. Now, let $x_{s,t} \in \mathbb{Z}$, where s denotes the row entry and t denotes the column entry. Then,

$$|J^1, \ldots, J^n| = |F^1, \ldots, F^n| \cdot \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}.$$

Hence, $|F^1, \ldots, F^n| \Big| |J^1, \ldots, J^n|$, since $x_{s,t} \in \mathbb{Z}$ implies $\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \in \mathbb{Z}$. □

Similarly, we will not prove a generalized version of Lemma 4.3.8 in depth.

Lemma 5.1.5. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$. If $F^1, \ldots, F^n$ form a basis for $\mathcal{S}_n(\boldsymbol{A})$ and $J^1, \ldots, J^n$ is another basis for $\mathcal{S}_3(\boldsymbol{A})$, then $|F^1, \ldots, F^n| = \pm|J^1, \ldots, J^n|$.*

Proof. Let $|F^1, \ldots, F^n| = D$. Then, from Lemma 5.1.4, $D \Big| |J^1, \ldots, J^n|$. In other words, for some $x \in \mathbb{Z}$, $Dx = |J^1, \ldots, J^n|$. A similar application of Lemma 5.1.4 shows that for some $y \in \mathbb{Z}$, $|J^1, \ldots, J^n|y = |F^1, \ldots, F^n|$. That is to say, $(Dx)y = D \Rightarrow xy = 1 \Rightarrow y = \pm 1$. Hence, $|F^1, \ldots, F^n| = \pm|J^1, \ldots, J^n|$. □

We now generalize Lemma 4.3.1.

Lemma 5.1.6. *Fix the edge labels on $(G, \boldsymbol{A})$, where $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$. Let $Q = \dfrac{a_1, a_2, \ldots a_n}{(a_1, a_2, \ldots, a_n)}$ and let $F^1, \ldots, F^n, D \in \mathcal{S}_n(\boldsymbol{A})$. Suppose $|F^1, \ldots, F^n| = \pm Q$. Then, QD is in the span of $\{F^1, \ldots, F^n\}$.*

Proof. We need to show that there exist $x_1, \ldots, x_n \in \mathbb{Z}$ such that, $QD = x_1F^1 + \cdots + x_nF^n$.

Now,

$$x_1 = \frac{\left|QD, \quad F^2, \quad F^3, \quad \cdots \quad, F^n\right|}{\left|F^1, \quad F^2, \quad F^3, \quad \cdots \quad, F^n\right|} \qquad \text{By Cramer's Rule over } \mathbb{Q} \tag{5.1.5}$$

$$= Q\frac{\left|D, \quad F^2, \quad F^3, \quad \cdots \quad, F^n\right|}{\pm Q} \qquad \text{Properties of Determinants} \tag{5.1.6}$$

$$= \pm\left|D, \quad F^2, \quad F^3, \quad \cdots \quad, F^n\right|. \tag{5.1.7}$$

Hence, x_1 exists and is in $\mathbb{Z}$ since the entires in the determinant are also in $\mathbb{Z}$. A reapplication of Cramer's Rule over $\mathbb{Q}$, shows that there exist $x_2, \ldots, x_n \in \mathbb{Z}$. Hence, QD is in the $span_{\mathbb{Z}}\{F^1, \ldots, F^n\}$. □

We now show that Theorem 4.3.9 can be generalized for *n-cycles*.

Theorem 5.1.7. *Fix the edge labels on* $(G, \boldsymbol{A})$*, where* $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$*. Let* $Q = \dfrac{a_1a_2\cdots a_n}{(a_1, a_2, \ldots, a_n)}$ *and let* $G^1, \ldots, G^n \in \mathcal{S}_n(\boldsymbol{A})$*. Then,* $G^1, G^2, \ldots, G^n$ *form a module basis for* $\mathcal{S}_n(\boldsymbol{A})$ *if and only if* $\left|G^1, G^2, \ldots, G^n\right| = \pm Q$*.*

Proof. $\Rightarrow$ From Theorem 3.3.9, we know that the smallest flow-up classes, $F_0, \ldots, F_{n-1}$ form a module basis for $\mathcal{S}_n(\mathbf{A})$. From Corollary 4.2.4, we know that

$$|F_0, \ldots, F_{n-1}| = \frac{a_1a_2\cdots a_n}{(a_1, a_2, \ldots, a_n)}.$$

From Lemma 5.1.5, we know that if $G^1, G^2, \ldots, G^n$ is another basis for $\mathcal{S}_n(\mathbf{A})$, then

$$|F_0, \ldots, F_{n-1}| = \pm|G^1, G^2, \ldots, G^n| \Rightarrow \pm|G^1, G^2, \ldots, G^n| = \frac{a_1a_2\cdots a_n}{(a_1, a_2, \ldots, a_n)}.$$

Therefore, $|G^1, G^2, \ldots, G^n| = \pm Q$.

$\Leftarrow$ Since $\left|G^1, G^2, \ldots, G^n\right| = \pm Q \neq 0$, then $G^1, G^2, \ldots, G^n$ are linearly independent. Now, to show that $G^1, G^2, \ldots, G^n$ span $\mathcal{S}_n(\mathbf{A})$, we need to show that for $D \in \mathcal{S}_n(\mathbf{A})$, D is a linear combination of $G^1, G^2, \ldots, G^n$.

Now, from Lemma 5.1.6, we know that for some $y_1, y_2, \ldots, y_n \in \mathbb{Z}$

$$QD = y_1G^1 + \cdots + y_nG^n. \tag{5.1.8}$$

Now,

$$\pm y_1 Q = y_1 \big| G^1, G^2, \ldots, G^n \big| \tag{5.1.9}$$

$$= \big| y_1 G^1, G^2, \ldots, G^n \big| \quad \text{Properties of Determinants} \tag{5.1.10}$$

$$= \big| y_1 G^1 + y_2 G^2 + \cdots + y_n G^n, G^2, \ldots, G^n \big| \quad \text{Properties of Determinants} \tag{5.1.11}$$

$$= \big| QD, G^2, \ldots, G^n \big| \quad \text{By Lemma 5.1.6} \tag{5.1.12}$$

$$= Q \big| D, G^2, \ldots, G^n \big| \quad \text{Properties of Determinants.} \tag{5.1.13}$$

Hence, $y_1 = \pm|D, G^2, \ldots, G^n|$. From Lemma 5.1.3, we also know that $Q \big| |D, G^2, \ldots, G^n| \Rightarrow y_1 = \pm k_1 Q$, where $k_1 \in \mathbb{Z}$. If we repeat Equations 5.1.9 - 5.1.13 for y_i, where $2 \leq i \leq n$, we see that for $k_i \in \mathbb{Z}$, $y_i = \pm k_i Q$. Hence, plugging this result in Equation 5.1.8, we get

$$QD = y_1 G^1 + y_2 G^2 + \cdots + y_n G^n$$
$$\Rightarrow QD = \pm Q k_1 G^1 \pm Q k_2 G^2 \pm \cdots \pm Q k_n G^n$$
$$\Rightarrow D = \pm k_1 G^1 \pm k_2 G^2 \pm \cdots \pm k_n G^n.$$

Hence, D is in the span of $G^1, \ldots, G^n$. Therefore, $G^1, \ldots, G^n$ form a module basis for $\mathcal{S}_n(\mathbf{A})$. □

While we omit the proof, it is easy to show that the smallest flow-up classes, $F_0, \ldots, F_{n-1}$, form a module basis for $\mathcal{S}_n(\mathbf{A})$.

6

Future Work

If we had more time we would have looked at the following:

1. Handschy et al. [1] show that star splines exist as long as they fulfill a certain condition. Can we find the smallest flow-up classes for star splines?

2. Can Theorem 5.1.7 be generalized for any Euclidean Domain?

3. Can Theorem 5.1.7 be generalized for any $(G, \mathbf{A})$, where G is not an *n-cycle* graph?

Bibliography

[1] Julie Melnick Madelina Handschy and Stephanie Reinders., *Integer Generalized Splines on Cycles* (2014), 1-24.

[2] Kenneth H. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley Publishing Company **3rd edition** (1993), 4-156.

[3] Lindsey Scoppetta, *Modules of Splines with Boundary Conditions*, Senior Projects Spring 2012 **Paper 81** (2012), 1-86.

[4] Richard M. Foote David S. Dummit, *Abstract Algebra*, Prentice-Hall Inc. **Second Edition** (1999).

[5] Lauren L. Rose, *Module Bases for Multivariate Splines*, Journal of Approximation Theory 86 **Article No. 0052** (1996), 13-20.

[6] Thomas W. Hungerford, *Graduate Texts in Mathematics*, Springer **Book 73** (2012).