2016

Envy-Free Fair Division With Two Players and Multiple Cakes

Justin J. Shin
Bard College

Envy-Free Fair Division With Two Players and Multiple Cakes

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Justin Shin

Annandale-on-Hudson, New York
May, 2017

Abstract

When dividing a valuable resource amongst a group of players, it is desirable to have each player believe that their allocation is at least as valuable as everyone else's allocation. This condition, where nobody is envious of anybody else's share in a division, is called envy-freeness. Fair division problems over continuous pools of resources are affectionately known as cake-cutting problems, as they resemble attempts to slice and distribute cake amongst guests as fairly as possible. Previous work in multi-cake fair division problems have attempted to prove that certain conditions do not allow for guaranteed envy-free divisions. In this paper, we examine and attempt to generalize a series of proofs by Cloutier, Nyman, and Su regarding the existence of envy-free divisions of multiple cakes amongst two players.

Contents

Dedication

The teachers and professors that taught
This young academic have not
Spared effort nor patience
Despite the frustrations
Of dealing with my oft off thoughts

Acknowledgments

I must thank all of my family and professors for affording me my undergraduate education, but special thanks must be given to my adviser, Professor Maria Belk. Without Maria, the results of this project would not exist. Maria's expertise, patience, and good humor made it easy for me to be excited about working on this project, and have helped me through times of confusion and disappointment.

I must also give special thanks to the other members of my board, Professor Jim Belk and Professor Amir Barghi, who shed new light on a problem that I thought I had understood.

Beyond the math department, I also would like to acknowledge my fellow students for their support.

Thank you Adrian, for listening to my Senior Project woes.

Thank you Alexis, Chanae, Darren, and Jessica for keeping me company and in good spirits, and also for maintaining the fan club.

1
Introduction

1.1 Fair Division Problems

Fair division problems ask how we can divide a valuable resource amongst a group of people such that each person receives an appropriate share. Fair division problems are of immediate practical interest, as mathematical models of fairness are useful when, say, dividing up possessions during a divorce, or determining how to split a plot of land amongst a group of inheritors. What constitutes an appropriate share is determined by the use of different fairness criterion, meant to try and capture intuitive ideas of fairness.

One additional characteristic of fair division problems in Game Theory is that a fair division cannot rely on an arbitrator to decide on a division. Although arbitrators are commonly used in real life to assist with the division of valuable resources, a division in Game Theory only falls under fair division when the division relies on the evaluation of the players to the resource that is to be divided. For instance, if Alice, Bob, and Carl are dividing candies amongst themselves, in order for them to achieve a fair division, they cannot call David over and agree to accept whatever portions David hands them, even if David is an impartial arbitrator.

There are many different fairness criterion used in fair division, but to illustrate what a fair division might look like, let us consider proportional division. *Proportional division* is perhaps the simplest fairness criterion used in Game Theory, all it asks is that each player receive $\frac{1}{n}^{th}$ of the resource by their own evaluation, where n is the number of players. Suppose Alice, Bob, and Carl are dividing six candies amongst themselves, 2 vanilla candies, 2 chocolate candies, and 2 strawberry candies. Alice and Bob value each type of candy equally, but Carl values chocolate candies twice as much as vanilla candies, and strawberry candies three times as much as vanilla candies. They divide the candies so that Alice receives two vanilla candies, Bob receives strawberry chocolate candies, and Carl receives two chocolate candies. Observe that both Alice and Bob believe they received $\frac{1}{3}$ of the resource as they value each type of candy equally. Carl also believes that he received $\frac{1}{3}$ of the resource as he is left with 4 vanilla candies worth of value out of a pool of 12 vanilla candies worth of value. Thus, this division qualifies as a *proportional division*, even though Carl believes that Bob was given a more valuable share than his own share.

Fair division Problems, beyond being classified by their selected fairness criterion, can be classified by whether or not the resource is divisible or indivisible, and whether or not the resource is homogenous or heterogenous. A resource is divisible if we can allocate parts of each item to each player. For instance, dividing a chocolate cake and a vanilla cake between players is divisible because I can assign a slice of chocolate cake and a slice of vanilla cake to one player. A resource is indivisible if each item must be assigned entirely to one person. For instance, dividing candies is indivisible given the assumption that I cannot split each candy into smaller candies. A homogenous resource is a resource where the players only care about the amount of the resource that they receive. One common example is money. A heterogenous resource is a resource, where there are other factors that govern the desirability of an allocation, such as flavor, or toppings. One common example is cake flavors that might be preferred by only some players.

Our paper will focus on dividing divisible heterogenous resources, commonly referred to as cakes in Game Theory.

1.2 Two-player Envy-Free Multi-Cake Division

Our fairness criterion is slightly more involved than the proportional division criterion presented earlier. The condition we are interested is envy-freeness. Intuitively, an envy-free division is a division such that every player does not want to trade any of their pieces for a piece that another player has. The envy-free condition is valuable because it represents a measure of stability. With an envy-free solution, we are assured that no player can complain that someone else got a better allocation than they did.

Additionally, we have a maximizing condition where each player receives their most valuable allocation given a particular cake slicing. This distinguishes envy-free divisions from envy-free piece selections, and we will provide a rigorous definition to these terms later.

In this paper, we will focus on finding envy-free divisions when dividing multiple cakes amongst two players. Particularly, we will examine results proposed by Nyman, Cloutier, and Su in their paper *Two-Player Envy-Free Multi-Cake Division*[1]. We claim that their attempted proof of a conjecture is incorrect, generalize a kind of preference used in their paper, and demonstrate that this preference does indeed behave as Nyman, Cloutier, and Su expected, but does not prove what they had intended to prove.

1.3 Contents

In Chapter 2, we will go over some preliminary topics that we need to understand multi-cake fair-division. In Section 2.1, we go into more detail to illustrate the unique complexities of dealing with multi-cake division as opposed to single-cake division. Section 2.2 is where we will rigorously define cakes, cake slicings, cake allocations, and cake divisions, and introduce some useful notation. In Section 2.3, we go over some terms relating to polytopes. In Section 2.4, we define what we mean by envy-free piece selection as opposed to envy-free division. In Section 2.5, we introduce the Polytope of Slicings, a polytope that represents every cake slicing as a unique point within its bounds, and demonstrate its usefulness in determining if envy-free selections are possible. In Section 2.6 we rigorously define what a partial order is and introduce Hasse diagrams, which we will use in Section 2.7 to rigorously define what a preference is.

In Chapter 4, we begin our review of previous work on our problem starting with Section 3.1, where we state some Theorems and Conjectures made by Cloutier, Nyman, and Su. In Section 3.2, we identify an error in a previously accepted proof. In Sections 3.3 we define what a triangulation on a polytope is, and introduce Sperner's Lemma, so we are prepared for Section 3.4, where we demonstrate the use of Sperner labelings to prove that the existence of a cake slicing that allows for envy-free piece selection is guaranteed under certain conditions.

In Chapter 4 we discuss results, starting with Section 4.1, where we generalize a behavior proposed in Section 3.1, and in Sections 4.2 and 4.3, we prove that this generalization behaves as expected by Cloutier, Nyman, and Su[1] with an arbitrarily high number of cakes. In Section 4.4, we end our results by demonstrating that this generalized behavior behaves as expected by a Conjecture proposed in Section 3.1.

Chapter 4 is dedicated to a short description of our results, and discussion of future research.

2
Preliminaries

2.1 Our Problem

One commonly used example of a fair division algorithm that produces an envy-free division is the I-cut-you-choose method. Suppose Alice and Bob would like to divide a cake. Alice is handed the knife, and she gets to cut the cake into two slices in any way that she would like. After Alice makes the cut, Bob selects which piece he would like, and Alice takes the leftover slice. Alice, because Bob gets to select his slice first, is incentivized to cut the cake in such a way that to her own evaluation, the slices are equal, as if they are not, Bob would grab the larger slice. Once Alice cuts the cake into two equal slices to her best evaluation, Bob selects whichever piece he prefers. If Bob shares Alice's evaluation of the slices, it does not matter which piece he selects, as Alice and Bob both think their assigned slices are exactly half of the cake. However, if Bob has a different evaluation of the value of the slices, then he will take whichever one is larger to him. This does not bother Alice, as she believes that the two slices are equal. In this division Bob believes he got more than his fair share, while Alice believes she has exactly her fair share.

This is another reason why envy-free divisions are valuable. By considering that we only need each player to believe that they have their fair share, differences in evaluations of the cake to be divided often result in one or more of the players believing that they got more than what they deserve, without making anyone else jealous of their share.

While the procedure for two players dividing one cake is fairly simple, fair division algorithms get complicated very quickly when more participants and more cakes are involved. There is an additional question as to whether or not there necessarily exists an envy-free division given a certain number of participants, cakes, and slices on each cake. In this paper, we are interested in finding sufficient conditions for the existence of envy-free divisions with 2 players over c cakes and s slices, where each player receives one slice of each cake. Note that with the I-cut-you-choose method amongst two participants and one cake, there will always be an envy-free result. However, there are cases where envy-free divisions are not guaranteed.

We also cannot simply use the I-cut-you-choose method over every cake and expect an envy-free result, as it might be the case that the players have preferences that span multiple cakes. To understand this, imagine that instead of slicing cakes, we are dividing work shifts between Alice and Bob. We are dividing Monday and Tuesday, and each day will have a morning and evening shift. Alice prefers to have either both evening shifts or both morning shifts, while Bob prefers to have one morning shift and one evening shift. For simplicity, we suppose that these are their only preferences. We will examine what might happen if we attempt to implement the I-cut-you-choose method over these cakes. Suppose Alice cuts. When Alice divides Monday into morning and evening shifts, she divides it as evenly as possible. Because neither Alice nor Bob have selected a day yet, she has no preference over which shift she gets. Note that whatever Bob chooses, Alice would not be envious of his slice yet, as the player preferences only kick in when we have two slices. Suppose Bob selects the Monday morning shift, leaving the Monday evening shift to Alice. They then move onto slicing the second cake. No matter how Alice slices the cake, Bob will always move to take the evening shift, satisfying his one preference, leaving Alice with morning shift. Now, Alice is left with the Monday evening shift and the Tuesday morning shift, thus not satisfying her one preference. She is envious of both of Bob's selections. She wants to steal either Bob's Monday morning shift, so she can have both morning shifts, or Bob's Tuesday evening shift, so she can have both evening shifts. Bob of course, is very happy with the division, and so will not consent to swap shifts with Alice as that would result in him not satisfying his one preference.

So we know that we cannot simply repeat division algorithms for a single cake over a series of cakes, but we have yet to prove that there does not exist an envy-free division for Alice and Bob. We will do this later, as the proof that there exists conditions that admit to no envy-free divisions requires some more set-up. There are cases where it is possible to simply repeat division algorithms for a single cake, but this problem only becomes interesting when players have linked preferences over their cakes, in which the slice of one cake that a player prefers is influenced by the slice of another cake that they might also obtain.

It is trivial to demonstrate that there exists some conditions that do not admit to an envy-free solution. The simplest is that if we have two players, one cake, and we must slice it into one piece. Then one player is saddled with the entire cake while the other sits in envy.

2.2 Cakes and Cake Slicings

A *cake* is a continuous heterogenous resource, and will be represented in our paper as a continuous closed interval $[0, 1]$. A *cake slicing on a single cake* is a tuple that identifies where each cut lies on our cake. For instance, a cake slicing on one cake that is perfectly even is denoted as $(.5)$, and a cake slicing on one cake into three pieces where the first slice is one fourth of the cake, and the second slice is one third of the cake, is denoted as $(.25, .58\bar{3})$.

A *cake slicing* is a c-tuple of $(s-1)$-tuples, where c is the number of cakes and s is the number of slices on each cake. For instance, slicing two cakes where the first cake is cut in to two slices evenly, and the second cake is cut into three slices where the first slice is one fifth of the cake and the last two slices are two fifths of the cake will be represented as $\big((.5), (.2, .6)\big)$.

A *cake allocation* consists of one piece of every cake being assigned to a player. We will notate a cake allocation of c cakes cut into s slices as an n-tuple that consists of the first s letters of the alphabet. For instance, if we were to cut 2 cakes into 3 slices, the allocation (abb) to Player 1 would represent giving Player 1 the a slice of the first cake, the b slice of the second cake, and the b slice of the third cake. For sake of clarity, we can say that the letter type of the slice represents how far the slice is from the leftmost piece, so the a slice would be the leftmost piece, the b slice would be the second-most leftmost piece, and so on.

A *cake division* consists of a cake allocation for every player. We will represent cake divisions by stacking the cake allocations on top of one another, where the topmost allocation maps to Player 1's allocation, and the second-most topmost allocation maps to Player 2's allocation, and so on. For instance, $\left\{ {aaaa \atop bcdc} \right\}$ represents a division where Player 1 receives all type a slices and Player 2 receives a b slice on the first cake, c slice on the second cake, a d slice on the third cake, and a c slice on the fourth cake. So, the first cake is divided into four pieces, and the a slice goes to Player 1 and the b slice goes to Player 2, the second cake is divided into four pieces, and the a slice goes to Player 1 and the c slice goes to Player 2, and so on.

2.3 Convex Polytopes

In order to represent all possible divisions of n cakes amongst 2 players, we will make use of convex polytopes.

A *convex set* is a region such that, for every pair of points within the region, the line segment connecting those points is completely contained within the region.

A *convex hull* of a set X is the smallest convex set that contains X.

A *polytope* is a generalization of a polygon into n-dimensions, but we are interested in a specific type of polytope, a *convex polytope.*

A *convex polytope* is the convex hull of a finite set of points.

Intuitively, A convex polytope can be thought of as a solid that doesn't cave in anywhere that has flat sides.

As an example, a disc in $\mathbb{R}^2$ is a convex set, which is the convex hull of the set of points in the disc's boundary. It is not a Convex Polytope, as it is not the Convex Hull of a finite number of points. A solid square in $\mathbb{R}^2$ is a convex polytope as it is the convex hull of its vertices, and we have a finite number of vertices.

We will be identifying our convex polytopes by the number of dimensions they reside in and the number of vertices they have. A $P(v, d)$ polytope is a polytope that has v vertices and resides in d dimensions. For instance, our solid square in $\mathbb{R}^2$ is 2-dimensional and has 4 vertices, so we will represent it as $P(4, 2)$. For the purposes of this paper, every polytope we invoke will be convex.

2.4 Envy-Freeness in Our Problem

Because we are working with multiple cakes, we have two notions of envy-freeness that we must consider.

A cake division is an *envy-free piece selection* if both players receive their most preferred allocation, and their most preferred allocations are disjoint, meaning that no particular slice of cake is in both allocations.

A cake division is an *envy-free allocation* if neither of our players benefits by exchanging their allocation with the other player.

To illustrate this distinction, we will provide an example of a division that is not an envy-free piece selection, but is an envy-free allocation.

Example 2.4.1. As before, suppose Alice, Bob, and Carl are dividing six candies amongst themselves, 2 vanilla candies, 2 chocolate candies, and 2 strawberry candies. Alice values each type of candy equally, Bob prefers to have one vanilla and one strawberry candy, and Carl values chocolate candies twice as much as vanilla candies, and strawberry candies three times as much as vanilla candies. They divide the candies so that Alice receives one vanilla candy and one chocolate candy, Bob receives one strawberry chocolate candy and one vanilla candy, and Carl receives one strawberry candy and one chocolate candy. Observe that Alice is not envious of either Bob's or Carl's allocation, as she believes all allocations are equal. Observe that Bob is not envious of Alice or Carl as his one preference has been satisfied. Observe that Carl is not envious of either Alice or Bob's allocations, as he values his allocation at five vanilla candies, and values Alice and Bob's allocations at three and four vanilla candies respectively. However, Carl can swap one of his chocolate candies for one of Bob's strawberry candies and improve his position. Thus Carl's most preferred allocation is not disjoint with Bob's most preferred allocation, and so this division is not an envy-free piece selection, but is an envy-free allocation. ♢

We are interested in conditions for envy-free piece selections. Note that all envy-free piece selections are envy-free allocations, but as we have shown in our example, not all envy-free Allocations are envy-free piece selections.

2.5 Polytope of Slicings

We will represent a proposed slicing of c cakes into s slices using a polytope. We will construct the polytope as follows.

Suppose that we have c cakes of length 1 which must be sliced into s slices. We will denote the j^{th} piece of the i^{th} cake as x_{ij}. Then a slicing of one cake can be represented as $x_{i1} + x_{i2} + ... + x_{is}$. A slicing of all c cakes into s slices each can then be represented as a series of these expressions shortened to a matrix like so:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1s} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{c1} & x_{c2} & x_{c3} & \dots & x_{cs} \end{bmatrix}$$

If we suppose that the cakes have been cut in such a way that one slice holds the entire cake and all other slices have nothing in them, our matrix would be such that every row has one 1 and the rest of the entries would be 0. A *convex combination* is a linear combination whose coefficients are non-negative and sum to 1. Any slicing of these cakes can be represented as a convex combination of pure slicings, and so it is natural to think of the possible slicings as existing as points in a polytope whose vertices represent pure slicings. The dimension of the polytope is $c(s-1)$ as the last entry on each row is determined by its other components. We have s^c vertices, one for each pure slicing. So our polytope is a $P(s^c, c(s-1))$ polytope. Note that this matrix notation works differently than our previously defined tuple notation as the matrix notation's entries correspond to the size of the slice the entry represents, while the entries in the tuple notation correspond to the location of the slices. To illustrate the difference, a cake slicing of two cakes into three pieces where the first cake is cut into even thirds and the second cake is cut such that the first two slices are of size .1 and the last slice is of size .8 can be represented in matrix form as $\begin{bmatrix} .3 & .3 & .3 \\ .1 & .1 & .8 \end{bmatrix}$ or in tuple form as $\left((\frac{1}{3}, \frac{2}{3}), (.1, .2)\right)$.

Example 2.5.1. Consider the Polytope of slicings of 2 cakes and 2 slices. $\Diamond$

Our polytope is of $P(2^2, 2(2-1))$, or $P(4,2)$, which is a square. Our matrix of slicings looks like: $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ A slicing that cuts each cake perfectly in half would be represented by $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ while a slicing that cuts the first cake into a 1/4 slice and a 3/4 slice and the second cake into a 2/3 and a 1/3 slice would be $\begin{bmatrix} .25 & .75 \\ .\bar{6} & .\bar{3} \end{bmatrix}$. We can now draw the polytope of slicing itself and label our example slicings as points, as shown in Figure 2.5.1.

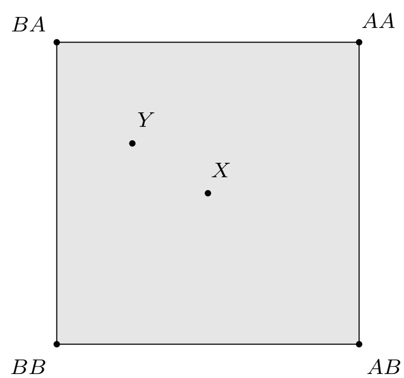


Figure 2.5.1.

In Figure 2.5.1, point X represents the division $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ and point Y represents the division $\begin{bmatrix} .25 & .75 \\ .\bar{6} & .\bar{3} \end{bmatrix}$.

We can see that the vertices of our square represent the pure slicings, and every point in the square corresponds to a unique slicing of the two cakes, and every slicing of the two cakes has a unique corresponding point on our square. The vertices of the square are labelled to annotate the Pure Division. For instance, the lower left vertex is labelled BB because that vertex corresponds to a slicing where the first slice of both cakes is of size 0, and thus both cakes are of type B. The preferences of our players can be represented by coverings of this square composed of 4 pieces, one piece for each vertex. We use the labelled vertices aa, bb, ab, ba to represent possible piece selections. The covering is constructed by placing point p in our polytope in the set that contains vertex xy if in the slicing of the cakes represented by p, our player would prefer the selection x, y.

Note that the last column of our matrix is unnecessary because we are not throwing away any cake and so knowing how large every piece of the cake is except for the last one is enough to determine the size of the last piece, as we represent the whole cake as having a length of 1.

We can now consider a hypothetical covering that represents Alice and Bob's preferences. Recall that Alice prefers to have both morning shifts and Bob prefers to have one of each shift. Cloutier, Nyman, and Su provide a polytope in figure Figure 2.5.2 with hypothetical coverings for both players.

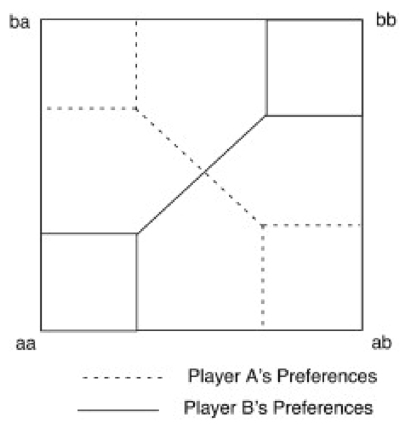


Figure 2.5.2. [1]

Observe that Alice's preferences are expressed by having her only prefer the *ab* and *ba* slices only if we cut the cake in such a way that we have one morning or evening shift reasonably sized and the opposite shift very small. Similarly, Bob only prefers the *aa* and *bb* slices when both morning shift or both evening shift are very large. In order for Alice and Bob to prefer disjoint pieces selections, there must be some point p in our polytope of slicings that resides in the *aa* preference of one and the *bb* preference of the other, or the *ab* preference of one and the *ba* preference of the other. A quick check of the sketch shows that there is no such point, and so given 2 cakes, each cut into 2 slices, there exists preferences such that there does not exist a cake slicing that admits to envy-free piece selections.

2.6 Partial Orders and Hasse Diagrams

We will be using partial orders to help define what exactly a preference is in our problem. Intuitively, a partial order is an arrangement of elements in a set where some elements precede other elements. In this paper, we will be using partial orders to describe how players prefer some allocations over others.

A binary relation $\leq$ on a set P is *reflexive* if for all $x \in P$ it is true that $x \leq x$.

A binary relation $\leq$ on a set P is *anti-symmetric* if for all $x, y \in P$ it is true that if $x \leq y$ and $y \leq x$ then $x = y$.

A binary relation $\leq$ on a set P is *transitive* if for all $x, y, z \in P$ it is true that if $x \leq y$ and $y \leq z$ then $x \leq z$.

Formally, a *partial order* is a binary relation $\leq$ on a set P that is reflexive, anti-symmetric, and transitive.

As an example, the binary relation of being a subset $\subseteq$ imposes a partial order on the power set of $\{1, 2, 3\}$. It is reflexive, as every set is a subset of itself, it is anti-symmetric, as if set A is a subset of set B, and set B is a subset of set A, then sets A and B are the same set. It is transitive because if set A is a subset of set B, and set B is a subset of set C, then set A is a subset of set C.

To help visualize partial orders, we will be making use of Hasse Diagrams. Let $\leq$ be a partial order on set P. A Hasse diagram represents every element of P as a vertex on a graph. A vertex x is connected to another vertex y above it if $x \leq y$, and there is no element $z \in P$ such that $x \leq z \leq y$. In Figure 2.6.1 we have the Hasse diagram of the partial order on $\{1, 2, 3\}$ using the subset relation that we used as an example.

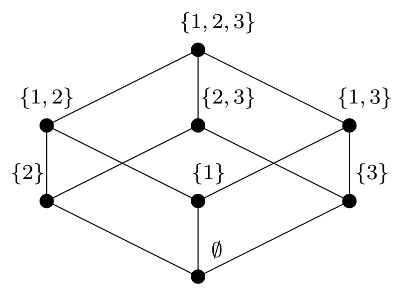


Figure 2.6.1.

2.7 Preferences in Our Problem

Definition 2.7.1. A *preference* in our problem is a function p from the set of all possible cake slicings to the set of all partial orders of allocations. that satisfy the following three criterion.

1. *Independence of preferences* - A player's preferences cannot be affected by choices made by other players.

2. *Hungry Players* - Every player would prefer a non-empty piece to an empty piece.

3. *Preference sets are closed* If an allocation is preferred for a convergent sequence of cake slicings in the polytope of divisions, then that piece selection will be preferred in the limiting cake slicing.

△

For instance, in Figure 2.7.1, the preference function p has mapped a cake slicing where the first cake is cut at .2 and .6 and the second cake is cut at .3 and .5 to a partial ordering of all allocations. As a reminder, slice a refers to the leftmost slice, slice b to the second leftmost slice, and so on.

$$p\Big(\big((.2, .6), (.3, .5)\big)\Big) =$$

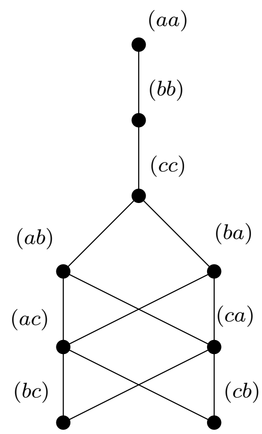


Figure 2.7.1.

We inherit the three criterion stated in our definition of preference from Cloutier, Nyman, and Su[1]. Further discussion of the third condition is needed, as we claim that the preference stated in Cloutier, Nyman, and Su's paper for Conjecture 2.1.2 does not follow the criterion that preference sets are closed.

The three criterion originate from Stromquist's *How to Cut a Cake Fairly*[5] where he proposes a procedure to produce an envy-free division for three people dividing a single cake, in an attempt to capture our intuitions on what a preference over a cake would look like. The first criterion removes the possibility of a player forming their preferences in response to another player, which would suggest that they don't have preferences that come out of an evaluation of the actual cake. The second criterion makes sure that every player actually wants a piece of the cake. The last criterion ensures that arbitrarily small, normally undetectable variations in where the knife lands do not change what allocation we prefer.

3
Review of Previous Work

3.1 Previous Work on Envy-Free Multi-cake Fair Division

Cloutier, Nyman, and Su considered multi-cake division in their paper *Two-Player Envy-free Multi-cake Division*[1]. They proved:

Theorem 3.1.1. *Given 2 players and 2 cakes, there does not necessarily exist a division of the cakes into 2 slices each that contains disjoint envy-free piece selections for those players.*

Conjecture 3.1.2. *Given 2 players and 3 (or more) cakes, there does not necessarily exist a division of the cakes into 3 slices each that contains disjoint envy-free piece selections for the two players.*

The above conjecture is stated as a theorem in the original paper, and was the result we had set out to generalize to n cakes and $n+1$ slices, but the proof presented in Cloutier, Nyman, and Su's paper contains an error which we will address later, and so this remains a conjecture.

Theorem 3.1.3. *Given 2 players and 2 cakes, there is a division of the cakes, one cut into 2 (or more) pieces, the other cut into 3 (or more) pieces, so that the players have disjoint envy-free piece selections.*

Theorem 3.1.4. *Given 2 players and 3 cakes, there exists a division of the cakes into 4 (or more) pieces each that contains disjoint envy-free piece selections for both players.*

From these theorems, it is natural to form two conjectures.

Conjecture 3.1.5. *Given 2 players and* n *cakes, there does not necessarily exist a division of the cakes into* n *pieces each that contains disjoint envy-free piece selections for those players.*

Conjecture 3.1.6. *Given 2 players and* n *cakes, there exists a division of the cakes into* $n+1$ *(or more) pieces each that contains disjoint envy-free piece selections for both players.*

Intuitively Conjecture 2.8.6 can be understood as saying "If I know the preferences of my two players, and I am allowed to slice n cakes into at least $n + 1$ slices, there always exists a way I can slice the cakes so that if each player reached for their most preferred allocation that consists of one slice out of each cake, the two players would not conflict." Theorem 2.8.3 and Theorem 2.8.4 can be understood analogously.

Intuitively, Conjecture 2.8.5 can be understood as saying "If I have two players and n cakes, and I am only allowed to slice each cake into n slices, my two players might have preferences such that no matter how I cut the cakes, if each player reached for their most preferred allocation that consists of one slice out of each cake, they would reach for the same slice one or more times."

In order to outline the general method of proving that there does not necesarily exist an envy-free division under certain conditions, I provide a restatement of Cloutier, Nyman, and Su's attempted proof for Conjecture 3.1.2.

Proof. Their method, in brief, is by contradiction. They suppose that there does exist a disjoint envy-free division under a specified preference, and demonstrate that this produces a contradiction. In their proof, they provide the following preference description:

Each cake is divided into three slices, label each left slice a, each middle slice b, and each right slice c. Let $\epsilon > 0$.

Player 1 prefers, in descending order of preference:

1. Three pieces of the same type. (i.e., aaa, bbb, ccc)
2. Two pieces of the same type. (i.e., aab, bbc)
3. Pieces of all different types. (i.e., abc)

Player 2 prefers the reverse. In descending order of preference:

1. Pieces of all different types. (i.e., abc)
2. Two pieces of the same type. (i.e., aab, bbc)
3. Three pieces of the same type. (i.e., aaa, bbb, ccc)

Neither player will accept a piece selection if any of its pieces are of size less than ϵ. Each player, given multiple selections in the same-preference category, will select the piece selection that has the greatest total size, and in the case where multiple piece selections are of the same size and preference category, they will select the slice that corresponds to the lexicographic first option.

We show that for any piece selection chosen by Player 1, Player 2 prefers a piece selection which is not disjoint from Player 1.

Case 1: Suppose that Player 1's piece selection consists of all pieces of the same type. Without loss of generality, suppose Player 1's piece selection is (aaa). Then Player 2's piece selection, because it must be disjoint, cannot contain any a slices, and so must contain either a doubled b or a doubled c by the pigeonhole principle. Then Player 2 is envious of the a slice on the cake where he has a doubled preference. For instance, if the division was $\left\{\substack{aaa\\bbc}\right\}$, then Player 2 would be envious of Player 1's slice of the first cake, as swapping their pieces would result in $\left\{\substack{baa\\abc}\right\}$, a better result for Player 2.

Case 2: Suppose that Player 1's piece selection consists of two pieces of the same type. Without loss of generality, suppose Player 1's piece selection is (aab). Then the a slice of the thrid cake must be of a size less than ϵ, as if it was not, then Player 1 would have preferred the a slice over the b slice in the third cake. And so Player 2 must prefer the c slice of the third cake, as it cannot accept the $a$slice as it is too small, and cannot take the b slice because Player 1 and Player 2 have disjoint piece selections. Player 2 also cannot prefer the a slice on the first and second cakes, because their piece selections are disjoint, and so Player 2's piece selection must contain either a doubled b or a doubled c by the pigeonhole principle.Then Player 2 is envious of the a slice on the cake where he has a doubled preference that is not the third cake. For instance, if the division was $\left\{\substack{aab\\bcb}\right\}$ then Player 2 would be envious of Player 1's slice of the first cake, as swapping their pieces would result in $\left\{\substack{bab\\abc}\right\}$, which is a better result for Player 2.

Case 3: Suppose that Player 1's piece selection consists of pieces of all different types. Without loss of generality, suppose Player 1's piece selection is (abc). Then it must be the case that the c slice of the third cake is the only slice on that cake that is of size greater than or equal to ϵ, because if the a or b slices on the third cake were of size greater than or equal to ϵ, Player 1 would have preferred those slices over the c slice on the third cake. Then Player 2 must be envious of Player 1's c slice on the third cake as it is the only one of size greater than or equal to ϵ.

In each case, Player 2 cannot have a disjoint piece selection, and so given 2 players and 3 (or more) cakes, there does not necessarily exist a division of the cakes into 3 pieces each that contains disjoint envy-free piece selections for the two players. □

3.2 Problems with the proposed proof of Conjecture 2.1.2

The problem lies within the description of the player preferences, provided below:

Each cake is divided into three slices, label each left slice a, each middle slice b, and each right slice c. Let $\epsilon > 0$.

Player 1 prefers, in descending order of preference:

1. Three pieces of the same type. (i.e., aaa, bbb, ccc)
2. Two pieces of the same type. (i.e., aab, bbc)
3. Pieces of all different types. (i.e., abc)

Player 2 prefers the reverse. In descending order of preference:

1. Pieces of all different types. (i.e., abc)
2. Two pieces of the same type. (i.e., aab, bbc)
3. Three pieces of the same type. (i.e., aaa, bbb, ccc)

Neither player will accept an allocation if any of its pieces are of size less than ϵ. Each player, given multiple selections in the same preference category, will select the piece selection that has the greatest total size, and in the case where multiple piece selections are of the same size and preference category, they will select the slice that corresponds to the lexicographic first option.

The issue is that the ϵ characteristic can render preferences not closed. As an example, consider a simpler case: dividing a single cake where $\epsilon = .01$, and a player that prefers the first slice of the cake unless it is of size less than ϵ. (For this example we depart from the preference outlined by Cloutier, Nyman, and Su). We can construct a convergent sequence of cake slicings under which an allocation is preferred, but a different allocation is preferred in the limiting cake slising. The simplest is the sequence of cake slicings $(a_1), (a_2), (a_3)$... where $\lim_{n \longrightarrow \infty} a_n = .01$ and where the sequence $\{a_n\}_{n=1}^{+\infty}$ is strictly increasing and each a_n is positive. Note that in each (a_n) the first slice would be of size less then ϵ, and so our player would prefer the second slice, but in the limiting cake slicing, (.01), our player prefers the first slice.

Using the preferences in the attempted proof provided in Section 3.1, a similar convergent sequence can be constructed that demonstrates that the preference used by Cloutier, Nyman, and Su is not closed.

Example 3.2.1. Suppose $\epsilon = .01$, and let the preferences of our players be the same as the preferences used in in Cloutier, Nyman, and Su's attempted proof of Conjecture 3.1.2[1]. Consider the sequence of cake slicings $\big((a_1, a_1 + .5), (.5, .5 + a_1), (.5, 1 - a_1)\big)$, $\big((a_2, a_2 + .5), (.5, .5 + a_2), (.5, 1 - a_2)\big)$, $\big((a_3, a_3 + .5), (.5, .5 + a_3), (.5, 1 - a_3)\big)$..., pictured in Figure 3.2.1, where $\lim_{n \longrightarrow \infty} a_n = \epsilon$ and the sequence $\{a_n\}_{n=1}^{+\infty}$ is strictly increasing and each a_n is positive.

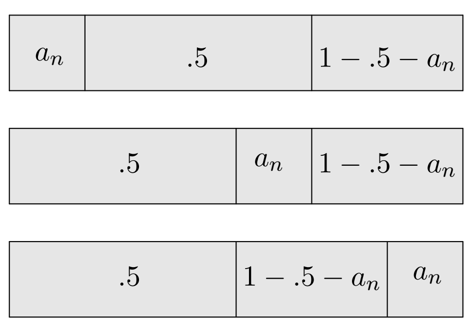


Figure 3.2.1.

Observe that in each of the divisions in the sequence of cake slicings, Player 1 prefers the allocation (baa), as Player 1 seeks to maximize the number of common piece types, but cannot choose (aaa), (bbb), or (ccc) due to the distribution of pieces of size less than ϵ.

However, at the limiting cake slicing of this sequence of cake slicings, the a_n terms approach ϵ, and so the limiting cake slicing is simply $\big((.01, .01 + .5), (.5, .5 + .01), (.5, 1 - .01)\big)$, as pictured in Figure 3.2.2.

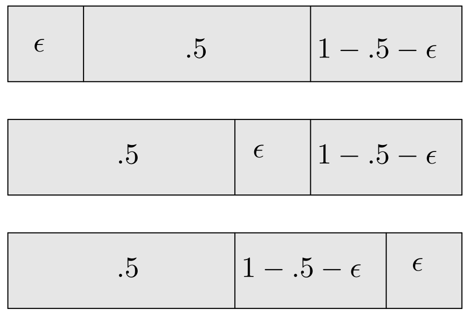


Figure 3.2.2.

In this limiting cake slicing, Player 1 is no longer restricted by the "less than ϵ" condition and so is allowed to select (aaa), which he prefers over the allocation preferred in all cake slicings in the sequence of cake slicing, (baa). Because there exists a sequence of cake slicings such that Player 1 prefers one allocation in all divisions in the sequence, but prefers a different allocation at the limiting division, the preferences used in the attempted proof is not closed, and thus the proof does not hold. $\Diamond$

Note that a convergent sequence of limiting cake slicings can be seen as a convergent sequence of points on the corresponding Polytope of divisions. In our first example with one cake sliced into two parts, out Polytope is of $P(2^1, 1(2-1))$, or a line segment from 0 to 1. On this polytope, the sequence we defined is a series of points on this line segment that approach ϵ.

On our example using the cakes and preferences of the attempted proof of Conjecture 2.2.2, our Polytope is of $P(3^3, 3(3-1)$, or a $(27, 6)-$polytope. Each cake slicing corresponds to a point in this polytope that converges to the point precisely in the center. It is difficult to see this convergence in a six dimensional polytope, so I will introduce one more example using a division of two cakes into two slices each.

As stated earlier, the Polytope of slicings for two cakes and two slices is a square. Suppose $\epsilon = .25$. Our convergent sequence of cake slicings is as follows, $\big((a_1),(1-a_1)\big), \big((a_2),(1-a_2)\big), \big((a_3),(1-a_3)\big)$... where $a_n = \epsilon - \frac{\epsilon}{2n}$. As we can see in Figure 3.2.3, this sequence of divisions converges to $\big((\epsilon),(1-\epsilon)\big)$. In this case, a Player with analogous preferences to the preferences of Player 1 in the attempted proof of Conjecture 2.2.2 will prefer allocation (ba) in all divisions that compose the sequence, but in the limiting division, our Player will prefer the allocation (aa).

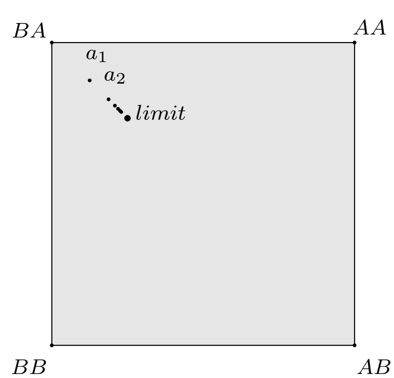


Figure 3.2.3.

3.3 Triangulation and Sperner's Lemma

Here we introduce some tools that will help us use the Polytope of division.

A *k-simplex* is a convex hull of $k+1$ vertices which are affinely independent. Affine independence means that if we suppose our $k+1$ vertices are $p_0, P_1, ..., p_k$, it is true that $p_1 - p_0, p_2 - p_0, ..., p_k - p_0$ are linearly independent. A k-simplex is a generalization of a triangle into arbitrary dimensions.

A *triangulation* of a polytope P is a collection of distinct simplices that satisfy the following criterion:

- The union of all simplices in the triangulation is P.
- The intersection of any two simplices in the triangulation is either empty, or a face shared by both simplices.
- Every face of a simplex in the triangulation is also in the triangulation.

A *sperner labelling* of a triangulation T of a polytope P is a labelling of the vertices of T that satisfy the following criterion:

- All of the vertices of P have distinct labels.
- The label of any vertex of T which lies on a facet of P matches the label of one of the vertices of P that spans that facet.

For a triangulation T of a (v, d)-polytope P, a *full cell* is any $d-$dimensional simplex in T such that each of its $d+1$ vertices have distinct labels.

Cloutier, Nyman, and Su provide two examples of a Sperner labelling, which are presented in Figure 3.3.1.

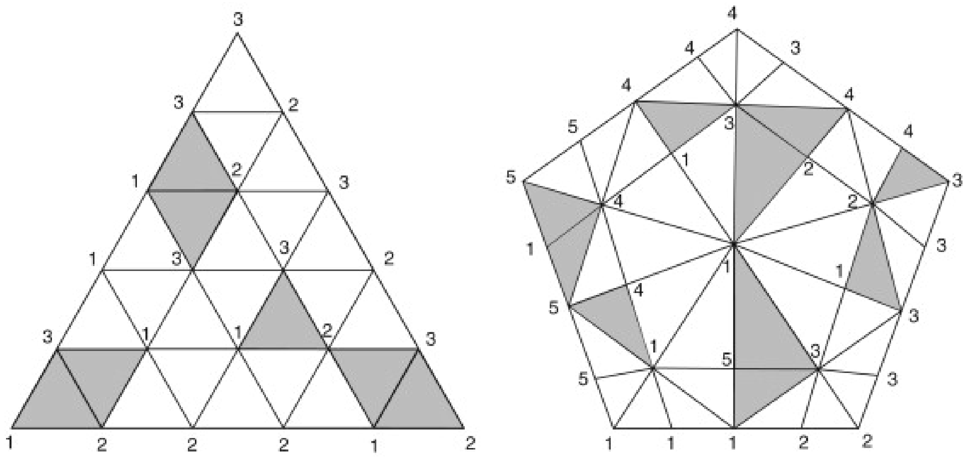


Figure 3.3.1. [1]

Sperner's Lemma states that any Sperner-labeled triangulation of a simplex contains an odd number of full cells.3.3.1 In order to generalize to polytopes, we will use a theorem by De Loera, Peterson, and Su.

Theorem 3.3.1. *Any Sperner-labeled triangulation of an $(v, d)-$polytope P must contain at least $(v - d)$ full cells.* [2]

We will use Sperner labelings on polytopes of division in order to prove that there always exist a cake slicing that admits to envy-free piece selections under certain conditions.

3.4 Sperner-Labelings on Polytopes of Divisions

We will now describe how we will is Sperner-labelings on polytopes of division to find envy-free divisions.

Suppose there are two players, Player A and Player B. Let T be a triangulation on Polytope P.

An *owner-labelling* of a triangulation is a labelling where each vertex in the triangulation is labelled with an a or a b. An owner-labelling is *uniform* if in each simplex, the number of vertices labelled for one player differs by at most one from any other player. Any polytope has a triangulation of arbitrarily small mesh size that can be given a uniform owner-labelling.[1]

A *preference-labelling* of T assigns to every vertex v of T, the most preferred allocation in the division that v represents by the owner of the vertex. Because each allocation corresponds to a pure division where the selected slices are the only non-zero selections, we can obtain a preference labeling of the vertices of T by pure divisions.

This new labeling is a Sperner labeling. By theorem 3.3.1 there exists $(v - d)$ $d-$dimensional full cells in T. The owner labelling of each of these cells is uniform, and so a full cell represents $d + 1$ divisions in which players A and B choose different allocations. if we repeat this procedure for a sequence of finer and finer triangulations of P, we would create a sequence of smaller and smaller full cells.

Since P is compact, and thus contains all of its limit points, there must be a convergent sequence of full cells that converges to a single point. Since each full cell in the convergent subsequence also has a uniform owner labeling and since there are only finitely many ways to choose an allocation, there must be an infinite subsequence of our convergent sequence for which the allocations of each player remain unchanged. Because preference sets are closed, the allocations will not change at the limit point of these full cells. So, at this limit point, the players choose different allocations just as they did in the cells of the sequence. Because each vertex is labeled with a players most preferred allocation given a division, they represent a player's piece selection.

To demonstrate how this method can guarantee the existence of envy-free piece selections, we present a sketch of a proof in Cloutier, Nyman, and Su's paper.[1] One additional theorem we will use comes from De Loera, Peterson, and Su, stated below. [2]

Theorem 3.4.1. *Let P be an $(v, d)-$polytope with a Sperner-labeled triangulation T. Let $f : P \to P$ be the piecewise-linear map that takes each vertex of T to the vertex of P that* *shares the same label, and is linear on each $d-$simplex of T. The map f is surjective, and thus the collection of full cells in T forms a cover of P under f.*

We proceed with the sketch of the proof provided by Cloutier, Nyman, and Su.[1]

Theorem 3.4.2. *Given* 2 *players and* 2 *cakes, one cut into 2 pieces, the other cut into 3 pieces, there always exists a cake slicing such that the players have disjoint envy-free piece selections.*

The polytope of divisions for this case is a $(6, 3)-$polytope called P, where the vertices correspond to the pure divisions aa, ab, ac, ba, bb, bc. The polytope can be represented with adjacent vertices containing conflicting piece selections, as shown in Figure 3.4.1 provided by Cloutier, Nyman, and Su.

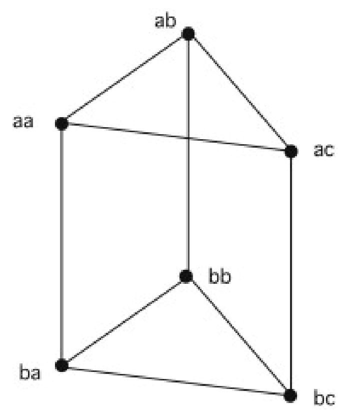


Figure 3.4.1. [1]

Let T be a triangulation of P with a uniform labeling. By Theorem 3.3.1 and Theorem 3.4.1, we know there exists a fully-labeled $3-$simplex s whose image $f(s)$ is one of the simplices on the cover of P, using the function f from Theorem 3.4.1. Observe $f(s)$ is non-degenerate, and its four vertices do not lie on a common face of P.

In this case, one vertex v of $f(s)$ must be non-adjacent in P, to two other vertices, w and y. This means w and y correspond to piece selections that are disjoint from the piece selection of v. Without loss of generality, if A owns v, B must own at least one of w and y

because the owner labeling is uniform. By finding finer owner and preference labelings for our fully labeled simplex s, we can find a convergent sequence of full cells that converge to a single point, as described earlier in this section. Because there will always exist two vertices of each full cell where A and B's piece selections are disjoint, there is an infinite convergent subsequence within our sequence of full cells for which the piece selections of each player remains unchanged. Because preference sets are closed, their selections will not change at the limit point of these full cells. So at this limit point, the players choose disjoint piece selections, as they did in the cells of the sequence.

4
Results

4.1 The Same and Different Preferences

We will generalize the preference set used in the attempted proof of Conjecture 3.1.2 to n cakes, and demonstrate that this preference set forces envious cake sharing in all cases where we have two players sharing n cakes sliced into n slices.

As before, we let $a, b, c, d...$ represent the left most slice, the second most left most slice, the third most left most slice and so on until we have labelled all slices in all cakes with a letter from a to the n^{th} letter of the alphabet.

So if $n = 6$, a preference for the leftmost slice of each cake is represented as $aaaaaa$. A preference for the n^{th} slice of the n^{th} cake is represented as $abcdef$. The set of all preferences of n slices on n cakes is therefore all strings of letters from a to the n^{th} letter that are n characters long. We will call the set of all preferences for n cakes and n slices P_n.

In order to describe the preferences of our two players, we make use of some functions.

Let $m \in \mathbb{N}$ with $m \leq n$. We define the function $f_m : P_n \to \mathbb{N}$ as $f_m(p) =$ the amount of times the m^{th} most commonly preferred letter appears in p. For instance, if $p = aabc$, then $f_1(p) = 2$, and $f_2(p) = 1$, $f_3(p) = 1$, $f_4(p) = 0$.

We define $g_m : P_n \to$ the alphabet $\cup\gamma$, with γ being the last letter of the alphabet, as $g_m(p) =$ the letter that is m^{th} most often preferred in p. If there are multiple letters that are m^{th} most often preferred, then the letter with alphabetical priority is assigned first. If there is no letter that is m^{th} most commonly preferred, $g_m(p) = \gamma$.For instance, if $p = aabc$, then $g_1(p) = a$, and $g_2(o) = b$, $g_3(p) = c$, $g_4(p) = \gamma$.

We will say that Player 1, given two allocations X and Y, will determine their preference ordering for X and Y using the following procedure, called the *same-preference*:

Let $\epsilon > 0$.

1. If X contains a slice of size less than ϵ, and Y does not, then $X < Y$. If Y contains a slice of size less than ϵ, and X does not, then $Y < X$. If neither X nor Y or both X and Y contains a slice of size less than ϵ, move to the next step.

2. If $f_1(X) > f_1(Y)$ then Player 1 prefers X over Y, and if $f_1(Y) > f_1(X)$ then Player 1 prefers Y over X.

3. Let $l \in \mathbb{N}$ and let $l \leq n$. If $f_i(X) = f_i(Y)$ for all $i \leq l$, then if $f_{l+1}(X) > f_{l+1}(Y)$ then Player 1 prefers X over Y, and if $f_{l+1}(Y) > f_{l+1}(X)$ then Player 1 prefers Y over X.

4. If for all functions f_m, it is true that $f_m(X) = f_m(Y)$, then if $g_1(X)$ has higher alphabetical priority then $g_1(Y)$, Player 1 prefers X over Y, and if $g_1(Y)$ has higher alphabetical priority then $g_1(X)$, Player 1 prefers Y over X.

5. Let $l \in \mathbb{N}$ and let $l \leq n$. If $g_i(X) = g_i(Y)$ for all $i \leq l$, then if $g_{l+1}(X) > g_{l+1}(Y)$ then Player 1 prefers X over Y, and if $g_{l+1}(Y) > g_{l+1}(X)$ then Player 1 prefers Y over X.

6. If for all $k \in \mathbb{N}$ such that $k \leq n$, it is true that $f_k(X) = f_k(Y)$ and $g_k(X) = g_k(Y)$, then Player 1 selects the piece whose preference is written with the highest lexical priority. For instance *aabbcd* would be preferred over *bbaacd*.

Player 2 has preferences opposite that of Player 1, except for their lexical preferences. We will say that Player 2, given two preference selections X and Y, will determine their preference ordering for X and Y using the following procedure, called the *different-preference*:

1. If X contains a slice of size less than ϵ, and Y does not, then $X < Y$. If Y contains a slice of size less than ϵ, and X does not, then $Y < X$. If neither X nor Y or both X and Y contains a slice of size less than ϵ, move to the next step.

2. If $f_1(X) > f_1(Y)$ then Player 1 prefers Y over X, and if $f_1(Y) > f_1(X)$ then Player 1 prefers X over Y.

3. Let $l \in \mathbb{N}$ and let $l \leq n$. If $f_i(X) = f_i(Y)$ for all $i \leq l$, then if $f_{l+1}(X) > f_{l+1}(Y)$ then Player 1 prefers Y over X, and if $f_{l+1}(Y) > f_{l+1}(X)$ then Player 1 prefers X over Y.

4. If for all functions f_m, it is true that $f_m(X) = f_m(Y)$, then if $g_1(X)$ has higher alphabetical priority then $g_1(Y)$, Player 1 prefers X over Y, and if $g_1(Y)$ has higher alphabetical priority then $g_1(X)$, Player 1 prefers Y over X.

5. If for all $k \in \mathbb{N}$ such that $k \leq n$, it is true that $f_k(X) = f_k(Y)$ and $g_k(X) = g_k(Y)$, then Player 1 selects the piece whose preference is written with the highest lexical priority. For instance *aabbcd* would be preferred over *bbaacd*.

We will show that this set of preferences for two players will never produce an envy-free division for n cakes sliced into n slices.

As an example, Figure 2.7.1 accurately depicts the preference of Player 1 when $\epsilon < .2$, and we are slicing 2 cakes into 3 slices.

4.2 Results for 4, 5, and 6 cakes

In this section, we present confirmation of Cloutier, Nyman, and Su's suspicion that their defined player behavior in their attempted proof of Conjecture 3.1.2 can generalize to show similar results with more cakes.

Theorem 4.2.1. *If we have* 4 *cakes sliced into* 4 *slices being divided amongst 2 people with the same-preference and the different-preference, there does not exist a cake slicing that admits to a disjoint envy-free piece selection.*

Proof. Our Players have the preferences outlined in section 3.1, where $n = 4$.

We will use a proof by contradiction. Suppose that there exists a cake slicing that admits to a disjoint envy-free piece selection for players 1 and 2 under the stated preferences. We have five cases:

Case 1: Suppose Player 1 prefers all selections of the same type, like $(aaaa)$.

Then Player 2's preferred allocation, if it is disjoint from Player 1's, consists of at most three unique letters and at least one copied letter, as it must select pieces out of four cakes but can only select out of three slice types, as Player 2 cannot use the slice type Player 1 prefers anywhere. Player 2 envies Player 1's slice in whatever cake houses a letter that is used twice in Player 2's piece selection.

Case 2: Suppose Player 1 prefers three selections of the same type, and one selection of a different type, like $(aaab)$.

Without loss of generality, suppose Player 1 prefers (*aaab*). Then Player 2's selection for the fourth cake must either be c or d if the choices are disjoint, as if a in the fourth cake had a size larger than ϵ, Player 1 would have chosen it. Suppose without loss of generality that Player 2 prefers the c slice of the fourth cake. Then Player 2's preferences for the first three cakes cannot be *bbb* or *ccc* as either of them would cause a to prefer either *bbbb* or *cccc* as their preferred piece selection. Suppose Player 2's preferences are *dddc*. Then Player 2 is envious of the a slice in the first, second, and third cakes. Suppose Player 2's preferences involves a c in the first, second, or third cakes. Then Player 3 envies the a selection of whatever cake the second c slice appears in. So then Player 2's preferences for the first three cakes must have either 2 b's or 2 d's in them. In either case Player 2 is envious of the a slice of whatever cakes contain the doubled b or d selections.

Case 3: Suppose Player 1 prefers two selections of the same type, and two selections of a different type, like (*aabb*).

Without loss of generality, suppose Player 1 prefers (*aabb*). The only reason that Player 1 would prefer b over a in the third and fourth cake is if the a slice of the third and fourth cake have a size less than ϵ, so Player 2 cannot accept them either. Additionally, the only reason why Player 1 would prefer a over b in the first or second cakes is because the b slices had a size less than ϵ.

Subcase 3-1: Suppose B prefers c, and d in the third and fourth cakes. Then if Player 2 prefers d in the first cake, he is envious of Player 1's a slice in the first cake. If Player 2 prefers c in the first cake, then he can prefer either c or d in the second cake. In either case he is envious of Player 1's a slice on the second cake.

Subcase 3-2: Suppose B prefers d and d in the third and fourth cakes. Then if Player 2 prefers d in the first cake he is envious of Player 1's a slice in the first cake. If Player 2 prefers c in the first cake, then Player 2 can prefer either c or d in the second cake. In either case Player 2 is envious of Player 1's a slice of the second cake.

Case 4: Suppose Player 1 prefers four selections of the same type, and two selection of two different types, like ($aabc$).

Without loss of generality, suppose Player 1 prefers ($aabc$). Then Player 2's preferences in the third cake cannot be c, because if c in the third cake had a length greater than ϵ, Player 1 would prefer it. It also cannot be a, because if a in the third cake had a length greater than ϵ, Player 1 would prefer it. So Player 2, if they have disjoint preferences, must select piece d for the third slice. By a similar argument, Player 2 must select d for the fourth slice as well. However, if both the d slices of the third and fourth cake have a length greater than ϵ, Player 1 would have preferred ($aadd$), so one of the d slices is too short. Thus, in either the third or fourth cake, Player 1 has selected the only viable piece, and so Player 2 must be envious of that piece.

Case 5: Suppose Player 1's preferred allocation consists of all unique slice types, like ($abcd$).

The only reason why Player 1 would prefer d on the fourth cake over c on the fourth cake is if c had a size less than ϵ. This is also true for the a and b slices. So there is only one valid selection on the fourth cake, and it is being taken by Player 1, and so Player 2 must be envious of that piece.

In all cases, Player 2 is envious of a slice in Player 1's piece selection, and so the division is not disjoint. □

Theorem 4.2.2. *If we have* 5 *cakes sliced into* 5 *slices being divided amongst 2 people with the same-preference and the different-preference, there does not exist a cake slicing that admits to a disjoint envy-free piece selection.*

Proof. Our Players have the preferences outlined in section 3.1, where $n = 5$.

We will use a proof by contradiction. Suppose that there exists a cake slicing that admits to a disjoint envy-free piece selection for players 1 and 2 under the stated preferences.

Case 1: Player 1 prefers all selections of the same type, like ($aaaaa$).

Then Player 2's preferred allocation, if it is disjoint from Player 1's, consists of four unique letters and one copied letters, as it must fill five places using only four letters, as Player 2 cannot use whatever slice type Player 1 prefers anywhere. Player 2 envies Player 1's selection in whatever cake houses a letter that is used twice in Player 2's piece selection.

Case 2: Suppose Player 1 prefers four selections of the same type, and one selection of a different type, like ($aaaab$).

Without loss of generality suppose Player 2 prefers the c slice on the fifth cake. Then Player 2 cannot prefer the c slice on any other cake, because then Player 2 would prefer to change that c slice to an a slice. So we must fill the remaining four slots with only three options, and so we have duplicate selections, and Player 2 is envious of the a slice in one of the duplicate places.

Case 3: Suppose Player 1 prefers three selections of the same type, and two selection of a different type, like ($aaabb$).

Without loss of generality, suppose Player 1 prefers ($aaabb$). Then we have two cases, either Player 2 prefers cc or cd in the last two cakes.

Subcase 3-1: Suppose Player 2 prefers cc. Note that two of the b slices in the first three cakes must have size less than ϵ because if they did not then Player one would have chosen four b selections. So there is only one cake where Player 2 can have a b selection. Observe that if any letter appears twice in the first three cakes, then Player 2 is immediately envious of the a selection in one of the cakes that has duplicate preferences. If no three of the cakes have a slice in common, then Player 2 is envious of the b selection in either of the last two cakes, as swapping to b would raise the lexical priority of Player 2's preferences.

Subcase 3-2: Now suppose Player 2 prefers cd. Then he only has two available choices for three spots, and must produce a duplicate choice in two of the three remaining cakes, and is envious of the a slice in at least one of them.

Case 4. Suppose Player 1 prefers three selections of the same type, and two selection of two different types, like (*aaabc*).

Without loss of generality, suppose that Player 1 prefers (*aaabc*). Then we have two cases.

Subcase 4-1: Suppose Player 2 prefers different types in the last two cakes. Then, we have 2 options for three slots in the first three cakes, and Player 2 is envious of Player 1's slice in one of the cakes with a duplicate slice type.

Subcase 4-2: Suppose Player 2 prefers the same type in the last two cakes. Then we must distribute the remaining three slice types in the first three cakes. No matter how these three slices are arranged, Player 2 would like to simultaneously swap the a slice of the cake with Player 2's c preference and the c slice of Player 1.

Case 5: Suppose Player 1 prefers two selections of one type, two selections of another type, and one selection of still another type, like (*aabbc*).

Without loss of generality, suppose Player 1 prefers (*aabbc*). Suppose Player 2 prefers slice type γ in the last cake. Not that γ cannot be a, b, or c. Notice that none of the b slices on player one's a cakes can be acceptable, and none of the a slices on player one's b cakes are acceptable. Observe that Player 2 cannot prefer γ in any of the first four cakes, as then they would benefit by swapping their first γ slice with Player 1's corresponding a or b slice. Then our first four cakes can only contain two different slice types, and any place where a duplicate slice type occurs, Player 2 is envious of the corresponding slice that Player 1 prefers.

Case 6: Suppose Player 1 prefers two selections of one type, and three selections of all different types, like (*aabcd*)..

Without loss of generality, suppose Player 1 prefers (*aabcd*). Then Player 2 must prefer the e slice on the last cake. But the fourth cake cannot have its a, b, or d slices be larger than epsilon, and the third cake cannot have its a, c, or d slices be larger than epsilon and so Player 2 must prefer the e slice in the last three cakes. And so Player 2 prefers at minimum three common slices. The first two cakes cannot both have a slices larger than epsilon, or both b slices larger than epsilon, or both d slices larger than epsilon, and so Player 2 is envious of Player a's selection for the last three cakes, as taking them would force at most two common slices.

Case 7: Suppose Player 1 prefers all different types, like ($abcde$).

The only reason why Player 1 would prefer e on the fifth cake over d on the fifth cake is if d had a size less than ϵ. This is also true for the a, b, and c slices. So there is only one valid selection on the fifth cake, and it is being taken by Player 1, and so Player 2 must be envious of that piece.

In all cases, Player 2 is envious of a slice in Player 1's piece selection, and so the division is not disjoint. □

Theorem 4.2.3. *Given 2 players and 6 cakes, there does not necessarily exist a division of the cakes into 5 pieces each that contains disjoint envy-free piece selections for the two players.*

Proof. Our Players have the preferences outlined in section 3.1, where $n = 6$.

We will use a proof by contradiction. Suppose there exists a disjoint envy-free division between the two players. We have eight cases that Player 1's preference can fall into.

Case 1: Player 1 prefers all selections of the same type, like ($aaaaaa$).

Then Player 2's preferred allocation, if it is disjoint from Player 1's, consists of five unique letters and one copied letter, as it must fill six places using only five letters, as Player 2 cannot use a anywhere. Player 2 envies Player 1's a selection in whatever cake houses a letter that is used twice in Player 2's piece selection.

Case 2: Player 1 prefers five selections of the same type, and one selection of a different type, like($aaaaab$).

Without loss of generality suppose Player 2 prefers the c slice on the fifth cake. Then Player 2 cannot prefer the c slice on any other cake, because then Player 2 would prefer to change that c slice to an a slice. So we must fill the remaining four slots with only three options, and so we have duplicate selections, and Player 2 is envious of the a slice in one of the duplicate places.

Case 3: Player 1 prefers four selections of the same type, and two selection of a different type, like($aaaabb$).

Without loss of generality, suppose Player 1's allocation is ($aaaabb$).Then we have two cases, either Player 2 prefers slices of the same type, such as cc or slices of a different type, such as cd in the last two cakes.

Case 3-1: Suppose Player 2 prefers slices of the same type, called type γ in the last two cakes. Our division looks like $\left\{ {AAAABB \atop ???\gamma\gamma} \right\}$ where A is the most preferred slice type for Player 1 and B is the second most preferred slice type for Player 1. Observe that Player 2 must have exactly one B preference in the first three cakes, as if he did not have any in the first three cakes, he would be envious of the B slice above one of his γ slices, and if he had multiple B preferences in the first three cakes, he would be envious of the A slice above one of his B slices. And so our division now looks like $\left\{ {AAAABB \atop ?B?\gamma\gamma} \right\}$. Observe that Player 2 can improve their position by simultaneously swapping their B slice for Player 1's A slice and their γ slice for one of Player 1's B slices, giving us a division that looks like $\left\{ {ABAA\gamma B \atop ?A?B\gamma} \right\}$ where the two remaining unknown slots in Player 2's preference must be of a different type from each other and from A, B, and γ.

Case 4: Player 1 prefers four selections of the same type, and two selection of two different types, like ($aaaabc$).

If Player 2 prefers different piece types under Player 1's unique typed piece selections, Player 2 is forced to place three types into four spaces.

If Player 2 prefers the same slice, slice d, under Player 1's unique typed piece selections, then Player 1 should have preferred four of the same type, and two d slices.

Case 5: Player 1 prefers three selections of two different types, like $(aaabbb)$.

Then Player 2 cannot prefer any of the two types taken by Player 1, as that would imply that Player 1 could have had more than three of the same type.

Case 6: Player 1 prefers three selections of one type, two selections of another type, and one selection of another type, like $(aaabbc)$.

Then there are two cases, Player 2 can prefer pieces of the same type under Player 1's pair selection, or Player 2 can prefer different types under Player 1's pair selection.

If Player 2 prefers the same type, then observe that Player 2 prefers one and only one slice of the same type as Player 1's pair selection under Player 1's triple selection. Then Player 2 is envious of one of Player 1's pair selections.

If Player 2 prefers different types, then we must force either a pair in Player 1's triple, in which case Player 2 is envious of one of Player 1's triple slices, or we force a pair with one piece under Player 1's triple, and the other in Player 1's unique selection, in which case Player 2 is envious of Player 1's triple selection.

Case 7: Player 1 prefers two selections of three different types, like $(aabbcc)$.

Then Player 2 cannot prefer any of the types taken by Player 1, as that would imply that Player 1 could have selected a triple.

Case 8: Player 1 prefers selections of all different types, like $(abcdef)$.

By a similar argument from Case 7 in Theorem 3.2.2, Player 2 is envious of at least one of Player 1's slices.

In all cases, Player 2 is envious of a slice in Player 1's piece selection, and so the division is not disjoint. □

4.3 General result for n cakes and $n+1$ slices

In this section, we present a proof that confirms Cloutier, Nyman, and Su's suspicion that their defined player behavior in their attempted proof of Conjecture 3.1.2 can generalize to show similar results with an arbitrary number of cakes.

Theorem 4.3.1. *If we have n cakes sliced into n slices being divided amongst 2 people with the same-preference and the different-preference, there does not exist a cake slicing that admits to a disjoint envy-free division.*

Proof. We will use a proof by contradiction. Suppose it is always possible, no matter the player preferences, to find a disjoint envy-free division. Then it is possible to find a disjoint envy-free division under our described pathological preference. Let p_1 be Player 1's allocation, and let p_2 be Player 2's allocation.

Lemma 4.3.2. *The preference p_2 cannot contain any pieces of the type that is most commonly preferred in p_1.*

Proof. We use a proof by contradiction. Suppose that p_2 contains a slice of type α, which is also the type of piece most commonly preferred in p_1. Then Player 2 prefers an α slice in a cake where Player 1 does not prefer an α slice because Player 1 and Player 2's assigned piece selections are disjoint. Then Player 1 is envious of the α slice that Player 2 is assigned, because it is in a cake where Player 1 is not assigned the α slice, and by swapping with Player 2 in this cake, Player 1 raises the value of $f_1(p_1)$. Thus we have a contradiction, as Player 1 cannot benefit by swapping pieces with Player 2 in an envy-free division. So our supposition is false, and Player 2's assigned piece selection cannot contain any pieces of the type that is most commonly preferred in Player 1's assigned piece selection. □

Because of lemma 3.1.8 and the pigeonhole principle, we know that p_2 must contain at least two slices of the same type, as we must distribute $n-1$ types to fill n places.

Lemma 4.3.3. *A slice type that is m^{th} most often preferred in p_1, called type M, cannot be the allocated slice for Player 2 of a cake where Player 1 prefers a slice type that is k^{th} most often preferred in p_1, called type K, where $k > m$ and $k, m \in \mathbb{N}$.*

Proof. We use a proof by contradiction. Suppose type M is the allocated slice for Player 2 of a cake where p_1 prefers a slice type that is k^{th} most often preferred in p_1. Then swapping the M and K slices in the same cake would result in raising the value of $f_m(p_1)$, without affecting any $f_l(p_1)$, where $l < m$. So Player 1 can benefit from swapping pieces, and thus our division is not envy-free. □

Consider the cakes where p_2 has multiple slice types. Let us call this slice type γ. Our division then looks like $\left\{ {AAA...BBB...CCC... \atop ....\gamma.........\gamma} \right\}$, where $A = f_1(p_1), B = f_2(p_2), C = f_3(p_2)....$ Suppose, without loss of generality, that the the leftmost gamma slice is under a cake where Player 1 prefers C. Our division is now $\begin{Bmatrix} AAA & \cdots & BBB & \cdots & CCC & \cdots \\ ??? & \cdots & ??? & \cdots & ?\gamma? & \gamma \end{Bmatrix}$

Lemma 4.3.4. *Slice type C must appear at least once in p_2 in a cake where Player 1 prefers a slice type that is k^{th} most often preferred in p_1, called type K, where $k < m$ and C is the m^{th} most preferred slice type in p_1.*

Proof. I use a proof by contradiction. Suppose C appears no times in p_2 of a cake where Player 1 prefers a K slice. Then Player 2 is jealous of the leftmost γ slice that appears in the same cake where Player 1 prefers a C slice, because by Lemma 3.1.6 and our supposition in this case, there are no C slices in p_2.

□

Our division is now $\begin{Bmatrix} AAA & \cdots & BBB & \cdots & CCC & \cdots \\ ??? & \cdots & ?C? & \cdots & ?\gamma? & \gamma \end{Bmatrix}$ We have two cases.

Case 1: Suppose there are no B slices in p_2 in a cake where Player 1 prefers a slice type that is more often preferred then B. Then with Lemma 3.1.6 we know that there are no B slices in p_2, and so Player 2 can improve their position by swapping their C slice with

Player 1's B slice, and their γ slice with Player 1's C slice. Through this swap the number of C slices represented in p_2 does not change, but the number of γ slices represented in p_2 drops by one, and Player 2 receives a unique B slice. After the this swap is made our division is then $\begin{Bmatrix} AAA & \cdots & BCB & \cdots & C\gamma C & \cdots \\ ??? & \cdots & ?B? & \cdots & ?C? & \gamma \end{Bmatrix}$ where Player 2 has essentially traded a γ slice for a B slice.

Case 2: Suppose there is a B slice in p_2 in a cake where Player 1 prefers a slice type that is more often preferred then B. Then our division is now

$$\begin{Bmatrix} \cdots & AAA & \cdots & BBB & \cdots & CCC & \cdots \\ \cdots & ?B? & \cdots & ?C? & \cdots & ?\gamma? & \gamma \end{Bmatrix}$$

.

We have two cases.

Subcase 2-1: Suppose there are no A slices in p_2 in a cake where Player 1 prefers a slice type that is more often preferred then A. Then with Lemma 3.1.6 we know that there are no A slices in p_2, and so Player 2 can improve their position by swapping their B slice with Player 1's A slice, their C slice with Player 1's B slice, and their γ slice with Player 1's C slice. Through this swap the number of C and B slices represented in p_2 does not change, but the number of γ slices represented in p_2 drops by one, and Player 2 receives a unique A slice. After the this swap is made our division is then $\begin{Bmatrix} \cdots & ABA & \cdots & BCB & \cdots & C\gamma C & \cdots \\ \cdots & ?A? & \cdots & ?B? & \cdots & ?C? & \gamma \end{Bmatrix}$ where Player 2 has essentially traded a γ slice for an A slice.

Subcase 2-2: Suppose there is an A slice in p_2 in a cake where Player 1 prefers a slice type that is more often preferred then B. In a similar argument to Case 1, we have again two cases, case 2-2-1 and case 2-2-2. Observe that eventually, because we have a finite number of cakes, we will hit a point where there are no slice types that are more often preferred then the slice type examined in the case. Therefore, in all of our cases, Player 2 will always be envious of at least one slice in p_1. And so, if we have n cakes sliced into n

slices being divided amongst 2 people, there always exists preferences that do not admit to a disjoint envy-free division. □

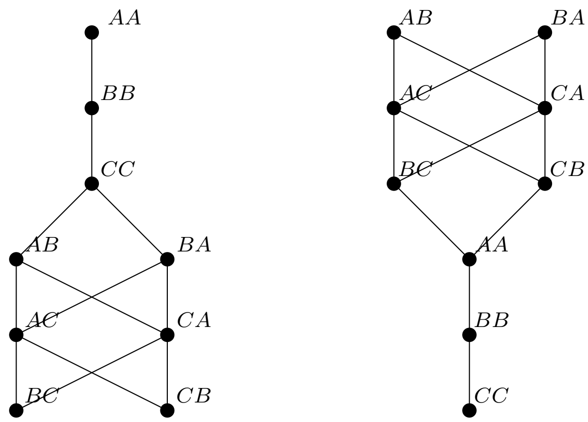
4.4 Guaranteed Cake Slicings

Not only do the same-preference and different-preference produce the result that envy-free piece selections are impossible when cutting n cakes into n slices, it is not difficult to prove that they always allow for cake slicings that produce envy-free divisions, and envy-free piece selections when cutting n cakes into $n+1$ slices. We provide an example using 2 cakes and 3 slices before we prove the more general result for envy-free divisions, and then envy-free selections.

Theorem 4.4.1. *With two players, Player 1 having the same-preference and Player 2 having the different-preference, there exists a cake slicing that allows for envy-free divisions when 2 cakes are sliced into 3 pieces.*

Proof. Suppose that the two cakes are sliced in such a way that every slice is of size greater than ϵ. I propose that Player 1 receiving the selection cc and Player 2 receiving the selection ab constitutes an envy-free division. Observe that neither player is willing to exchange their entire allocations, and so we need only to attend to swapping individual slices. Observe that Player 1 is unwilling to swap their first or second piece because it would result in two different types, which is a strictly worse outcome than what they currently have, which is one type in both cakes. Observe that Player 2 is unwilling to swap their first or second piece because doing so would result in a lexicographically lesser preferred piece selection, while keeping the $f_n(p_2)$ functions the same. □

In Figure 4.4.1, we can see that Player 2 is receiving one of his most preferred piece selections, and therefore cannot recover any benefit from swapping pieces. This division is envy-free, as Player 1 cannot improve their piece selection by stealing pieces from Player 2.



Player 1 Hasse Diagram Player 2 Hasse Diagram

Figure 4.4.1.

With brief reflection, we can see that slicing the cakes such that no slice is of size less than ϵ will always result in the same-preference and different preference producing similar Hasse Diagrams. For the same-preference, the Hasse diagram will be composed of all allocations with the same slice type on top in descending lexical order, followed by a transition to all of the allocations composed of different slice types, each layer representing the different allocations that, when arranged lexically, produce the same string of letters. From this, we can prove the general case.

Theorem 4.4.2. *With two players, Player 1 having the same-preference and Player 2 having the different-preference, there always exists a cake slicing such that there exists an envy-free division when n cakes are divided into $n+1$ pieces.*

Proof. We claim that any cake slicing where each slice is of size less than ϵ is such a cake slicing.

Suppose the cakes are cut such that each slice is of size less than ϵ. Then the division where we give Player 1 all slices of the last lexicographically preferred piece and we give Player 2 one slice of each slice type except the last lexicographically preferred piece constitutes an envy-free division. Observe that Player 1 is not envious of any of Player 2's slices, as any trade would result in $f_1(a_1)$ dropping in value, where a_1 is Player 1's allocation. Observe that Player 2 is not envious of any of Player 1's slices, as any trade would result in a_2 dropping in lexical priority, where a_2 is os Player 2's allocation. Thus slicing each cake such that every slice is of size less than ϵ admits to an envy-free division. □

Now that we have attended to the existence of cake slicings that guarantee the existence of envy-free division when the same-preference and different-preference are used, we will now show that there also exists cake slicings that guarantee the existence of envy-free piece selections when the same-preference and different-preference are used. Once again, we begin with a particular case before we generalize.

Theorem 4.4.3. *With two players, Player 1 having the same-preference and Player 2 having the different-preference, there exists a cake slicing that allows for envy-free piece selections when 2 cakes are sliced into 3 pieces.*

Proof. Our cake slicing is one in which for the n^{th} cake, only the n^{th} leftmost slice is of size less than ϵ. In this case this means our first cake is sliced so that the first leftmost slice is of size less than ϵ, and in the second cake, the second cake is sliced so that the second leftmost slice is of size less than ϵ, as pictured in Figure 4.4.2.

Observe that Player 1's piece selection is (cc) and Player 2's piece selection is (ba). Thus, this slicing allows for envy-free piece selections. □

Once again, the generalization comes naturally.

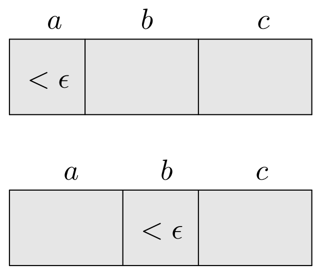


Figure 4.4.2.

Theorem 4.4.4. *With two players, Player 1 having the same-preference and Player 2 having the different-preference, there exists a cake slicing that allows for envy-free piece selections when* n *cakes are sliced into n+1 pieces.*

Proof. Our cake slicing is one in which for the n^{th} cake, only the n^{th} leftmost slice is of size less than ϵ, and all other slices are of equal size in each cake. Observe that because we have one fewer cakes than we do slices, in all cakes, the last slice is of size greater than ϵ. Additionally, the only allocation a where $f_1 = n+1$ is the allocation that selects the last slice from every piece. Therefore, Player 1's piece selection consists of this unique allocation. Observe that Player 2 can always have an allocation that consists of one slice of each type except the type preferred by Player 1 by selecting, for the nth cake, the $n+1^{th}$ left most piece, except for the last cake, where they prefer the first slice. This minimizes the maximum $f_n(a_2)$, where a_2 is Player 2's allocation, and also is of maximum lexical priority, and thus Player 2 selects this allocation. Because Player 1's piece selection consists of only slices of the slice type with the least lexical priority, and Player 2's piece selection consists of only pieces that are not the slice type with the least lexical priority, this cake slicing allows for envy-free piece selections. □

5

Discussion and Future Research

Here we have shown that Cloutier, Nyman, and Su's attempted proof of 3.1.2 is flawed as the proposed player behavior does not constitute a preference as defined in their paper and other foundational papers on cake division. However, Cloutier, Nyman, and Su were correct in their expectation that this behavior could be generalized to an arbitrary number of cakes as an example of a preference that will never admit to an envy-free piece selection. Additionally, it is fairly simple to demonstrate that this behavior follows the expectations of Conjecture 3.1.6, as always produces a cake slicing that allows for both envy-free divisions and envy-free piece selections.

Future research can take two directions depending on interest in Conjecture 3.1.5 or Conjecture 3.1.6

Further research on Conjecture 3.1.5 would include discovering a generalizable preference that satisfies the criterion outlined for preferences in this paper, thus proving Conjecture 3.1.5 and Conjecture 3.1.2. One natural direction would be to try and force the same-preference and different-preference closed by having players be indifferent to a slice of size ϵ as compared with a slice of size greater than ϵ, or less than ϵ. Cursory attempts to find cake slicings that force this forced closure to admit to an envy-free piece selection were not fruitful, so this method remains a plausible direction for a proof of Conjecture 3.1.5.

Further research into Conjecture 3.1.6 may build on Lebert, Meunier, and Carbonnaux's[3] attempt to find the number of slices each cake must be sliced into in order to be guaranteed an envy-free piece selection amongst two players, no matter what the preferences are of those two players. In their paper they have proved $c(c-1)+1$ is an upper bound for this number, if c is the number of cakes. Further research might aim to find a tighter bound for this number. Our own attempts to prove Conjecture 3.1.6 ran into problems when using the method outlined by Cloutier, Nyman, and Su[1] in their paper, and so investigating the method used by Lebert, Meunier, and Carbonnaux may be fruitful.

Bibliography

[1] John Cloutier, Kathryn L. Nyman, and Francis Edward Su, *Two-player Envy-free Multi-cake Division*, Mathematical Social Sciences **59** (2010).

[2] Jesus De Loera, Elisha Peterson, and Francis Edward Su, *A Polytopal Generalization of Sperner's Lemma*, Journal of Combinatorical Theory, Series A **100** (2002).

[3] Nicolas Lebert, Frédéric Meunier, and Quentin Carbonneaux, *Envy-Free Two-Player* m-*Cake and Three-Player Two-Cake Divisions*, Operations Research Letters **41** (2013).

[4] Emanuel Sperner, *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Mathematische Zeitschrift **27** (1928).

[5] Walter Stromquist, *How to Cut a Cake Fairly*, The American Mathematical Monthly **87** (1980).