Graph Replacement Systems for Julia Sets of Quadratic Polynomials

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by
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Abstract

Belk and Forrest construct a specific class of graph replacement systems that give sequences of graphs that converge to fractals. Given a polynomial, we have an algorithm that gives a replacement system that leads to a graph sequence which we conjecture converges to the Julia set. We prove the conjecture for the quadratic polynomial $z^2 + c$ where c is a real number and the critical point is in a three cycle. We present some additional results and observations on replacement systems obtained from certain polynomials.

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Dedication

For my grandfather.

Acknowledgments

I would like to thank Lauren for all her support over the years.

Thanks and sorry to all the professors who have seen me cry.

Thanks to the following friends for putting up with me and believing in me:
Adrian, Alexis, Anna, Arti, Dani, Darren, Diljit, Justin, Kara, Leanna,Raina, Sarah, and Victoria.
You guys were the best part of college.
And thanks to my other friends whom I also love and appreciate.

This project would not have been possible with Jim's excellent and enthusiastic guidance and understanding.

Introduction

Thompson's groups are three groups F,T, and V are finitely presented infinite groups that are of interest to group theorists because of their unusual properties. These groups act as piece-wise linear homeomorphisms on the circle, the interval, and the Cantor set. In [4], Belk and Forrest presented a Thompson-like group that acts by homeomorphisms on the Basilica, which is the Julia set of the quadratic polynomial z^2-1. In their paper, they mention that their eventual goal is to associate a Thompson-like group to the Julia set of every quadratic polynomial, in a way such that the original three Thompson's groups are associated to the quadratic polynomials with Julia sets that are homeomorphic to the circle, the interval, and the Cantor set. Two previous senior projects at Bard, [8] [10], have been written to further this program. Jasper Weinhart-Burd developed a Thompson-like group for the Julia set of the post-critically finite quadratic rational function with both critical points in the same three cycle, and Will Smith developed a Thompson-like group for the function $z^2 + i$.

Belk and Forrest's subsequent paper, [5], acts as a more general intermediate step towards the goal of associating a Thompson-like group to every quadratic polynomial. In it, they define rearrangement groups of fractals, which are a generalization of Thompson's groups. The construction works by approximating each fractal with a sequence of finite graphs, called the **expansion sequence**, such that the fractal is homeomorphic to the **limit space**. The sequence is given by the first graph of the sequence, the **base graph**, and graph rewriting rules called **replacement rules**. Each graph in the sequence is constructed by applying the replacement rules to the previous graph. Each element of the rearrangement group associated to a given fractal acts as as a piecewise homeomorphism of one of the graphs in the expansion sequence.

Now to associate a rearrangment group to an arbitrary polynomial, we need some way of finding the expansion sequence that has a limit space that is homeomorphic the correct Julia set. The essential information about the structure and dynamics of a post-critically finite polynomial is captured by a certain tree connecting the post-critical points within the Julia set, called the **Hubbard tree**. This construction is described in [7]. Belk and Forrest have developed an unpublished algorithm to obtain a base graph and replacement rules starting with just the Hubbard tree, which they conjecture gives a limit space that is homeomorphic to the starting Julia set. In chapter 2 we give a proof of this conjecture for the Airplane Julia set, the Julia set for the quadratic polynomial with a critical point that is in a three-cycle where all three points are on the real line.

Additionally, in chapter 3 we present finite state automata that give some progress towards how one might go about defining rearrangement groups for matings, which are rational functions that can be constructed from two quadratic polynomials.

1

Background

1.1 Introduction to Dynamical Systems

Most of the material of this section is taken from [1]. The theory of dynamical systems is the study of how physical systems evolve over time. It has its origins in Newton's study of planetary motions, and has since found applications in virtually every scientific field. Mathematically, a dynamical system refers to a set of possible states, together with a rule that determines the present state in terms of past states [**??**]. For example, population models in biology are generally modeled as dynamical systems, such that a state corresponds to a population size, and the rule is something like $f(x) = 2x$ (if the population size doubles each generation).

Until the second half of the 20th century, scientists were generally under the impression a dynamical system starting from an initial state will eventually settle down into a steady state or end up oscillating among a subset of possible states, like the motion of the planets. In the 1970s, a third option came to the world's attention: "chaos". In Maxwell's investigation of gas laws, he showed that if two particles with random positions and equal but opposite velocities collide, then all direction of travels will be equally likely after the collision. This behavior is distinct from that of systems that simply have quasi-periodic behavior with a large number of periods, in that chaotic systems exhibit **sensitive dependence on initial conditions**, meaning that

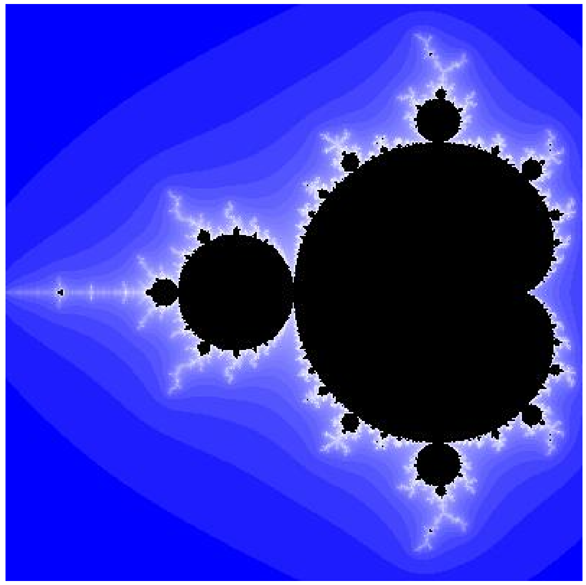


Figure 1.1.1. The Mandelbrot Set

an arbitrarily small change of the initial state will have large, essentially unpredictable affects on the long term behavior of the system.

The introduction of "chaos"[1] into the vernacular of dynamical systems theorists coincided with the introduction of computer visualization to the field. Edward Lorenz, the meteorologist who gave the famous talk "Predictability: does the flap of a butterfly's wing in Brazil set off a tornado in Texas?" , in the same talk presented a computer graphic of a **strange attractor**, a subset of states the dynamical system tends to and in which the system behaves chaotically. [2] Strange attractors have fractal structures, meaning that they exhibit self similarity at increasingly small scales.

One of the most well known mathematical structures in the popular consciousness is the Mandelbrot set, a fractal first visualized by Benoit Mandelbrot on an IBM computer in 1980. The Mandelbrot set is the set of c values for which the critical point of the function $z^2 + c$ does not diverge, and so in this sense it indexes the set of complex quadratic polynomials, which will be the objects of focus in this project.

[1]The term "chaos theory" was coined by James Yorke, one of the authors of [1]

[2]The existence of the Lorenz attractor would not be proved analytically until the year 2000.

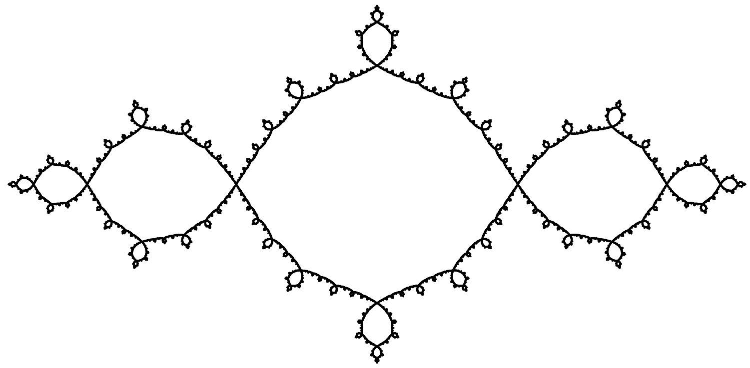


Figure 1.2.1. The Basilica Julia set

1.2 Dynamics on the Complex Plane

We begin by presenting some basic definitions from complex dynamics, taken from [3].

Definition 1.2.1. The **orbit** of a point p on the complex plane under a function $f : \mathbb{C} \to \mathbb{C}$ refers to the set

$$\{p, f(p), f(f(p)), f(f(f(p))), \ldots\}.$$

The **filled Julia set** of f is the set of points that have a bounded orbit.

Example 1.2.2. Let $f = z^2$. The orbits of points under iteration of f include:

$$\{i, -1, 1, 1, \ldots\}, \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}, \{2i, -4, 16, 256, \ldots\}$$

It general, if $|p| \leq 1$, then the orbit of p is bounded, and if $|p| > 1$, then the orbit of p is unbounded. Thus the filled Julia set of z^2 is the closed unit disc.

Definition 1.2.3. The **Julia set** of a polynomial is defined as the boundary of the closed Julia set. (Equivalently, it is the closure of the set of repelling periodic points.)

1.3 Hubbard Trees

Most of the material in this section is taken from [7], and many of the pictures are from Belk.

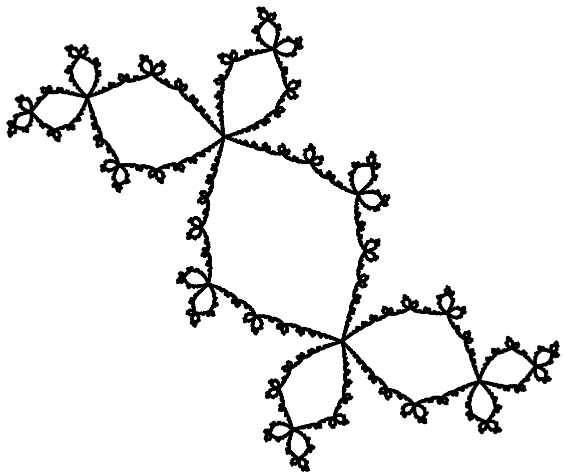


Figure 1.2.2. The Douady Rabbit Julia set

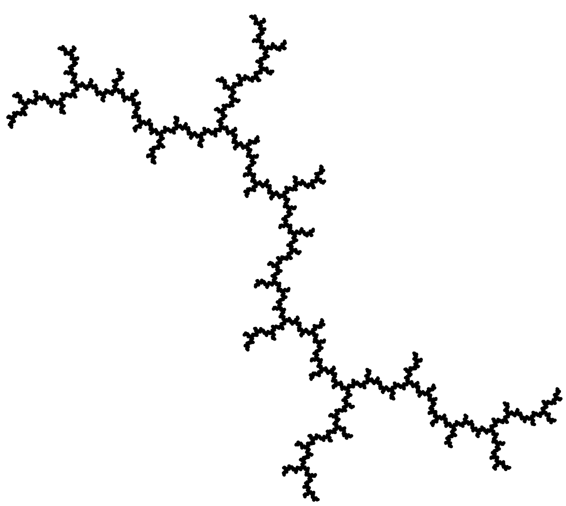


Figure 1.2.3. $z^2 + i$

Recall that a critical point of a polynomial f is a point z such that $f'(z) = 0$. If a point p is in the orbit of a critical point, we call p a **postcritical point**. If f has a finite number of postcritical points, we say that f is **postcritically finite**. In other words,

Definition 1.3.1. A polynomial function p is **postcritically finite** if every critical point of p is periodic or pre-periodic.

In this project we will only be concerned with post-critically finite polynomials. The Julia sets and filled Julia sets of post-critically finite polynomials are compact, connected, locally connected. Moreover, there is a simple criterion for when the Julia set has interior.

Theorem 1.3.2. *Let f be a postcritically finite polynomial. If f has no periodic critical points, then the Julia set has empty interior. If f has at least one periodic critical point, then the interior is non-empty.*

When the interior of f is empty, then the Julia set is called a **dendrite**. One example is the Julia set for $z^2 + i$. When the interior of f is nonempty, then the interior of the Julia set is the union of all points whose orbits converge to a cycle containing a critical point. Properties of interior components are summarized in the following theorem:

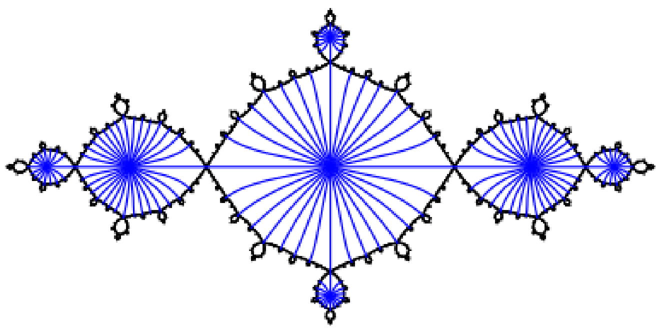
Theorem 1.3.3. *Suppose f is postcritically finite and has Julia set K_f with non-empty interior. Then:*

1. *Each interior component of K_f is homeomorphic to an open disk.*
2. *If U is an interior component of K_f, then so is $f(U)$. If U does not contain a critical point, then f maps U homeomorphically to $f(U)$.*
3. *For each interior component $U \subset K_f$, there exists a unique point $p \in U$ whose forward orbit a periodic critical point. We call $p \in U$ the **center point** of U.*

Definition 1.3.4. Let K_f be a Julia set with interior, and let U be an interior component of K_f. Then there is a family of Riemann maps $\phi : D^2 \to U$ that maps 0 to the center point of U.

The **internal rays** of U are the images of radial lines under the Riemann maps. Note that the definition does not depend on which map you choose.

Example 1.3.5. Some internal rays of the Basilica.

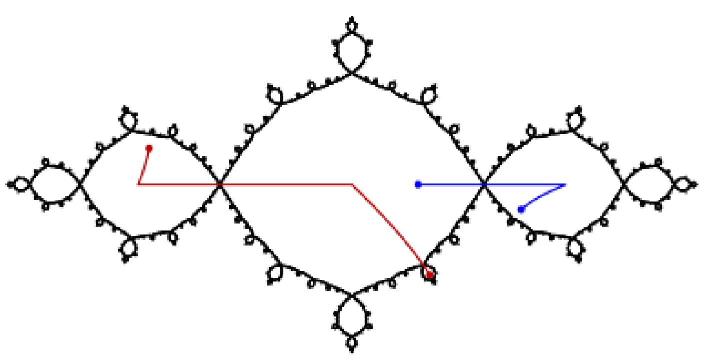


Definition 1.3.6. An arc $A \subset K_f$ is regulated if for each interior component U of K_f, $A \bigcap U$ is contained in at most two internal rays of U.

Theorem 1.3.7. *Let K_f be the filled Julia set of a post-critically finite polynomial. For any two points p and q in K_f, there is a unique regulated arc with p and q as endpoints.*

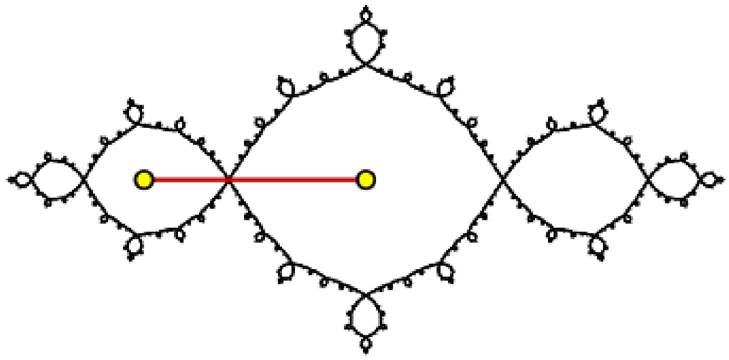
Definition 1.3.8. Let K_f be a filled Julia set. The **restricted hull** of a set of points $P \in K_f$ is the union of all regulated arcs with endpoints in P.

Example 1.3.9. The restricted hull for some points in the Basilica.



Definition 1.3.10. Let f be a post-critically finite polynomial. The **Hubbard tree** H^f is the restricted hull of the postcritical points of f.

Example 1.3.11. The Basilica together with its Hubbard tree (the regulated hull of 0 and -1).



We will remember on a Hubbard tree H the following information:

Definition 1.3.12. Let H be the Hubbard tree of a post-critically finite quadratic polynomial f. We will remember on H the structure given by a set of data called the **primary structure**, consisting of:

1. the topology
2. the cyclic order of the branches at branch points
3. the set P of post-critical points $\bigcup_{n\geq 0} f^{\circ n}(0)$
4. the dynamics on P, i.e. $f_{|}P : P \to P$.
5. The degree of elements of P under the mapping f (i.e. 0 has degree 2 and all other post-critical points have degree 1)

Theorem 1.3.13 (Corollary 6.2 of the Orsay notes)**.** *Let $f : z \mapsto z^2 + c_1$ and $g : z \mapsto z^2 + c_2$ be two quadratic polynomials. If there exists a homeomorphism from H_f to H_g preserving the primary structure, then we have $c_1 = c_2$.*

Thus quardratic polynomials are uniquely determined by the primary structure of their Hubbard trees.

1.4 Replacement Systems

Using graph replacement systems, we can recover the topology of a Julia set from the combinatorial information given by the Hubbard tree. Informally, a graph replacement system can be thought of an algorithm to "draw" a sequence of representations of fractals that have increasing levels of accuracy. We start by drawing a base graph. Given an graph G_n in the graph replacement sequence, we obtain G_{n+1} by replacing edges of G_n with other graphs, adding "detail" to the representation. This is formalized in the following definition:

Definition 1.4.1. An **(edge) replacement rule** is a pair of the form $e \to R$, where

1. e is a single edge with two vertices, with one marked as the initial vertex and the other marked as the terminal vertex, and
2. R is a finite directed graph with two distinct vertices marked as the initial vertex and the terminal vertex.

Given a directed graph G and a replacement rule $e \to R$, we can **expand** any edge ϵ of G by removing ϵ and pasting in a copy of R, such that the initial vertex of R is attached to the initial vertex of ϵ and the terminal vertex of R is attached to the terminal vertex of ϵ.

Definition 1.4.2. A colored graph replacement system $\mathcal{R}$ consists of the following data:

1. A finite set of colors C.
2. A base graph G_0, which is a finite directed graph with all edges colored by elements of C.
3. For every $c \in C$, a directed graph R_c with two distinguished vertices (the initial vertex and the terminal vertex).

From the initial data, we construct a sequence of directed graphs called the **expansion sequence**:

Definition 1.4.3. Given a colored replacement system R, the **full expansion** of the base graph G_0 is the graph G_1 obtained by expanding each edge ϵ of G_0 with a replacement rule $e \to R_c$, such that c is the color of ϵ.

Iterating this process gives us the *full expansion sequence* $\{G_n\}_{n=0}^{\infty}$, where G_n is the full expansion of G_{n-1} for every n.

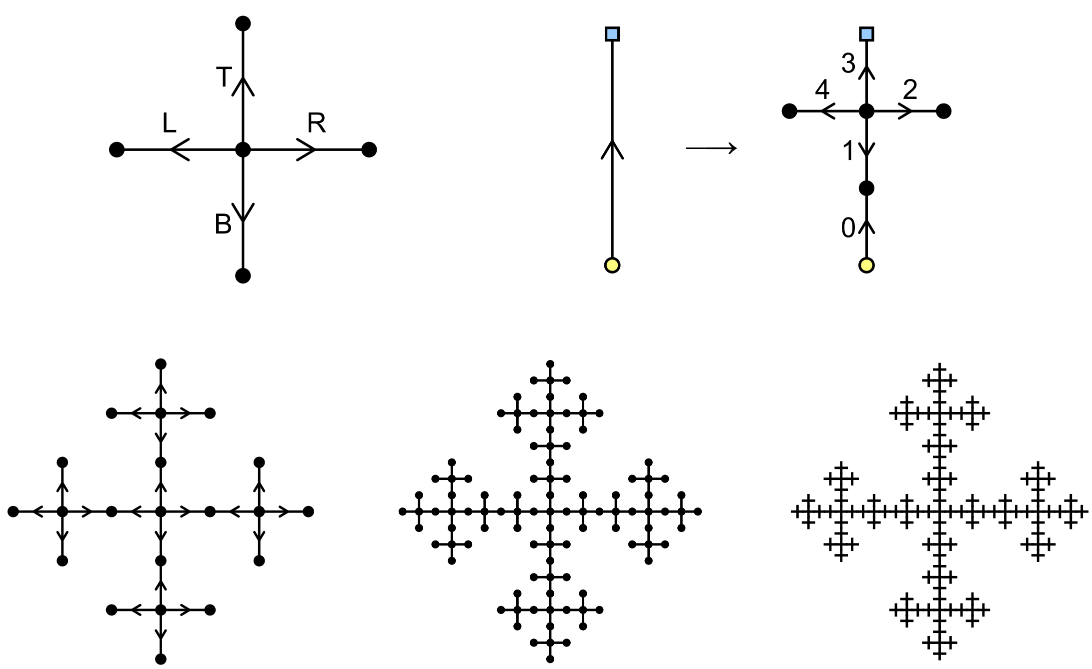
Note that each edge of every graph G_n of the expansion can be written as a finite sequence of edges

$$\epsilon_0\epsilon_1\epsilon_2\dots\epsilon_n$$

where ϵ_0 is an edge of the base graph and if ϵ_k has color c then ϵ_{k+1} is an edge of R_c.

The graphs in the replacement sequence converge to the **limit space**, which we will define as a quotient of an infinite product of edges.

Example 1.4.4. The replacement system for the Vicsek snowflake, along with some graphs from its expansion sequence.



Definition 1.4.5. Let R be a colored replacement system. The **symbol space** Ω of R is the space of infinite sequences

$$\epsilon_0\epsilon_1\epsilon_2\epsilon_3\dots$$

where ϵ_0 is an edge of the base graph and if ϵ_k has color c then ϵ_{k+1} is an edge of R_c. Alternatively, it is the space of infinite sequences with the property that every finite sub-sequence $\epsilon_0 \ldots \epsilon_n$ is an edge in some graph G_n of the expansion sequence.

Definition 1.4.6. The **gluing relation** on Ω is the equivalence relation $\sim$ such that two sequences $\epsilon_0\epsilon_1\epsilon_2\ldots$ and $\epsilon_0'\epsilon_1'\epsilon_2'\ldots$ are equivalent if for all n the edges $\epsilon_0\epsilon_1\ldots\epsilon_n$ and $\epsilon_0'\epsilon_1'\epsilon_2'\ldots\epsilon_n'$ share a vertex in G_n.

The **limit space** is the quotient $\Omega/\sim$.

In order for the gluing relation to be an equivalence relation, the following restrictions must be placed on the replacement system:

Definition 1.4.7. A colored graph replacement system R is said to be **expanding** if the following conditions are satisfied:

1. Neither G_0 or any of the R_c have any isolated vertices.
2. None of the R_c have an edge connecting the initial and terminal vertices.
3. Each R_c has at least three vertices and two edges.

Theorem 1.4.8. *If the colored graph replacement system R is **expanding**, then the gluing relation is an equivalence relation and the limit space $\Omega/\sim$ is compact and metrizable.*

Given a polynomial f, we construct a replacement system from the Hubbard tree H. This construction is due to Belk and Forrest (unpublished).

Definition 1.4.9. The replacement system derived from the post-critically finite polynomial f is obtained according the following procedure:

Step 1. Compute the Hubbard tree H for f.

Step 2. If T is a tree, then the **blow up** of T, denoted $\overline{T}$, is the graph obtained by replacing every vertex of T with a counterclockwise directed cycle.

Construct the blow up of H, denoted $\overline{H}$. This is the base graph. **Step 3.** Let H_1 be the pre-image of H under f. Construct the blowup of H_1, $\overline{H_1}$.

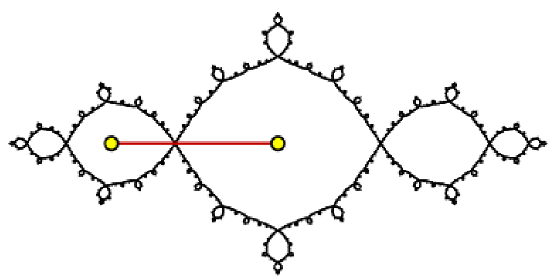
Step 4. Orient the edges of the blowup $\overline{H}$. Choose a distinct color for each edge.

Step 5. The branched covering map from $H_1 \to H$ induces a covering map $\overline{H_1} \to H$. Use this covering to lift the orientations and colors of the edges of H to the edges of H_1, so that the covering map preserves orientations and colors.

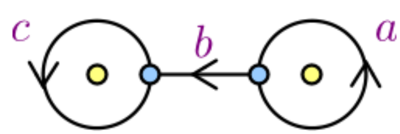
Step 6. The set obtained by *cutting* $\overline{H}$ at F is the disjoing union of the closures of the connected components of $\overline{H} \setminus F$. Since every vertex of $\overline{H}$ is a vertex of $\overline{H_1}$, removing the vertices of $\overline{H}$ from $\overline{H}$ and $\overline{H_1}$ cuts them into corresponding pairs of components, which are the replacement rules.

Example 1.4.10. Using Definition 1.4.9, we compute a replacement system for the Basilica.

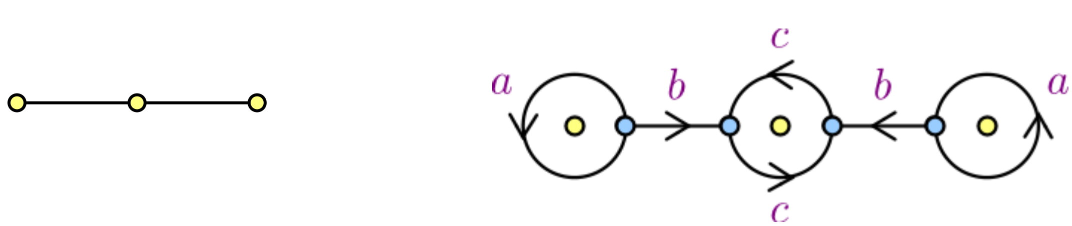
Step 1: The Basilica Hubbard tree:



Step 2: The blow-up of the Hubbard tree

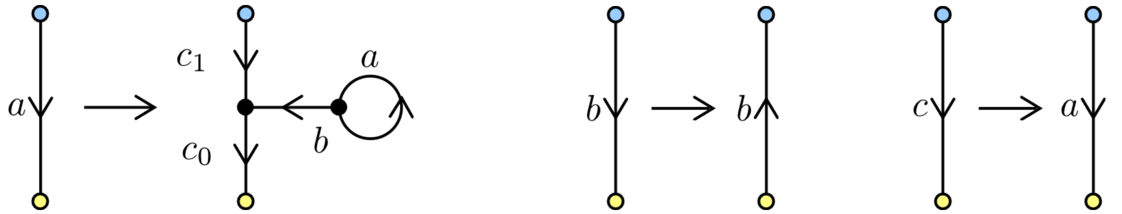


Step 3: The pre-image of the Basilica Hubbard tree, and the blow up:



Step 5: The edges are already oriented, and the edge labels are the colors.

Step 6: We obtain the following replacement rules:



Note that the subscripts are necessary in the first replacement rule to differentiate between the two edges labeled with c.

In general, the gluing relation we originally defined is not an equivalence relation on replacement systems obtained from the above procedures. The following definition for the gluing relation turns out to be an equivalence relation for this class of replacement systems, even though they are not necessarily expanding:

Definition 1.4.11. The **gluing relation** for replacement systems obtained from Julia sets is the relation $\sim$ such that two edge addresses $\epsilon_0\epsilon_1\epsilon_2\ldots$ and $\epsilon'_0\epsilon'_1\epsilon'_2\ldots$ are related if there is a finite upper bound on the distance between $\epsilon_0\epsilon_1\epsilon_2\ldots\epsilon_n$ and $\epsilon'_0\epsilon'_1\epsilon'_2\ldots\epsilon'_n$ for all n.

We can make the following simplifications to the replacement system.

Step 1: If for any replacement rule $e \to R_c$, R_c is a single edge with color c' (where $c \neq c'$), recolor every edge in the replacement system that was colored with c with c' instead.

Step 2: Once all the redundant colors from Step 1 are removed, if there are trivial replacements where R_c is a single edge with color c, then contract all edges colored with c.

Lemma 1.4.12. *For any replacement system obtained from a Julia set according to Definition 1.4.9, procedure outlined in Step 1 and Step 2 above gives an expanding replacement system with limit space homeomorphic to the limit space of the original replacement system.*

1.5 Laminations

The material in this section is adapted from [9] and [2]. A more topological perspective on Julia sets is to study them as quotients of a circle. This idea was introduced by Thurston. Suppose we have a monic polynomial p with degree $d \geq 2$, such that its Julia set K_p is connected[3]. The trick is to study the dynamics on the Julia set by associating points of the Julia set with points on S^1, the unit circle a way such that the map conjugates the dynamics of p on K_p to to z^d on S^1. This map is known as the **Carathéodory loop**, and in general it is defined as the extension of a Riemann map from the open disc to the interior of K_p. The existence of such a Riemann map follows from the **Riemann mapping theorem**:

Theorem 1.5.1. *If U is a non-empty simply connected open subset of the complex plane that is not all of C, then there exists a biholomorphic mapping from U onto the open disc, known as a* ***Riemann mapping****.*

Then since the complement of K_p is locally connected, Carathéodory's theorem lets us extend the inverse of the Riemann map onto the boundary of the disc. This extension maps the unit circle to K_p.

Theorem 1.5.2. *Let f be a Riemann map from an open subset U to the open disc D^2, and let $g = f^{-1}$. The map g extends to a continuous map $\bar{g}$ defined on the closed disc $\bar{D^2}$ if and only if the complement $S^2 - U$ is locally connected.*

Now let's suppose we have a Carathéodory loop σ that maps the unit circle to $\bar{U}$, where $\bar{U}$ is the boundary of some open subset U. Unless $\bar{U}$ is topologically equivalent to the unit circle, σ will not be injective, and some points in the unit circle will be identified with one another under the mapping. It turns out that the structure of the set of identifications cannot be too complicated - if x and y are two points in $\bar{U}$, then the convex hull of the preimages of x is disjoint from the convex hull of the preimages of y. Given this fact, it is convenient to draw a picture of the identifications by drawing the convex hulls (called **leaves**) of each pair of points that get identified under the mapping σ. The set of leaves is called the **lamination**.

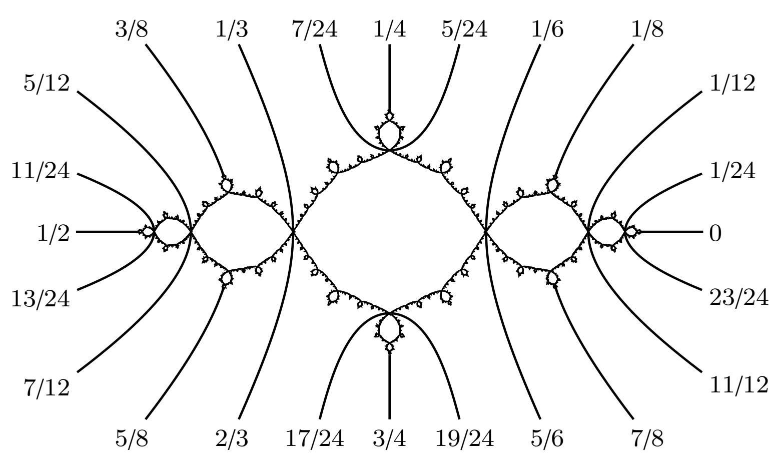
[3]Recall that K_p is connected if and only if the forward orbits of the critical points (aside from infinity) are bounded.

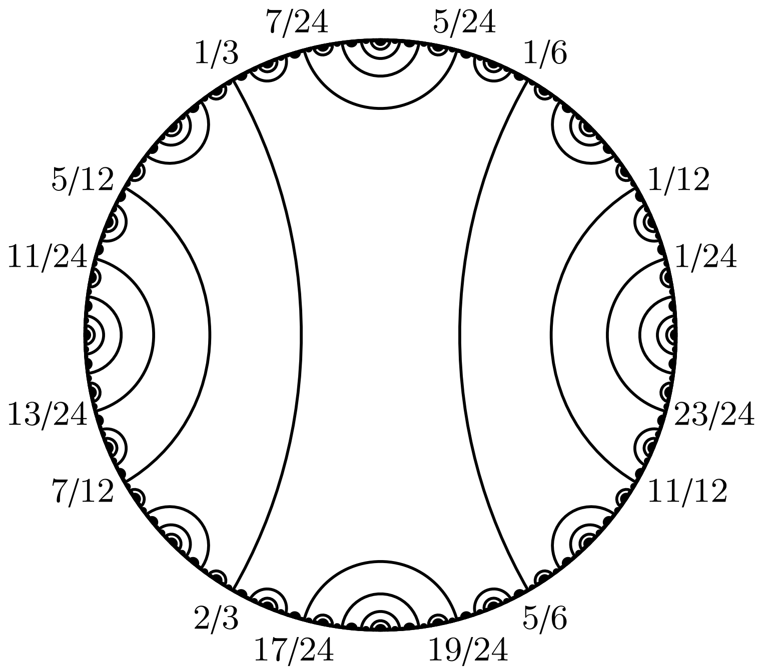
If the Carthéodory loop mapping $\mathbb{R}/\mathbb{Z}$ to K_p maps α to a point $z \in K_p$, then we say that α is an external ray that lands at K_p.

If a point in K_p has two external angles α_1 and α_2, then we stipulate that there is a leaf in the lamination connecting $e^{i\alpha_1}$ and $e^{i\alpha_2}$. If a point in K_p has more than two external angles, then we draw leaves connecting points on the circle that correspond to adjacent angles, so that all external angles corresponding to the point are the vertices of a single polygonal gap in the lamination.

From the lamination, we obtain a model of the Julia set as a quotient of the circle by identifying points that are connected by leaves. This is called the **pinched disc model** of the Julia set.

Example 1.5.3. Below we have a picture of the Basilica with external rays marked, as well as a lamination for the pinched disc model of the Basilica.





A natural question to ask is which quotients of the circle can be realized by Julia sets. Thurston approached this problem by first defining laminations more abstractly as disjoint sets of chords on the unit disc, and then classifying the set of laminations that are forwards and backwards invariant under the function $z \mapsto z^2$. The definitions and results below are from Thurston's manuscript on invariant laminations:

Definition 1.5.4. A **geodesic lamination** L is a closed set of chords, called **leaves**, on the closed unit disc $\overline{D^2}$ that are disjoint, except possibly at the endpoints.

A **gap** of a lamination is the closure of a component of the complement of $\bigcup L$. Any gap is the convex hull of its intersection with a boundary of the disc.

Definition 1.5.5. A geodesic lamination L is **quadratic invariant**, i.e. invariant under the map $f(z) = z^2$, if it satisfies the following conditions:

1. Forward invariance: If a leaf $\overline{pq}$ is in L, then either $f(p) = f(q)$ or $\overline{f(p)f(q)}$ is a leaf in L.

2. Backwards invariance: If $\overline{pq}$ is a leaf in L, then there are two disjoint leaves connecting a preimage of p with a preimage of q.

3. Gap invariance: For any gap G, the convex hull of $G \bigcap S^1$ is either
   (a) a gap,
   (b) a leaf,
   (c) or a single point

   If the image of G is a gap, then the image of a point moving clockwise around G must move clockwise around the image of G. In other words, the boundary of G must map locally monotonically to the boundary of the image gap with positive orientation.

Definition 1.5.6. If the two endpoints of a leaf map to the same point, then that leaf is a **critical leaf**. If the degree of the map from a gap to its image gap is not 1, then that gap is a **critical gap**

Definition 1.5.7. The **length** of a leaf in a lamination is the length of the smaller arc between the two endpoints on the circle. The perimeter of the circle is defined to have length 1, so the length of a leaf is at most 1/2, and a leaf with length 1/2 is a critical leaf.

The following definitions are from Definition II.6.2 of Thurston's manuscript.

Definition 1.5.8. Let L be a quadratic invariant lamination, and let M be the longest length of a leaf of L. Unless $M = 1/2$, there are two leafs with length M. We will call any leaf of length M a major leaf. If there are two major leaves, then their images coincide. We will call the image of a major leaf the **minor leaf** of the lamination.

The following is from Definition II.6.3 of Thurston's manuscript.

Definition 1.5.9. A lamination is **clean** if no three leaves have a common endpoint.

The following theorem is part (a) of Proposition II.6.7 of Thurston.

Theorem 1.5.10. *Any invariant lamination which is clean is the minimal lamination with its given minor leaf.*

Corollary 1.5.11. *If a clean invariant lamination has the same minor leaf as the lamination that gives the pinched disc model of a Julia set, then the two laminations are the same.*

Bandt and Keller take a more abstract view of the pinched disc model, defining it more directly as an equivalence relation on the circle, given by identifying itineraries of points on the circle under the angle doubling map. The notation is summarized in the following definition:

Definition 1.5.12. $T = \mathbb{R}/\mathbb{Z}$ is the unit circle, and $h : \beta \mapsto (2\beta \mod 1)$ is the angle doubling map.

Fix $\alpha \in T$. The diameter (critical leaf) $(\frac{\alpha}{2}, \frac{\alpha+1}{2})$ divides T into the open semi-circles T_1^α and T_0^α such that $\alpha \in T_1^\alpha$.

Definition 1.5.13. An equivalence relation on $\{0,1\}*$ is forward invariant under the shift σ if $s \sim t$ implies $\sigma(s) \sim \sigma(t)$.

Theorem 1.5.14. *Let α be a periodic angle. Among all forward invariant equivalence relations on T that do not identify T_0^α with T_1^α, there is a largest lamination $\sim_\alpha$.*

Let f be a periodic quadratic polynomial of period p with critical value c and Julia set J_f. Suppose that the fixed point under f^p on the basin containing c has external angle α. Then (J_f, f) is conjugate to $(T/\sim_\alpha, h)$.

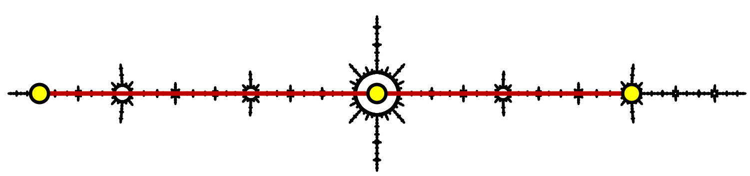
Corollary 1.5.15. *Let f be a periodic quadratic polynomial. The lamination of the pinched disc model for f is the largest clean lamination that does not have a leaf connecting a point from T_0^α to a point from T_1^α. The points $\frac{\alpha}{2}$ and $\frac{\alpha+1}{2}$ are each an endpoint of a major leaf.*

2
The Airplane

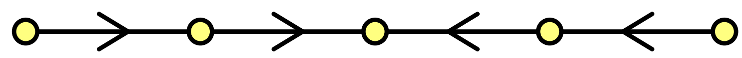
2.1 The Replacement System

The Airplane is the Julia set of the polynomial z^2+c, where c is the real root of c^3+2c^2+c+1 (which turns out to be around -1.75488).

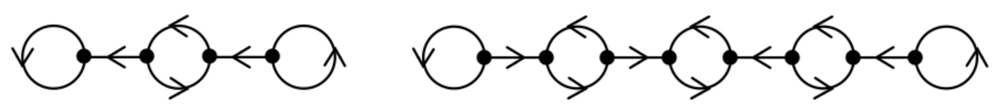
The critical point is periodic with period three. All three points in the orbit are on the real axis, and the Hubbard tree is a path.



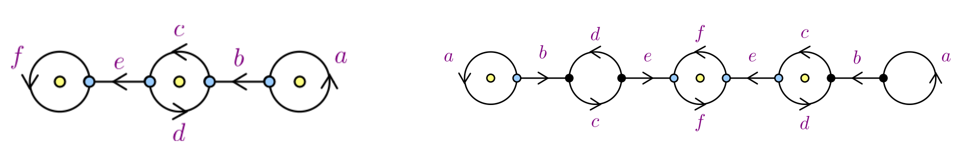
The pre-image of the Hubbard tree H^1 is also a path.



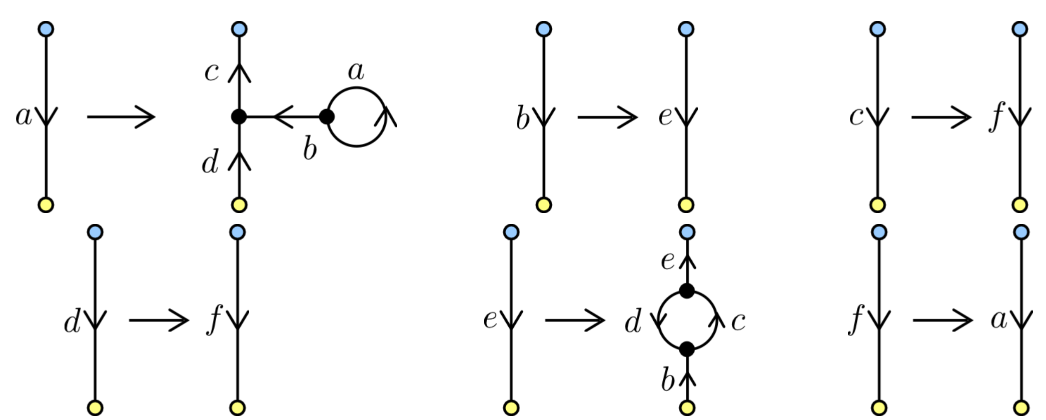
Below we have the blow-ups of H_0 and H_1.



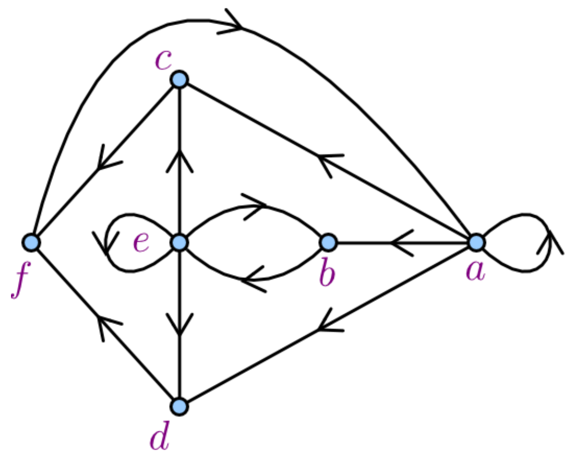
We construct H_1 by gluing together two copies of H_0. Then we use the vertices of H_0 to cut H_0 and H_1 into corresponding pairs of components, and assign addresses to H_1 accordingly.



From the cuts we get our first set of replacement rules:



Our function f maps every edge in H_1 to an edge in H. Our second set of replacement rules are of the form $e_1 \to e_2$ if $e_1 e_2$ is an address of H_1. The set of possible edge addresses in the symbol space correspond to paths in the following graph:



There are four gluing vertices on the base graph, which have addresses:

$$\overline{ebe} = \overline{fac} = \overline{fad}$$

$$\overline{eeb} = \overline{cfa} = \overline{dfa}$$

$$\overline{bee} = \overline{acf} = \overline{adf}$$

$$b\overline{ebe} = d\overline{fac} = c\overline{fad}$$

Note that the replacement system can actually be simplified to just two colors: a and e, making it into an expanding replacement system.

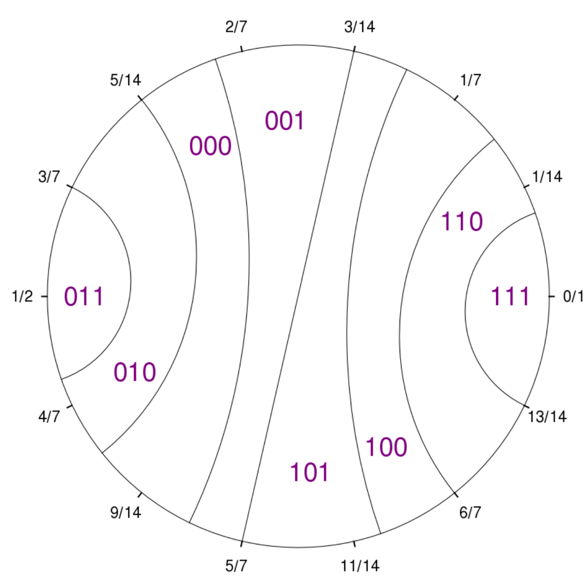
2.2 The Lamination

The airplane is in the component of the Mandelbrot set that has an external ray of $\alpha = \frac{3}{7}$. We construct the corresponding lamination S_0^α according to the methods of Bandt and Keller. The diameter of the circle that maps to α under the angle doubling map has endpoints at $\frac{5}{7}$ and $\frac{3}{14}$. This diameter divides the circle $T = \mathbb{R}/\mathbb{Z}$ into two semicircles T_0 and T_1, where $0 = 1$ is in T_1. The itinerary of a point $\beta \in T$ with respect to α is defined as

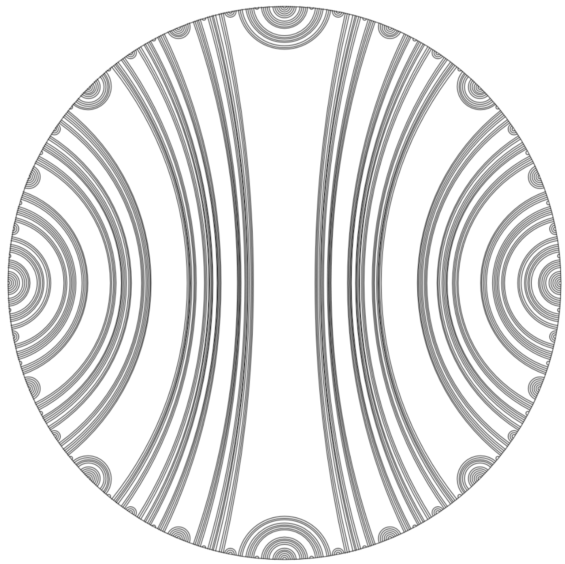
$$I(\beta) = s_1 s_2 s_3 \ldots \text{ with } s_i = \begin{cases} 0 & h^{i-1}(\beta) \in T_0 \\ 1 & h^{i-1}(\beta) \in T_1 \\ * & h^{-1}(\beta) \in \{\frac{3}{14}, \frac{5}{7}\} \end{cases}$$

Note that our α is periodic with period three, where $h(\alpha) = \frac{6}{7}$ and $h^2(\alpha) = \frac{5}{7}$. Our kneading sequence, which is the itinerary of α, is $\overline{01*}$. The characteristic symbol of α is 1, so we consider the pre-image of α which is periodic to be in T_1 and the pre-image of α which is pre-periodic to be in T_0. I.e., $\frac{5}{7} \in T_1$ and $\frac{3}{14} \in T_0$.

Now all points of T have 01 itineraries. Now we say that two 01 sequences β and γ are equivalent if and only if either $I(\beta) = I(\gamma)$, or $\{I(\beta), I(\gamma)\} = \{w101\overline{001}, w001\overline{101}\}$ for any 01 word w, or $\{I(\beta), I(\gamma)\} = \{w\overline{001}, w\overline{101}\}$ for any 01 word w that does not end in 001 or 101.

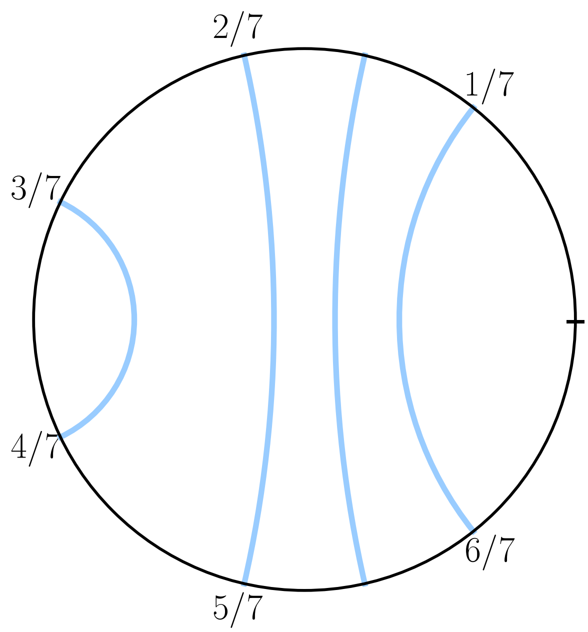


Below is the lamination of the airplane, L:



The minor leaf m of the airplane is the leaf connecting $3/7$ and $4/7$, and the major leaves are $(2/7, 5/7)$ and $(3/14, 11/14)$.

Let L_0 be the lamination containing the major leaves and all forward images of the major leaves. L_0 is shown below.

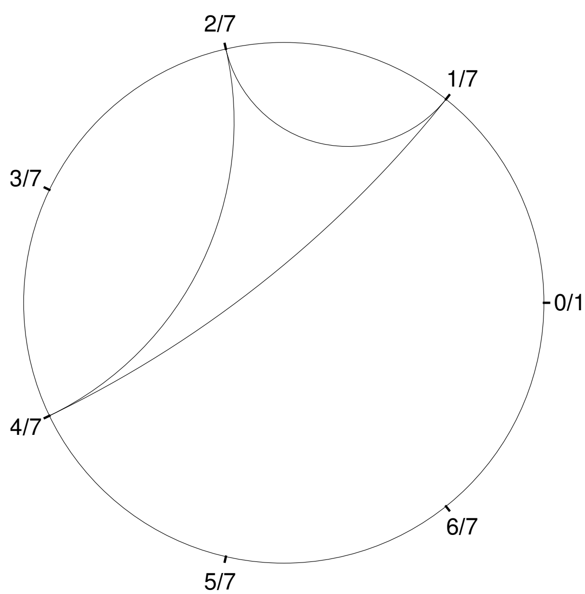


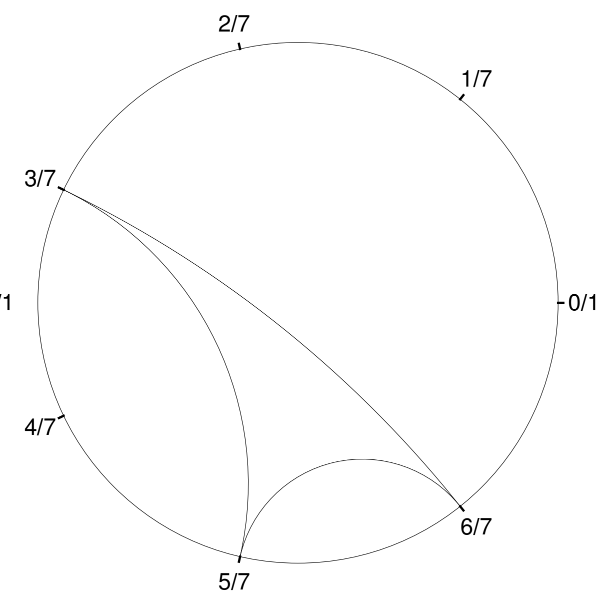
Note that from Thurston, we know that the airplane lamination has the property that any other lamination having minor leaf $(3/7, 4/7)$ contains the airplane lamination. From Bandt and

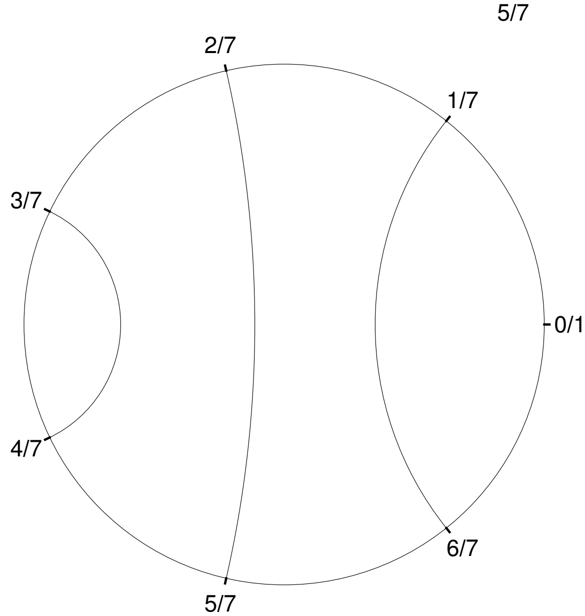
Keller, we know that the airplane lamination is the largest forward invariant lamination that does not have a leaf connecting a point of T_0 with a point ofT_1.

Thus to check that a lamination is equal to the airplane lamination, it is enough to check that the lamination is invariant, has minor leaf $(3/7, 4/7)$, and does not identify a point of T_0 with a point of T_1.

Here are some observations that are helpful in determining whether or not a lamination has minor leaf $(3/7, 4/7)$. If a leaf is in a three-cycle, there are only three possibilities for what the three cycle looks like. Those possibilities are shown below:





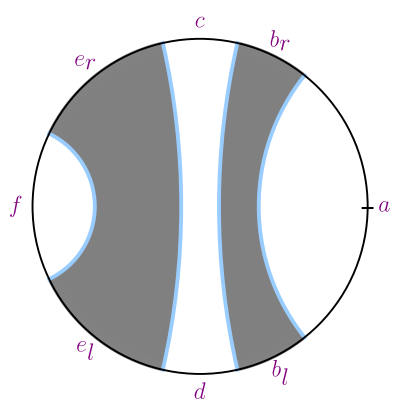


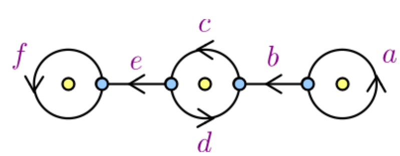
Definition 2.2.1. Notice that the cycle containing the leaf $(3/7, 4/7)$ is the only three cycle with disjoint leaves. Call the forward invariant lamination consisting of the three leaves in this cycle L_F.

2.3 Laminations of the Replacement System

Next, we construct a sequence of laminations that give the graphs of the expansion sequence defined in the previous section.

Definition 2.3.1. Let L_0 be the following lamination. We also show the base graph $\overline{H}$ of the airplane replacement system again for reference:



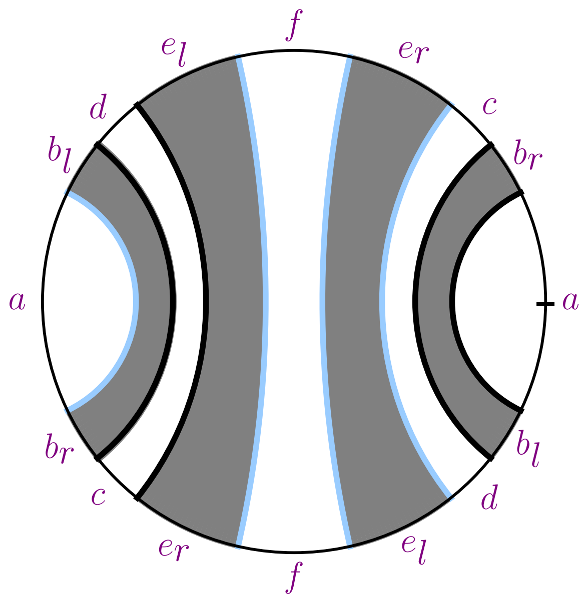


It will be important for us to distinguish between the lamination L_0 on S^1 and the graph consisting of an edge for each labeled arc on S^1 in L_0, where two edges are adjacent if the corresponding arcs are adjacent on S^1. Clearly this graph is homeomorphic to a circle. We will call it C_0.

L_0 is exactly the set of leaves that identifies each point on the closure of e_r or b_r with its conjugate on e_l or b_l. H_0 is homeomorphic to the quotient of the circle obtained by the points connected by the leaves.

C_0 is the boundary of the external face of $\overline{H}$.

Similarly, the preimage of the base graph, $\overline{H_1}$, is given by identifying endpoints of leaves the lamination $\overline{L_1}$ shown below. Note that the degree 2 covering map from $\overline{H_1}$ to $\overline{H}$ extends to a degree 2 covering map from C_1, the boundary of the external face of $\overline{H_1}$, to C_0. That map extends to a degree 2 covering map from L_1 to L_0. The labels in the below figure correspond to the labels that each arc maps to under the map.



Lemma 2.3.2. *$\overline{L_0}$ contains L_F.*

Proof. From Definition 1.4.9, we know that the map from f H_1 to H extends to a covering map $\hat{f}$ from $\overline{H_1}$ to $\overline{H}$. Thus $\hat{f}$ maps the vertices of $\overline{H_1}$ to the vertices of $\overline{H}$. We can restrict the map to the vertices of $\overline{H}$. Since $\overline{f}$ was an extension of f, it must map the two vertices that are the endpoints of c and d to the vertex on f, the vertex on f to the vertex on a, and the vertex on a to one of the endpoints of c and d. Each one of these vertices is an endpoint of e or b, and thus corresponds a leaf of the lamination L_0 (marked in blue in the image). Also note that all the leaves of L_0 are disjoint. Thus this three-cycle must be L_F. $\square$

Definition 2.3.3. We define the following replacement system (which we will refer to as the **circle replacement system**:

Step 1: Let C_0 from Definition 2.3.1 be the base graph.

Step 2: As we do with Julia sets in 1.4.9, use the vertices of C_0 to partition C_1 into paths. The corresponding pairs of components (where each component in C_0 is an edge and each component in C_1 is an edge or a path) are the replacement rules.

The limit space of this replacement system is homeomorphic to a circle, and we will refer to it as the **circle limit space**, C_∞.

We would like to define an equivalence relation $\sim_C$ on the circle limit space such that $C_\infty / \sim_C$ is homeomorphic to the limit space of the airplane replacement system.

Definition 2.3.4. For each finite expansion graph C_n, there is a surjection σ_n from the set of edge addresses of C_n to the set of edge addresses graph G_n of the airplane expansion sequence, such that two edge addresses map to the same edge addresses if they are equal if you drop all subscripts from edge labels in the edge addresses.
Similarly, we can define σ_∞, such if s is an edge address of $C\infty$, $\sigma(s) = t$ t is the sequence obtained by removing the subscripts from any letter of s with a subscript.

This gives us an equivalence relation $\sim_C$ such that two edge labels are equivalent if they are the same edge label, and additionally,

$$e_r \sim_C e_l \qquad \text{and} \qquad b_r \sim_C b_l.$$

Two edge addresses $\epsilon_1\epsilon_2\ldots$ and $\epsilon'_1\epsilon'_2\ldots$ are equivalent under $\sim_C$ if $\epsilon_k \sim_C \epsilon'_k$ for all $k \in \mathbb{N}$.

In other words, two edge addresses are equivalent if they map to the same point under σ.

We state the following proposition without proof:

Proposition 2.3.5. *The shift map on edge addresses of the circle limit space is a degree* 2 *covering map on the circle. Furthermore, it is conjugate to the angle doubling map* $z \mapsto z^2$.

Definition 2.3.6. $\overline{L_\infty}$ be the lamination such that two points are connected by a leaf if is the image under the conjugation from the previous proposition of two points of $C\infty$ that are equivalent by $\sim_C$.

Lemma 2.3.7. *All leaves of* $\overline{L_\infty}$ *are disjoint.*

Proof. First we want to show that no two leaves cross within the disc. This is equivalent to the statement that if a and b are two edge addresses of C_∞ such that there are two other edge addresses a', b', $a \sim_C a'$ and $b \sim_C b'$, a and a' are on the same arc connecting b and b'. This is true in the base graph C_0, and the expansion rules preserve this property, thus it is true for all C_n by induction. Since the limit space is a Hausdorff space, we can find 4 disjoint cells that contain a, b, a', and b', so it is enough to prove the statement for the finite expansions.

If two points are connected by a leaf, then they are in the same equivalence class under the relation $\sim_C$. Each equivalence class of points on the circle limit space under the equivalence relation from contains at most two points. Thus if two leaves share at least one endpoint, they must be the same leaf. $\square$

Lemma 2.3.8. *L_F is a subset of $\overline{L_\infty}$.*

Proof. Under σ_0 from Definition 2.2.4, the endpoints of each leaf in L_f map to vertices of G_0 which in the limit space can be represented by an edge addresses containing only e and b. Thus in G_∞, the vertices in a three cycle must have two pre-images under σ_∞, and they must correspond to leaves. $\square$

Lemma 2.3.9. *$\overline{L_\infty}$ contains all pre-images of the leaves in L_F.*

Proof. Each edge address in the symbol space of G_∞ has two preimages (see the graph of edge addresses in Section 1 of this chapter). If an point in G_∞ has two preimages under σ_∞, then its pre-image also has two preimages under σ_∞. $\square$

Lemma 2.3.10. *$\overline{L_\infty}$ is closed.*

Proof. Since the endpoints of e and b correspond to leaves, for each L_n, L_n is closed in the disc and $\mathbb{D}/L_n$ is open in the disc. The complement of L_∞ is the set in $S^1 \times S^1$ for which a leaf connecting the two coordinates would cross some preimage of the critical leaf. This is an open set, so $\overline{L_\infty}$ is a closed lamination. $\square$

Lemma 2.3.11. *$\overline{L_\infty}$ does not identify a point of $T_0^{3/7}$ with a point of $T_1^{3/7}$.*

Proof. We know that C_0 does not identify any points of $T_0^3/7$ with $T_1^{3/7}$, since we have a complete description of the identifications.Thus we can partition C_0 into $T_0^{3/7}$ and $T_1^{3/7}$. Now note that the expansion rules of the circle replacement system preserve the property in the statement of the lemma. □

Lemma 2.3.12. $\overline{L_\infty}$ *has minor leaf* $(3/7, 4/7)$.

Proof. $\overline{L_\infty}$ contains the two pre-images of $(3/7, 4//7)$, and does not identify a point of $T_0^{3/7}$ with a point of $T_1^{3/7}$. Thus the two preimages must be the major leaves. □

Theorem 2.3.13. G_∞ *is homeomorphic to the airplane.*

Proof. Claim: The lamination L_∞ is the lamination for the pinched disc model of the airplane. The lamination for the airplane is the closure of the union of the set of pre-images of L_F. We've proven that L_∞ contains that union and L_∞ is closed, so it must contain the airplane lamination. The airplane lamination is the largest clean lamination that does not have leaves that connect points from $T_0^{3/7}$ points from $T_1^{3/7}$, so it must contain L_∞. □

2.4 The Correspondence

We define a map σ from the set of edge labels to $\{0, 1\}$ by by

$$\sigma(a) = \sigma(b) = \sigma(d) = 0$$

$$\sigma(c) = \sigma(e) = \sigma(c) = 1$$

This extends to a map from the set of address in the symbol space to $\{0, 1\}^\infty$ where for $x_1x_2x_3 \cdots \in \Omega$

$$\sigma(x_1x_2x_3 \dots) = \sigma(x_1)\sigma(x_2)\sigma(x_3) \dots$$

Note that the gluing vertex $[\overline{ebe}]$ maps to α.

3
Polynomial Matings

3.1 Background

Quadratic rational functions are functions of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ have degrees of at most 2 and at least one has degree 2. Rational functions act on the Riemann sphere $\hat{C}$, which can be thought of as the complex plane together with a point at infinity.

In general the structure of Julia sets of rational functions are not well understood. However, when a post-critically finite rational map has two critical points and the two critical points are attracted to disjoint cycles, the dynamics of the rational map can be understood as the dynamics of two post-critically finite polynomials combined in a specific way, and the rational map is then called the **mating** of the two polynomials.

Definition 3.1.1. Let f, g be two post-critically finite quadratic polynomials with pinched disc models given by laminations L_f and L_g.

Let $\sim_{lam}$ be the smallest equivalence relation on S^1 such that two points p and q are identified if they are connected by a leaf in L_f *or* $1-p$ and $1-q$ are connected by a leaf in L_g.

If the quotient $S^1/\sim_{lam}$ is a topological two sphere then we say that f and g are **topologically matable**, and the action of the angle doubling map on L_f and L_g extends to a map on the quotient, which we call the **topological mating**.

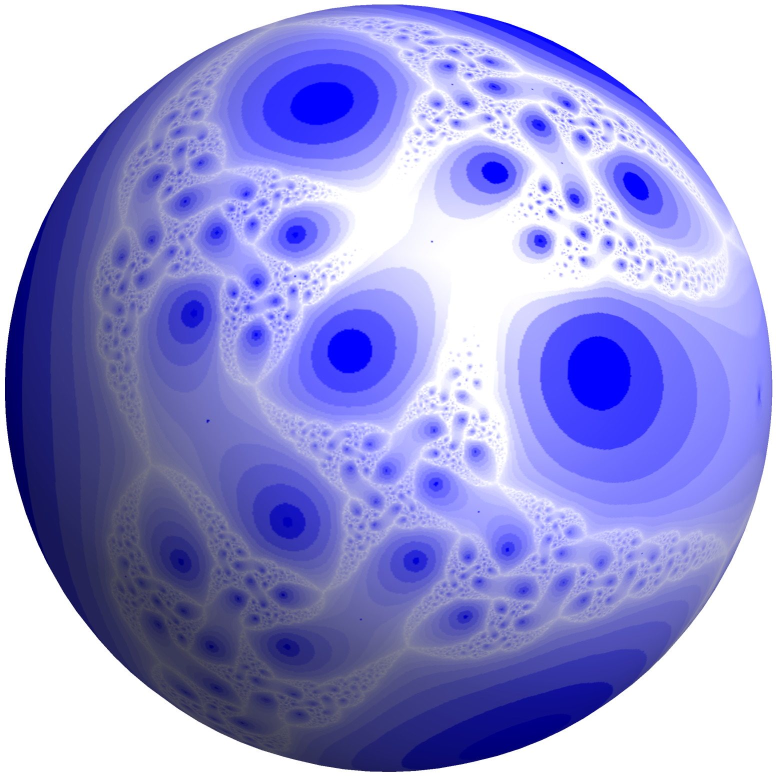


Figure 3.1.1. A Julia set of a quadratic rational function on the Riemann sphere.

If there is a orientation preserving homeomorphism that conjugates the topological mating to a quadratic rational function R on $\hat{\mathbb{C}}$, then we say that f and g are **geometrically matable** and R is the **geometric mating** of f and g.

The original intuition for a mating comes from thinking of putting one Julia set on the top hemisphere of the sphere and putting one Julia set on the bottom, and then stretching both towards the equator. This is why the definition of $\sim_{lam}$ identifies p and q if $1-p$ and $1-q$ are identified in L_g. When we attach the two laminations, L_g is in a sense "upside down".

Matings of quadratic polynomials are well-understood by the following theorem:

Theorem 3.1.2. *Suppose f and g are post-critically finite quadratic polynomials. Then the following are equivalent:*

1. *f and g are topologically matable.*

2. *f and g are geometrically matable.*

3. f and g do not lie in conjugate limbs of the Julia set.

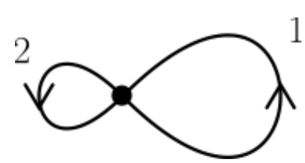
Now if we have graph replacement systems for quadratic polynomial Julia sets, its natural to try to extend the construction to quadratic matings, which would hopefully lead to rearrangement groups on matings. In the following two sections we present, without proof, finite state automata for the Rabbit Julia set and the Basilica Julia set that describe how the limit spaces should be attached to give the Julia set of a mating. The finite state automata we present here were conjectured after extensive experimentation in Mathematica.

3.2 Replacement Systems for the Rabbit and the Basilica

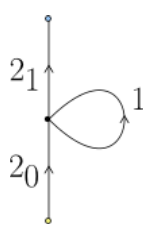
The replacement system for the Basilica is given in the background section on replacement systems. Here we present the simplified version, as well as the simplified version of the replacement system for the Rabbit Julia set. The replacement systems given in this section are similar to ones to the ones in Rearrangement Groups of Fractals. However, we do not do all of the simplifications to get to the ones in that paper, as the "delay" in the expansions of some edges is essential information.

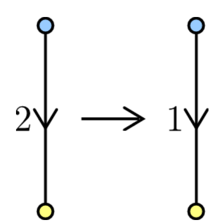
Definition 3.2.1. The components of the replacement system for the basilica are given below:

The base graph:



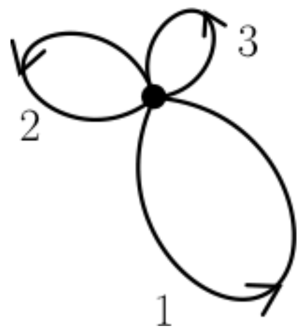
Edges labeled with 1 get replaced with the first graph below, and edges labeled with 2 get replaced with an edge labeled with 1:



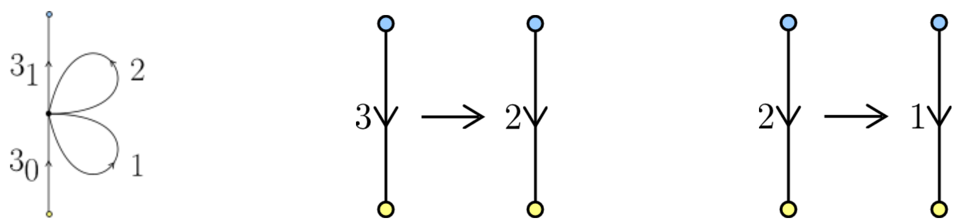


Definition 3.2.2. The components of the replacement system for the rabbit are given below:

The base graph:



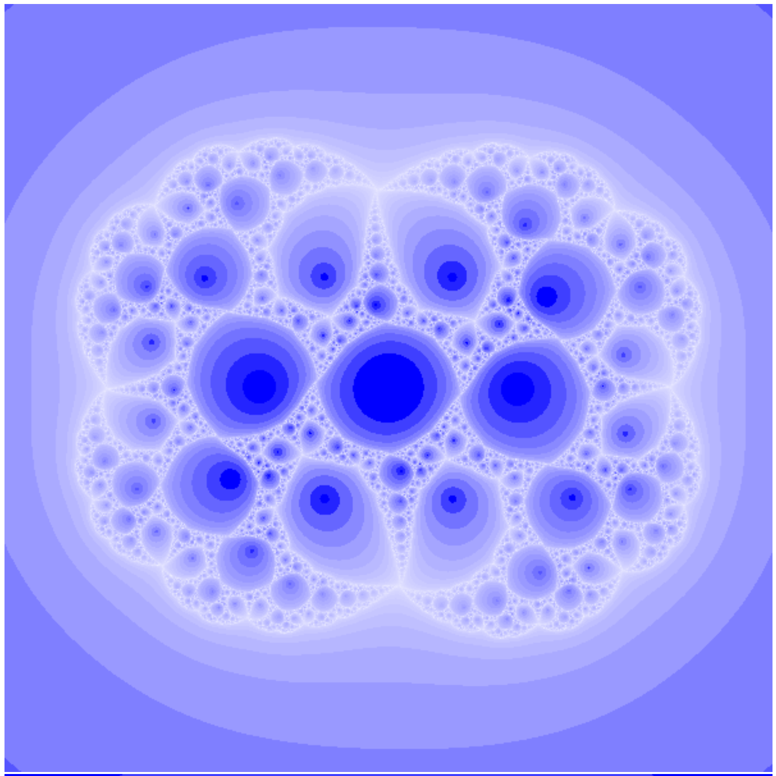
Edges labeled with 1 get replaced with the first graph below, edges labeled with 2 get replaced with an edge labeled with 1, and edges labeled with 3 get replaced with edges labeled with 2:

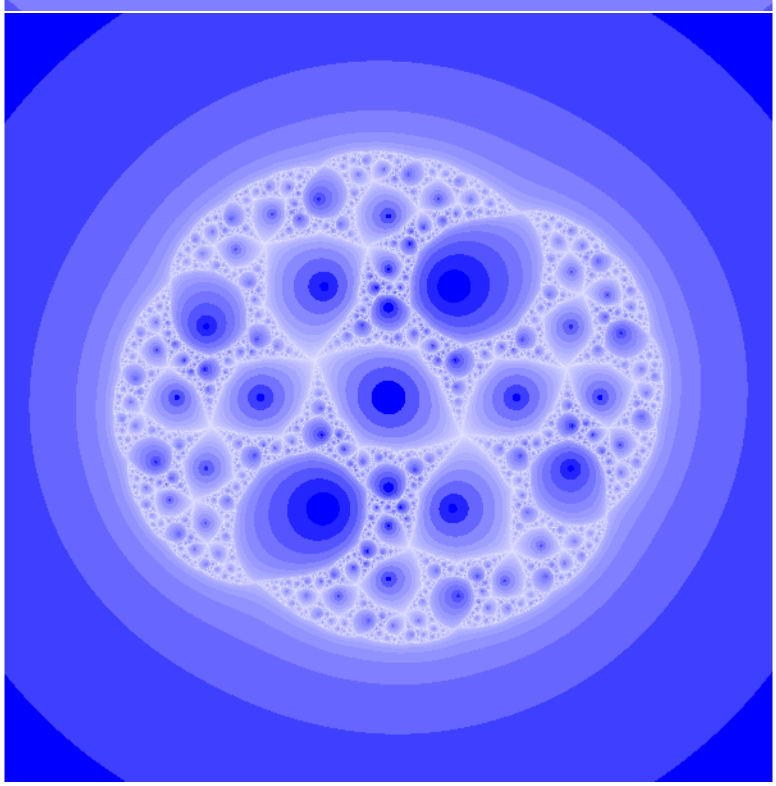


3.3 Mating of the Rabbit and the Basilica

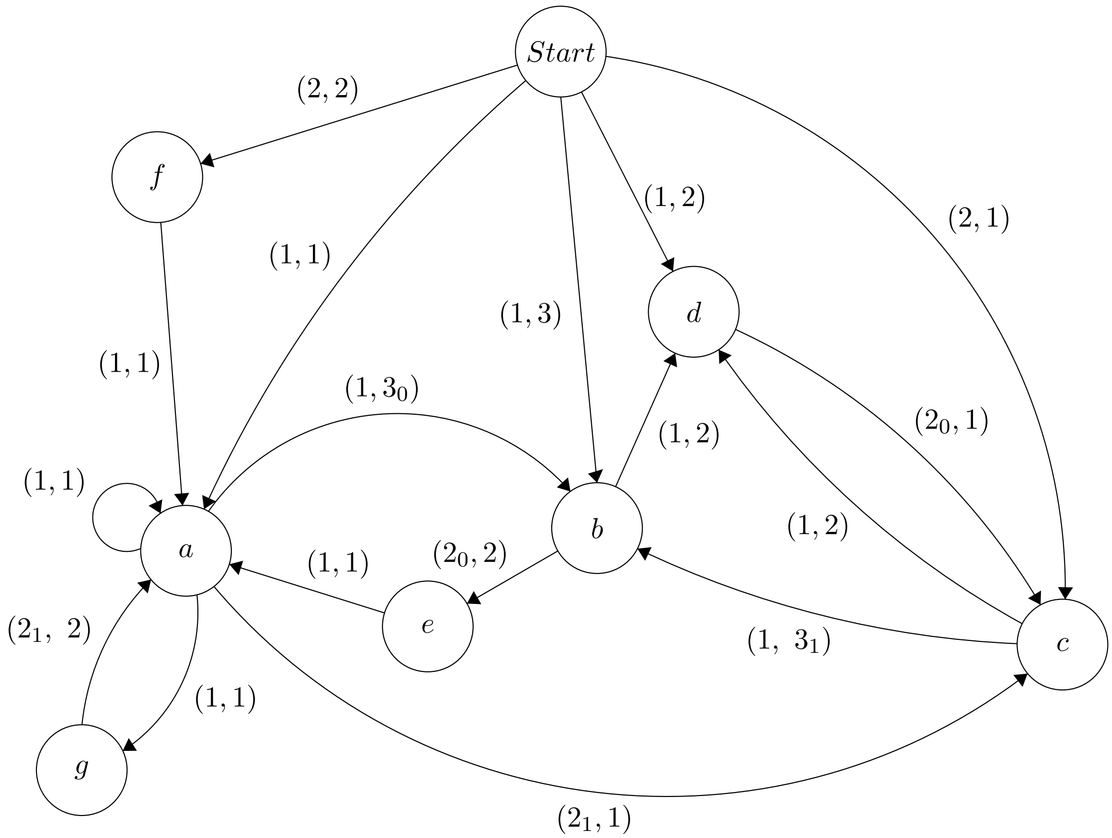
The goal of this section is to present a relation from the set of points in the Basilica limit space to the set of points in the Rabbit limit space, in a way such that gluing together the two limit spaces at points that are related gives the Julia set of the topological mating of the Rabbit and the Basilica, and the shift map on the glued together limit spaces is conjugate to the action of the topological mating on its Julia set.

Below are two pictures of the mating of the Basilica and the Rabbit. In the first one the critical point in the two cycle is 0, and the critical point in the three cycle is ∞, and in the second one the critical point in the three cycle is ∞ and the critical point in the first cycle is 0. We can see in these pictures that the Julia set of the mating of the Rabbit and the Basilica looks like the Julia sets for the polynomials stuck together. Indeed, one can construct the mating by gluing together two laminations such that points with the same binary angle are glued together, and then identifying points connected by leaves as usual.



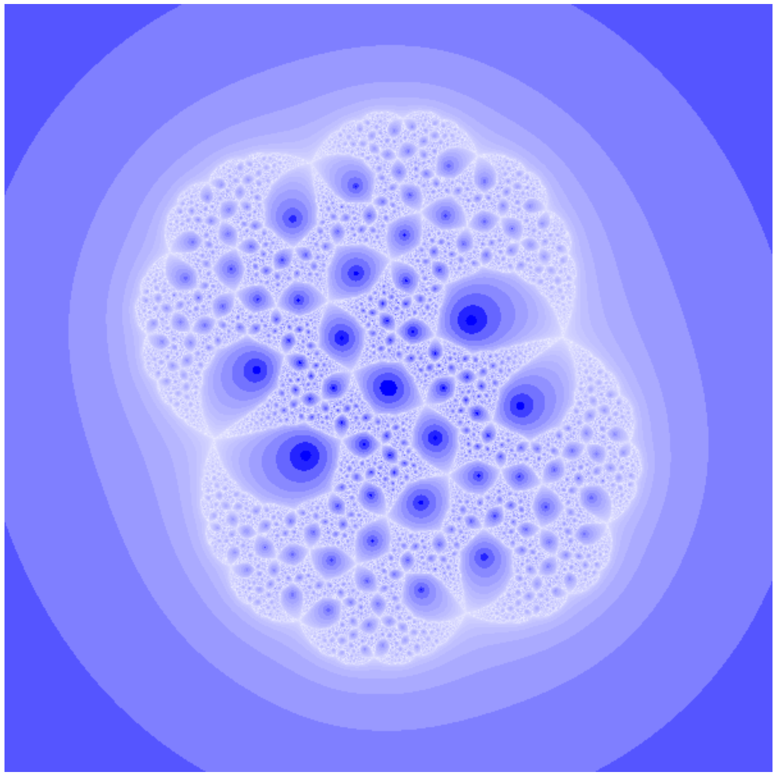


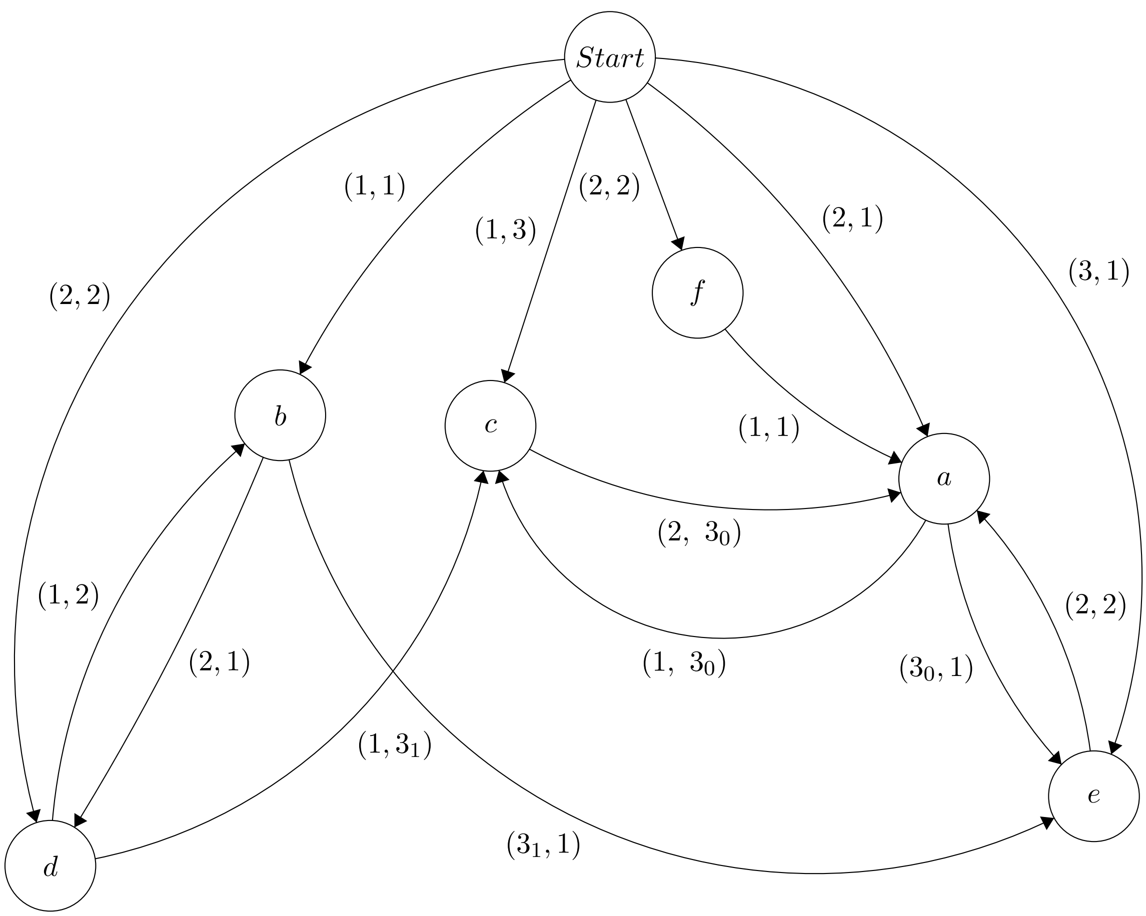
The finite state automata works below as follows. For a pair of edge addresses $e_1e_2e_3\ldots$ and $e'_1e'_2e'_3\ldots$ where $e_1e_2e_3\ldots$ represents a point in the Basilica limit space and $e'_1e'_2e'_3\ldots$ represents a point in the Rabbit limit space, the two points of the limit spaces are identified if and only if $(e_1, e'_1), (e_2, e'_2), (e_3, e'_3)\ldots$ is a valid sequence of state transitions in the diagram.



3.4 Mating of Two Rabbits

The automata for the mating of two rabbits works the same as the automata from the section above.




Start
(1, 1)
(2, 2)
(1, 3)
(2, 1)
(3, 1)
(2, 2)
f
b
c
a
(1, 1)
(2, 3₀)
(1, 2)
(2, 1)
(1, 3₀)
(3₀, 1)
(2, 2)
(1, 3₁)
e
d
(3₁, 1)

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