Homeomorphisms of the Sierpinski Carpet

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Abstract

The Sierpiński carpet is a fractal formed by dividing the unit square into nine congruent squares, removing the center one, and repeating the process for each of the eight remaining squares, continuing infinitely many times. It is a well-known fractal with many fascinating topological properties that appears in a variety of different contexts, including as rational Julia sets. In this project, we study self-homeomorphisms of the Sierpiński carpet. We investigate the structure of the homeomorphism group, identify its finite subgroups, and attempt to define a transducer homeomorphism of the carpet. In particular, we find that the symmetry groups of platonic solids and $D_n \times \mathbb{Z}_2$ for $n \in \mathbb{N}$ are all subgroups of the homeomorphism group of the carpet, using the theorem of Whyburn that any two S-curves are homeomorphic.

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Dedication

To Liam, with all of my love.

Acknowledgments

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1
Introduction

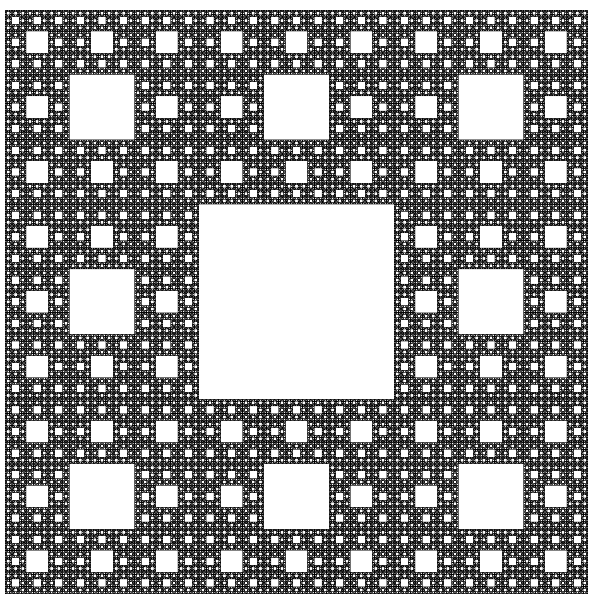


Figure 1.0.1: The Sierpiński carpet

The Sierpiński carpet, shown in Figure 1.0.1, is a fascinating fractal that is of interest in a wide variety of fields. Originally an attempt to generalize the Cantor set, it now shows up in engineering contexts, such as antennae for mobile phones. It is one of the simplest fractals to define, yet has many unusual topological properties. The study of these properties has inspired countless mathematicians, and is the main subject of this project.

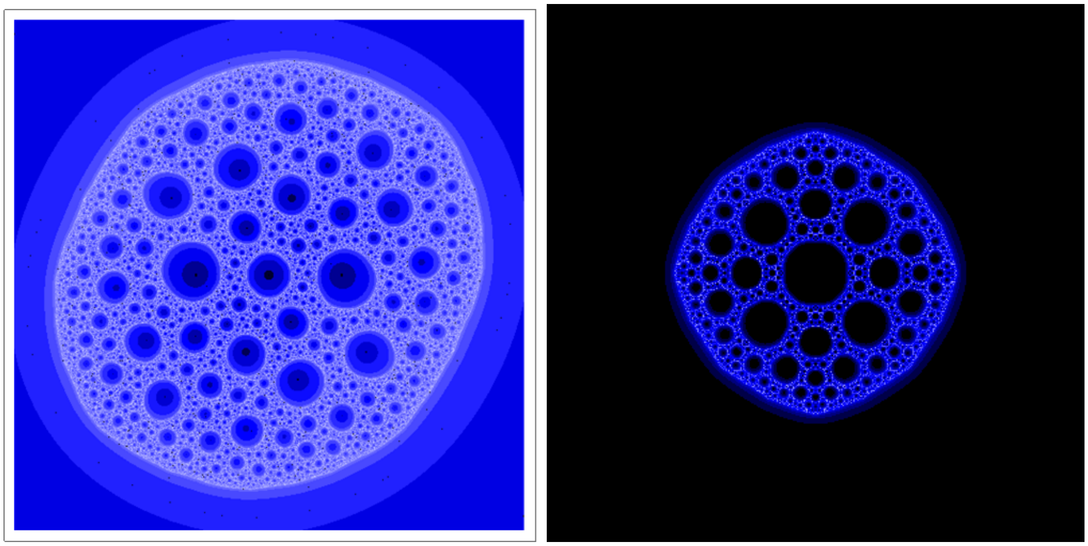


Figure 1.0.2: Two rational Julia sets homeomorphic to the Sierpiński Carpet

The Sierpiński carpet was first defined by Wacław Sierpiński in 1916 [9], who proved that the carpet was what is known as a universal plane curve — that is to say, any subset of $\mathbb{R}^2$ with topological dimension 1 is homeomorphic to a subset of the carpet. Gordon Whyburn [10] went further, describing a class of sets that he referred to as "S-curves" (see Definition 4.2.1) and proving that the Sierpiński carpet is homeomorphic to any S-curve. Such sets show up in a variety of surprising situations, such as the rational Julia sets shown in Figure 1.0.2.

Studying Whyburn's methods, one can easily infer that the set of self-homeomorphisms of the carpet is quite large. It follows naturally that a number of mathematicians have an interest in studying these homoemorphisms. Previous work has been done by Jmel, Salhi, and Vago [6] on self-homeomorphisms of the carpet with various periodic properties, including a proof that any self-homeomorphism of the carpet must extend to a self-homeomorphism of the sphere. There has also been work done by Charatonik [3], building on Whyburn's original paper, studying mappings other than homeomorphisms from the carpet into itself. However, constructing explicit homeomorphisms of the carpet has proven difficult. Aside from the natural symmetries of a square, constructing homeomorphisms of the carpet can become rather complicated, and thus it is of some interest to find such homeomorphisms.

The main technique for constructing homeomorphisms that prompted this project is through the use of finite state machines known as transducers. The use of transducers in such a way is described by Grigorchuk, Nekrashevich, and Sushchanskii [5] – these techniques are further used by Belk, Bleak, and Matucci [1], and it is also true that the rearrangements described by Belk and Forrest [2] on the basilica Julia set can be descrbied by transducers. However, no homeomorphisms of the carpet aside from the natural symmetries of the square have yet been defined using transducers under the classic address system on the carpet, and so the initial goal of this project was to find such a homeomorphism. The second goal of this project was to classify all of the finite subgroups of the homeomorphism group of the carpet. By Jmel et. al. [6], it follows that any group that acts on the carpet by homeomorphism must act on the sphere by homeomorphism, so it is a matter of showing which such groups also act on the carpet.

This project is organized in four sections. First we present preliminary information that will be necessary for the rest of the project. Second, we discuss the attempted rational homeomorphisms on the carpet. Third, we describe the theorem of Whyburn and discuss its implications. Fourth, we prove various results regarding the finite subgroups of the homeomorphism group, as well as discuss ways in which to further use such techniques.

2
Preliminaries

2.1 The Sierpiński carpet

2.1.1 Definitions

The classic definition for the Sierpiński carpet is as follows.

Definition 2.1.1. Letting I be the unit interval $[0,1] \subseteq \mathbb{R}$, we define the set of functions $F_0, \ldots, F_7 \colon I \times I \to I \times I$ as follows:

$$
\begin{aligned}
F_0(\mathbf{x}) &= \frac{1}{3}\mathbf{x}, \\
F_1(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(\frac{1}{3}, 0\right), \\
F_2(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(\frac{2}{3}, 0\right), \\
F_3(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(0, \frac{1}{3}\right), \\
F_4(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(\frac{2}{3}, \frac{1}{3}\right), \\
F_5(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(0, \frac{1}{3}\right), \\
F_6(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(\frac{1}{3}, \frac{2}{3}\right), \\
F_7(\mathbf{x}) &= \frac{1}{3}\mathbf{x} + \left(\frac{2}{3}, \frac{2}{3}\right).
\end{aligned}
$$

Such a set of functions is known as an iterated function system in dynamics. Now, let $S_1 = I \times I$, and let us define a sequence of subsets $\{S_n\}$ in $I \times I$ by $S_{n+1} = \bigcup_{i=0}^{7} F_i(S_n)$ for all $n \in \mathbb{N}$. The **Sierpiński carpet** is the set $S = \bigcap_{n \in \mathbb{N}} S_n$. $\triangle$

We now list a few important propositions regarding the Sierpiński carpet.

Proposition 2.1.2. *The Sierpiński Carpet is compact, locally connected, and one-dimensional.*

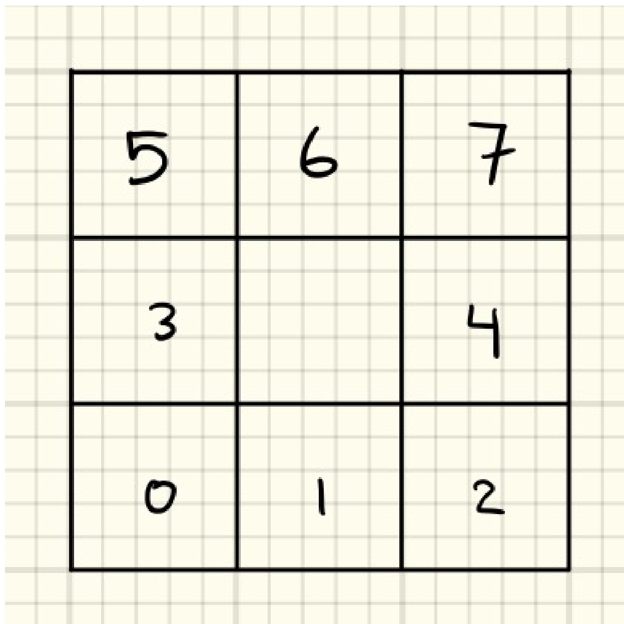


Figure 2.1.1: Labeling of self-similar components

We can see that the Sierpiński carpet is comprised of eight self-similar parts, each of which corresponds to one of the functions of the iterated function system above, as shown in Figure 2.1.1. For example, F_0 maps the full carpet to a scaled version of itself contained in the square $\left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{3}\right]$. We can label each of the these self-similar parts with the integer that labels the corresponding function in the iterated function system. Thus, for any point in the carpet, we can say that it is in "square i_0" – referring to the self-similar portion that corresponds to F_{i_0} – for some $0 \leq i_0 \leq 7$. Now, within each of these "squares," we see that they too can be divided up into eight self-similar components labeled in the same manner, so x must be in one of these smaller squares labeled $i_0 i_1$ for some $0 \leq i_1 \leq 7$. We can repeat this process infinitely many times, so that we have an infinite sequence $i_0 i_1 i_2 \ldots$ such that $0 \leq i_n \leq 7$ for all $n \geq 0$.

We will refer to an infinite sequence $i_0 i_1 \ldots$ as an **address** for x if x is in "square $i_0 \ldots i_n$" for all $n \geq 0$.

Proposition 2.1.3. *Let $i = i_0 i_1 i_2 \ldots$ be an infinite sequence such that $0 \leq i_n \leq 7$ for all $n \geq 0$. Then i defines a unique point x in the Sierpiński carpet S for which it is the address.*

Proof. Let $i = i_0 i_1 \ldots$ be an address. We see that for each $n \in \mathbb{N}$, the set $(F_{i_0} \circ \cdots \circ F_{i_n})(I \times I)$ is compact, as well as that $(F_{i_0} \circ \cdots \circ F_{i_{n+1}})(I \times I) \subseteq (F_{i_0} \circ \cdots \circ F_{i_n})(I \times I)$. Because this is a nested sequence of compact sets, it follows that $\bigcap_{n \in \mathbb{N}} (F_{i_0} \circ \cdots \circ F_{i_n})(I \times I)$ is nonempty. Given that $x \in S$ has the address i if and only if $x \in (F_{i_0} \circ \cdots \circ F_{i_n})(I \times I)$ for all $n \in \mathbb{N}$, we see that there must exist a point $x \in S$ such that x has the address i.

Suppose both x and y are distinct points in S with the address i. Then, let $r = d(x, y)$ under the Euclidean distance d. Because the diameters of the squares $(F_{i_0} \circ \cdots \circ F_{i_n})(I \times I)$ tend to 0, it follows that there exists some $N \in \mathbb{N}$ such that $(F_{i_0} \circ \cdots \circ F_{i_N})(I \times I)$ has diameter less than r. However, this is a contradiction, because $(F_{i_0} \circ \cdots \circ F_{i_N})(I \times I)$ must have diameter greater than or equal to r if it contains both x and y. Therefore, the address i determines a unique point in S. ■

2.2 Necessary concepts in topology

We present a number of fairly elementary but essential concepts in topology.

Definition 2.2.1. Let X be a topological space and let $S \subseteq X$. The **interior** of S, denoted $\mathring{S}$, is the union of all subsets of S that are open in X. △

Definition 2.2.2. Let X be a topological space and let $S \subseteq X$. The **boundary** of S is $\overline{S} \setminus (\mathring{S})$. △

Definition 2.2.3. A subset S of a topological space X is **nowhere dense** if its closure has an empty interior. △

Definition 2.2.4. A topological space X is **locally connected** if for every $x \in X$ and every neighborhood U of x, there exists a connected neighborhood V of x such that $V \subseteq U$. △

Definition 2.2.5. Let X be a topological space. The **topological dimension** of X is the smallest n such that every open cover of X has a refinement such that no point of x is contained in more than $n+1$ sets. If such an n does not exist, then X is said to have an infinite dimension. $\triangle$

Definition 2.2.6. Let X be a topological space, and let $Y \subseteq X$ be a subspace of X. A **complementary domain** of Y is a connected component of $X \setminus Y$. $\triangle$

Definition 2.2.7. Let X and Y be topological spaces, let $A \subseteq X$ and $B \subseteq Y$, and let $h\colon A \to B$ be a homeomorphism. An **extension** of h is a homeomorphism $H\colon X \to Y$ such that $H|_A = h$. $\triangle$

Definition 2.2.8. A subset $S \subseteq \mathbb{R}^2$ is called an **elementary closed region** if it is closed, connected, and the boundary of S is a finite, disjoint union of simple closed curves. $\triangle$

The following definitions develop the notion of cellular subdivisions, as they relate to this project.

Definition 2.2.9. A **graph** is a topological space G with the following properties:

1. G is compact and Hausdorff;
2. There exist finitely many points $v_1, \ldots, v_n \in G$ and finitely many subspaces $e_1, \ldots, e_m \subseteq G$ such that
   (a) each e_i is homeomorphic to $[0, 1]$ (is a 1-cell);
   (b) each e_i contains exactly two v_j's, which are the endpoints;
   (c) two e_i's intersect only at endpoints;
   (d) and $G = \{v_1, \ldots, v_n\} \cup \bigcup_{i=1} e_i$.

$\triangle$

Definition 2.2.10. Let $G \subseteq \mathbb{R}^2$ be a graph, let $F_1, \ldots, F_n$ be some of the connected components of $\mathbb{R}^2 \setminus G$ that are homeomorphic to D^1, and let $H = G \cup \bigcup_{i=1}^n F_i$. We then refer to each F_i as a **face** of H with respect to its underlying graph structure G. $\triangle$

If G is a graph, the points $v_1, \dots, v_n$ are called the **vertices** of G, and the subspaces $e_1, \dots, e_m$ are called the **edges** of G.

Definition 2.2.11. Let $S \subseteq \mathbb{R}^2$. A graph $G \subseteq S$ gives a **cellular subdivision** of S if

1. Each connected component of $S \setminus G$ is homeomorphic to an open disk.

2. The boundary in $\mathbb{R}^2$ of each connected component of $S \setminus G$ is a subset of G.

△

If G gives a cellular subdivision of S, then the graph G is called the **1-dimensional structure** associated with the subdivision. The **2-cells** of the subdivision are the closures of the connected components of $S \setminus G$. Each 2-cell is homeomorphic to a closed disk.

Definition 2.2.12. Let $S \subseteq \mathbb{R}^2$ and let $G \subseteq S$ be a graph that gives a cellular subdivision of S. The **mesh** of S is equal to $\sup \{\operatorname{diam}(S_i) \mid S_i \text{ is a 2-cell of the subdivision given by } G\}$. △

Theorem 2.2.13 ([8])**.** *Let X and Y be metric spaces such that Y is complete, let $D \subseteq X$ be dense in X, and let $f : D \to Y$ be a uniformly continuous function. Then f extends uniquely to a continuous function $F : X \to Y$.*

3

Rational homeomorphisms of the Sierpiński carpet

A topic of interest regarding the Sierpiński carpet are its self-homeomorphisms. In particular, we are interested in explicitly defining homeomorphisms that are not rotations or reflections, or compositions of these. Here, we turn to a finite state machine known as a Mealy machine to construct such a homeomorphism.

3.1 Background Information

Definition 3.1.1. Let $Q = \{q_0, q_1, \ldots, q_n\}$ be a finite set of states, let q_0 denote the start state, let Σ_1 and Σ_2 be alphabets, and let $\delta\colon Q \times \Sigma_1 \to Q$ and $\gamma\colon Q \times \Sigma_1 \to \Sigma_2$ be functions. We define a synchronous Mealy machine to be the six-tuple $M = (Q, \Sigma_1, \Sigma_2, \delta, \gamma, q_0)$, which then defines the function $M\colon \Sigma_1^\omega \to \Sigma_2^\omega$. The inputs for a synchronous Mealy machine are the elements of Σ_1^ω, where for some $\mathbf{x} \in \Sigma_1^\omega$ such that $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ for each $x_i \in \Sigma_1$, we have that the output $M(\mathbf{x}) = (\gamma(q_0, x_1), \gamma(\delta(q_0, x_1), x_2), \gamma(\delta(\delta(q_0, x_1), x_2), x_3), \ldots)$. $\triangle$

Theorem 3.1.2. *Let $M = (Q, \Sigma, \Sigma, \delta, \lambda, q_0)$ be a Mealy machine, and suppose that for each $q_i \in Q$, we have that the function $\lambda_i\colon \Sigma \to \Sigma$ defined by $\lambda_i(x) = \lambda(q_i, x)$ is a bijection. Then, M defines a bijection from the set Σ^ω of all infinite sequences over Σ to itself, and $M^{-1} = (Q, \Sigma, \Sigma, \delta, \lambda', q_0)$ where $\lambda'(q_i, x) = \lambda_i^{-1}(x)$.*

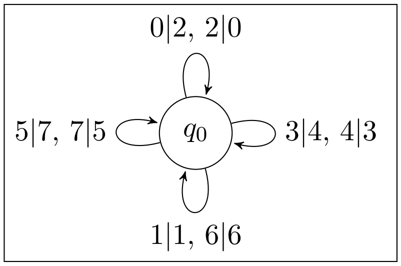
Proof. Let $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots) \in \Sigma^\omega$ such that $\mathbf{x} \neq \mathbf{y}$ and $M(\mathbf{x})$ and $M(\mathbf{y})$ have the same output. Then, there exists some $n \in \mathbb{N}$ such that $x_n \neq y_n$ but $x_i = y_i$ for all $i < n$. We see that for all $i < n$, the transition function δ must take both $\mathbf{x}$ and $\mathbf{y}$ to the same state, and let us call the last of these states q_j. Then, we have the output for the nth position $\lambda(q_j, x_n) = \lambda(q_j y_n)$, or $\lambda_j(x_n) = \lambda_j(y_n)$. However, this is a contradiction, because λ_j must be bijective, so it cannot have the same output for x_n and y_n. Therefore, $\mathbf{x} = \mathbf{y}$, so M is injective.

Let $\mathbf{z} = (z_1, z_2, \dots) \in \Sigma^\omega$. We will show by induction that for all $n \in \mathbb{N}$, there exists some output with the prefix $z_1 z_2 \cdots z_n$. Let $n = 1$. Because the Mealy machine begins at q_0, we see that the output is determined by the function λ_0, which is bijective so there exists some $w_1 \in \Sigma$ such that $\lambda_0(w_1) = z_1$. Now, suppose that there exists an output with the prefix $z_1 z_2 \cdots z_n$, so we can find an input with the prefix $w_1 w_2 \cdots w_n$. Then, we are at some state, which we will name q_j. The function λ_j is bijective, so there exists some w_{n+1} such that $\lambda_j(w_{n+1}) = z_{n+1}$. Therefore, there exists an output with the prefix $z_1 z_2 \cdots z_{n+1}$. Since we can find an output with the prefix $z_1 z_2 \cdots z_n$ for all $n \in \mathbb{N}$, it follows that there exists some $\mathbf{w} \in \Sigma^\omega$ such that $M(\mathbf{w})$ outputs $\mathbf{z}$, so M is surjective, and thus a bijection. ■

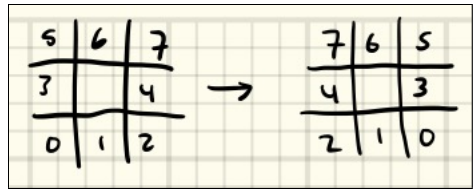
3.2 Rational Homeomorphisms

Definition 3.2.1. Let $f\colon S \to S$ be a homeomorphism of the Sierpiński carpet. We define f to be a **rational homeomorphism** of S if there exists a Mealy machine M with input and output alphabet $\{0, \dots, 7\}$ such that $f \circ \varphi = \varphi \circ M$, where $\varphi\colon \{0, \dots, 7\}^\omega \to S$ is the quotient map defined by the quotient topology of S. △

We can see a very simple example of a rational homeomorphism of the Sierpiński carpet in Figure 3.2.1. The graph in Figure 3.2.1a describes the Mealy machine for the reflection across the line $x = \frac{1}{2}$, where a given edge labeled $x|y$ represents the output y associated with the input x. We have collapsed some of the edges for the sake of space and readability. Another rational homeomorphism can be seen in Figure 3.2.2, this time for a rotation of order 4. The subgroup of $\mathrm{Homeo}(S)$ generated by the two homeomorphisms in Figures 3.2.1 and 3.2.2 is isomorphic to

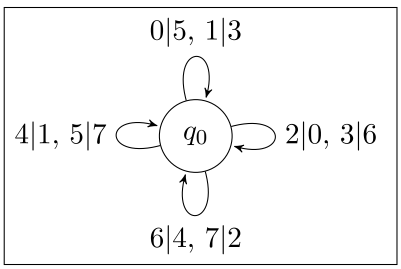


(a) Mealy machine

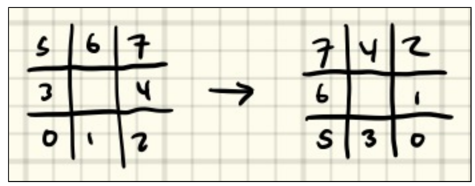


(b) Visual representation

Figure 3.2.1: Reflection across $x = \frac{1}{2}$



(a) Mealy machine

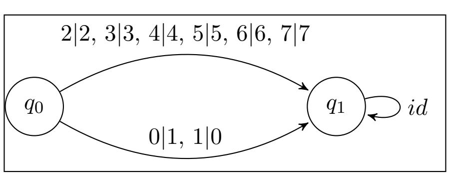


(b) Visual representation

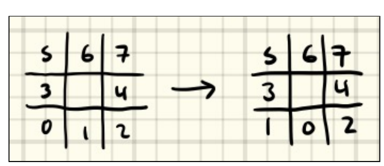
Figure 3.2.2: Counterclockwise rotation of order 4

D_4, the dihedral group on 4 vertices. The elements of this subgroup are, in fact, currently the only known rational self-homeomorphisms of the Sierpiński carpet.

Of course, there are a large number of bijections defined by Mealy machines that do not define homeomorphisms of the carpet. An example of one can be seen in Figure 3.2.3, which swaps the bottom left square with the one to its right by translation. (Here, the edge labeled *id* refers to an edge that gives the same output as input for all possible input values.)

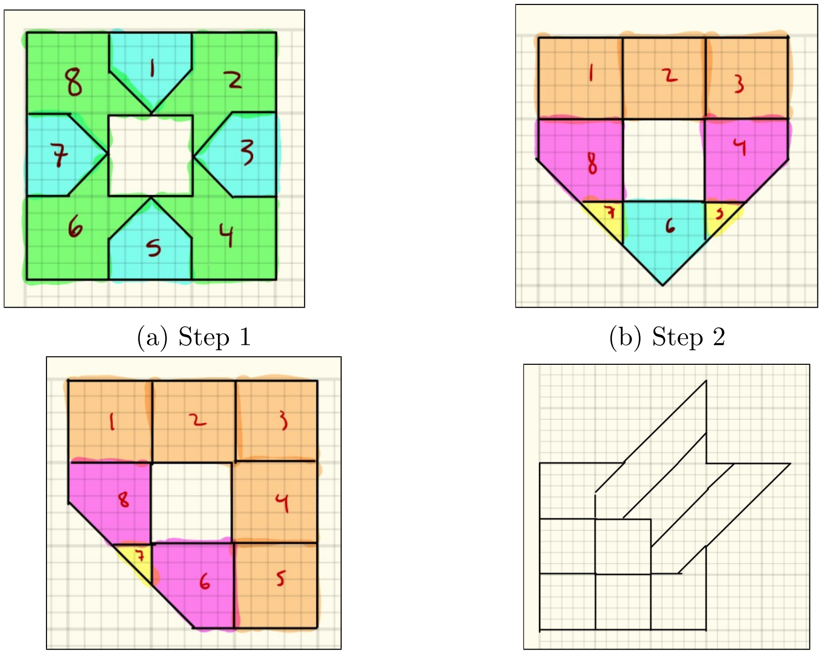


(a) Mealy machine



(b) Visual representation

Figure 3.2.3: Example of a bijection that is not a homeomorphism



(a) Step 1

(b) Step 2

(c) Step 3

(d) Step 4

Figure 3.3.1: Cutting up the carpet

3.3 A "Failed" Attempt

Unfortunately, as of now, we have been unable to find a rational homeomorphism of the Sierpiński carpet. The techniques used have primarily been finding ways to cut the carpet into pieces to which we can map the original eight squares. A successful rational homeomorphism would have finitely many such types of pieces, which can eventually be decomposed into one another. However, all attempts so far have devolved into increasingly complicated pieces that seem to regularly require new ways to cut the carpet. While it is very possible that a rational homeomorphism can be found in one (or all) of these attempts, there is no good way of telling when to stop trying.

Let us take a look at one such example. We can cut up the carpet as shown in Figure 3.3.1a. In Figure 3.3.1b, we can see a possible decomposition of the light blue piece. In Figure 3.3.1c, we see a decomposition of the pink piece as well, which, save for the yellow triangle, uses the same shapes as previous pieces that we already know how to decompose. However, the green piece gives us trouble, as seen in the attempted decomposition shown in Figure 3.3.1d. It should be noted that Figure 3.3.1d is not compatible with the structure of the divisions of the other sections. Though they are not shown here, attempts to decompose the yellow piece have also been unsuccessful.

It is important to note that the diagonal lines through the carpet have been proven to exist by Chen and Niemeyer [4].

4
Whyburn's original paper

4.1 Preliminary

Previously, we defined the Sierpiński carpet in terms of a specific iterated function system. However, there are other ways to understand the carpet from a topological perspective. Wacław Sierpiński proved that the carpet was a universal plane curve; that is, a compact subset of $\mathbb{R}^2$ with a topological dimension of 1. Gordon Whyburn [10] went further and characterized the Sierpiński carpet as what he referred to as an "S-curve," defined by a compact, connected, locally connected, nowhere dense subset of $\mathbb{R}^2$ with at least two points such its boundary is a disjoint union of simple closed curves. In the same paper, he proved the following significant theorem regarding S-curves.

Theorem 4.1.1 (Whyburn [10])**.** *Any two S-curves are homeomorphic.*

We present a proof of a stronger version of this theorem later on. This particular result is very important, as it states that any S-curve is homeomorphic to the Sierpiński carpet, and thus we can use other S-curves to understand the structure of $\mathrm{Homeo}(S)$.

4.2 Definitions

Definition 4.2.1. An ***S*-curve** is a compact, connected, locally-connected, and nowhere-dense set $S \subseteq \mathbb{R}^2$ with at least two points with the property that the boundary of any two connected components of $\mathbb{R}^2 \setminus S$ do not intersect and that each such boundary is a simple closed curve. △

We will use the following terminology for an S-curve S:

- The unbounded connected component of $\mathbb{R}^2 \setminus S$ is called the **outside** of S.
- The bounded connected components of $\mathbb{R}^2 \setminus S$ are called **holes** of S.
- The **boundary circles** of S are the components of the boundary of S, i.e. the boundaries of the complementary domains.
- In particular, the **outer circle** of S is the boundary of the outside, and the **inner circles** of S are the boundaries of the holes.
- A **frame** for S is an elementary closed region bounded by the outer circle of S and finitely many inner circles.

Definition 4.2.2. Let S be an S-curve, and let $\{C_n\}_{n\in\mathbb{N}}$ be the set of boundary circles of S. We define the **outer points** of S as $\mathrm{Ext}(S) = \bigcup_{n\in\mathbb{N}} C_n$. △

It should be noted that $\mathrm{Ext}(S)$ is not the same as $\mathrm{Bd}(S)$, as $\mathrm{Bd}(S) = S$.

Definition 4.2.3. Let S be an S-curve. A graph $G \subseteq S$ gives a **subdivision** of S if there exists a frame F for S such that:

1. G gives a cellular subdivision of F, and
2. $G \cap \mathrm{Ext}(S) = \mathrm{Bd}(F)$.

△

Proposition 4.2.4. *Let S be an S-curve, let $G \subseteq S$ be a graph that gives a subdivision of S, let F be a corresponding frame, and let $R_1, \ldots, R_n$ be the 2-cells of F under the subdivision given by G. Then each $R_i \cap S$ is an S-curve.*

Proof. [Proof of Proposition 4.2.4] The only hard part should be getting $R_i \cap S$ to be locally connected. ■

Definition 4.2.5. Let $f\colon A \to B$ be a continuous function where (A, d) is a metric space and B is a topological space, and let $\varepsilon > 0$. Then f is an **ε-mapping** if $f^{-1}(\{b\})$ has a diameter of less than ε for all $b \in B$. △

4.3 Whyburn's proofs

Lemma 4.3.1. *Suppose that C is a plane curve with diameter D and area A. Then there exist $p, q \in C$ such that $d(p,q) < \frac{2A}{D}$ for the Euclidean metric on $\mathbb{R}^2$, but the smallest connected set containing p and q in the exterior of C has diameter greater than $\frac{D}{4}$.*

Proof. [Proof of Lemma 4.3.1] Let C be a plane curve with diameter D and area A. Then, by compactness, there exist points $a, b \in C$ such that $d(a,b) = D$. Without loss of generality, we will let $a = (0,0)$ and $b = (0, D)$ in $\mathbb{R}^2$. Now, suppose that there exist no points p and q in C that meet the conditions described in the statement of the lemma. Let $S = \mathbb{R} \times [\frac{D}{4}, \frac{3D}{4}]$. Since for any point $x \in C \cap S$, it must be true that both $d(x,a)$ and $d(x,b)$ are greater than $\frac{D}{4}$, it follows that $d(p,q) \geq \frac{2A}{D}$ for all $p, q \in C \cap S$. Now, let $K_y = \{(x,y) \mid (x,y) \text{ is in the interior of } C\}$. Since $\mu(K_y) \geq \frac{2A}{D}$ for all $y \in [\frac{D}{4}, \frac{3D}{4}]$, it follows that $\int_{[\frac{D}{4}, \frac{3D}{4}]} \mu(K_y) d\mu > \frac{2A}{D}\frac{D}{2} = A$, which is a contradiction, as the area of C is A. Therefore, there must exist points $p, q \in C$ such that $d(p,q) < \frac{2A}{D}$ but the smallest connected set containing p and q in the exterior of C has diameter greater than $\frac{D}{4}$. ■

Lemma 4.3.2. *Suppose that S is an S-curve, and let $\varepsilon > 0$. Then there exist only finitely many holes of S with diameter greater than ε.*

Proof. [Proof of Lemma 4.3.2] Suppose that there are infinitely many holes $\{C_n\}$ with diameters greater than ε. For each n, we will let A_n be the area of C_n and D_n be the diameter of C_n. Given that S is compact, it follows that $\lim\limits_{n\to\infty} A_n = 0$. Now, by Lemma 4.3.1, we know that for each n, there exists points $p_n, q_n \in C_n$ such that $d(p_n, q_n) < \frac{2A_n}{D_n}$ and the smallest connected set containing p_n and q_n in the exterior of C_n has diameter greater than $\frac{D_n}{4}$. Now, let p be a limit point of $\{p_n\}$, and by the compactness of S, it must be true that $p \in S$. Now, let $\varepsilon' > 0$, and let $U = B(p, \varepsilon') \cap S$. ■

Whyburn proved the following two lemmas in the process of proving his main theorem.

Lemma 4.3.3 (Whyburn [10]). *Let A and B be compact metric spaces and let $f\colon A \to B$ be an ε-mapping. Then, there exists some $\delta > 0$ such that if $B_0 \subseteq B$ has a diameter less than δ then $f^{-1}(B)$ has a diameter less than ε.*

The following proof has been revised from Whyburn's original proof, but we were unfortunately unable to verify which theorem of R. L. Moore [7] was being cited, and so were not able to adequately revise the proof. Furthermore, Whyburn assumes several points which were difficult to verify, though they are undoubtedly true, so it would be of some interest to clarify Whyburn's original proof in the context of modern topology.

Lemma 4.3.4 (Whyburn [10]). *Let S and S' be S-curves, let F and F' be frames of S and S' respectively such that there exists a homeomorphism $h\colon \operatorname{Bd} F \to \operatorname{Bd} F'$, and let $\varepsilon > 0$. Then, there exist ε-subdivisions of S and S' such that their respective graph structures K and K' are homeomorphic under an extension of h.*

Proof. Let $C_1, \ldots, C_k$ be the boundary circles of F, and let $C_i' = h(C_i)$ for all $1 \leq i \leq k$. It follows that every boundary circle C of F' is of the form $C = C_i$ for some $1 \leq i \leq k$.

Now, let m be an integer such that there are at most m holes of S and S' each that have diameter greater than or equal to ε. Now, for $n = k + m$, let $C_{k+1}, \ldots, C_n$ and $C_{k+1}', \ldots, C_n'$ be distinct circles of S and S' respectively such that for any circle C of S and C' of S' with diameters $\geq \varepsilon$, we have that $C = C_i$ for some $1 \leq i \leq n$ and $C' = C_j'$ for some $0 \leq j \leq n$.

Now, let us define the relation $\sim$ on S such that $x \sim x$ and for $x, y \in C$ where C is some circle of S not in $\{C_i\}_{i=1}^n$, we have $x \sim y$. It is easy to see that $\sim$ is an equivalence relation, so let us define the quotient space $W = S/\sim$. We can similarly define an equivalence relation $\sim'$ on S' such that $x \sim' x$ and for $x, y \in C'$ where C' is some circle of S' not in $\{C_i'\}_{i=1}^n$, we have

$x \sim y$, and we can let $W' = S'/\sim'$. Let $\varphi\colon S \to W$ and $\varphi'\colon S' \to W'$ be the quotient maps, with $\varphi(x) = [x]$ and $\varphi'(y) = [y]$ for $x \in S$ and $y \in S'$.

According to Whyburn, it follows by a theorem of R. L. Moore [7] that W and W' are homeomorphic to a closed plane elementary region with $n+1$ circles. Then, the map $\varphi' \circ h \circ \varphi^{-1}$ is a homeomorphism from $\varphi(C_0)$ to $\varphi'(C_0')$ that can then be extended to a homeomorphism $t\colon W \to W'$.

Now, let $Q = \{w \in W' \mid (\varphi')^{-1}(w) \text{ or } (t \circ \varphi)^{-1}(w) \text{ contains more than one point}\}$, and we see that since S and S' have countably many circles, it follows that Q is countable. Let $\delta > 0$, and let G be a 1-dimensional graph in W' that does not intersect Q but contains all circles of W', such that G gives a cellular subdivision Σ of W' that has mesh $< \delta$. Then, both $(\varphi')^{-1}$ and $(t \circ \varphi)^{-1}$ are homeomorphisms from G to their respective images. Let $K' = (\varphi')^{-1}(G)$ and $K = (t \circ \varphi)^{-1}(G)$. Then, K and K' give subdivisions σ of S and σ' of S' respectively such that there is a 1-1 correspondence between the subdivisions of σ and Σ (and the subdivisions of σ' and Σ). Further, $(\varphi')^{-1} \circ t \circ \varphi$ is a homeomorphism from K to K' such that $(\varphi')^{-1} \circ t \circ \varphi$ restricted to C_0 is equal to h, since t is equal to $\varphi' \circ h \circ \varphi^{-1}$ when restricted to $\varphi(C_0)$ and we then have

$$\left((\varphi')^{-1} \circ \varphi' \circ h \circ \varphi^{-1} \circ \varphi\right)(C_0) = h(C_0) = C_0'.$$

Therefore, h can be extended to a homeomorphism from K to K'.

Finally, we see that both $t \circ \varphi$ and φ' are ε-mappings, so by Lemma 4.3.3 it follows that there exists some $\delta > 0$ such that σ and σ' have mesh $< \varepsilon$. ■

4.4 Implications of Whyburn's theorem

The following is a stronger version of Theorem 4.1.1, assuming Lemma 4.3.4 first.

Theorem 4.4.1. *Let S and S' both be S-curves and let F and F' be frames for S and S' respectively such that there exists a homeomorphism $h_0\colon \operatorname{Bd} F \to \operatorname{Bd} F'$. Then h_0 extends to a homeomorphism $h\colon S \to S'$.*

Proof. [Proof of Theorem 4.4.1] Let $\varepsilon = 1$. We see by Lemma 4.3.4 that there exist 1-subdivisions σ_1 and σ_1' of S and S' respectively such that their respective graph structures K_1 and K_1' are homeomorphic under a homeomorphism $h_1 \colon K_1 \to K_1'$ that extends h_0, and that the faces correspond in a 1-1 fashion under h_1.

Now, suppose that for some $n \in \mathbb{N}$, we have that for any $2 \leq k \leq n$, there exist $\frac{1}{k}$-subdivisions σ_k and σ_k' of S and S' respectively that are a refinement of the existing $\frac{1}{k-1}$-subdivisions σ_{k-1} and σ_{k-1}'. Further, let us suppose that for any k there exists a homeomorphism $h_k \colon K_k \to K_k'$ (where K_k and K_k' are the graph structures of the subdivisions σ_k and σ_k' respectively) such that h_k extends h_{k-1} and the faces of σ_k and σ_k' correspond in a 1-1 fashion under h_k. Now, let S_n and S_n' be an arbitrary pair of corresponding faces of σ_n and σ_n'. By Lemma 4.3.4, there must exist $\frac{1}{n+1}$ subdivisions of S_n and S_n' such that we can find a homeomorphism that extends h_n and maps the faces of the $\frac{1}{n+1}$ subdivision of S_n to the faces of the $\frac{1}{n+1}$ subdivision of S_n' in a 1-1 fashion. Since this is true for all pairs of corresponding faces S_n and S_n', it follows that by putting them all together, we have the $\frac{1}{n+1}$-subdivisons σ_{n+1} and σ_{n+1}' of S and S' that are refinements of σ_n and σ_n' respectively. Furthermore, letting K_{n+1} and K_{n+1}' be the graph structures of σ_{n+1} and σ_{n+1}', we can find a homeomorphism $h_{n+1} \colon K_{n+1} \to K_{n+1}'$ that is an extension of h_n. By induction, we see that this must be true for all $n \in \mathbb{N}$.

Now, let $K = \bigcup_{n\in\mathbb{N}} K_n$ and $K' = \bigcup_{n\in\mathbb{N}} K_n'$. Let us define $h \colon K \to K'$ such that for each $n \in \mathbb{N}$ and for each $x \in K_n$, we have $h(x) = h_n(x)$. This is well-defined because h_n is always an extension of h_{n-1}, so $x \in K_i$ implies that $h_i(x) = h_j(x)$ for all $j \geq i$. Further, we see that h must be 1-1, as $h(x) = h(y)$ would imply that $h_n(x) = h_n(y)$ for some $n \in \mathbb{N}$ and thus $x = y$.

We can now show that h is uniformly continuous. Let $\varepsilon > 0$. Then there exists some integer n such that $n > \frac{2}{\varepsilon}$. Let us then choose some $\delta > 0$ such that given two different faces of σ_n, they must either intersect or be at a distance greater than δ apart. Now, let $x, y \in K$ such that $\rho(x, y) < \delta$. Let S_x, S_y be faces of σ_n such that $x \in S_x$ and $y \in S_y$. Then, S_x and S_y must be the same or intersect at some point. Then, the corresponding faces S_x' and S_y' must also intersect or be the same. Thus, by the triangle inequality, it follows that $S_x' \cup S_y'$ must have a diameter that is less than ε, as each has a diameter that is less that $\frac{1}{n} < \frac{\varepsilon}{2}$. Then, $\rho\left(h(x), h(y)\right) < \varepsilon$, as $h(x) \in S'_x$ and $h(y) \in S'_y$. Therefore, h is uniformly continuous, and we can show similarly that h^{-1} is also uniformly continuous.

We see that K and K' are dense in S and S' respectively, so by Theorem 2.2.13 there must exist unique continuous extensions [6] of h to S and h^{-1} to S'. Further, the extension of h^{-1} is the inverse of the extension of h, so the extension of h is a homeomorphism from S to S'. ∎

It is clear to see that the implications of this version of Whyburn's original theorem are quite significant. In particular, given some n circles of an S-curve, it is possible to find a homeomorphism that maps those n circles to any permutation of them. Thus, the following corollary must hold true.

Corollary 4.4.2. *For every $n \in \mathbb{N}$, there exists a surjective homomorphism $\varphi\colon \operatorname{Homeo}(S) \to S_n$.*

5

Subgroups of the homeomorphism group

The implications of Theorem 4.4.1 spark a number of questions regarding the structure of Homeo(S). In particular, we wish to consider the finite subgroups of Homeo(S)

5.1 Finite subgroups through symmetry

One method of constructing finite subgroups of Homeo(S) is to construct other S-curves and consider the obvious homeomorphisms of those. For example, we can construct S-curves that have the symmetries of D_n. It is also possible to find S-curves that are subsets of platonic solids with the same symmetry.

Let us, for example, imagine a cylinder constructed out of n Sierpiński carpets of equal size lined up next to one another and circling around. Immediately, it is clear that this structure is homeomorphic to some planar set and is thus an S-curve itself, as well as that it has the symmetry of D_n. In fact, by swapping the "top" and the "bottom" of the cylinder, it is possible to find a subgroup of Homeo(S) that is isomorphic to $D_n \times Z_2$ for all $n \in \mathbb{N}$. A similar principle of "gluing" carpets together allows us to construct regular n-gons (shown in Figure 5.1.1), which we can then place on the faces of any platonic solid. By choosing one face and defining the center circle on that face to be the "outer circle" of the shape, we see that it is in fact an S-curve. Then, any of the symmetry groups of platonic solids are then subgroups of Homeo(S).

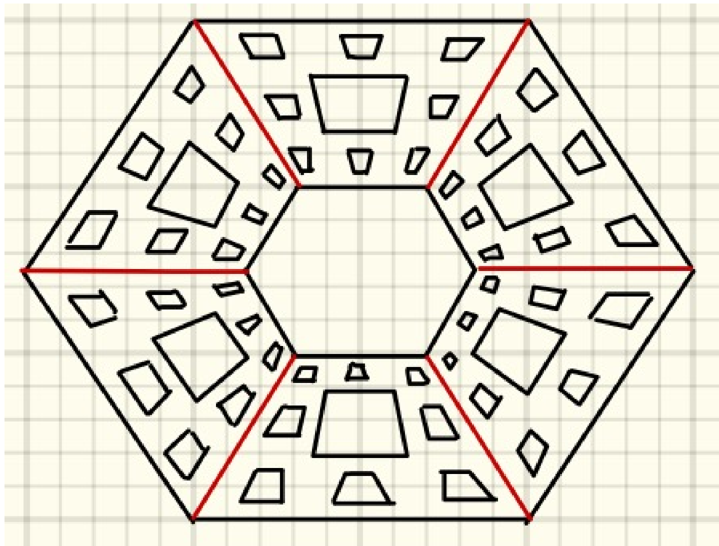


Figure 5.1.1: An S-curve embedded in a regular hexagon.

Lemma 5.1.1. *Let S be an S-curve, and let $A \subseteq S$ be a boundary circle of S. Then, there exists some S curve S' and a homeomorphism $h\colon C \to S'$ such that $h(A)$ is the outer circle of S'.*

Proof. If A is the outer circle of S, then we are done. Let us suppose that A is not the outer circle of C. Let A_0 be the outer circle of S. Now, let D be the annulus bounded by A and A_0, and it is clear that $S \subseteq D$. There exists a homeomorphism $h\colon D \to D$ such that $h(A) = A_0$ and $h(A_0) = A$. We see that $h(S)$ is compact, connected, locally connected, and nowhere dense, as well as that $h(S) \subseteq \mathbb{R}^2$. Every complementary domain with boundaries other that A and A_0 must be a subset of D, and thus for any such complementary domain K, it follows that the boundary of $h(K)$ must be a simple closed curve that does not intersect with any other such boundary. Now, the two remaining complementary domains are those whose boundaries are A and A_0, which we see must not intersect with any of the other boundaries. Therefore, $h(S)$ is an S-curve, and $h|_S$ is a homeomorphism onto its image such that $h(A)$ is the outer circle of $h(S)$. ■

We first prove the following lemma about "gluing" together two S-curves.

Lemma 5.1.2. *Let S_1 and S_2 be two S-curves in $\mathbb{R}^2$.*

1. *Let $A_1 \subseteq S_1$ be an arc of a boundary circle of S_1, let $A_2 \subseteq S_2$ be an arc of a boundary circle of S_2, and let $h\colon A_1 \to A_2$ be a homeomorphism. Identify points in S_1 with points in C_2 by $x \sim h(x)$ for all $x \in A_1$. Then the quotient space $S_1 \uplus S_2/\sim$ is homeomorphic to an S-curve.*

2. *Let $A_1 \subseteq S_1$ be a boundary circle of S_1, let $A_2 \subseteq S_2$ be a boundary circle of S_2, and let $h\colon A_1 \to A_2$ be a homeomorphism. Identify points in S_1 with points in S_2 by $x \sim h(x)$ for all $x \in A_1$. Then the quotient space $S_1 \uplus S_2/\sim$ is homeomorphic to an S-curve.*

3. *Let $A_1, A_2 \subseteq S_1$ be disjoint counterclockwise arcs of the same boundary circle $A \subseteq S_1$, and let $h\colon A_1 \to A_2$ be a homeomorphism that reverses direction. Identify points in S_1 by $x \sim h(x)$ for all $x \in A_1$. Then the quotient space $S_1/\sim$ is homeomorphic to an S-curve.*

Proof. 1. Let $A_1 \subseteq S_1$ be an arc of a boundary circle of S_1, let $A_2 \subseteq S_2$ be an arc of a boundary circle of S_2, and let $h\colon A_1 \to A_2$ be a homeomorphism. By Lemma 5.1.1, we see that there exist S-curves S_1' and S_2' as well as homeomorphisms $g_1\colon S_1 \to S_1'$ and $g_2\colon S_2 \to S_2'$ such that $g_1(A_1)$ is on the outer circle of S_1' and $g_2(A_2)$ is on the outer circle of S_2'. Furthermore, there exist disks D_1 and D_2 bounded by the outer circles of S_1' and S_2' respectively such that $S_1' \subseteq D_1$ and $S_2' \subseteq D_2$. Now, we see that $g_2 \circ h \circ {g_1}^{-1}$ must extend to a homeomorphism $h'\colon D_1 \to \mathbb{R}^2$ such that $h'(D_1) \cap D_2 = g_2(A_2)$. Now, we define the function $k\colon (S_1 \uplus S_2) \to (h'(S_1') \cup S_2')$ by $k|_{S_1} = h' \circ g_1$ and $k|_{S_2} = g_2$. We see that k is a continuous surjection from a compact to a Hausdorff space, and must be a quotient map. Furthermore, we see that for $x \in A_1$, it follows that $k(x) = (g_2 \circ h)(x)$, and thus $x \sim h(x)$ for all $x \in A_1$. We see that $k(S_1 \uplus S_2)$ must be locally connected, as it is the quotient of a locally connected space. It is also clearly compact and connected, since $k(S_1)$ and $k(S_2)$ are compact and connected and their intersection is nonempty. We also see that since their intersection is homeomorphic to an interval, $k(S_1 \uplus S_2)$ is nowhere dense. Finally, we see that since D_1 and D_2 contain all of the complementary components of S_1' and S_2' respectively aside from their unbounded components, it is clear that the bounded complementary components of S_1' are preserved under h' and thus continue to have simple closed curve boundaries that are pairwise disjoint. Thus, all the bounded complementary components of $h'(S_1') \cup S_2'$ have boundaries as simple closed curves that do not intersect with one another. We also know that the outer circles of $h'(S_1')$ and S_2' do not intersect with any of the other circles, since $h'(S_1') \cap S_2' = g_2(A_2)$. However, we also see that the unbounded complementary component of $h'(S_1') \cup S_2'$ is equal to $\mathbb{R} \setminus (h'(D_1) \cup D_2)$, which has a boundary of a simple closed curve. Therefore, $k(S_1 \uplus S_2)$ is an S-curve.

2. Let $A_1 \subseteq S_1$ be a boundary circle of S_1, let $A_2 \subseteq S_2$ be a boundary circle of S_2, and let $h\colon A_1 \to A_2$ be a homeomorphism. By Lemma 5.1.1, there exist S-curves S_1' and S_2' with homeomorphisms $g_1\colon S_1 \to S_1'$ and $g_2\colon S_2 \to S_2'$ such that $g_1(A_1)$ is not the outer circle of S_1' and $g_2(A_2)$ is the outer circle of S_2'. Let D_1 be the closed disk bounded by $g_1(A_1)$ and D_2 be the closed disk bounded by $g_2(A_2)$. Clearly, there exists a homeomorphism $k\colon D_2 \to D_1$ such that $k|_{g_2(A_2)} = g_1 \circ h^{-1} \circ g_2{}^{-1}$. Letting $x \sim h(x)$ for all $x \in A_1$, it is clear that $(S_1 \uplus S_2)/\sim$ is homeomorphic to $g_1(S_1) \cup (k \circ g_2)(S_2)$. Now, it is evident that $g_1(S_1) \cup (k \circ g_2)(S_2)$ must then be compact, connected, and locally connected, and it is also clear that it is nowhere dense. We see that every complementary component of S_1' other than the one bounded by A_1 does not change under the quotient, as well as that every bounded complementary component of S_2' is preserved under k. All these complementary components must have simple closed curves as boundaries, and these boundaries must not intersect with one another. However, we see that $g_1(A_1) = (k \circ g_2)(A_2)$, so the complementary components associated with those two boundaries no longer exist. Thus, all the complementary components of $g_1(S_1) \cup (k \circ g_2)(S_2)$ have already been shown to have disjoint simple closed curves as boundaries. Therefore, $g_1(S_1) \cup (k \circ g_2)(S_2)$ is an S-curve.

3. Let $A_1, A_2 \subseteq S_1$ be disjoint counterclockwise arcs of the same boundary circle $A \subseteq S_1$, and let $h\colon A_1 \to A_2$ be a homeomorphism that reverses direction. By Lemma 5.1.1, there exists some S-curve S and homeomorphism $g\colon S_1 \to S$ such that $g(A)$ is the outer circle of S. Now, let D be the closed disk bounded by $g(A)$, so $S \subseteq D$. Now, let us define $\sim$

on D such that $x \sim h(x)$ for all $x \in g(A_1)$. We see that $D/\sim$ forms an annulus, and we will let $j\colon D \to D/\sim$ be the corresponding quotient map. We also see that there exists an embedding $k\colon D/\sim \to \mathbb{R}^2$. Now, $j(S)$ must be compact, connected, and locally connected, and so $(k \circ j)(S)$ must also have all these properties. We also see that $S/\sim$ and thus $(k \circ j)(S)$ must be nowhere dense. Finally, we see that every bounded complementary component of S is contained in D, and because $j(D \setminus g(A))$ is a homeomorphism onto its image, it follows that for any bounded complementary component $B \subseteq D$ of S, the image $(k \circ j)(B)$ is homeomorphic to B. Thus, for each such B, the boundary must be a simple closed curve. Now, we see that any remaining complementary components of $(k \circ j)(S)$ must also be complementary components of $(k \circ j)(D)$.

■

Given this lemma, it is easy to construct S-curves with a wide variety of symmetries, and the following theorem is apparent.

Theorem 5.1.3. *$D_n \times \mathbb{Z}_2$ is a subgroup of* $\mathrm{Homeo}(C)$ *for all $n \in \mathbb{N}$.*

Proof. Let $C_1, \ldots, C_n$ be n disjoint copies of the square carpet C. For C_i and C_{i+1} for $1 \leq i < n$, let us identify $(1, y) \in C_i$ with $(0, y) \in C_{i+1}$ for all $y \in I$. Similarly, let us identify $(1, y) \in C_n$ with $(0, y) \in C_1$ for all $y \in I$. The resulting quotient space, which we will call C', is a subset of the surface of a prism with a regular n-gon as its base and squares as its sides. By Lemma 5.1.2, C must be an S-curve. The symmetry group of this prism is $D_n \times \mathbb{Z}_2$. Clearly, any symmetry of the prism results in a symmetry of C', and each symmetry of C' corresponds to a homeomorphism of C'. Thus, the subgroup of $Homeo(C')$ generated by these symmetries is isomorphic to the symmetry group of the prism, so $D_n \times \mathbb{Z}_2$ is a subgroup of $\mathrm{Homeo}(C)$. ■

In order to construct an S-curve in a regular n-gon, we can remove a smaller regular n-gon from its center and divide the remaining structure into n isosceles trapezoids. By the following lemma, we see that we can place S-curves with corresponding symmetry in each of the trapezoids (shown in Figure 5.1.2), and the remaining result is thus an S-curve as well.

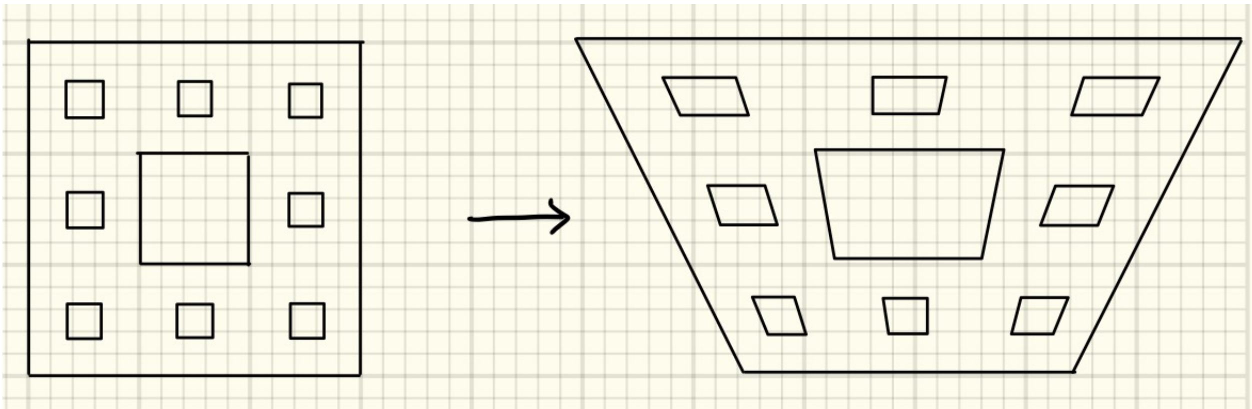


Figure 5.1.2: Taking the standard carpet to a trapezoidal carpet.

Lemma 5.1.4. *Given any isosceles trapezoid T, there exists an S-curve with T as its outer circle that also has the same symmetry as T.*

Proof. Let T be an isosceles trapezoid. We can embed it in $\mathbb{R}^2$ such that its vertices are the following points: $(a, 0)$, (b, c), $(-b, c)$, and $(-a, 0)$. We will define the map $h\colon I \times I \to T$ by

$$h(x, y) = ((2x - 1)(a + (b - a)y), cy).$$

It is clear that h must be continuous. We now wish to show that h is a bijection. Let us define the function $g\colon T \to \mathbb{R}^2$ by

$$g(x, y) = \left(\frac{x}{2\left(a + (b - a)\frac{y}{c}\right)} + \frac{1}{2},\ \frac{y}{c} \right).$$

We can see that

$$\begin{aligned}
(g \circ h)(x, y) &= \left(\frac{(2x - 1)(a + (b - a)y)}{2\left(a + (b - a)\frac{cy}{c}\right)} + \frac{1}{2},\ \frac{cy}{c} \right) \\
&= \left(\frac{(2x - 1)(a + (b - a)y)}{2\left(a + (b - a)y\right)} + \frac{1}{2},\ y \right) \\
&= \left(\frac{2x - 1}{2} + \frac{1}{2},\ y \right) \\
&= (x, y)
\end{aligned}$$

by the fact that $a + (b - a)y \neq 0$, as well as that

$$\begin{aligned}(h \circ g)(x, y) &= \left(\left(2\left(\frac{x}{2\left(a+(b-a)\frac{y}{c}\right)} + \frac{1}{2}\right) - 1\right)\left(a + (b-a)\frac{y}{c}\right),\ c\frac{y}{c}\right) \\ &= \left(\left(\frac{x}{\left(a+(b-a)\frac{y}{c}\right)} + 1 - 1\right)\left(a + (b-a)\frac{z}{c}\right),\ y\right) \\ &= \left(\frac{x}{\left(a+(b-a)\frac{y}{c}\right)}\left(a + (b-a)\frac{y}{c}\right),\ y\right) \\ &= (x, y)\end{aligned}$$

by the same fact above combined with the fact that $z \in [0, c]$. It is then clear that g is an inverse for h, so h must be bijective. Now, since $I \times I$ is a compact subset of $\mathbb{R}^2$ and T is Hausdorff, it follows that h must be a homeomorphism.

Now, we observe that there exists a line of symmetry at $x = \frac{1}{2}$ for the usual carpet $C \subseteq I \times I$. We also see that the line of symmetry for T is at $x = 0$. Letting $(x, y) \in I \times I$, we see that the reflection $r\colon I \times I \to I \times I$ across $x = \frac{1}{2}$ is defined by $r(x, y) = (1 - x, y)$, and the reflection $t\colon T \to T$ across $x = 0$ is defined by $t(x, y) = (-x, y)$. Then,

$$\begin{aligned}(h \circ r)(x, y) &= ((2(1 - x) - 1)(a + (b - a)y), cy) \\ &= (-(2x - 1)(a + (b - a)y), cy) \\ &= (t \circ h)(x, y),\end{aligned}$$

so symmetry across $x = \frac{1}{2}$ for $A \subseteq I \times I$ implies symmetry across $x = 0$ for $h(A) \subseteq T$.

Finally, it is evident that $h(C) \subseteq T$ is an S-curve. As we showed above, the S-curve $h(C)$ has the same symmetry as the isosceles trapezoid T, as C is symmetric across $x = \frac{1}{2}$. ■

With the trapezoids, we can now put together an n-gon and prove one more theorem regarding subgroups of $\mathrm{Homeo}(C)$.

Theorem 5.1.5. *$S_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, and $A_5 \times \mathbb{Z}_2$ are subgroups of* $\mathrm{Homeo}(C)$.

Proof. It is known that the groups mentioned (and some subgroups) are exactly the symmetry groups of the platonic solids. Thus, it suffices to show that one can embed an S-curve in a symmetric manner in any given platonic solid.

Let P be a platonic solid with faces comprised of regular n-gons. By Lemma 5.1.4, it is possible to create n copies of an isosceles trapezoidal S-curve with mirror symmetry, and by taking a smaller regular n-gon out of a larger one, the larger one can be divided into n isosceles trapezoids. By Lemma 5.1.2, we can glue together n copies of a trapezoidal S-curve and embed it in the regular n-gon such that the resulting S-curve has all the symmetries of the n-gon. Now, by Lemma 5.1.2 once again, we can glue together several copies of such an S-curve to form another S-curve S that is contained in the surface of P. It is obvious that the symmetry group of P acts on S by homeomorphism. Therefore, any symmetry group of a platonic solid is a subgroup of Homeo(S). ■

It is of note that the finite groups described in Theorems 5.1.3 and 5.1.5 (and their subgroups) are in fact the only finite subgroups of Homeo(C). It was shown in a paper by Jmel, Salhi, and Vago [6] that any homeomorphism of the carpet must extend to a homeomorphism of the sphere S^2, the survey by Zimmermann [11] details a proof that the only finite groups that act by homeomorphism on the sphere are the ones mentioned above. Thus, we have a full classification of all finite subgroups of Homeo(C).

5.2 Using deck transformations

In our efforts to study the subgroups of Homeo(S), we considered looking at finite-sheated covers of the carpet and studying their deck transformations.

Conjecture 5.2.1. *Let S be an S-curve, and let X be a connected planar finite-sheeted cover of S. Then X is an S-curve as well.*

Given that any two S-curves are homeomorphic by Theorem 4.1.1, it is clear that one can use homeomorphisms of a given cover to create homeomorphisms of S by conjugation. We use this technique in the following corollary.

Conjecture 5.2.2. *Let S be an S-curve, let X be a connected planar finite-sheeted cover of S, and let G be the deck transformation group of X. Then G is isomorphic to a subgroup of* $\mathrm{Homeo}(S)$.

We can prove this if we assume Conjecture 5.2.1

Proof. [Proof of Conjecture 5.2.2] We see by Conjecture 5.2.1 and Theorem 4.1.1 that there must exist some homeomorphism $h\colon S \to X$. We can then define a function $\varphi\colon G \to \mathrm{Homeo}(S)$ such that $\varphi(g) = h^{-1} \circ g \circ h$ for $g \in G$. It is clear that φ is a homomorphism, since $\varphi(g_1 \circ g_2) = h^{-1} \circ g_1 \circ g_1 \circ h = h^{-1} \circ g_1 \circ h \circ h^{-1} \circ g_1 \circ h = \varphi(g_1) \circ \varphi(g_2)$. Furthermore, suppose that there exist $g_1, g_2 \in G$ such that $\varphi(g_1) = \varphi(g_2)$. Then $h^{-1} \circ g_1 \circ h = h^{-1} \circ g_2 \circ h$, so $h \circ h^{-1} \circ g_1 \circ h \circ h^{-1} = h \circ h^{-1} \circ g_2 \circ h \circ h^{-1}$ and $g_1 = g_2$. Thus, the homomorphism φ must be injective, so G is isomorphic to a subgroup of $\mathrm{Homeo}(S)$. ∎

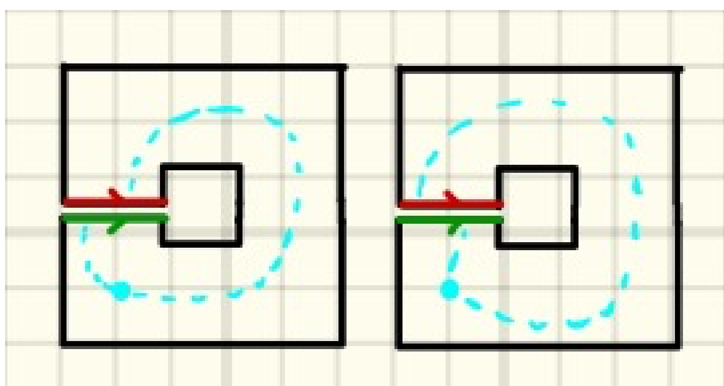


Figure 5.2.1: A simple cover of the carpet

It is possible to construct a simple cover of the Sierpiński carpet by taking a frame with a single hole and constructing a cover of the frame, as shown in Figure 5.2.1. It is also possible to generalize this to other frames, with an easy one being a frame with two holes. We present such a cover in Figure 5.2.2. Both covers shown are planar. Each also corresponds to planar graphs that cover a loop and a figure-eight respectively. While it is evident that no cover corresponding to a non-planar graph is planar, we are attempting to work out what conditions guarantee a planar cover.

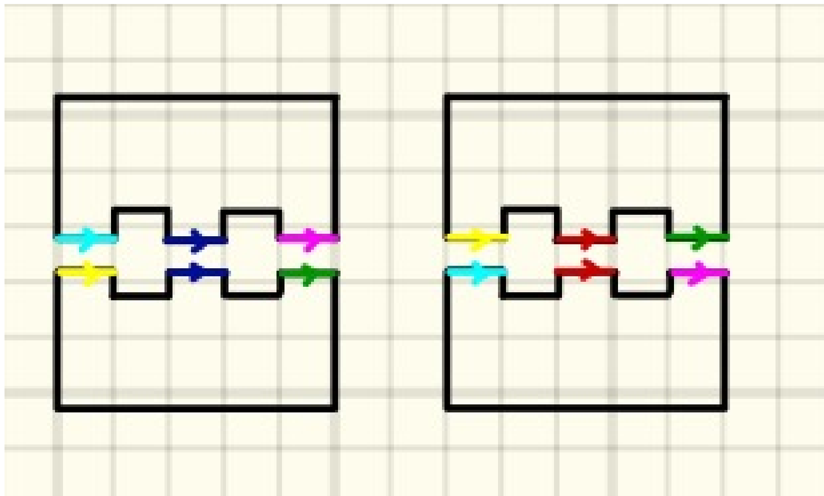


Figure 5.2.2: A slightly more complicated cover of the carpet

5.3 Further questions

Beyond the questions brought up by the work using deck transformations, we also wish to explore other ways to study the subgroups of Homeo(S). In particular, we are interested in proving that any free group is a subgroup of Homeo(S), by embedding a carpet in the hyperbolic plane with boundary using ideal squares. Whether this is in fact true is still far from known, but such techniques may lead to other discoveries regarding the structure of Homeo(S).

In addition, we are interested in generalizing from plane carpets to “carpet-like” structures that are embedded in various surfaces. We suspect that the deck transformation techniques may prove useful for this, particularly when showing relationships between different carpet-like structures.

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