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A Variational Approach to the Moving Sofa Problem

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Abstract

The moving sofa problem is a two-dimensional idealisation of real-life furniture moving problems, and its goal is to find the biggest area that can be maneuvered around a L-shape hallway with unit width. In this project we will learn about Hammersly's sofa ,Gerver's sofa and adapt Hammersly's sofa to non-right angle hallways. We will also use calculus of variations to maximize the area and find out Gerver's sofa satisfied several conditions that the best sofa satisfies.

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Dedication

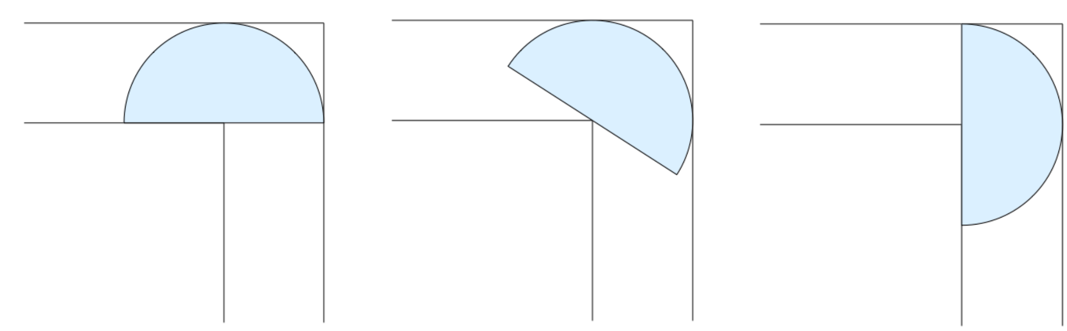
To all my friends, without whom I could have graduated two years earlier.

Acknowledgments

I would like to thank my advisor, Jim Belk. This project would be impossible without his guidance. I would also like to thank all the professors along the way. Bard has taught me the most important two things of life: love and logic.

Introduction

The moving sofa problem asks us to find the largest possible area of a shape in the plane that can be maneuvered around an L-shaped hallway with unit width. For example, a semicircle with radius of 1 can be moved around an L-shaped hallway, as shown in Figure 0.0.1. This “sofa” has an area of $\pi/2$, but larger areas are possible. The best sofa so far was discovered by Joseph Gerver and it has an area of 2.2195 (Figure 0.0.2).Gerver also conjectured this is the best possible sofa. However, this remains an open problem.



(a) (b) (c)

Figure 0.0.1: How a semi circle turns around a 90 degree corner

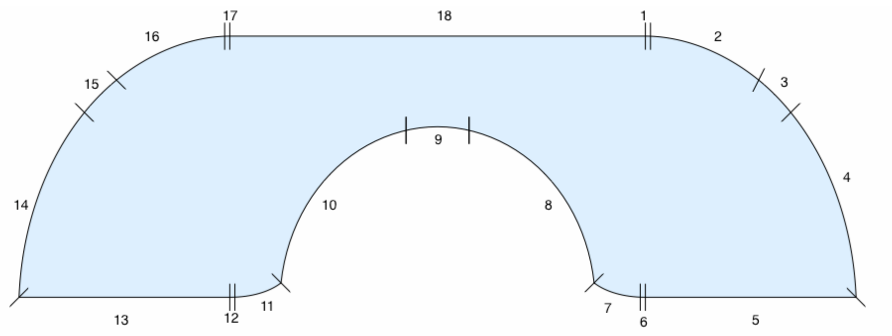


Figure 0.0.2: Gervers sofa

The moving sofa problem was formulated by Leo Moser in 1966. One simple family of sofas is the Hammersley's family sofas, which are constructed from straight lines and circular arcs. The best of them is the "Shephard piano" shown in Figure 0.0.3, with area of fairly large area of $\pi/2 + 2/\pi \approx 2.2074$, ,which is a good first approximation to the optimum. C. Francis and Richard Guy managed to find a slightly larger area of 2.2156 [1]. Later on Joseph Gerver further increased the the sofas area to approximately 2.2195[2]. Gerver also gave some conditions that an optimal sofa must satisfy,and showed his sofa satisfied these conditions. However he was not able to prove that was the best sofa. Philip Gibbs[3] did a numerical study on this problem and he implemented his computation in Java. His results agreed with Gerver's.

The boundary of Gerver's sofa has 18 sections and with four of them being circular arcs, six of them being involute of circles and two of them being involute of involute of circles. In Gervers paper, a specific formula of his sofa was not given. Therefore , at the end of chapter 1 I will give the formula of Gervers sofa and show how it rotates around the 90-degree hallway.

In this project I generalized the 90-degree hallways. First I defined non-right angle corners. Then I adapted Hammersly's sofa to non-right corner hallways and found out which one is the best one for every angle. Notice the sofas we have so far all rotated around the corner clockwise. So I introduced the RHAM sofa, which has the surprising

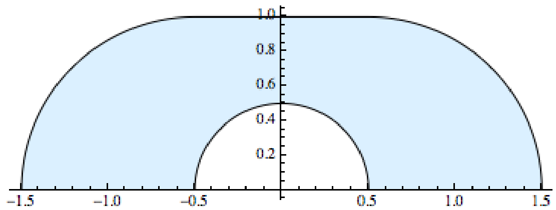


Figure 0.0.3: The “Shephard piano”

property that it rotates counterclockwise and its area surpass other shapes of sofa we have so far when it rotates very small angle corners. The way RHAM sofa was discovered is worth mentioning: instead of rotating the sofa, we rotate the hallway. And when the hallway is rotating, its two ”tails” leaves traces. The blank area that the traces dont touch is the RHAM sofa. However, RHAM sofa has bigger area than circular sectors for small angles corner hallways. The family of Hammersly’s sofas are the biggest sofa so far for angles larger than $\pi/2$.

During the study of RHAM sofa I was fascinated by how rotation of the hallway forms a sofa. The boundary curve of the sofa is the envelop to a family of lines, which are the four walls of the hallways. This would derive an equation for the boundary curve. Then I used Green’s theorem to obtain the area bounded by these curves in the form of integrals. Now our goal is to maximize these integrals.

To achieve this goal, we applied a technique called calculus of variations. Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function, which means H is a function that is twice differentiable and whose second derivative is continuous. Calculus of variations is a technique to find a function that maximize or minimize the integral

$$\int_a^b H(f(x), f'(x), f''(x), x)dx$$

which can be obtained by finding functions where the derivative of functions is equal to zero. And this is where we solve the associated Euler-Lagrange equation:

$$\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f(x), f'(x), x)\right] = \frac{\partial H}{\partial y}(f(x), f'(x), x) \quad for \quad all \quad x \in (0,1)$$

where $f : [0,1] \to \mathbb{R}$ is a $\mathbb{C}^2$ solution to the basic function optimization problem.

For example, we will minimize the the following integral by this technique

$$I(t) = \int_{-1}^{1} (f(x)^2 + f'(x)^2)dx$$

subject to the constraint that $f_t(-1) = 1$ and $f_t(1) = 1$. Then the calculus of variations tells us that the optimum f is a solution to the "Euler-Lagrange equation"

$$\frac{d}{dx}\left[2f'(x)\right] = 2f(x)$$

And it follows that $f(x) = \frac{\cosh x}{\cosh 1}$.

Now it will be easier to explain how calculus of variation works in simple sentences: first, calculus of variation gives a differentiable equation of the target integral we want to maximize or minimize, then we solve the corresponding Euler-Lagrange equation to get the solutions. In chapter 3, we calculated the functional of the integrals associated to the boundary curves of the sofa. Given some boundary curves, if we want to maximize the region bounded by them, we only have to solve the corresponding Euler-Lagrange equation. This will give several conditions that a area-maximized sofa must satisfy.

Applying the technique is challenging because the results strongly depends on which walls the sofa is touching at the same time. So far we have discovered when the sofa is only touching the top and bottom wall of the hallway, the boundary of the sofa indeed needs to be a circular arcs with radius of 1 sharing the same center, which agrees with Gerver's result. This project would be more interesting if it could have covered other stages of Gerver's sofa's movement.

In the first chapter some basic concepts of sofa and its transformation rotating around 90-degree corner hallways are defined. At the end of this chapter the formula of Gerver's sofa are given. In chapter two non-right angle corners hallways are defined. Then generalized Hammersly's sofa, RHAM sofa and a circular arc shape sofa are defined and their areas are compared. In the last chapter calculus of variations is first introduced and proved it can be also used for three-time differentiable functions. Then it is applied to maximize the area and a few conditions that an area- maximized sofa must satisfy were discovered.

1
The Original Moving Sofa Problem

In this chapter we will provide a formal definition of the moving sofa problem. To precisely describe the movement of a sofa in the hallway, the 90-degree-hallway and rotation path. We say a sofa can be rotated around the 90-degree-hallway if and only if there exists a rotation path .Later an example of semicircle with radius of one was given as an example of illustrating the difference of rotating paths. Then we would talk about the conversion between sofa's coordinates and hallway's coordinates. We then introduce Hammersly's sofa and explain why it can be rotated around the 90-degree corner. At the end of this chapter, the formula of Gerver's sofa will be given.

1.1 The Original Moving Sofa Problem

The problem is to find the largest area that can be moved around a 90-degree corner with width of 1. It is called "moving sofa" because turning around a geometry shape around a corner looks like moving a sofa around a narrow corner in real life. Notice in this project, we have a different definition of sofa.

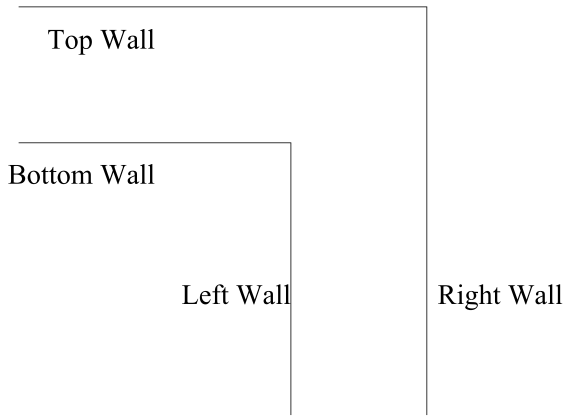


Figure 1.1.1: A 90 degree hallway

Definition 1.1.1. A **sofa** is a closed, bounded region in a plane whose boundary is a simple closed curve. $\triangle$

Now we define the 90-degree hallway in the plane.

Definition 1.1.2. The **90 degree hallway** is the set of points (a, b) such that $a \leq 1$ and $b \leq 1$ and such that either $a \geq 0$ or $b \geq 0$ as shown in Figure 1.1.2. $\triangle$

Later in this project we will use (a, b) for the points in the hallway or in the reference of hallway coordinates, we refer these (a, b) as hallway coordinates.

The boundary of the hallway is the union of four infinite rays that lie along the lines $a = 0$, $a = 1$ and $b = 0$, and $b = 1$. Naturally, $b = 1$ is the **top wall**, $b = 0$ is the **bottom wall**, $a = 1$ is the **right wall**, $a = 0$ is the **left wall**. And $(0, 0)$ is the **corner**.

First we define two notations for transformation:

1. $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a clockwise rotation about the origin by an angle of θ.

2. Translation of S by $\vec{v}$: If $S \subseteq \mathbb{R}^2$ and $\vec{v} \in \mathbb{R}^2$, $S + \vec{v} = \{\vec{s} + \vec{v} | \vec{s} \in S\}$

Definition 1.1.3. Let $S \subseteq \mathbb{R}^2$ be a sofa, and let H be the 90 degree hallway. A **rotation path** for S is a continuous path $\vec{O} : [0, \pi/2] \to \mathbb{R}^2$ such that

$$R_\theta(S) + \vec{O}(\theta) \subseteq H$$

for all $\theta \in [0, \pi/2]$.

We say that S **can be rotated** in H if S has a rotation path. $\triangle$

Instead of moving the sofa, we can rotate the hallway, which means the sofa stays we can also rotate the hallway with $\vec{O}(\theta)$ being the rotation path.

Proposition 1.1.4. *Let S be a sofa, $\vec{O} : [0, \pi/2] \to \mathbb{R}^2$ be a continuous path. $\vec{O}(\theta)$ is a rotation path if and only if*

$$S \subseteq \bigcap_{\theta \in [0,\pi/2]} R_{-\theta}(H - \vec{O}(\theta))$$

for all $\theta \in [0, \pi/2]$.

Proof. The proof is straightforward. $R_\theta(S) + \vec{O}(\theta) \subseteq H \Leftrightarrow R_\theta(S) \subseteq H - \vec{O}(\theta) \Leftrightarrow S \subseteq R_{-\theta}(H - \vec{O}(\theta))$. Since $\theta \in [0, \pi/2]$, we have

$$S \subseteq \bigcap_{\theta \in [0,\pi/2]} R_{-\theta}(H - \vec{O}(\theta))$$

$\square$

This motivates the following definition.

Definition 1.1.5. Given a $\vec{O}(\theta) : [0, \theta] \to \mathbb{R}^2$, the corresponding **maximal sofa** is defined by

$$S = \bigcap_{\theta \in [0,\pi/2]} R_{-\theta}(H - \vec{O}(\theta))$$

$\triangle$

Example 1.1.6. Let H be a hallway, $\vec{O} : [0, \pi/2] \to \mathbb{R}^2$ be the path $\vec{O}(\theta) = (0, 0)$. If we rotate the hallway along $\vec{O}(\theta)$, we will get our sofa S: a semi circle with radius of 1 defined by $x^2 + y^2 \leq 1$ and $x \geq 0$. See figure 1.1.2. $\Diamond$

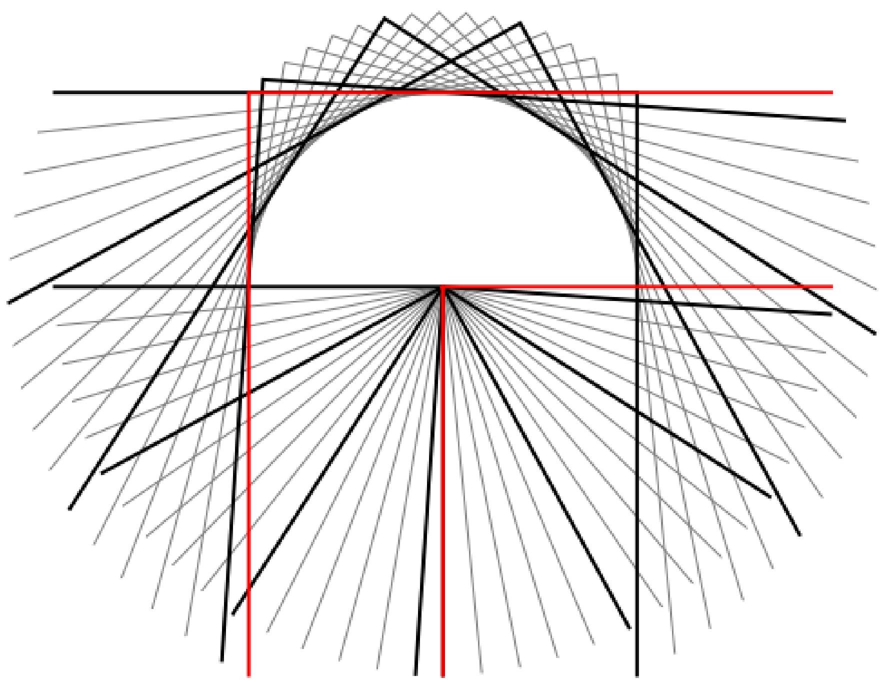


Figure 1.1.2: A semicircle obtained by rotating the hallway

In this project a rotation path $\vec{O}$ will be written in the form of $\vec{O} = (\vec{p}(\theta), \vec{q}(\theta))$. If $P = (x, y)$ is the point on the sofa, then for each $\theta \in [0, \pi/2]$ the coordinates of P in the hallway are $S \subseteq \bigcap_{\theta\in[0,\pi/2]} R_{-\theta}(H - \vec{O}(\theta))$.

$$(a, b) = R_\theta(x, y) + \vec{O} \tag{1.1.1}$$

where R_θ is the rotation matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Plug in the matrix, we have

$$-x\sin\theta + y\cos\theta + q(\theta) = a \tag{1.1.2}$$

$$y\sin\theta + x\cos\theta + p(\theta) = b \tag{1.1.3}$$

.

Rearrange it we can have the formula for (x, y)

$$y = (a - q(\theta))\cos\theta + (b - p(\theta))\sin\theta \tag{1.1.4}$$

$$x = (b - p(\theta))\cos\theta - (a - q(\theta))\sin\theta \tag{1.1.5}$$

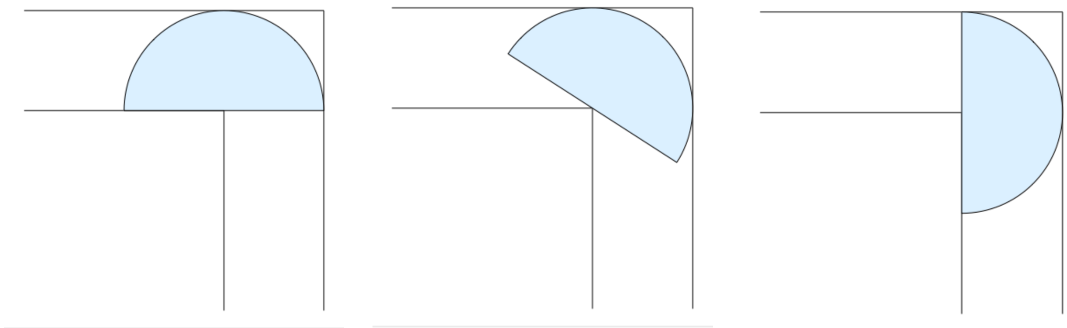
Later in this porject, we refer to (a, b) as **hallway coordinates**, (x, y) as **sofa coordinates**

Example 1.1.7. Let S be a semi circle with radius of 1 defined by $x^2 + y^2 \leq 1$ and $y \geq 0$. Let $\vec{O} : [0, \pi/2] \to \mathbb{R}^2$ be the path $\vec{O}(\theta) = (0, 0)$. Geometrically, this means the semicircle stays in the 90-degree corner and rotates around $(0, 0)$.Then $\vec{O}$ is a rotation path for S.

Notice if we have chose the rotation path differently, the semicircle would still go around the 90-degree-hallway. Let $\vec{O}' : [0, \pi/2] \to \mathbb{R}^2$ be the path

$$\vec{O}'(\theta) = (\cos\theta, -\sin\theta)$$

Geometrically, this means the same but the rotation path describes the movement of the point $(1, 0)$. $\Diamond$



(a) (b) (c)

Figure 1.1.3: How a semi circle turns around a 90 degree corner

Definition 1.1.8. Let S be a sofa. If S is tangent to the top wall for $\theta_1 < \theta < \theta_2$, the corresponding **top curve** is the curve formed by tangents. We define **bottom curve**, **left curve**, **right curvecorenr curve** in the similar way. △

1.2 Hammersley's Sofa

In this section we will give a specific definition of the family of Hammersley's sofa. The best sofa, the one with the biggest area, is knows as "Shephard Piano". And we will point out its rotation path with a proof following up. In the end we will compute its area.

Definition 1.2.1. Given any $R \in (0, 1)$, the corresponding **Hammersley sofa** is bounded by the following six curves:

1. A semicircular arc with radius R goes from 0 to π, whose center is $(0, 0)$. The semi circle starts at $(R, 0)$, ends at $(-R, 0)$.

2. A line segment goes from $(R, 0)$ to $(R + 1, 0)$.

3. A quarter circle arc goes from $\pi/2$ to 0, whose center is $(R, 0)$. This quarter circle starts at $(R + 1, 0)$, ends at $(R, 1)$.

4. A line segment goes from $(R, 1)$ to $(-R, 1)$.

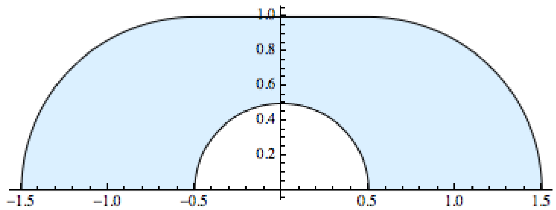


Figure 1.2.1: Hammersley's sofa when $R = 0.5$

5. A quarter circle arc goes from π to $\pi/2$, whose center is $(-R, 0)$. This quarter circle starts at $(-R, 1)$, ends at $(-R-1, 0)$.

6. A line segment goes from $(-R-1, 0)$ to $(-R, 0)$. △

Proposition 1.2.2. *Hammersley's sofa with* $R \in [0, 1]$ *can be moved around a 90-degree hallway.*

Proof. We claim the path is $R(-cos(\theta), -sin(\theta))$. Since the sofa is only touching top wall, the corner and right wall when turning around the corner, we only have to prove these three parts are tangent to the corner. H_3 is easy to check since we only have to show that O is the midpoint of AB. The coordinates of A and B are : $(-2R\cos(\theta), 0), (0, -2R\sin(\theta))$ Let θ be the angle that sofa rotating by, O be the center of rotation. Then $\angle OAC = \angle ACO = \theta$ since OA and OC are the radius of the semi circle. One can easily check that the coordinates of O is $R(-cos(\theta), -sin(\theta))$.

We only need to check H_1 that is always tangent to the hallway since H_1 and H_4 are mirror images. H_1 is a quarter circle with radius 1 rotaing around A, where A is a moving point sliding along $x = 0$ since we just proved H_3 is always tangent to the 90 degree corner. □

Proposition 1.2.3. *Hammersley's sofa reaches the biggest area when* $R = 2/\pi$.

Proof. $A = \pi/2 + 2R - \pi \times R^2/2$ where A is the area of Hammersley's sofa and $R \in [0, 1]$.$A' = 2 - \pi \times R$. Thus when $R = \pi/2$,A reaches maximum: $\pi/2 + 2/\pi$. □

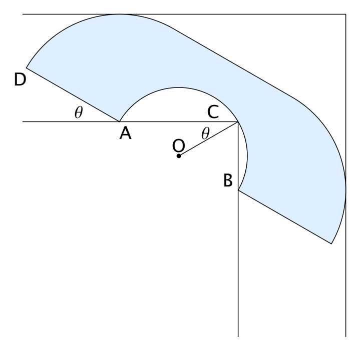
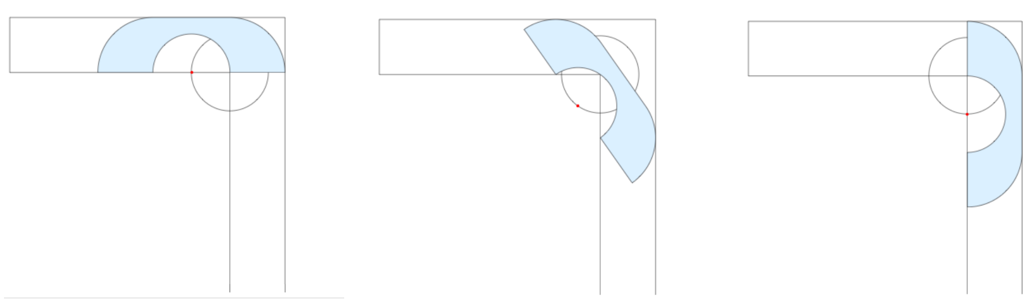


Figure 1.2.2: Hammersley's sofa rotating around a 90 degree corner



(a) (b) (c)

Figure 1.2.3: How a semicircle turns around a 90 degree hallway with the red point being its rotation center

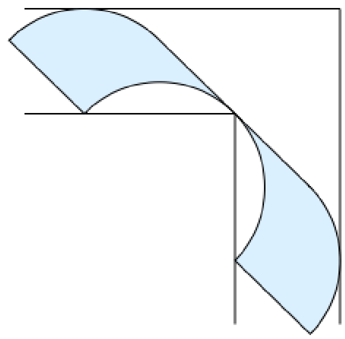


Figure 1.2.4: Hammersley's sofa in the hallway when $R = 1$

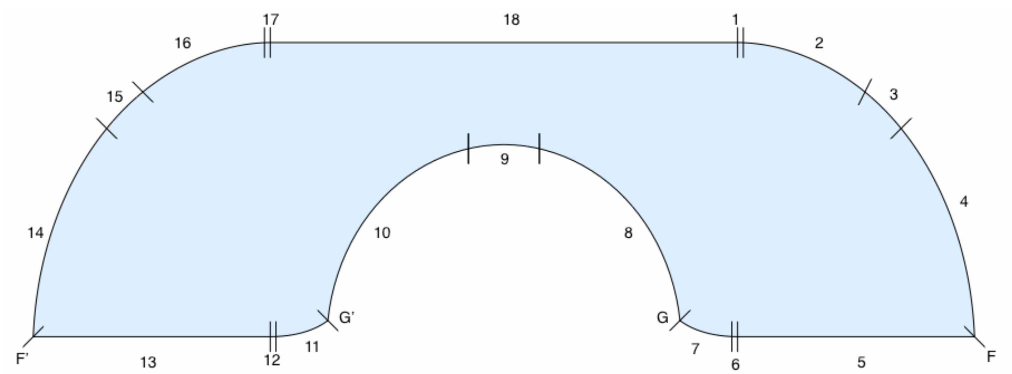


Figure 1.3.1: Gerver's sofa

The best one in the family of Hmmersley's sofa is also referees as "Shephard Piano". This is obviously not the best sofa. In the next section, Gerver will give a better sofa.

1.3 Gerver's sofa

Gervers sofa is bounded by 18 curves as shown in figure 1.3.1. Section 5,13,18 are straight lines, 1,6,11,17 are circular arcs of radius a half, 2,3,7,11,15 and 16 are involutes of circles, and 5 and 14 are involutes of involutes of circles. An involute of circles is a curve obtained from a circle by attaching an imaginary taut string to the given circle and tracing its free end as it is wound onto the given circle. An involute of involute of circles is a curve obtained from another given involute of circles by attaching an imaginary taut string to the given involute of circles and tracing its free end as it is wound onto that given involute of circles.

Notice in this project we usually refer the angle a sofa rotates around the hallway by as θ. However, in Gerver's paper, he called it α and a constant θ. We will use his notations only in this section.

The radius of curvature of section 1 to 4 are: $[k(\alpha) = \begin{cases} 1/2 \\ 1/2(1+A+\alpha-\varphi) \\ A+\alpha-\varphi \\ B-1/2(\pi/2-\alpha-\varphi)(1+A)-(\pi/2-\alpha-\varphi)^2/4 \end{cases}$

where A, B, φ and θ are constant:

A = 0.094426560843652813441435452024889346 47

B =1.399203727333547341233029034704534129 6

φ = 0.039177364790083594549708477915423860 94

θ = 0.681301509382724537268223151601367945 52

The curvature of part 6 and 7 is $1-k(\alpha)$ when $0<\alpha<\varphi$ and $\varphi<a<\theta$. Part 5,13, 18 are line segments.

Before we get to the bottom curve of Gerver's sofa, we will talk about its movement first. Gerver's sofa moves differently around the corner than Hammersly's sofa. When the sofa goes around the hallway by θ of 0 through $\pi/2$, it happens in five stages. It touches top and bottom wall along the hallway throughout the straight sections 18 and 13 and goes right until it touches at point F. The first stage of rotation for $0<\alpha\leq\phi$, the sofa touches the hallway in section 12 and 17. Using a fixed angle in this range, the sofa moves horizontally by small amounts until it reaches the hallway at F or in section 7. When $\alpha=\phi$, the end of the first stage, the sofa touches the corner of the hallway at G and the

Table 1.3.1: Parts Gerver's sofa touching the hallway at different stages

$0<\alpha<\varphi$	12,17
$\alpha = \varphi$	point G
$0<\alpha<\theta$	11,8,4,16
$\alpha = \theta$	point G'
$\theta<\alpha<\pi/2-\theta$	15,9,3

right wall tangentially at F. The second stage , $0 < \alpha \leq \theta$, the sofa will touch the walls in four different places in sections 11, 8, 4 and 16. The sofa will be separated from the wall by a distance larger than 0.0012 at point G. At the end of stage two, point G touches the wall. The third stage, $\theta < \alpha < \pi/2\theta$, the sofa touches in sections 15, 3 and 9, for a total of three times in this section. Stages four and five are mirror images in reverse of stages two and one, respectively. The first three stages are shown in table 1.3.1.

In stage 3, the sofa is only touching the top, right walls and the corner. Recall in the previous section , equation 1.1.2 and 1.1.3 converse the hallway coordinates to sofa coordinates. In this stage, $(a, b) = (1, 1)$. Thus, we have

$$-x\sin\theta + y\cos\theta + q(\theta) = 1$$

$$y\sin\theta + x\cos\theta + p(\theta) = 1$$

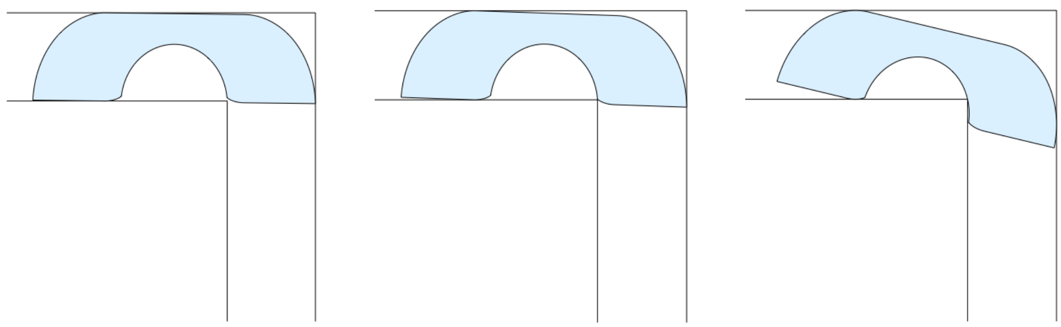
. Since we already know the formula of part 15 and 3, we can compute $p(\alpha)$ and $q(\alpha)$. Then we only have to plug them in

$$-x\sin\theta + y\cos\theta + q(\theta) = 0$$

$$y\sin\theta + x\cos\theta + p(\theta) = 0$$

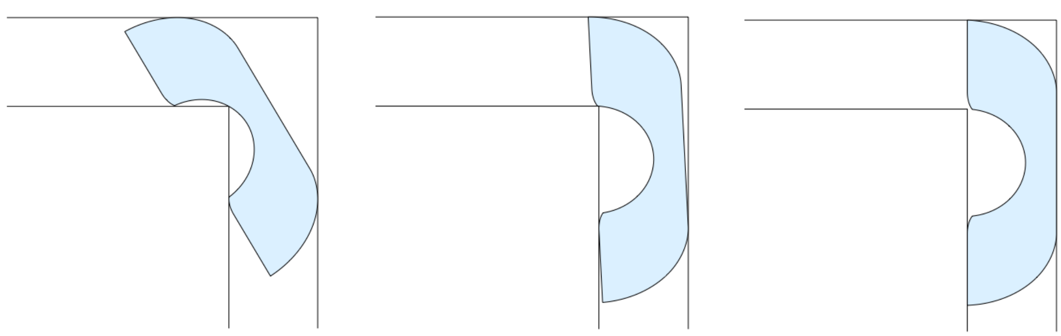
. to obtain the corner part 9.

Part 8 was computed in a similar way.



(a) When $0 < \alpha < \varphi$ (b) When $\alpha = \varphi$ (c) When $\alpha = \theta$

Figure 1.3.2: Gerver's sofa rotating around the hallway (1)



(a) When $\theta < \alpha < \pi/2 - \theta$ (b) When $\alpha = \pi/2 - \varphi$ (c) When $\alpha = \pi/2$

Figure 1.3.3: Gerver's sofa rotating around the hallway (2)

2
Non-Right Angles

In this chapter we are going to explore several different sofas that can be rotated around non-right angle corners.

2.1 Non-Right Angle Corners

Definition 2.1.1. Let $\alpha \in [0, \pi]$. The **hallway with angle** α is the region H_α in the plane bounded by four rays $\vec{BA}$,$\vec{BC}$,$\vec{ED}$ and $\vec{EF}$ where A,B,C,D,E,F are the points

1. $A = (-1, 0)$
2. $B = (0, 0)$
3. $C = (\cos\alpha, \sin\alpha)$
4. $D = (-1, 1)$
5. $E = (\csc\alpha + \cot, 1)$
6. $F = (\cos\alpha + \sin\alpha, \sin\alpha - \cos\alpha)$

△

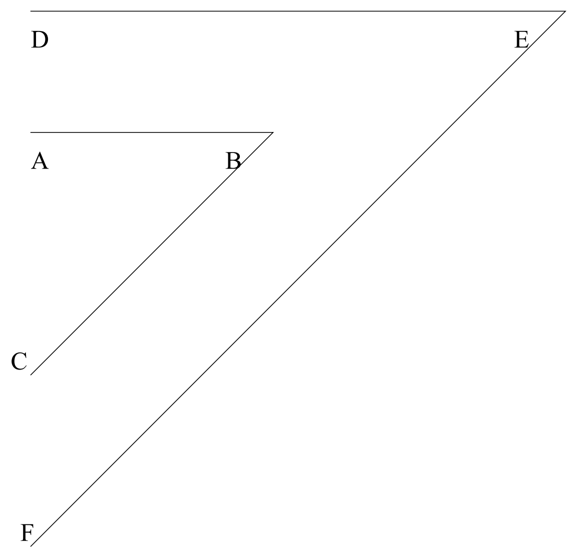


Figure 2.1.1: Non Right Angle Hallway

α is the angle between (picture).Notice when α is close to zero, the turn is very sharp; when α is close to π, the hallway is almost straight.

Now we define the rotation path for H_α similarly.

Definition 2.1.2. Let $S \subseteq \mathbb{R}^2$ be a sofa, and let H_α be the α angle hallway. A **rotation path** for S is a continuous path $\vec{O} : [0, \pi - \alpha] \to \mathbb{R}^2$ such that

$$R_\theta(S) + \vec{O}(\theta) \subseteq H_\alpha$$

for all $\theta \in [0, \pi - \alpha]$.

We say that S **can be rotated** in H_α if S has a rotation path. $\triangle$

Example 2.1.3. Let S be a semi circle with radius of 1 defined by $x^2 + y^2 \leq 1$ and $y \geq 0$. Let $\vec{O} : [0, \pi/2] \to \mathbb{R}^2$ be the path, when $\vec{O}(\theta) = (0, 0)$, S can rotate around H_α.See figure 2.1.3. $\diamondsuit$

Another naive approach is to let the sofa be a parallelogram.

Example 2.1.4. Let P be a parallelogram with four vertices being

$$(0, 0), (\csc\alpha, 0), (\csc\alpha + \cot\alpha, 1), (\cot\alpha, 1)$$

P can be moved around H_α. See figure 2.1.4. It has an area of $\csc\alpha$. $\Diamond$

2.2 Generalized Hammersley's sofa

Now with the definition of generalized hallway, we are ready to generalize Hammersley's sofa to go around H_α

Definition 2.2.1. Let$0 < R < \frac{1}{1+\cos\alpha}$, $\alpha \in [0, \pi]$. The **generalized Hammersley's sofa** is a region bounded by 6 curves:

1. Part 1 is a circular arc with radius R goes from point $F = (-R\sin\alpha, -R\cos\alpha)$ to $A = (R\sin\alpha, -R\cos\alpha)$, whose center is $(0, 0)$.
2. Part 2 is a line segment goes from point A to point $B = (R\sin\alpha + 1, -R\cos\alpha)$.
3. Part 3 is a quarter circle arc goes from B to $C = (R\sin\alpha, -R\cos\alpha + 1)$ with radius 1, whose center is A.
4. Part 4 is a line segment goes from C to $D = (-R\sin\alpha, -R\cos\alpha + 1)$.
5. Part 5 is a quarter circle arc from D to $E = (-R\sin\alpha - 1, -R\cos\alpha)$ with radius, whose center is F.
6. Part 6 is a line segment from E to F.

We call this sofa $GH_{R,\alpha}$. $\triangle$

A $GH_{R,\alpha}$ rotates around H_α with the rotation path being

$$(-R\sin(\alpha + \theta), R\cos(\alpha + \theta))$$

.

This means $GH_{R,\alpha}$ rotates along a circular arc. See figure 2.2.2.

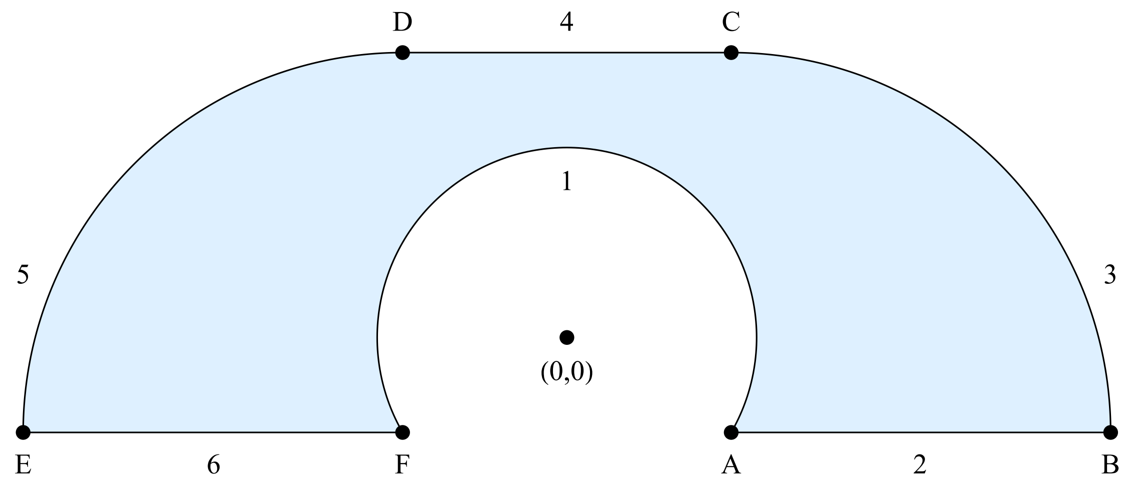


Figure 2.2.1: Generalized Hammersley's sofa

Proposition 2.2.2. *$GH_{R,\alpha}$ reaches its biggest area when*

$$R = \frac{\sin\alpha}{\sin\alpha\cos\alpha + \pi - \alpha}$$

.

Proof. The area A of $GH_{R,\alpha}$ is

$$A = \frac{\pi}{2} + 2R\sin\alpha - R^2(\pi - \alpha) - R^2\sin\alpha\cos\alpha$$

Now take the derivative of A in term of *alpha*

$$A' = 2\sin\alpha - 2(\sin\alpha\cos\alpha + \pi - \alpha)R$$

Now we set A' to zero

$$R = \frac{\sin\alpha}{\sin\alpha\cos\alpha + \pi - \alpha}$$

. □

Notice when $\alpha > \pi/2$, there is more room on the right foot of $GH_{R,\alpha}$ for sofa to turn around. It will be more interesting if we include the red part in $GH_{R,\alpha}$ when $\alpha > \pi/2$ in the future. See the red part in figure

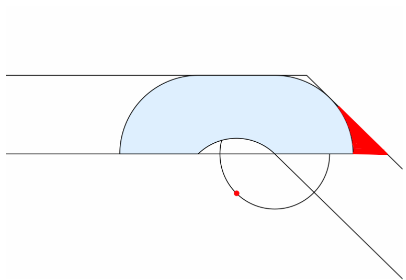
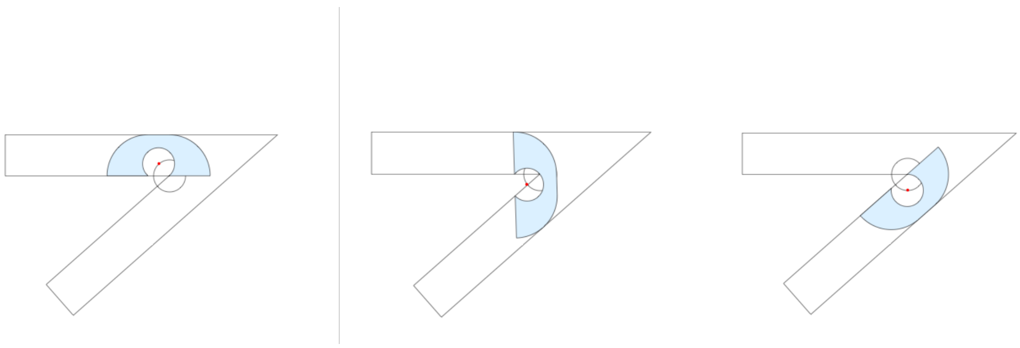


Figure 2.2.2: The red part is the extra room when sofa turns around H_α when $\alpha > \pi/2$



(a) (b) (c)

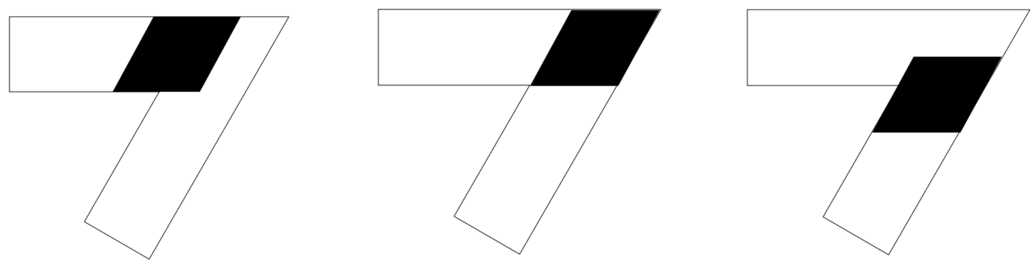
Figure 2.2.3: When Corner Angle is smaller than $\pi/2$

2.3 The RHAM sofa

The goal is to find the biggest area that can be moved around a corner. Notice how the family of Hammersley's sofa rotate clockwise. What if the sofa rotates counterclockwise?

Before we define any other sofas, a generalized definition of sofa being rotated around hallways is needed.

Definition 2.3.1. Let $H_{start,\alpha}$ be the area between $\vec{BA}$ and $\vec{ED}$, $H_{end,\alpha}$ be the area between $\vec{BC}$ and $\vec{EF}$.A **generalized rotation path** is a path $\vec{O} : [t_1, t_2] \to \mathbb{R}^2$ and



(a) (b) (c)

Figure 2.3.1: How a parallelogram turns around a hallway

$\theta : [t_1, t_2] \to \mathbb{R}$ so that

$$\mathbb{R}_{\theta(t)}(S) + \vec{O}(t) \subseteq H_\alpha$$

$$\mathbb{R}_{\theta(t_1)}(S) + \vec{O}(t_1) \subseteq H_{start,\alpha}$$

$$\mathbb{R}_{\theta(t_2)}(S) + \vec{O}(t_2) \subseteq H_{end,\alpha}$$

$\triangle$

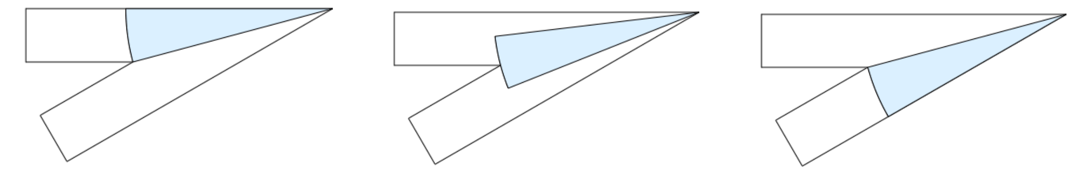
Example 2.3.2. Let P be a parallelogram with four vertices being

$$(0,0), (\csc\alpha, 0), (\csc\alpha + \cot\alpha, 1), (\cot\alpha, 1)$$

P can be moved around H_α. See figure 2.1.4. It has an area of $\csc\alpha$. $\diamond$

Now we are ready to introduce our first sofa that rotates around the hallway counter-clockwise.

Let F_α be a circular sector defined by $x^2 + y^2 \leq R^2$ and $x\tan(\frac{\alpha}{2}) \leq y \leq 0$ where $R = \frac{\csc(\frac{\alpha}{2})}{2}$. F_α can rotated around H_α with rotation path $\vec{O}(t) = -t$ where $[t_1, t_2] = [0, \alpha/2]$.



(a) (b) (c)

Figure 2.3.2: How a circular sector can be moved around the hallway

See figure 2.3.3. The area A of F_α is

$$\begin{aligned} A &= \frac{\alpha/2}{2\pi}\pi\frac{\csc(\frac{\alpha}{2})^2}{2} \\ &= \frac{\alpha\csc^2(\frac{\alpha}{2})}{8} \end{aligned} \quad \triangle$$

F rotates around the hallway H_α counterclockwise by angle $\alpha/2$. However, when α is very small the parallelogram we introduced before has a bigger area than F_α and F_α is better than generalized Hammersley's sofa.

Recall in chapter 1, we rotated the hallway instead of the sofa. The hallway leaves a trace during its rotation,and the blank space is the sofa that can be moved around this hallway.

First we assume the rotating path is any fixed point on the line segment from (0,0) to $(\csc\alpha+\cot\alpha, 1)$, which is the upper right corner of hallway. In example 1.1.6, the rotation path is $(0,0)$ and a semicircle with radius 1 is obtained.

When will the sofa have the biggest area? After a series of little experiments, we make the mid point of the line segment from (0,0) to $(\csc\alpha+\cot\alpha, 1)$ our rotation path. Then, we obtain the RHAM sofa.

Definition 2.3.3. Given $\alpha \in (0, \pi/2]$, the corresponding **RHAM sofa** is bounded by the following 7 parts:

1. Part 1 is a line segment from G $(-\cot\frac{\alpha}{2}/2, -1/2)$ to A $(0, -1/2)$.

2. Part 2 is a circular arc with radius of $1/2$ from A to B $(\sin\alpha/2, -\cos\alpha/2)$ with its center O $(0,0)$.

3. Part 3 is a line segment from B to C $(\csc\alpha/2, 0)$.

4. Part 4 is a line segment from C to D $(\sin\alpha/2, \cos\alpha/2)$.

5. Part 5 is a circular arc with radius of $1/2$ from D to E $(0, 1/2)$ with its center O.

6. Part 6 is a line segment from E to F $(-\cot\frac{\alpha}{2}/2, 1/2)$.

7. Part 7 is a circular arc with radius of $\csc\frac{\alpha}{2}/2$from F to G with its center O.

△

Now with definition of RHAM sofa, the area A is easy to compute since it is the union of four triangles and three circular sectors

$$A = \theta(1+\cos(\theta))/(4\sin\theta^2) + (\cot\theta + \csc\theta)/4 + \cot\theta/2 + \theta/4$$

To computer the area, intuitively, we can count the number of the white pixels. Notice there are some white spots in the shades that will affect the accuracy of computation. (Figure 1.9).

$RHAM_\alpha$ can be moved around a hallway of corner angle α with its rotation path being $(\csc\alpha + \cot\alpha)/2, 0.5)$.

To explain it another way,by how we get a $RHAM_\alpha$ sofa (the blank space after rotating the hallway), we can easily know a $RHAM_\alpha$ sofa can be moved around a hallway of corner angle α.

To sum up the sofas we have so far, we plot the areas of the family of generalized Hammersley's sofa,F_α and $RHAM_\alpha$, the parallelogram, and the semicircle. in terms of the corner angle α. When $0 < \alpha < 0.0.873207$, a RHAM sofa is bigger than F_α and the best generalized Hammersley's sofa that can turn around H_α. F_α is never a better solution

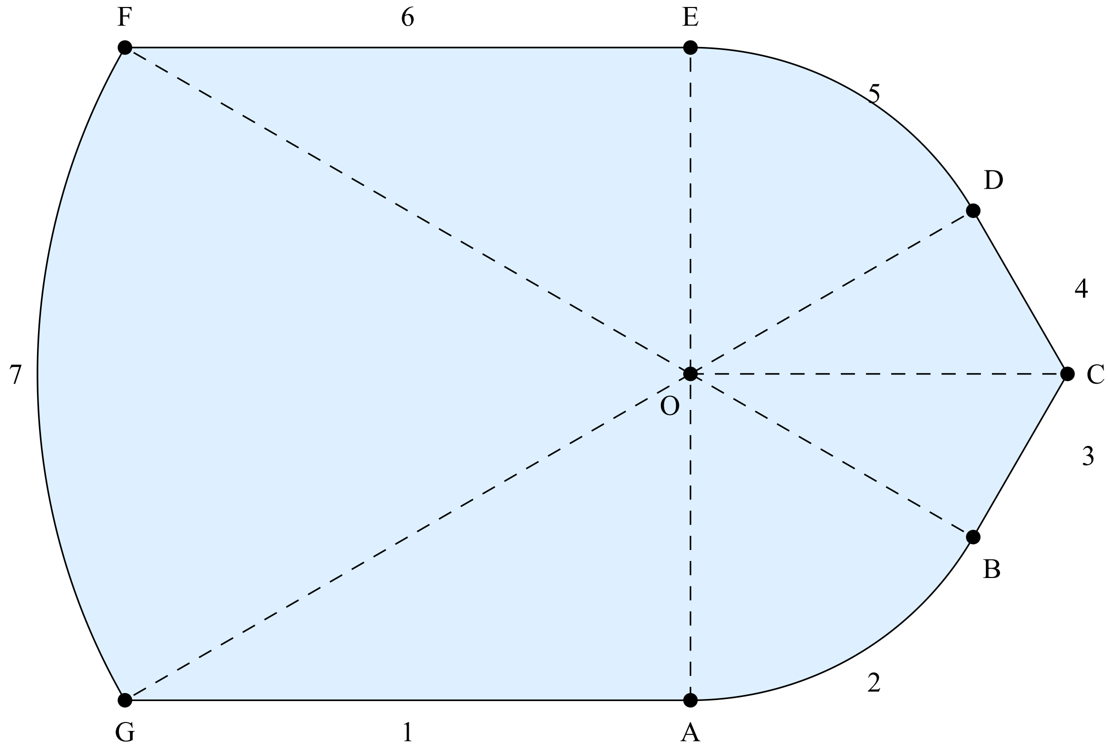


Figure 2.3.3: RHAM sofa

since we came up with this shape in only 5 seconds.

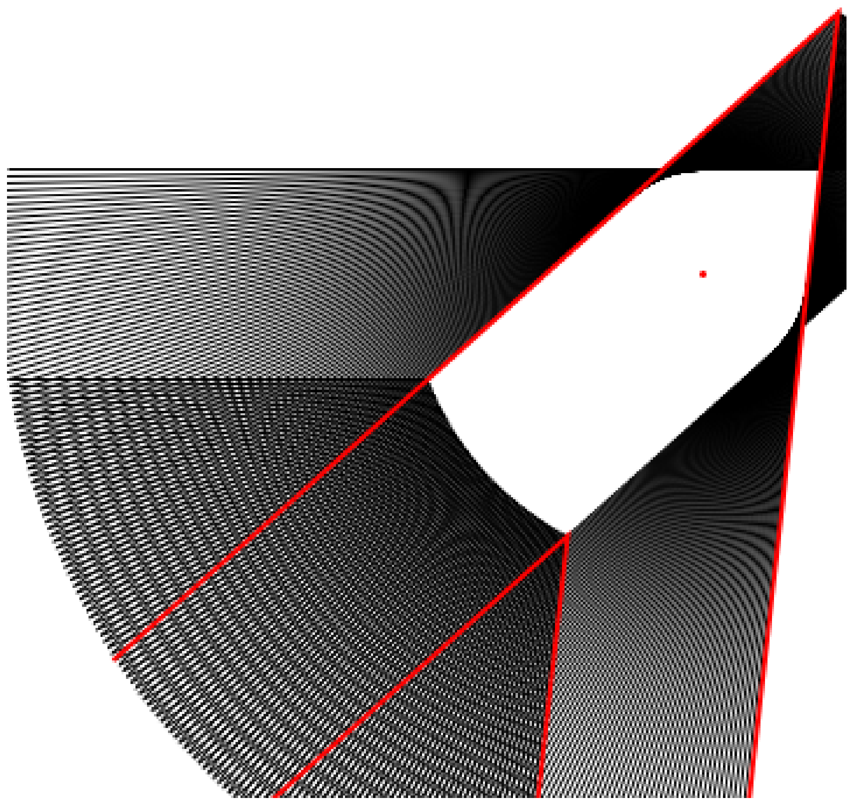
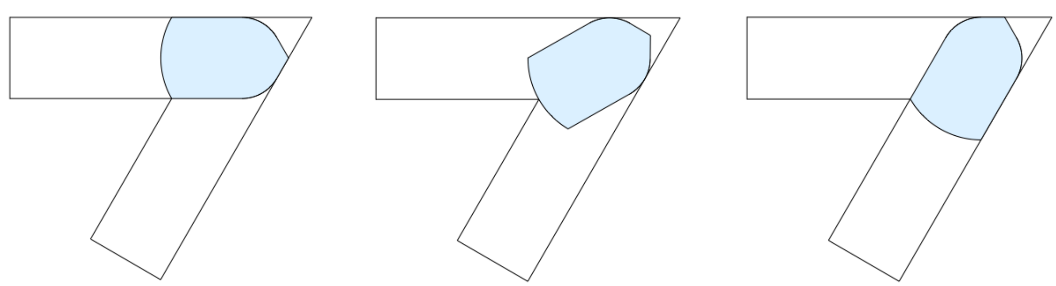
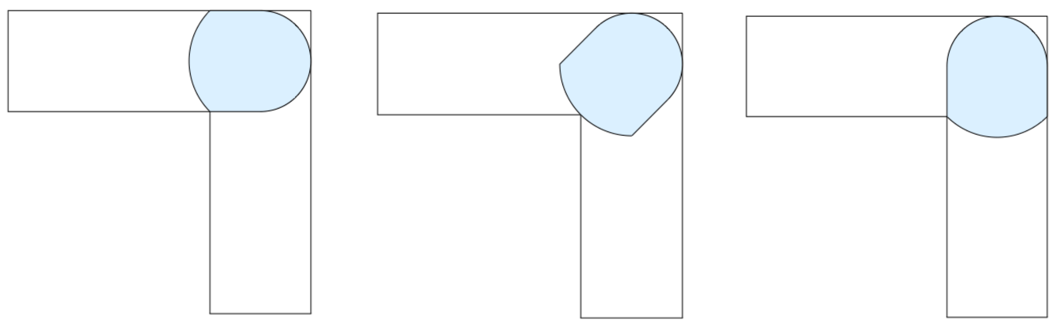


Figure 2.3.4: White pixels in the shade



(a) (b) (c)

Figure 2.3.5: How a RHAM sofa moved around the hallway when $\alpha = \pi/3$



(a) (b) (c)

Figure 2.3.6: How a RHAM sofa moved around the hallway when $\alpha = \pi/2$

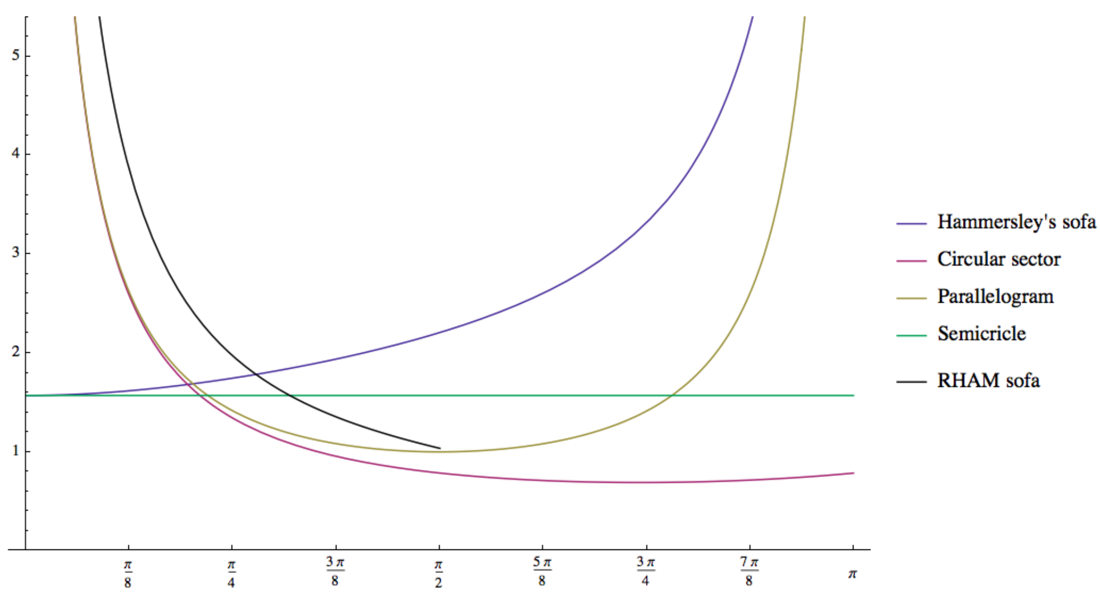


Figure 2.3.7

3

Calculus of Variations

During the formation of a sofa by rotating the hallway, the boundary curve of the sofa is the envelop to a family of lines, which are the four walls of the hallways.In this chapter we will first define envelope and give its parametrization, then calculate the area bounded the boundary curve by Green's theorem. Then we apply calculus of variations to find the conditions an area-maxmized sofa must satisfy.

3.1 Background on Envelopes

In this section we will define envelop and the formula of envelop to a family of lines. Then an example of using the formula will be given.

Definition 3.1.1. A **parametrized curve** in $\mathbb{R}^2$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ for some α,β with $-\infty \leq \alpha < \beta \leq \infty$ △

Definition 3.1.2. A **parametrized family of lines** is a function L :$(\alpha, \beta) \to \mathbb{L}(\mathbb{R}^2)$ for some interval $(\alpha, \beta) \in \mathbb{R}$. △

Definition 3.1.3. Let $L(t)$ be a parameter family of lines and can be written as $a(t)x + b(t)y = c(t)$ where $a(t), b(t), c(t) : (\alpha, \beta) \to \mathbb{R}$ and $a(t)$,$b(t)$ and $c(t)$ are differentiable. An **envelope** for $L(t)$ is a regular parametrized curve: $\gamma(\alpha, \beta) \to \mathbb{R}^2$ such that $L(t)$ is the tangent line to (t) for each t, where regular means is differentiable and $\gamma(t) \neq 0$ △

Proposition 3.1.4. *Let $L(t) : a(t)x + b(t)y = c(t)$ be a parameter family of lines where $a(t)$,$b(t)$ and $c(t)$ are differentiable. The envelope can be parametrized as*

$$\left(\frac{c(t)b'(t) - b(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}, \frac{-c(t)a'(t) + a(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}\right).$$

Proof. Since $a(t)$,$b(t)$ and $c(t)$ are differentiable, by chain rule we have

$$a'(t)x(t) + a(t)x'(t) + b'(t)y(t) + b(t)y'(t) = c'(t) \tag{1}$$

Meanwhile, notice the normal vector $\vec{U}(t)$ and the tangent vector $\vec{T}(t)$ of $L(t)$ are

$$\vec{U}(t) = (a(t), b(t))$$

$$\vec{T}(t) = (x'(t), y'(t))$$

Geometrically, $\vec{U}(t)$ and $\vec{T}(t)$ are perpendicular to each other. Thus

$$(a(t), b(t)) \cdot (x'(t), y'(t)) = 0$$

Therefore (1) can be simplified as

$$a'(t)x(t) + b'(t)y(t) = c'(t)$$

Now we intersect $L'(t)$ and $L(t)$. Now we use Cramer's rule

$$x(t) = \frac{\begin{vmatrix} c(t) & b(t) \\ c'(t) & b'(t) \end{vmatrix}}{\begin{vmatrix} a(t) & b(t) \\ a'(t) & b'(t) \end{vmatrix}}$$

Therefore

$$\left(\frac{-c(t)b'(t) + b(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}, \frac{c(t)a'(t) - a(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}\right)$$

□

Example 3.1.5. For $t \in [0, 1]$, let $L(t)$ be the line segment from the point $(0, 1-t)$ to the point $(t, 0)$. Find an equation for the line containing $L(t)$. And together, all of the line segment $L(t)$ fill a region in the plane. By previous proposition, we will find a parametric equations for the top boundary curve.(Figure 3.2.1)

The slope of $L(t)$ is $\frac{t-1}{t} = 1 - t^{-1}$, so the equation is

$$y = (1 - t^{-1})x + (1 - t)$$

To find a parametrization for the curve, we plug in $\left(-\frac{c(t)b'(t)-b(t)c'(t)}{b(t)a'(t)-a(t)b'(t)}, -\frac{-c(t)a'(t)+a(t)c'(t)}{b(t)a'(t)-a(t)b'(t)}\right)$ from the previous proposition. In this case

$$a(t) = 1 - t^{-1}$$
$$a'(t) = t^{-2}$$
$$b(t) = -1$$
$$b'(t) = 0$$
$$c(t) = t - 1$$
$$c'(t) = 1$$

Then

$$\vec{x}(t) = \left(\frac{c(t)b'(t) - b(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}, \frac{-c(t)a'(t) + a(t)c'(t)}{b(t)a'(t) - a(t)b'(t)}\right) = (t^2, (1-t)^2)$$

◊

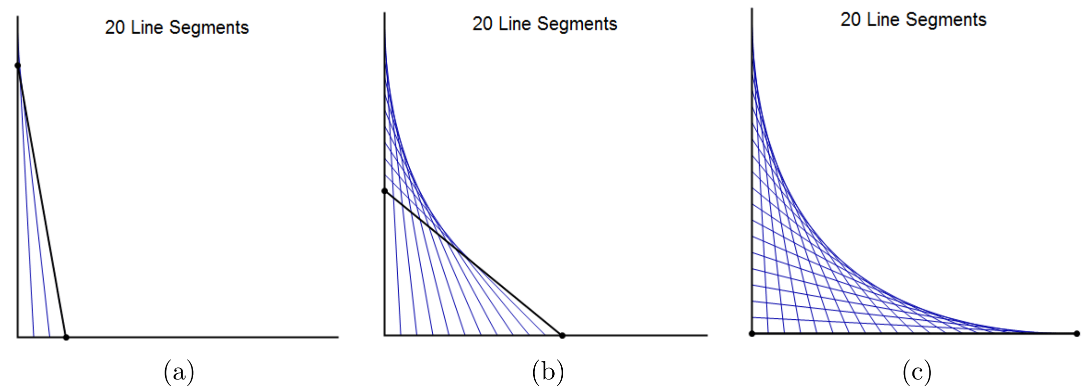


Figure 3.1.1

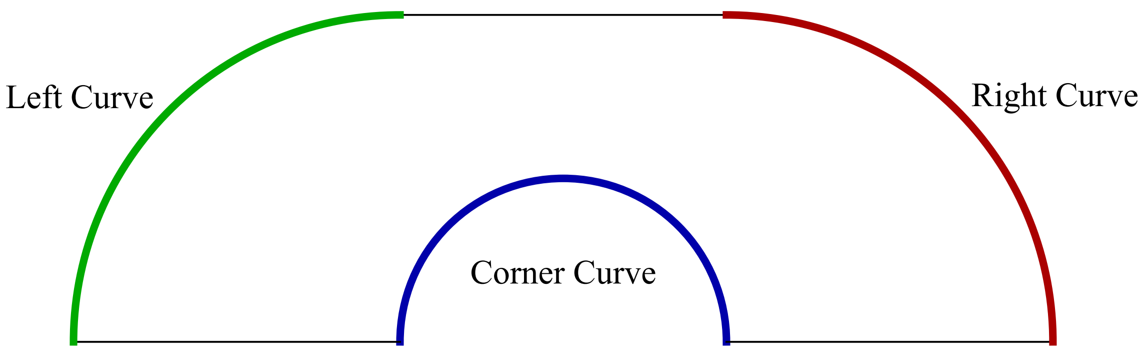


Figure 3.2.1: Top, right and bottom curve in Hammersley's sofa

3.2 The Boundary of the Sofa as an Envelope

This technique can be also used to find the formula of the shape of sofa. Lets start with the upper left wall of the hallway.

Definition 3.2.1. Let S be a sofa. If S is tangent to the top wall for $\theta_1 < \theta < \theta_2$, the corresponding **top curve** is portion of the boundary of the sofa traced out the point of tangency. We define **bottom curve**, **left curve**, **right curve**, and **corner curve** in the similar way. △

Example 3.2.2. The top, right and bottom curve in Hammersley's sofa is shown in figure 3.3.1. ◊

Proposition 3.2.3. *Let S be a sofa with rotation path $(p(\theta), q(\theta))$.*

1. *If S is tangent to the top wall for $\theta_1 \leq \theta \leq \theta_2$, then the corresponding top curve is*

$$H_T(\theta) = (-\sin(\theta) + q(\theta)\sin(\theta) + \cos(\theta)q'(\theta), \cos(\theta) - \cos(\theta)q(\theta) + \sin(\theta)q'(\theta))$$

2. *If S is tangent to the bottom wall for $\theta_1 \leq \theta \leq \theta_2$, then the corresponding top curve is*

$$H_B(\theta) = (q(\theta)\sin\theta + \cos\theta q'(\theta), -\cos\theta q(\theta) + \sin(\theta)q'(\theta))$$

3. *If S is tangent to the right wall for $\theta_1 \leq \theta \leq \theta_2$, then the corresponding right curve is*

$$H_R(\theta) = (-\cos\theta p(\theta) + \sin\theta p'(\theta), -p(\theta)\sin\theta - \cos\theta p'(\theta))$$

4. *If S is tangent to the left wall for $\theta_1 \leq \theta \leq \theta_2$, then the corresponding left curve is*

$$H_L(\theta) = (\cos\theta - \cos\theta p(\theta) + \sin\theta p'(\theta), \sin\theta - p(\theta)\sin\theta - \cos\theta p'(\theta))$$

5. *If S is touching the corner $(0,0)$ for $\theta_1 \leq \theta \leq \theta_2$, then the corresponding corner curve is*

$$H_C(\theta) = (-\cos\theta p(\theta) + q(\theta)\sin\theta, -\cos\theta(q(\theta + p(\theta)\tan\theta)))$$

Proof. We will prove H_1. The other proofs are similar to this one. Recall in chapter one we have talked about the conversion between hallway coordinates (a, b) and sofa coordinates (x, y). In terms of the upper left wall of the hallway, $a = 1$. Thus we have our line equation:

$$L_1(\theta) : -x\sin\theta + y\cos\theta + q(\theta) = 1$$

To use the proposition 3.2.4, $a(t)$, $b(t)$ and $c(t)$ are listed below

$$a(t) = -\sin\theta \qquad b(t) = \cos\theta \qquad c(t) = 1 - q(\theta)$$

$$a'(t) = -\cos\theta \qquad b'(t) = -\sin\theta \qquad c'(t) = -q'(\theta)$$

Then

$$\left(-\frac{c(t)b'(t)-b(t)c'(t)}{b(t)a'(t)-a(t)b'(t)}, -\frac{-c(t)a'(t)+a(t)c'(t)}{b(t)a'(t)-a(t)b'(t)}\right)$$
$$= (q(\theta)\sin\theta + \cos\theta q'(\theta), -\cos\theta q(\theta) + \sin(\theta)q'(\theta))$$

□

Proposition 3.2.4.

1. *Suppose the sofa is tangent to the top wall for $\theta_1 \le \theta \le \theta_2$, and let C_T be the corresponding top curve. Then $\int_{C_T} x\,dy = \int_{\theta_1}^{\theta_2} E_T(\theta)\,d\theta$, where*
$$E_T(\theta) = \big(q'(\theta)\cos\theta + q(\theta)\sin\theta - \sin\theta\big)\big(q''(\theta) + q(\theta) - 1\big)\sin\theta$$

2. *Suppose the sofa is tangent to the bottom wall for $\theta_1 \le \theta \le \theta_2$, and let C_B be the corresponding bottom curve. Then $\int_{C_B} x\,dy = \int_{\theta_3}^{\theta_4} E_B(\theta)\,d\theta$, where*
$$E_B(\theta) = \sin\theta(q(\theta)\sin\theta + \cos\theta q'(\theta))(q(\theta) + q''(\theta))$$

3. *Suppose the sofa is tangent to the left wall for $\theta_1 \le \theta \le \theta_2$, and let C_L be the corresponding left curve. Then $\int_{C_L} x\,dy = \int_{\theta_5}^{\theta_6} E_L(\theta)\,d\theta$, where*
$$E_L(\theta) = \cos\theta(p(\theta)\cos\theta - \sin\theta p'(\theta))(p(\theta) + p''(\theta))$$

4. *Suppose the sofa is tangent to the right wall for $\theta_7 \le \theta \le \theta_8$, and let C_R be the corresponding right curve. Then $\int_{C_R} x\,dy = \int_{\theta_7}^{\theta_8} E_R(\theta)\,d\theta$, where*
$$E_R(\theta) = \cos\theta(\cos\theta(-1 + p(\theta)) - \sin\theta p'(\theta))(-1 + p(\theta) + p''(\theta))$$

5. *Suppose the sofa is tangent to the corner for $\theta_9 \le \theta \le \theta_{10}$, and let C_C be the corresponding right curve. Then $\int_{C_C} x\,dy = \int_{\theta_9}^{\theta_{10}} E_C(\theta)\,d\theta$, where*
$$E_C(\theta) = (\cos\theta p(\theta) - q(\theta)\sin\theta)(\sin\theta(-q(\theta) + p'(\theta)) + \cos\theta(p(\theta) + q'(\theta)))$$

Now we apply calculus variation to find the maximum of this integral.

3.3 Calculus of Variations

Definition 3.3.1. A **functional** $I[f]$ is a function that takes functions as inputs and outputs numbers. △

Definition 3.3.2. Let $C^2([a,b])$ be the set of C^2 functions on $[a,b]$, and let $I : C^2([a,b]) \to \mathbb{R}$ be the functional

$$I[f] = \int_a^b H(f(x), f'(x), x)dx$$

where $H(y, y', x)$ is a C^2 function $\mathbb{R}^3 \to \mathbb{R}$.

The **functional derivative** of I is the function

$$\frac{\delta I[f]}{\delta f} : [a,b] \to \mathbb{R}$$

defined by

$$\frac{\delta I[f]}{\delta f}(x) = \frac{\partial H}{\partial y}(f(x), f'(x), x) - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f(x), f'(x), x)\right]dx$$

△

Proposition 3.3.3. *Let* $\{f_t\}\, t \in \mathbb{R}$ *be a* $\mathbb{C}^2$ *family of functions* $f_t : 0, 1 \to \mathbb{R}$ satisfying

$$f_t(0) = y_0 \qquad \text{and} \qquad f_t(1) = y_1$$

for all $t \in \mathbb{R}$, and let $I : \mathbb{R} \to \mathbb{R}$ be the function

$$I(t) = \int_0^1 H(f_t(x), f_t'(x), x)dx$$

Then I is differentiable, with

$$I'(t) = \int_0^1 \dot{f}_t(x) \frac{\delta I[f_t]}{\delta f_t}(x)dx$$

where $\dot{f}_t(x)$ donates the derivative of $f_t(x)$ with respect to t.

Proof. We have

$$I'(t) = \frac{d}{dt}\left[\int_0^1 H(f_t(x), f_t'(x), x)dx\right])dx = \int_0^1 \frac{d}{dt}\left[H(f_t(x), f_t'(x), x)dx]\right) dx$$

By the chain rule,

$$\frac{d}{dt}\left[H(f_t(x), f_t'(x), x)dx\right] = \dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), x) + \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)$$

And thus

$$I'(t) = \int_0^1 \left(\dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), x) + \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right) \tag{3.3.1}$$

$$= \int_0^1 \dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), x)dx + \int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)dx \tag{3.3.2}$$

But the integration by parts on the second integral gives

$$\int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)dx$$

$$= \left[\dot{f}_t(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right]_0^1 - \int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right] dx$$

Now, since $f_t(0) = y_0$ and $f_t(1) = y_1$ for all s, we know that $\dot{f}_t(0) = \dot{f}_t(1) = 0$ for all t, so the boundary term vanishes. Thus

$$\int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)dx = -\int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right] dx$$

Substituting this into equation 3.42 gives

$$\begin{aligned} I'(t) &= \int_0^1 \dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), x)dx - \int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right] dx \\ &= \int_0^1 \dot{f}_t(x)\left(\frac{\partial H}{\partial y}(f_t(x), f_t'(x), x) - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), x)\right]\right) dx \end{aligned}$$

□

Definition 3.3.4. Let $C^3([a,b])$ be the set of C^3 functions on $[a,b]$, and let $I : C^3([a,b]) \to \mathbb{R}$ be the functional

$$I[f] = \int_a^b H(f(x), f'(x), f''(x), x)dx$$

where $H(y, y', y'', x)$ is a C^3 function $\mathbb{R}^4 \to \mathbb{R}$.

The **functional derivative** of I is the function

$$\frac{\delta I[f]}{\delta f} : [a, b] \to \mathbb{R}$$

defined by

$$\frac{\delta I[f]}{\delta f}(x) = \frac{\partial H}{\partial y}(f(x), f'(x), f''(x), x) - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f(x), f'(x), f''(x), x)\right] dx + \frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}(f(x), f'(x), f''(x), x)\right] dx$$

△

Proposition 3.3.5. *Let* $\{f_t\} t \in \mathbb{R}$ *be a* $\mathbb{C}^3$ *family of functions* $f_t : 0, 1 \to \mathbb{R}$ satisfying

$$f_t(0) = y_0 \qquad \textit{and} \qquad f_t(1) = y_1$$

for all $t \in \mathbb{R}$, and let $I : \mathbb{R} \to \mathbb{R}$ be the function

$$I(t) = \int_0^1 H(f_t(x), f_t'(x), f_t''(x), x) dx$$

Then I is differentiable, with

$$I'(t) = \int_0^1 \dot{f}_t(x) \frac{\delta I[f_t]}{\delta f_t}(x) dx$$

Proof. We have

$$I'(t) = \frac{d}{dt}\left[\int_0^1 H(f_t(x), f_t'(x), f_t''(x), x) dx]\right) dx = \int_0^1 \frac{d}{dt}\left[H(f_t(x), f_t'(x), f_t''(x), x) dx\right] dx$$

By the chain rule

$$\frac{d}{dt}\left[H(f_t(x), f_t'(x), f_t''(x), x) dx\right] = \dot{f}_t(x) \frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x) + \dot{f}_t'(x) \frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x) + \dot{f}_t''(x) \frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)$$

and thus

$$I'(t) = \int_0^1 \Bigg(\dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x) \quad (3.3.3)$$

$$+\dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x) + \dot{f}_t''(x)\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)dx \quad (3.3.4)$$

$$= \int_0^1 \dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x)dx + \int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)dx \quad (3.3.5)$$

$$+ \int_0^1 \dot{f}_t''(x)\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)dx \quad (3.3.6)$$

By integration by parts on the second integral give

$$\int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)dx = \left[\dot{f}_t(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]_0^1$$
$$- \int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$

Now, since $f_t(0) = y_0$ and $f_t(1) = y_1$ for all s, we know that $\dot{f}_t(0) = \dot{f}_t(1) = 0$ and $\dot{f}_t'(0) = \dot{f}_t'(1) = 0$ for all t, so the boundary term vanishes. Thus

$$\int_0^1 \dot{f}_t'(x)\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)dx = -\int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$

And by integration by parts on the third integral give

$$\int_0^1 \dot{f}_t''(x)\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)dx = \left[\dot{f}_t'(x)\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]_0^1$$
$$- \int_0^1 \dot{f}_t'(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$
$$= - \int_0^1 \dot{f}_t'(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$

Now we do the integration by parts again on the third interval

$$-\int_0^1 \dot{f}_t'(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]dx = -\left[\dot{f}_t(x)\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]_0^1$$
$$+ \int_0^1 \dot{f}_t(x)\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$
$$= \int_0^1 \dot{f}_t(x)\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$

Now plug them back in equation 3.4.3 we have

$$I'(t) = \int_0^1 \dot{f}_t(x)\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x)dx$$
$$- \int_0^1 \dot{f}_t(x)\frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$
$$+ \int_0^1 \dot{f}_t(x)\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]dx$$
$$= \int_0^1 \dot{f}_t(x)\left(\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x)dx - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]\right.$$
$$\left.+\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right]dx\right.$$

□

Now we are ready to prove any solution to the basic function optimization problem satisfies the Euler-Lagrange equation.

Theorem 3.3.6. *Let $C^3([a,b])$ be the set of C^3 functions on $[a,b]$, and let $I : C^3([a,b]) \to \mathbb{R}$ be the functional*

$$I[f] = \int_a^b H(f(x), f'(x), f''(x), x)dx$$

where $H(y, y', y'', x)$ is a C^3 function $\mathbb{R}^4 \to \mathbb{R}$.

Suppose that $f : [a,b] \to \mathbb{R}$ is a C^3 function with $f(a) = y_0$ and $f(b) = y_1$ that maximizes the value of $I[f]$. Then f is a solution to the equation

$$\frac{\delta I[f]}{\delta f} = 0.$$

Proof. Suppose that $f(x)$ is a solution to the basic function optimization problem, which means that $f : [0,1] \to \mathbb{R}$ is a $\mathbb{C}^2$ solution with $f(0) = y_0$ and $f(1) = y_1$ that minimizes the value of the integral

$$\int_0^1 H(f(x), f'(x), f''(x), x)dx$$

Suppose $\{f_t\}_{t\in\mathbb{R}}$ be any $\mathbb{C}^2$ family of functions $f_t : [0,1] \to \mathbb{R}$ satisfying

1. $f(0) = y_0$ and $f(1) = y_1$ for all $t \in \mathbb{R}$ and

2. $f_0 = f$

And let $I : \mathbb{R} \to \mathbb{R}$ be the function

$$I(t) = \int_0^1 H(f(x), f'(x), f''(x), x)dx$$

Then $I(t)$ must reach its minimum value with $t = 0$. Then we have $I'(0) = 0$. From the previous proposition, we have

$$\int_0^1 \dot{f}_0(x) \left(\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x)dx - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right] \right. \tag{3.3.7}$$

$$+\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right] = 0 \tag{3.3.8}$$

This holds for every $\mathbb{C}^2$ family $\{f_t\}_{t\in\mathbb{R}}$ of function for which $f_0 = f$ and $f_t(0) = f_t(1)$ are constant. For such a family, the function $\dot{f}_0(x)$ is essentially arbitrary subject only to the constant that $\dot{f}_0(0) = \dot{f}_0(1) = 0$. The only way for to hold for all such functions is that

$$\frac{\partial H}{\partial y}(f_t(x), f_t'(x), f_t''(x), x)dx - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}(f_t(x), f_t'(x), f_t''(x), x)\right]$$

$$+\frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}(f_t(x), f_t'(x), f_t''(x), x)\right] = 0$$

This is the Euler-Lagrange equation. □

Definition 3.3.7. Let $I : C^3([a,b]) \times C^3([a,b]) \to \mathbb{R}$ be the functional

$$I[f,g] = \int_a^b H(f(x), f'(x), f''(x), g(x), g'(x), g''(x), x)dx$$

where $H(y, y', y'', z, z', z'', x)$ is a C^3 function $\mathbb{R}^4 \to \mathbb{R}$.

The **functional derivative** of I is the function

$$\frac{\delta I[f,g]}{\delta f} : [a,b] \to \mathbb{R} \qquad and \qquad \frac{\delta I[f,g]}{\delta g} : [a,b] \to \mathbb{R}$$

defined by

$$\frac{\delta I[f,g]}{\delta f}(x) = \frac{\partial H}{\partial y} - \frac{d}{dx}\left[\frac{\partial H}{\partial y'}\right] dx + \frac{d^2}{dx^2}\left[\frac{\partial H}{\partial y''}\right] dx$$

and

$$\frac{\delta I[f,g]}{\delta g}(x) = \frac{\partial H}{\partial z} - \frac{d}{dx}\left[\frac{\partial H}{\partial z'}\right] dx + \frac{d^2}{dx^2}\left[\frac{\partial H}{\partial z''}\right] dx$$

$\triangle$

Theorem 3.3.8. *Let $I : C^3([a,b]) \times C^3([a,b]) \to \mathbb{R}$ be the functional*

$$I[f,g] = \int_a^b H(f(x), f'(x), f''(x), g(x), g'(x), g''(x), x)dx$$

where $H(y, y', y'', z, z', z'', x)$ is a C^3 function $\mathbb{R}^4 \to \mathbb{R}$.

Suppose that $f, g : [a,b] \to \mathbb{R}$ is C^3 functions with $f(a) = y_0$ and $f(b) = y_1$ that maximizes the value of $I[f,g]$. Then f ,g is a solution to the equations

$$\frac{\delta I[f,g]}{\delta f} = 0 \qquad \textit{and} \qquad \frac{\delta I[f,g]}{\delta g} = 0.$$

Theorem 3.3.9. *Let $I_T[p,q]$,$I_B[p,q]$,$I_C[p,q]$,$I_L[p,q]$,$I_R[p,q]$be the functional*

$$I_T[p,q] = \int_{\theta_1}^{\theta_2} E_T(\theta)d\theta$$
$$I_B[p,q] = \int_{\theta_3}^{\theta_4} E_B(\theta)d\theta$$
$$I_C[p,q] = \int_{\theta_5}^{\theta_6} E_C(\theta)d\theta$$
$$I_L[p,q] = \int_{\theta_7}^{\theta_8} E_L(\theta)d\theta$$
$$I_R[p,q] = \int_{\theta_9}^{\theta_{10}} E_R(\theta)d\theta$$

where $E_T(\theta)$, $E_B(\theta)$, $E_C(\theta)$, $E_L(\theta)$, $E_R(\theta)$ were defined in the previous section. These functional derivatives are

$$\begin{aligned}
\frac{\delta I_T[p,q]}{\delta p} + \frac{\delta I_T[p,q]}{\delta q} &= -1 + q(\theta) + q''(\theta) \\
\frac{\delta I_B[p,q]}{\delta p} + \frac{\delta I_B[p,q]}{\delta q} &= -q(\theta) + q''(\theta) \\
\frac{\delta I_R[p,q]}{\delta p} + \frac{\delta I_R[p,q]}{\delta q} &= -1 + p(\theta) + p''(\theta) \\
\frac{\delta I_L[p,q]}{\delta p} + \frac{\delta I_L[p,q]}{\delta q} &= -p(\theta) + p''(\theta) \\
\frac{\delta I_{C,x}[p,q]}{\delta p} + \frac{\delta I_{C,x}[p,q]}{\delta q} &= p(\theta) + q'(\theta) \\
\frac{\delta I_{C,y}[p,q]}{\delta p} + \frac{\delta I_{C,y}[p,q]}{\delta q} &= q(\theta) - p'(\theta)
\end{aligned}$$

3.4 Top Curve and Bottom Curve

Theorem 3.4.1. *Let S be a sofa and $(p(\theta), q(\theta))$ be a rotation path for S.If S is tangent to both top and bottom walls and not touching anywhere else of the hallway for $\theta_1 < \theta < \theta_2$, the top and bottom curves in an area-maximised sofa are circular arcs with radius of 1/2 sharing the same center.*

Proof. By Green's theorem, the area of the sofa is

$$\int_{\partial S} xdy$$

Where ∂S is the boundary curve of S. Let C_T and C_B be the top and bottom curves for $\theta_1 < \theta < \theta_2$ and let $C' = -C_T - C_B$ Then

$$Area = \int_{C_T} xdy + \int_{C_B} xdy + \int_{C'} xdy = \int_{\theta_1}^{\theta_2} (E_T + E_B)d\theta + \int_{C'} xdy$$

$\int_C xdy$ does not depend on $\vec{O}$ during θ_1 and θ_2. $\vec{O}$ must be the path that maximizes the integral in the best sofa subject to the fact that $\vec{O}(\theta_1)$ and $\vec{O}(\theta_2)$ are fixed.

Therefore, our goal is to maximize the integral

$$I[p,q] = \int_{\theta_1}^{\theta_2} (E_T + E_B) d\theta$$

And $I[p,q] = I_T[p,q] + I_B[p,q]$. Therefore, from Theorem 3.3.9, we know

$$\frac{\delta I[p,q]}{\delta p}(\theta) = \frac{\delta I_T[p,q]}{\delta p}(\theta) + \frac{\delta I_T[p,q]}{\delta p}(\theta) = 0 + 0 = 0$$

and

$$\begin{aligned}\frac{\delta I[p,q]}{\delta q}(\theta) = \frac{\delta I_T[p,q]}{\delta q}(\theta) + \frac{\delta I_T[p,q]}{\delta q}(\theta) &= (-1 + q(\theta) + q''(\theta)) + (q(\theta) + q''(\theta)) \\ &= -1 + 2q(\theta) + 2q''(\theta) = 0\end{aligned}$$

Now we have

$$q(\theta) + q''(\theta) = \frac{1}{2}$$

Solutions to this second-order linear ordinary differential equation are

$$q(\theta) = c_1 \sin\theta + c_2 \cos\theta + \frac{1}{2}$$

Then we plug $q(\theta)$ back in top curve, we have

$$\left(c_1 - \frac{\sin\theta}{2}, -c_2 + \frac{\cos\theta}{2}\right)$$

The same to bottom curve, we have

$$\left(c_1 + \frac{\sin\theta}{2}, -c_2 - \frac{\cos\theta}{2}\right)$$

The top and bottom curves are circular arcs with radius $\frac{1}{2}$ sharing the same center. □

Theorem 3.4.2. *Let S be a sofa and $(p(\theta), q(\theta))$ be a rotation path for S. Suppose S is touching the corner $(0,0)$, is tangent to both the top and right walls, and is not touching any other walls for $\theta_1 < \theta < \theta_2$. Then for these values of θ, the rotation path is given by*

$$p(\theta) = c_1 \cos\theta + c_2 \sin\theta - c_3 \sin(3\theta/2) + c_4 \cos(3\theta/2) + 1/3$$

and

$$q(\theta) = -c_1 \sin\theta + c_2 \cos\theta + c_3 \cos(3\theta/2) + c_4 \sin(3\theta/2) + 1/3$$

The proof will be very similar to the previous theorem.

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