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McKay Graphs and Modular Representation Theory

Polina Aleksandrovna Vulakh
Bard College

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Polina Vulakh

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Abstract

Ordinary representation theory has been widely researched to the extent that there is a well-understood method for constructing the ordinary irreducible characters of a finite group. In parallel, John McKay showed how to associate to a finite group a graph constructed from the group's irreducible representations. In this project, we prove a structure theorem for the McKay graphs of products of groups as well as develop formulas for the graphs of two infinite families of groups. We then study the modular representations of these families and give conjectures for a modular version of the McKay graphs.

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1 Introduction

A representation of a finite group G is a homomorphism $\rho : G \to GL(V)$ for some finite-dimensional vector space V. In ordinary representation theory, V is defined over the complex numbers $\mathbf{C}$, while modular representation theory refers to representations in positive characteristic. This project focuses on certain graphs arising from the irreducible representations of a finite group.

We begin by leading the reader through definitions of ordinary representation theory. In particular, we describe what it means for a representation to be irreducible and highlight key points in character theory, where the character of a representation ρ of a finite group G is the function $\chi_\rho : G \to \mathbf{C}$ given by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. Throughout most of the project, we use character data to study the representation theory of the groups in question.

In 1981, John McKay associated a graph to each irreducible representation of a group [2]. Essentially, this graph visualizes the interaction between its defining representation and the other irreducibles of the group at hand. More precisely, since every representation can be decomposed into a direct sum of irreducible representations, one can decompose a tensor product into a direct sum of irreducibles. The decomposition data is what generates the McKay graph. We give a rigorous definition in Chapter 3, as well as construct this for several groups. We contrast the McKay graph with the Cayley graph, another structure which comes to mind when discussing graphs associated with groups.

Having presented necessary background in ordinary representation theory, we go on to prove several theorems in Chapter 4. We show that the McKay graphs of the product of two groups $G_1 \times G_2$ have a direct relationship to the McKay graphs of the groups taken individually. We also give formulas for the decomposition of tensor products of representations of the family of dihedral groups D_n. This tells us how to construct any McKay graph for a given dihedral group. Our last result in ordinary representation theory has to do with another family of groups, $\mathbf{SL}_2(p)$, the special linear group of 2×2 matrices over the field of p elements. Members of this family have a distinguished representation called the Steinberg representation. We provide formulas for the tensor product decomposition of this representation with itself.

In Chapter 5, we depart from ordinary representation theory and introduce modular representations. These send a finite group to $GL(V)$ in characteristic p, so we work with $K = \mathbf{F}_p$ or its algebraic closure, $\overline{\mathbf{F}}_p$. We introduce a notion of McKay graphs in characteristic p and show that the formulas for the McKay graphs of dihedral groups hold for D_p in characteristic 2 and characteristic p. Similarly, for $\mathbf{SL}_2(p)$ we give a conjecture for the tensor product decomposition of the Steinberg representation with itself in characteristic p. Throughout the project we used substantial computational techniques in Magma and Mathematica. Sample code appears in Appendix B.1.

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Preliminaries

In this chapter, we give a brief background of representation theory, with a focus on ordinary representations and their characters. We introduce key notation and illustrate several interesting properties by way of examples. In particular, attention is paid to the effect of a group's structure on its representations.

2.1 Definitions

Definition 2.1.1. Let G be a finite group and let V be a finite dimensional vector space defined over a field K. A **representation** of G is a homomorphism $\rho : G \longrightarrow GL(V)$ where $GL(V)$ is the general linear group over V. Each element $g \in G$ gives rise to an invertible $n \times n$ matrix $\rho(g)$ with entries in K. We call n the degree of ρ. $\triangle$

If the characteristic of K is 0 or prime to $|G|$, we call ρ an **ordinary** representation. Otherwise, ρ is a **modular** representation, which we will cover in a later chapter. For now, $K = \mathbf{C}$.

Example 2.1.2. The **trivial representation** is the homomorphism $\rho_{1_G} : G \to GL_1(K) = K^*$, sending all $g \in G$ to $1 \in K$. For ease of notation, we denote this as ρ_1 from here on.

Definition 2.1.3. If $\ker \rho$ is trivial, then ρ is a **faithful** representation of G. Otherwise, it is unfaithful. △

Example 2.1.4. For any group G, the trivial representation is unfaithful since $\ker(\rho_1) = G$.

Example 2.1.5. Let $G = S_3$, $V = \mathbf{C}^3$, and $\rho : G \to GL_3(\mathbf{C})$ such that $\rho(g)$ permutes the coordinates of V. Explicitly, we have

$$\rho(()) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho((13)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\rho((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho((123)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \rho((132)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\ker(\rho)$ contains just the trivial element $()$, making ρ a faithful representation.

Definition 2.1.6. Let W be a subspace of V. If for all $g \in G$ and $w \in W$ it is the case that $\rho(g) \cdot w \in W$, then W is a ***G*-invariant subspace**, or **stable subspace** of V. △

All representations have the stable subspaces $\{0\}$ and V. To see that $\{0\}$ is stable, note that matrix multiplication sends 0 to 0. Similarly, $W = V$ is a stable subspace as well since if $v \in V$, then $\rho(g)v \in V$ by definition of $\rho(g)$.

Example 2.1.7. Recall $\rho : S_3 \to GL_3(\mathbf{C})$ from 2.1.5 and let ℓ be the line spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then for any $w \in \ell$, $g \in S_3$, it is the case that $\rho(g) \cdot w \in \ell$ as well. (In fact, we find that $\rho(g) \cdot w = w$.) Thus ℓ is a stable subspace of $\mathbf{C}$.

Definition 2.1.8. For any stable subspace W of V, there exists a related stable subspace $W^{\perp}$ where $W^{\perp}$ is the **orthogonal complement** of $W \in V$. Here, orthogonality requires a suitably defined inner product on V. The proof of this can be found in Appendix A. $\triangle$

Remark 2.1.9. If the characteristic of the field divides the order of the group - which means ρ is a modular representation - then there does not exist such a suitably defined inner product.

The fact that stable subspaces come in pairs is especially clear when we consider the two trivial subspaces $\{0\}$ and V. For a nontrivial pair, let us further extend Examples 2.1.5 and 2.1.7.

Example 2.1.10. Let ρ be the permutation representation from Example 2.1.7. For this representation of S_3, it just so happens that the required inner product is the dot product as defined for Euclidean space, yielding the intuitive notion of orthogonality. Then the orthogonal complement of our stable subspace ℓ is the plane $\ell^{\perp} = \{(x, y, z)|x+y+z = 0\}$. One can see that for any $w = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \ell^{\perp}$, $\rho(g) \cdot w$ simply permutes the values of x, y, z. This means $\rho(g) \cdot w$ satisfies $x + y + z = 0$ which implies $\rho(g) \cdot w \in \ell^{\perp}$, verifying that $\ell^{\perp}$ is a stable subspace.

Definition 2.1.11. If the only stable subspaces of V are $\{0\}$ and V, we call ρ an **irreducible representation** of G. $\triangle$

Definition 2.1.12. Consider $\rho_1 : G \longrightarrow GL_n(V_1)$ and $\rho_2 : G \longrightarrow GL_m(V_2)$. Then their **direct sum**, $\rho_1 \oplus \rho_2 : G \longrightarrow GL_{n+m}(V_1 \oplus V_2)$, is defined as

$$(\rho_1 \oplus \rho_2)(g) \cdot (v_1, v_2) = (\rho_1(g) \cdot v_1, \rho_2(g) \cdot v_2).$$

If one chooses an appropriate basis, then $\rho_1 \oplus \rho_2$ can be expressed in matrix form like so: $(\rho_1 \oplus \rho_2)(g) \cdot (v_1, v_2) = \begin{bmatrix} \rho_1(g) \cdot v_1 & 0 \\ 0 & \rho_2(g) \cdot v_2 \end{bmatrix}$. $\triangle$

Definition 2.1.13. Let A and B be $n \times n$ and $m \times m$ matrices, respectively. Then the **Kronecker product** $A \otimes B$ of A and B is the $nm \times nm$ matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{bmatrix},$$

where $a_{ij}B$ denotes the $m \times m$ matrix obtained by multiplying all entries of B by a_{ij}. See Figure 4.1.1 for an expanded version of the above matrix. $\triangle$

Definition 2.1.14. Given $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$, we define the **tensor product** $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ such that for all $g \in G$, $(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$, where the latter tensor product denotes the Kronecker product of matrices. $\triangle$

Theorem 2.1.15. *The tensor product of two representations of G is a representation of G.*

Proof. It suffices to show that $\rho_1 \otimes \rho_2$ is a homomorphism of G. See that

$$\begin{aligned} (\rho_1 \otimes \rho_2)(gh) &= \rho_1(gh) \otimes \rho_2(gh) \text{ by definition} \\ &= \rho_1(g)\rho_1(h) \otimes \rho_2(g)\rho_2(h) \text{ since } \rho_1, \rho_2 \text{ are homomorphisms} \\ &= [\rho_1(g) \otimes \rho_2(g)] \cdot [\rho_1(h) \otimes \rho_2(h)] \text{ by multiplication of Kronecker products} \\ &= (\rho_1 \otimes \rho_2)(g) \cdot (\rho_1 \otimes \rho_2)(h) \end{aligned}$$

as desired. $\square$

Theorem 2.1.16. *If ρ_1 is the trivial representation of G, then $\rho_1 \otimes \rho_i = \rho_i$ where ρ_i is any representation of G.*

Proof. Let $g \in G$ and take $(\rho_1 \otimes \rho_i)(g)$. Then

$$\begin{aligned} (\rho_1 \otimes \rho_i)(g) &= \rho_1(g) \cdot \rho_i(g) \\ &= I_n \cdot \rho_i(g) \\ &= \rho_i(g). \end{aligned}$$

□

The following is Maschke's Theorem - one of the foundational theorems of ordinary representation theory.

Theorem 2.1.17. [1, 3.1, p.16] *Every representation is a direct sum of irreducible representations, and this decomposition is unique.*

It is not necessarily the case that the tensor product of two irreducible representations is itself irreducible. However, it can be decomposed into a direct sum where each component is irreducible. More explicitly, for any pair of irreducible representations ρ_i, ρ_j, it is the case that

$$\begin{aligned}\rho_i \otimes \rho_j &= \bigoplus_{k=1}^{n} a_k \rho_k \\ &= a_1\rho_1 \oplus a_2\rho_2 \oplus ... \oplus a_n\rho_n\end{aligned}$$

where $a_k \in \mathbf{N}$ denotes the multiplicity of the irreducible representation ρ_k.

These last few theorems are integral to the project and will be used frequently.

2.2 The Characters of a Representation

Let M be an $n \times n$ matrix. If a_{ij} denotes the ith entry in the jth row of M, then the trace $\mathrm{Tr}(M)$ is defined as $\sum\limits_{i=1}^{n} a_{ii}$. It is a nontrivial fact that $\mathrm{Tr}(M)$ also equals the sum of the eigenvalues.

Definition 2.2.1. For every representation $\rho : G \longrightarrow GL_n(V)$, there exists the function $\chi_\rho : G \longrightarrow K$, with $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. We call χ_ρ the **character** of a given representation. If ρ is irreducible, we say χ_ρ is an **irreducible character**. △

From the definition alone, we see that $\chi_\rho(1) = n$ where ρ is any representation of G since $\rho(1) = I_n$. It is also the case that for $g, h \in G$,

$$\chi_\rho(g) = \chi_\rho(hgh^{-1}). \tag{2.2.1}$$

Observe that

$$\begin{aligned}
\chi_\rho(hgh^{-1}) &= \mathrm{Tr}(\rho(hgh^{-1})) \\
&= \mathrm{Tr}\left([\rho(h)\rho(g)][\rho(h^{-1})]\right) \text{ since representations are homomorphisms} \\
&= \mathrm{Tr}\left([\rho(h)][\rho(h^{-1})\rho(g)]\right) \text{ by } \mathrm{Tr}(AB) = \mathrm{Tr}(BA) \\
&= \mathrm{Tr}(I_n\rho(g)) \\
&= \mathrm{Tr}(\rho(g)) \\
&= \chi_\rho(g).
\end{aligned}$$

This shows that χ_ρ is constant on conjugacy classes, which means we can talk about the action of ρ on G in terms of conjugacy classes alone. Related to this are some deep facts about representations and characters that we will use throughout the project. In particular, these are:

1. The number of conjugacy classes of G is equal to the number of irreducible representations G has.
2. If all the irreducible representations of G are $\rho_1, ..., \rho_\alpha$, then $\sum_{j=1}^{\alpha} \dim(\rho_j)^2 = |G|$.

Proving the above would require an in-depth excursion into heavily theoretical material, taking us farther than we can afford from our immediate topic. We instead outline several theorems and provide citations for those wishing to dive further into the related proofs. These are Lemma 2.2.4 and Theorems 2.2.7, 2.2.9, and 2.2.10.

Definition 2.2.2. Let $f : G \to \mathbf{C}$ be a function. If $f(g) = f(hgh^{-1})$ for all $g, h \in G$, then we call f a **class function** and $Z(L(G))$ the set of class functions. [7, 4.2, p. 36] △

Example 2.2.3. From its definition and Equation 2.2.1, we can see that χ_ρ is a class function.

Lemma 2.2.4. *The set of class functions forms a vector space over* $\boldsymbol{C}$ *under pointwise addition and scalar multiplication.*

Proof. Let f, g be two class functions and define $(f+g)(x) = f(x) + g(x)$. Under this definition, it is clear that the sum of two class functions is itself a class function. Moreover, for any scalar $a \in C$, defining $(af)(x) = af(x)$ shows that a scalar multiple of a class function is a class function itself. The zero function is a class function and serves as the additive identity. Thus set the class functions $Z(L(G))$ forms a complex vector space. $\square$

Remark 2.2.5. It is known that $Z(L(G))$ has dimension $|Cl(G)|$. For a proof of this, see [7, 4.3.8, p.37].

Definition 2.2.6. For a group G with $|Cl(G)| = \alpha$, we define the **inner product of two characters** $\chi_{\rho_i}, \chi_{\rho_j}$ as

$$\langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \tfrac{1}{|G|} \sum_{k=1}^{\alpha} |Cl_k| \big(\chi_{\rho_i}(g_k) \cdot \chi_{\rho_j}(g_k) \big).$$

The above differs from the standard inner product due to the fact that χ_ρ is a class function. Rather than sum over all $g \in G$, we iterate over g_k where the latter is a representative of Cl_k. $\triangle$

Theorem 2.2.7. [7, 4.3.9, p. 37] *The irreducible characters of G form an orthonormal set of class functions.*

Example 2.2.8. Let $G = S_3$. We present its irreducible characters in the table below, to be introduced formally in Definition 2.2.12. For now, it is an $\alpha \times \alpha$ matrix which encodes information about a group's irreducible characters.

	()	(12)	(123)
χ_{ρ_1}	1	1	1
χ_{ρ_2}	1	-1	1
χ_{ρ_3}	2	0	-1

We now compute $\langle \chi_{\rho_i}, \chi_{\rho_3} \rangle$. Observe that

$$\langle \chi_{\rho_1}, \chi_{\rho_3} \rangle = \tfrac{1}{6}\big(1(2 \cdot 1) + 3(0 \cdot 1) + 2(-1 \cdot 1)\big) = \tfrac{1}{6}\big(2 - 2\big) = 0$$

$$\langle \chi_{\rho_2}, \chi_{\rho_3} \rangle = \tfrac{1}{6}\big(1(2 \cdot 1) + 3(0 \cdot -1) + 2(-1 \cdot 1)\big) = \tfrac{1}{6}\big(2 - 2\big) = 0$$

$$\langle \chi_{\rho_3}, \chi_{\rho_3} \rangle = \tfrac{1}{6}\big(1(2\cdot 2) + 3(0\cdot 0) + 2(-1\cdot -1)\big) = \tfrac{1}{6}\big(4+2\big) = 1$$

as expected, verifying Theorem 2.2.7.

In fact, it will always be the case that $\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle = 1, \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = 0$ for $i \neq j$.

Corollary 2.2.9. [7, 4.3.10, p. 38] *There are at most $|Cl(G)|$ equivalence classes of irreducible representations of G.*

Two inequivalent irreducible representations will have inequivalent characters, and will thus belong to different equivalence classes in $Z(L(G))$. This means there are $|Cl(G)|$ distinct irreducible representations. Then for a finite group G, not only are there finitely many irreducible representations, but there are precisely $|Cl(G)|$ of them.

Theorem 2.2.10. [6, Corollary 2.4.2(a)] *Dimensionality Theorem: Let $\rho_1, \rho_2, ..., \rho_\alpha$ be the set of irreducible representations of G. Then it is the case that*

$$\sum_{k=1}^{\alpha} \dim(\rho_k)^2 = |G|$$

Example 2.2.11. We look at $G = S_3$ once more. Then $|G| = 6$ and G has three irreducible representations with dimensions $1, 1, 2$ respectively. (For more information on these, see 3.2.4.) See that $1^2 + 1^2 + 2^2 = 6 = |G|$.

From our knowledge about the number of irreducible representations of G and their characters, there exists a logical and efficient way of organizing data on G's irreducible representations.

Definition 2.2.12. To group G is associated an $\alpha \times \alpha$ **character table** of the form below. The columns correspond to conjugacy classes with column j denoting the jth conjugacy class Cl_j, while the rows correspond to characters of G's irreducible representations $\rho_1, \rho_2, ..., \rho_\alpha$. As defined, ρ_1 is the trivial representation. The ijth entry in this table tells us $\chi_{\rho_i}(Cl_j)$, the character value of ρ_i on the jth conjugacy class. By convention, we list the trivial representation and the identity class Cl_1 first. Through abuse of notation, we set $\chi_i := \chi_{\rho_i}$.

	Cl_1	Cl_2	Cl_3	$\dots$	Cl_α
χ_1	1	1	1	$\dots$	1
χ_2	$\dim(\rho_2)$	$\chi_{\rho_2}(Cl_2)$	$\chi_{\rho_2}(Cl_3)$	$\dots$	$\chi_{\rho_2}(Cl_\alpha)$
χ_3	$\dim(\rho_3)$	$\chi_{\rho_3}(Cl_2)$	$\chi_{\rho_3}(Cl_3)$	$\dots$	$\chi_{\rho_3}(Cl_\alpha)$
$\vdots$	$\vdots$				
χ_α	$\dim(\rho_\alpha)$	$\chi_{\rho_\alpha}(Cl_2)$	$\chi_{\rho_\alpha}(Cl_3)$	$\dots$	$\chi_{\rho_\alpha}(Cl_\alpha)$

$\triangle$

We now provide several examples of character tables, in order for the reader to familiarize themselves with the notation.

Example 2.2.13. *Representations of a cyclic group*

Let G be a finite abelian group. Then the number of conjugacy classes of G is $|G|$, since each element is its own conjugacy class. This means there are $|G|$ irreducible representations. By 2.2.10, it follows that $\dim(\rho) = 1$ for each irreducible representation ρ of G. Further, let $G = C_n$, the cyclic group of order n with elements of the form g^x for $0 \leq x \leq n-1$ where we set $g^0 = 1$. Then for any $\rho : C_n \longrightarrow GL_1(\mathbf{C})$, we have $\rho(1) = 1 = \rho(g^n) = (\rho(g))^n$. Let $\rho(g) = z$. It is the case that z must satisfy

$$1 = z^n \text{ which implies}$$

$$0 = z^n - 1$$

$$0 = (z-1)(z^{n-1} + z^{n-2} ... + z^2 + z + 1).$$

This yields the solutions $z = \rho(g) = 1$, which corresponds to the trivial representation, and $z = \rho(g) = \sigma$ where σ is a nontrivial nth root of unity. Then $\sigma = \omega^j$ for $\omega = e^{2\pi i/n}$, $1 \leq j \leq n-1$. There are $n-1$ possibilities for σ, each yielding one irreducible representation. Thus the only irreducible representations of C_n are of the form $\rho_{j+1}(g^x) = (\omega^j)^x$.

Now let $n = 5$, and let us examine the resulting character table.

	1	g	g^2	g^3	g^4
χ_1	1	1	1	1	1
χ_2	1	ω	ω^2	ω^3	ω^4
χ_3	1	ω^2	ω^4	ω	ω^3
χ_4	1	ω^3	ω	ω^4	ω^2
χ_5	1	ω^4	ω^3	ω^2	ω

In fact, we can construct a general character table of C_n, relying on the formula for $\rho_{j+1}(g)$. This is displayed below.

	1	g	g^2	g^3	$\dots$	g^{n-1}
χ_1	1	1	1	1	$\dots$	1
χ_2	1	ω	ω^2	ω^3	$\dots$	ω^{n-1}
χ_3	1	ω^2	$\omega^{4 \pmod n}$	$\omega^{6 \pmod n}$	$\dots$	$\omega^{2(n-1) \pmod n}$
χ_4	1	ω^3	$\omega^{6 \pmod n}$	$\omega^{9 \pmod n}$	$\dots$	$\omega^{3(n-1) \pmod n}$
$\vdots$	$\vdots$					
χ_n	1	ω^{n-1}	$\omega^{2(n-1) \pmod n}$	$\omega^{3(n-1) \pmod n}$	$\dots$	$\omega^{(n-1)^2 \pmod n}$

Theorem 2.2.14. *Let G be a finite nonabelian group and $\rho : G \longrightarrow GL(V)$ of degree 1. Then ρ is unfaithful.*

Proof. Let G be nonabelian with $\rho : G \longrightarrow GL_1(\mathbf{C}) = \mathbf{C}^\star$ of degree 1. Suppose ρ is faithful. Then ρ is injective, which means $\rho(G)$ is a subgroup of $\mathbf{C}^\star$ that is isomorphic to G. However, $\mathbf{C}^\star$ is abelian while $\rho(G)$ is not. Since an abelian group cannot have nonabelian subgroups, this is impossible. Thus ρ must be unfaithful. $\square$

Example 2.2.15. *Constructing the representations of D_4 and Q_8*

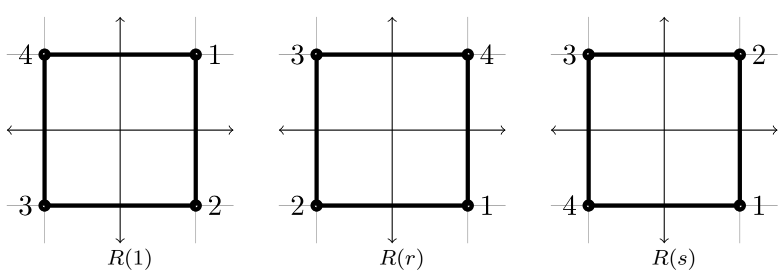
Consider $D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$ and $Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$ where $i^2 = j^2 = k^2 = ijk = -1$. It is clear that $|D_4| = |Q_8|$. Furthermore, they both have five conjugacy classes: $\{1\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}$ and $\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$, respectively. Then each of the two groups have five irreducible representations by Corollary 2.2.9. Let a, b, c, d, e denote the degrees of these representations. By 2.2.10, these must satisfy $8 = a^2 + b^2 + c^2 + d^2 + e^2$. Accounting for the trivial representation yields

$$8 = 1^2 + b^2 + c^2 + d^2 + e^2 \Longrightarrow 7 = b^2 + c^2 + d^2 + e^2.$$

The only solution set to the latter equation is $\{1, 1, 1, 2\}$. Thus both groups have four irreducible representations of degree 1 and one irreducible representation of degree 2. Note that these groups are nonisomorphic since Q_8 has more elements of order 4 than D_4. Let us now discuss the irreducible representations of both groups.

We refer to the four irreducible 1-dimensional representations of D_4 as $\chi_1, ..., \chi_4$ where χ_1 is the character of the trivial representation. Since D_4 is nonabelian, we know χ_i has a nontrivial kernel by 2.2.14. Then we can proceed by assigning normal subgroups of the group to $\ker(\chi_i)$. Said subgroups are $H_1 = \{1, r^2\}, H_2 = \{1, r, r^2, r^3\}, H_3 = \{1, r^2, s, sr^2\}, H_4 = \{1, r^2, sr, sr^3\}$, and D_4. The full group serves as $\ker(\chi_1)$, so we can dismiss it. Since $H_1 \subseteq H_i$ and we have four χ_i, it follows that $\chi_i(H_1) = 1$ for all i. Then we can distribute the remaining subgroups among our χ_i in the following manner: $\ker(\chi_2) = H_2$, $\ker(\chi_3) = H_3$, and $\ker(\chi_4) = H_4$. For any $g \in D_4$, we find that if $g \notin \ker(\chi_i)$, then $\chi_i(g) = -1$ since $g^2 \in \ker(\chi_i)$. Verification is left to the reader.

The last irreducible representation of D_4 is two dimensional - denote this as R. This representation corresponds to visualizing our group's action on the unit square in $\mathbf{R}^2$. For example, we have



We conclude from the previous pictures that $R(g) : R(r^m) = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)^m$, $R(s) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $R(sr^m) = R(s)R(r^m)$. Computing the traces for all $R(g)$ yields $\chi_R(1) =$

$2, \chi_R(r^2) = -2$, and $\chi_R(g) = 0$ for $g \neq 1, r^2$. Having now obtained all its characters, we construct the character table for D_4.

D_4	$\{1\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, sr^2\}$	$\{sr, sr^3\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_R	2	-2	0	0	0

We repeat this process with Q_8. Call its 1-dimensional irreducible representations $X_1, ..., X_4$ and let S be the irreducible representation of degree 2. Like D_4, Q_8 is non-abelian so we can find X_i in a similar fashion: we assign normal subgroups of Q_8 to $\ker(X_i)$. Said subgroups are $\{1\}, \{-1, 1\}, \langle i \rangle, \langle j \rangle, \langle k \rangle$, and Q_8. Let X_1 be the character of the trivial representation. By definition, $\ker(X_1) = Q_8$ so we move on to the nontrivial normal subgroups. Call any of them J. Since $\{-1, 1\} \subseteq J$ and we have exactly four X_i, we know $-1 \in \ker(X_i)$ for all i. Then we can assign $\langle i \rangle = \ker(X_2)$, $\langle j \rangle = \ker(X_3)$, and $\langle k \rangle = \ker(X_4)$. Now take X_2 and consider $X_2(ijk)$. Since $ijk = -1$ and $X_2(-1) = X_2(i) = 1$, we must have $X_2(j) = X_2(k) = -1$ so that the multiplicative group structure of Q_8 is preserved. By the same reasoning we have $X_3(i) = X_3(k) = -1$ and $X_4(i) = X_4(j) = -1$.

This gives us enough information to compute χ_S without explicitly defining the representation. We use 2.2.7, which tells us that for any inequivalent x_i, x_j, we have $\langle x_i, x_j \rangle = 0$ with the inner product operating as defined in 2.2.6. First, recall that S is of degree 2, which means $S(1) = I_2$ and so $X_S(1) = 2$. Together with the facts that $X_S(-1) \neq 2, (X_S(-1))^2 = 2$, this implies $X_S(-1) = -2$. To obtain the rest of X_S, we use 2.2.7 and 2.2.6 to generate equations of the form

$$\langle X_i, S \rangle = \big(|Cl_1|(X_i(1)X_S(1)) + |Cl_2|(X_i(Cl_2)X_S(Cl_2)) + ... + |Cl_5|X_i(Cl_5)X_S(Cl_5))\big).$$

These create the following system of linear equations,

$$\langle X_1, S\rangle = 2 + (-2) + 2(1)(\alpha) + 2(1)(\beta) + 2(1)(\delta) = 0$$

$$\langle X_2, S\rangle = 2 + (-2) + 2(1)(\alpha) + 2(-1)(\beta) + 2(-1)(\delta) = 0$$

$$\langle X_3, S\rangle = 2 + (-2) + 2(-1)(\alpha) + 2(1)(\beta) + 2(-1)(\delta) = 0$$

$$\langle X_4, S\rangle = 2 + (-2) + 2(-1)(\alpha) + 2(-1)(\beta) + 2(1)(\delta) = 0$$

which in turn yields the solutions $\alpha = \beta = \delta = 0$. Our result is $X_S = [2, -2, 0, 0, 0]$.

Now that we have a complete set of irreducible characters for both groups, we can compare them. We find that D_4 and Q_8 have identical character tables, overlaid below.

D_4 Q_8	$\{1\}$ $\{1\}$	$\{r^2\}$ $\{-1\}$	$\{r, r^3\}$ $\{i, -i\}$	$\{s, sr^2\}$ $\{j, -j\}$	$\{sr, sr^3\}$ $\{k, -k\}$
χ_1, X_1	1	1	1	1	1
χ_2, X_2	1	1	1	-1	-1
χ_3, X_3	1	1	-1	1	-1
χ_4, X_4	1	1	-1	-1	1
χ_R, X_S	2	-2	0	0	0

This example shows that non-isomorphic groups may have identical character tables.

Lemma 2.2.16. *Let ρ_r, ρ_s, ρ_t be representations of G such that $\rho_t = \rho_r \oplus \rho_s$. Then $\chi_{\rho_t} = \chi_{\rho_r} + \chi_{\rho_s}$.*

Recall from Definition 2.1.12 that $\rho_1 \oplus \rho_2$ may be put into block diagonal form by changing basis. Furthermore, the trace of a matrix is independent of basis and $\mathrm{Tr}(A) + \mathrm{Tr}(B) = \mathrm{Tr}(A + B)$. Then by definition of χ_ρ, the above holds.

This lemma extends Theorems 2.1.15, 2.1.16, 2.1.17, to the characters of a representation. This means that the product of two irreducible characters is a character of G - one which can be decomposed uniquely into a sum of irreducible characters. More explicitly, if G is a group with α irreducible representations, then it is the case that for any two irreducible characters χ_i, χ_j, their product $\chi_i \cdot \chi_j$ is also a character of G and

$$\begin{aligned}\chi_i \cdot \chi_j &= \sum_{k=1}^{\alpha} a_k \chi_k \\ &= a_1\chi_1 + a_2\chi_2 + ... + a_\alpha \chi_\alpha.\end{aligned}$$

The multiplicities a_k are one of the key components in constructing the McKay graph of a representation, which we will discuss in the next section.

3

Graphs of Representations

We introduce two types of graphs related to any given group: the Cayley graph and the McKay graph. Though the former has little to do with representation theory, its familiarity provides a nice setting for the more complicated McKay graph. We lead the reader through some graph theoretical definitions and introduce a method of constructing the McKay graph of a group's representation. This is to give background for the proof of Theorem 4.1.3.

3.1 The Cayley Graph

Definition 3.1.1. A **symmetric subset** of G is a set $S \subseteq G$ such that $1_G \notin S$ and if $g \in S$, then $g^{-1} \in S$ as well. $\triangle$

Example 3.1.2. Recall Q_8 from Example 2.2.15. All of its conjugacy classes, except for the identity, are symmetric subsets. This is evident upon writing them out: $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$, $\{k, -k\}$. Any union of these is also a symmetric subset of the group.

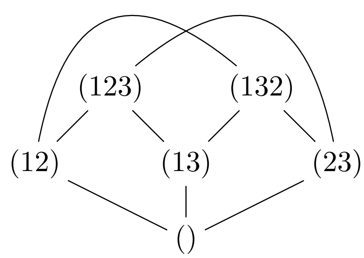
Example 3.1.3. The set $S = G - \{1_G\}$ is a symmetric subset of G for any group G.

Definition 3.1.4. Let G be a group and let S be a symmetric subset of G. Then the **Cayley Graph** Γ_S of G with respect to S is defined by the following rules:

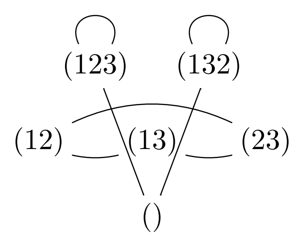
1. The vertex set $V(\Gamma)$ is equal to G, and
2. There exists an edge between vertices $g_1, g_2 \in V(\Gamma)$ if $g_1 g_2^{-1} \in S$.

$\triangle$

Example 3.1.5. Let $G = S_3$ and let $A = \{(12), (13), (23)\}$, $B = \{(123), (132)\}$. It is easy to see that A, B are each symmetric subsets. Then the Cayley graphs Γ_A, Γ_B of G with respect to A and B are

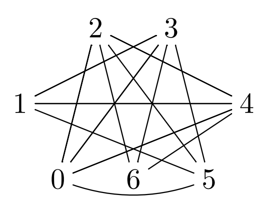


Γ_A



Γ_B

Example 3.1.6. Now let $G = \mathbf{Z}/7\mathbf{Z}$ and let $S = \{2, 3, 4, 5\}$. We take G to be the additive group, so $e = 0$. Then Γ_S of G is



Γ

3.2 The McKay Graph

Given a group G with irreducible representations $\rho_1, \rho_2, ..., \rho_\alpha$, each ρ_i has an associated **McKay graph** Γ_{ρ_i} which we derive through tensor product decompositions of the representations' characters [2]. This is best constructed through $M_{\Gamma_{\rho_i}}$, the adjacency matrix of Γ_{ρ_i}. For the sake of notation, let us rename this M_{ρ_i}.

Definition 3.2.1. The **adjacency matrix** M_Γ of a graph Γ is a square matrix of size $|V(\Gamma)| \times |V(\Gamma)|$ where the ijth entry is nonzero if there exists an edge between vertices v_i and v_j. $\triangle$

Suppose that for some G, we want to construct Γ_{ρ_i}, the McKay Graph of its irreducible representation ρ_i. The first step in creating the adjacency matrix M_{ρ_i} is to compute

$$\begin{aligned}\chi_{\rho_i} \cdot \chi_{\rho_j} &= \sum_{k=1}^{\alpha} m_{(j,k)} \chi_{\rho_k} \\ &= m_{(j,1)}\chi_{\rho_1} + m_{(j,2)}\chi_{\rho_2} + ... + m_{(j,\alpha)}\chi_{\rho_\alpha}\end{aligned}$$

for each irreducible ρ_j with $1 \leq j \leq \alpha$. Then the row vector $[m(j,1), m(j,2), ..., m(j,n)]$ becomes the jth row in M_{ρ_i}. This gives us a formula for M_{ρ_i} in terms of $m_{(j,k)}$:

$$M_{\rho_i} = \begin{bmatrix} m_{(1,1)} & m_{(1,2)} & \cdots & m_{(1,\alpha)} \\ m_{(2,1)} & m_{(2,2)} & \cdots & m_{(2,\alpha)} \\ \vdots & \ddots & & \vdots \\ m_{(\alpha,1)} & m_{(\alpha,2)} & \cdots & m_{(\alpha,\alpha)} \end{bmatrix}.$$

Theorem 3.2.2. *For a group G with α irreducible representations, the adjacency matrix $M_{\rho_i} = I_\alpha$ if and only if ρ_i is the trivial representation.*

Proof. Suppose for $\rho_i \neq \rho_1$ we have $M_{\rho_i} = I_\alpha$. Then $m_{(i,j)} = 0$ for $i \neq j$ and $m_{(i,j)} = 1$ otherwise. Since ρ_i is not the trivial representation, it is the case that $\chi_{\rho_i} \cdot \chi_{\rho_1} = \chi_{\rho_i}$ by Theorem 2.1.16. Then $m_{(1,i)} = 1$ but by hypothesis, $i \neq 1$. This contradicts $M_{\rho_i} = I_\alpha$. Now suppose $M_{\rho_1} \neq I_\alpha$. Then there exists some j such that $m_{(j,j)} \neq 1$ or $m_{(j,i \neq j)} = 0$. This means $\chi_1 \cdot \chi_{\rho_j} \neq \chi_{\rho_j}$, which contradicts the definition of ρ_1. $\square$

Corollary 3.2.3. *Γ_i is totally disconnected if and only if ρ_i is the trivial representation.*

We only consider nontrivial edges here. More concretely, the self-loops resulting from nonzero entries on the diagonal of M_{ρ_1} are ignored.

Example 3.2.4. Let $G = S_3$ and let us construct the McKay graphs for each of its irreducible representations. Below is its character table, for reference.

	()	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

We now compute $\chi_i \cdot \chi_j$ for fixed $i = \{1, 2, 3\}$. The symbol $\cdot$ will denote the dot product of rows (that is, $[a, b, c] \cdot [d, e, f] = [ad, be, cf]$). We derive the following:

$$\begin{aligned}
\chi_1 \cdot \chi_1 &= [1, 1, 1] \cdot [1, 1, 1] = [1, 1, 1] = \chi_1 \\
\chi_1 \cdot \chi_2 &= [1, 1, 1] \cdot [1, -1, 1] = [1, -1, 1] = \chi_2 \\
\chi_1 \cdot \chi_3 &= [1, 1, 1] \cdot [2, 0, -1] = [2, 0, -1] = \chi_3
\end{aligned}$$

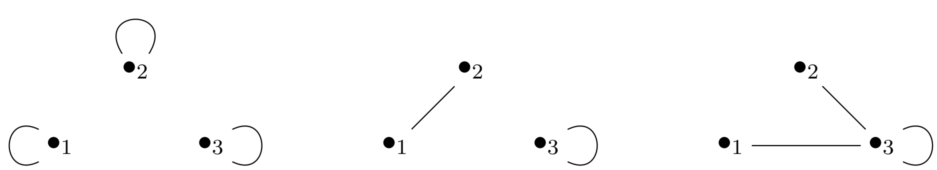
$$\begin{aligned}
\chi_2 \cdot \chi_1 &= [1, -1, 1] \cdot [1, 1, 1] = [1, -1, 1] = \chi_2 \\
\chi_2 \cdot \chi_2 &= [1, -1, 1] \cdot [1, -1, 1] = [1, 1, 1] = \chi_1 \\
\chi_2 \cdot \chi_3 &= [1, -1, 1] \cdot [2, 0, -1] = [2, 0, -1] = \chi_3
\end{aligned}$$

$$\begin{aligned}
\chi_3 \cdot \chi_1 &= [2, 0, -1] \cdot [1, 1, 1] = [2, 0, -1] = \chi_3 \\
\chi_3 \cdot \chi_2 &= [2, 0, -1] \cdot [1, -1, 1] = [2, 0, -1] = \chi_3 \\
\chi_3 \cdot \chi_3 &= [2, 0, -1] \cdot [2, 0, -1] = [4, 0, 1] = \chi_1 + \chi_2 + \chi_3
\end{aligned}$$

Recall the formula for the adjacency matrix M_{ρ_i} from the previous page. Using the above tensor product decompositions, we construct the adjacency matrices for the three representations of S_3.

$$\underset{M_{\rho_1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{M_{\rho_2}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{M_{\rho_3}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

These matrices yield the graphs below, respectively.



Γ_{ρ_1} Γ_{ρ_2} Γ_{ρ_3}

Example 3.2.5. Since the character table determines a group's McKay graphs, it is the case that D_4 and Q_8 have the same McKay graphs. We skip Γ_{ρ_1} since its adjacency matrix is I_5. For reference to the computations, here is the character table from Example 2.2.15:

D_4 / Q_8	$\{1\}$ / $\{1\}$	$\{r^2\}$ / $\{-1\}$	$\{r, r^3\}$ / $\{i, -i\}$	$\{s, sr^2\}$ / $\{j, -j\}$	$\{sr, sr^3\}$ / $\{k, -k\}$
χ_1, X_1	1	1	1	1	1
χ_2, X_2	1	1	-1	-1	1
χ_3, X_3	1	1	-1	1	-1
χ_4, X_4	1	1	1	-1	-1
χ_R, X_S	2	-2	0	0	0

Using this data, we compute the tensor product decompositions.

$$\begin{aligned}
\chi_2 \cdot \chi_1 &= \chi_2 \\
\chi_2 \cdot \chi_2 &= [1,1,-1,-1,1] \cdot [1,1,-1,-1,1] = [1,1,1,1,1] = \chi_1 \\
\chi_2 \cdot \chi_3 &= [1,1,-1,-1,1] \cdot [1,1,-1,1,-1] = [1,1,1,-1,-1] = \chi_4 \\
\chi_2 \cdot \chi_4 &= [1,1,-1,-1,1] \cdot [1,1,1,-1,-1] = [1,1,-1,1,-1] = \chi_3 \\
\chi_2 \cdot \chi_R &= [1,1,-1,-1,1] \cdot [2,-2,0,0,0] = [2,-2,0,0,0] = \chi_R
\end{aligned}$$

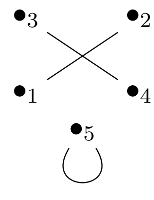
$$\begin{aligned}
\chi_3 \cdot \chi_1 &= \chi_3 \\
\chi_3 \cdot \chi_2 &= [1,1,-1,1,-1] \cdot [1,1,-1,-1,1] = [1,1,1,-1,-1] = \chi_4 \\
\chi_3 \cdot \chi_3 &= [1,1,-1,1,-1] \cdot [1,1,-1,1,-1] = [1,1,1,1,1] = \chi_1 \\
\chi_3 \cdot \chi_4 &= [1,1,-1,1,-1] \cdot [1,1,1,-1,-1] = [1,1,-1,-1,1] = \chi_2 \\
\chi_3 \cdot \chi_R &= [1,1,-1,1,-1] \cdot [2,-2,0,0,0] = [2,-2,0,0,0] = \chi_R
\end{aligned}$$

$$\begin{aligned}
\chi_4 \cdot \chi_1 &= \chi_4 \\
\chi_4 \cdot \chi_2 &= [1,1,1,-1,-1] \cdot [1,1,-1,-1,1] = [1,1,-1,1,-1] = \chi_3 \\
\chi_4 \cdot \chi_3 &= [1,1,1,-1,-1] \cdot [1,1,-1,1,-1] = [1,1,-1,-1,1] = \chi_2 \\
\chi_4 \cdot \chi_4 &= [1,1,1,-1,-1] \cdot [1,1,1,-1,-1] = [1,1,1,1,1] = \chi_1 \\
\chi_4 \cdot \chi_R &= [1,1,1,-1,-1] \cdot [2,-2,0,0,0] = [2,-2,0,0,0] = \chi_R
\end{aligned}$$

$$\begin{aligned}
\chi_R \cdot \chi_1 &= \chi_R \\
\chi_R \cdot \chi_2 &= [2,-2,0,0,0] \cdot [1,1,-1,-1,1] = [2,-2,0,0,0] = \chi_R \\
\chi_R \cdot \chi_3 &= [2,-2,0,0,0] \cdot [1,1,-1,1,-1] = [2,-2,0,0,0] = \chi_R \\
\chi_R \cdot \chi_4 &= [2,-2,0,0,0] \cdot [1,1,1,-1,-1] = [2,-2,0,0,0] = \chi_R \\
\chi_R \cdot \chi_R &= [2,-2,0,0,0] \cdot [2,-2,0,0,0] = [4,4,0,0,0] = \chi_1 + \chi_2 + \chi_3 + \chi_4
\end{aligned}$$

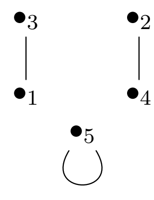
These yield the following nontrivial adjacency matrices and their respective graphs, with vertices labeled $1, 2, 3, 4$ and 5 corresponding to the irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$, and R:

$$\underset{M_{\rho_2}}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$



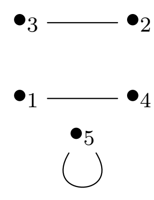
Γ_{ρ_2}

$$\underset{M_{\rho_3}}{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$



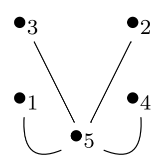
Γ_{ρ_3}

$$\underset{M_{\rho_4}}{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$



Γ_{ρ_4}

$$\underset{M_R}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}}$$



Γ_R

Example 3.2.6. *Multiplicities greater than 1*

Let $G = A_4$. Displayed below is its character table, which we use to compute $\chi_{\rho_4} \cdot \chi_{\rho_4}$. Here, $\omega = e^{2\pi i/3}$.

	()	(12)	(123)	(12)(34)
χ_1	1	1	1	1
χ_2	1	1	ω^2	ω
χ_3	1	1	ω	ω^2
χ_4	3	-1	0	0

See that

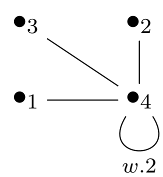
$$\chi_4 \cdot \chi_4 = [3, -1, 0, 0] \cdot [3, -1, 0, 0] = [9, 1, 0, 0].$$

Notice that the direct sum $\chi_1 + \chi_2 + \chi_3 + \chi_4$ yields $[5, 2, 0, 0]$. This tells us that for at least one χ_i, its multiplicity is greater than 1. We then obtain the correct decomposition,

$$\chi_4 \cdot \chi_4 = \chi_1 + \chi_2 + \chi_3 + 2\chi_4.$$

In order to show the McKay graph of this representation, we must compute $\chi_4 \cdot \chi_i$ for $i = 1, 2, 3$. From the character table, one can see that $\chi_4 \cdot \chi_i = \chi_4$, so we omit these calculations in favor of brevity. We show the resulting adjacency matrix M_{ρ_4} and McKay graph Γ_{ρ_4}. The vertex i corresponds to ρ_i and we label the edge of nontrivial weight.

$$M_{\rho_4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \rightarrow$$



Γ_{ρ_4}

4

Results in Ordinary Representation Theory

In this chapter we discuss two new theorems in ordinary representation theory. The first theorem focuses on the representations of direct products of groups and what this means for the McKay graphs of direct products of groups. The second theorem establishes formulas for the McKay graphs of D_n.

4.1 Direct Products of Groups

As we look at more complicated groups, the number of conjugacy classes grows, thus increasing the number of irreducible representations a given group has. This makes character tables difficult to generate by hand and increases the complexity of tensor product decompositions. Because of this, we rely on the Magma Calculator available at http://magma.maths.usyd.edu.au/calc/ to provide us with character tables. We also use Mathematica to aid in tensor product decompositions. An example of the two methods can be found in Appendices B.1 and B.2, respectively.

We now present a preexisting theorem having to do with the representations of the direct product of two groups, $G_1 \times G_2$, followed by our extension of this theorem.

Theorem 4.1.1. [6, Thm. 10, 3.2] *For any irreducible representations ρ_i, ρ_j of groups G_1, G_2 respectively, $\rho_i \otimes \rho_j$ is an irreducible representation of $G_1 \times G_2$. Furthermore, each irreducible representation of $G_1 \times G_2$ is isomorphic to some $\rho_i \otimes \rho_j$ where ρ_i, ρ_j are each some irreducible representation of G_1, G_2.*

A similar fact exists for the character of a representation. Specifically, for χ_1, χ_2 of G_1, G_2 respectively, it is the case that

$$\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \chi_1(g_1) \cdot \chi_2(g_2) \tag{4.1.1}$$

for $(g_1, g_2) \in G_1 \times G_2$ and $g_1 \in G_1, g_2 \in G_2$. [6, 3.2, p. 27]

Our goal is to extend the above to McKay graphs. First, let us show an example demonstrating the feasibility of this.

Example 4.1.2. Let $G = S_3$ and recall the adjacency matrices for its McKay graphs from 3.2.4:

$$\underset{M_{\rho_1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{M_{\rho_2}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{M_{\rho_3}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}.$$

Let us compute the nine Kronecker products $M_{\rho_i} \otimes M_{\rho_j}$ in order to compare these to M_{σ_k}, which correspond to the irreducible representations σ_k of $S_3 \times S_3$ for $k = 1, ..., 9$ and will be constructed afterwards. We obtain the following:

$$M_{\rho_1} \otimes M_{\rho_1} = I_9$$

$$M_{\rho_1} \otimes M_{\rho_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{\rho_1} \otimes M_{\rho_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{\rho_2} \otimes M_{\rho_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{\rho_2} \otimes M_{\rho_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{\rho_2} \otimes M_{\rho_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{\rho_3} \otimes M_{\rho_1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{\rho_3} \otimes M_{\rho_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{\rho_3} \otimes M_{\rho_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We must now compute M_{σ_k} of $S_3 \times S_3$. Below is the character table of $S_3 \times S_3$ generated by the Magma Calculator via the code CharacterTable(DirectProduct(Sym(3),Sym(3)));.

```
-----------------------------------

Class |    1  2  3  4  5  6  7  8  9
Size  |    1  3  3  9  2  2  4  6  6
Order |    1  2  2  2  3  3  3  6  6
-----------------------------------

X.1    +   1  1  1  1  1  1  1  1  1
X.2    +   1 -1  1 -1  1  1  1  1 -1
X.3    +   1  1 -1 -1  1  1  1 -1  1
X.4    +   1 -1 -1  1  1  1  1 -1 -1
X.5    +   2  2  0  0  2 -1 -1  0 -1
X.6    +   2 -2  0  0  2 -1 -1  0  1
X.7    +   2  0  2  0 -1  2 -1 -1  0
X.8    +   2  0 -2  0 -1  2 -1  1  0
X.9    +   4  0  0  0 -2 -2  1  0  0
```

This provides us with the data needed to decompose $\chi_{\sigma_n} \cdot \chi_{\sigma_m}$, which we leave out for brevity's sake. The above gives us

$$M_{\sigma_1} = I_9, \quad \underbrace{\begin{bmatrix} 0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_2}} \underbrace{\begin{bmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_3}}$$

$$\underbrace{\begin{bmatrix} 0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_4}} \underbrace{\begin{bmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&1&0&1&0&0&0&0\\ 0&1&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}}_{M_{\sigma_5}} \underbrace{\begin{bmatrix} 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}}_{M_{\sigma_6}}$$

$$\underbrace{\begin{bmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 0&0&0&0&1&1&0&0&1 \end{bmatrix}}_{M_{\sigma_7}} \underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&1&1&0&0&0&1&0\\ 1&1&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0&1 \end{bmatrix}}_{M_{\sigma_8}} \underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&1&1&0&0&1\\ 0&0&0&0&1&1&0&0&1\\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}}_{M_{\sigma_9}}$$

Upon inspection, it is evident that $M_{\rho_i} \otimes M_{\rho_j} \neq M_{\sigma_k}$ unless $i = j = k = 1$. However, the reader may have noticed patterns in M_{σ_k} similar to the earlier Kronecker products. For example, one can see that M_{σ_k} for $k = 7, 8, 9$ all have a 6×6 block of zeros in the same location as that of $M_{\rho_3} \otimes M_{\rho_i}$. This suggests that there may be more to M_{σ_k} than was evident upon first inspection.

For M_{σ_9} in particular, observe that swapping row 3 with row 4 and columns 3 with column 5 gives us $M_{\rho_3 \otimes \rho_3}$. Then a change of basis - which amounts to listing σ_k, Cl_ℓ in a different order within the character table of $S_3 \times S_3$ - may generate M_{σ_k} such that the equality to $M_{\rho_i \otimes \rho_j}$ holds.

We create a method of generating a character table of $S_3 \times S_3$ under the desired order. First we make a notation change, for if the order of the characters changes, then the subscripts of σ_k will change as well. To avoid confusion, let us call our newly ordered representations τ_m for $m = 1, ..., 9$. We create τ_m by exploiting the isomorphism property discussed in (2) of Theorem 4.1.1. More specifically, we compute $\chi_{\rho_i \otimes \rho_j}$ for each conjugacy class $Cl_i Cl_j$ of $S_3 \times S_3$, where Cl_i, Cl_j denote the conjugacy classes of S_3 and $\rho_i \otimes \rho_j \cong \sigma_m$. We can do this by Equation 4.1.1 and the fact that characters are class functions. For example,

$$\begin{aligned}\chi_{\rho_2 \otimes \rho_3} = [&\big(\chi_2(Cl_1) \cdot \chi_3(Cl_1)\big), \big(\chi_2(Cl_1) \cdot \chi_3(Cl_2)\big), \big(\chi_2(Cl_1) \cdot \chi_3(Cl_3)\big), \big(\chi_2(Cl_2) \cdot \chi_3(Cl_1)\big),\\ &\big(\chi_2(Cl_2) \cdot \chi_3(Cl_2)\big), \big(\chi_2(Cl_2) \cdot \chi_3(Cl_3)\big), \big(\chi_2(Cl_2) \cdot \chi_3(Cl_1)\big), \big(\chi_2(Cl_3) \cdot \chi_3(Cl_2)\big),\\ &\big(\chi_2(Cl_3) \cdot \chi_3(Cl_3)\big)]\\ = [&(1 \cdot 2), (1 \cdot 0), (1 \cdot (-1)), ((-1) \cdot 2), ((-1) \cdot 0), ((-1) \cdot (-1)), (1 \cdot 2), (1 \cdot 0), (1 \cdot (-1))]\\ = [&2, 0, -1, -2, 0, 1, 2, 0, -1].\end{aligned}$$

This corresponds to the sixth row of our new character table, shown on the following page. Of course, $\sigma_k = \sigma_m$ for some nine pairs (k, m). For the above example of $m = 6$, $k = 6$ satisfies this. However, it is not the case that $k = m$ for every pair (k, m). One can see this by noting the difference between the rows corresponding to $k = 3$ and $m = 3$ in the two character tables. We now present the character table resulting from this method of construction.

	Cl_1Cl_1	Cl_1Cl_2	Cl_1Cl_3	Cl_2Cl_1	Cl_2Cl_2	Cl_2Cl_3	Cl_3Cl_1	Cl_3Cl_2	Cl_3Cl_3
$\chi_1 \cdot \chi_1 = \sigma_1$	1	1	1	1	1	1	1	1	1
$\chi_1 \cdot \chi_2 = \sigma_2$	1	-1	1	1	-1	1	1	-1	1
$\chi_1 \cdot \chi_3 = \sigma_3$	2	0	-1	2	0	-1	2	0	-1
$\chi_2 \cdot \chi_1 = \sigma_4$	1	1	1	-1	-1	-1	1	1	1
$\chi_2 \cdot \chi_2 = \sigma_5$	1	-1	1	-1	1	-1	1	-1	1
$\chi_2 \cdot \chi_3 = \sigma_6$	2	0	-1	-2	0	1	2	0	-1
$\chi_3 \cdot \chi_1 = \sigma_7$	2	2	2	0	0	0	-1	-1	-1
$\chi_3 \cdot \chi_2 = \sigma_8$	2	-2	2	0	0	0	-1	1	-1
$\chi_3 \cdot \chi_3 = \sigma_9$	4	0	-2	0	0	0	-2	0	1

This table yields the following M_{σ_m}:

$$M_{\sigma_1} = I_9, \quad \underbrace{\begin{bmatrix} 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_2}} \underbrace{\begin{bmatrix} 0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}}_{M_{\sigma_3}}$$

$$\underbrace{\begin{bmatrix} 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_4}} \underbrace{\begin{bmatrix} 0&0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}}_{M_{\sigma_5}} \underbrace{\begin{bmatrix} 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}}_{M_{\sigma_6}}$$

$$\underbrace{\begin{bmatrix} 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}}_{M_{\sigma_7}} \underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}}_{M_{\sigma_8}} \underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&1&0&0&1&0&0&1 \\ 0&0&1&0&0&1&0&0&1 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}}_{M_{\sigma_9}}$$

It is clear that each M_{σ_m} created using this character table is of the form $M_{\rho_i \otimes \rho_j}$ where $\sigma_m = \rho_i \otimes \rho_j$. Furthermore, $m = 3(i-1) + j$.

We now present our extension of Theorem 4.1.1.

Theorem 4.1.3. *If ρ and σ are representations of groups G_1, G_2 with adjacency matrices M_1, M_2 respectively, then $M_1 \otimes M_2$ is the adjacency matrix of $\rho \otimes \sigma : G_1 \times G_2 \longrightarrow GL(V_1 \times V_2)$ where $M_1 \otimes M_2$ denotes the Kronecker product of M_1, M_2.*

Proof. Let G_1, G_2 be groups with irreducible representations ρ_1, ρ_2, ..., ρ_n and σ_1, σ_2, ..., σ_m respectively. Then for each ρ_i and σ_j, their adjacency matrices are

$$M_{\rho_i} = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} & a_{(1,4)} & \cdots & a_{(1,n)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} & a_{(2,4)} & \cdots & a_{(2,n)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} & a_{(3,4)} & \cdots & a_{(3,n)} \\ \vdots & \ddots & & & \vdots & \\ a_{(n,1)} & a_{(n,2)} & a_{(n,3)} & a_{(n,4)} & \cdots & a_{(n,n)} \end{bmatrix}$$

and

$$M_{\sigma_j} = \begin{bmatrix} b_{(1,1)} & b_{(1,2)} & b_{(1,3)} & b_{(1,4)} & \cdots & b_{(1,m)} \\ b_{(2,1)} & b_{(2,2)} & b_{(2,3)} & b_{(2,4)} & \cdots & b_{(2,m)} \\ b_{(3,1)} & b_{(3,2)} & b_{(3,3)} & b_{(3,4)} & \cdots & b_{(3,m)} \\ \vdots & \ddots & & & \vdots & \\ b_{(m,1)} & b_{(m,2)} & b_{(m,3)} & b_{(m,4)} & \cdots & b_{(m,m)} \end{bmatrix}.$$

These yield the Kronecker product $M_{\rho_i} \otimes M_{\sigma_j}$ of dimension nm:

$$
\begin{bmatrix}
a_{(1,1)}b_{(1,1)} & a_{(1,1)}b_{(1,2)} & \cdots & a_{(1,1)}b_{(1,m)} & a_{(1,2)}b_{(1,1)} & a_{(1,2)}b_{(1,2)} & \cdots & a_{(1,2)}b_{(1,m)} & \cdots & a_{(1,n)}b_{(1,1)} & a_{(1,n)}b_{(1,2)} & \cdots & a_{(1,n)}b_{(1,m)} \\
a_{(1,1)}b_{(2,1)} & a_{(1,1)}b_{(2,2)} & \cdots & a_{(1,1)}b_{(2,m)} & a_{(1,2)}b_{(2,1)} & a_{(1,2)}b_{(2,2)} & \cdots & a_{(1,2)}b_{(2,m)} & \cdots & a_{(1,n)}b_{(2,1)} & a_{(1,n)}b_{(2,2)} & \cdots & a_{(1,n)}b_{(2,m)} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & & \vdots & \ddots & & \vdots \\
a_{(1,1)}b_{(m,1)} & a_{(1,1)}b_{(m,2)} & \cdots & a_{(1,1)}b_{(m,m)} & a_{(1,2)}b_{(m,1)} & a_{(1,2)}b_{(m,2)} & \cdots & a_{(1,2)}b_{(m,m)} & \cdots & a_{(1,n)}b_{(m,1)} & a_{(1,n)}b_{(m,2)} & \cdots & a_{(1,n)}b_{(m,m)} \\
a_{(2,1)}b_{(1,1)} & a_{(2,1)}b_{(1,2)} & \cdots & a_{(2,1)}b_{(1,m)} & a_{(2,2)}b_{(1,1)} & a_{(2,2)}b_{(1,2)} & \cdots & a_{(2,2)}b_{(1,m)} & \cdots & a_{(2,n)}b_{(1,1)} & a_{(2,n)}b_{(1,2)} & \cdots & a_{(2,n)}b_{(1,m)} \\
a_{(2,1)}b_{(2,1)} & a_{(2,1)}b_{(2,2)} & \cdots & a_{(2,1)}b_{(2,m)} & a_{(2,2)}b_{(2,1)} & a_{(2,2)}b_{(2,2)} & \cdots & a_{(2,2)}b_{(2,m)} & \cdots & a_{(2,n)}b_{(2,1)} & a_{(2,n)}b_{(2,2)} & \cdots & a_{(2,n)}b_{(2,m)} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & & \vdots & \ddots & & \vdots \\
a_{(2,1)}b_{(m,1)} & a_{(2,1)}b_{(m,2)} & \cdots & a_{(2,1)}b_{(m,m)} & a_{(2,2)}b_{(m,1)} & a_{(2,2)}b_{(m,2)} & \cdots & a_{(2,2)}b_{(m,m)} & \cdots & a_{(2,n)}b_{(m,1)} & a_{(2,n)}b_{(m,2)} & \cdots & a_{(2,n)}b_{(m,m)} \\
\vdots & & & \ddots & & & & & & \vdots & & & \\
a_{(n,1)}b_{(1,1)} & a_{(n,1)}b_{(1,2)} & \cdots & a_{(n,1)}b_{(1,m)} & a_{(n,2)}b_{(1,1)} & a_{(n,2)}b_{(1,2)} & \cdots & a_{(n,2)}b_{(1,m)} & \cdots & a_{(n,n)}b_{(1,1)} & a_{(n,n)}b_{(1,2)} & \cdots & a_{(n,n)}b_{(1,m)} \\
a_{(n,1)}b_{(2,1)} & a_{(n,1)}b_{(2,2)} & \cdots & a_{(n,1)}b_{(2,m)} & a_{(n,2)}b_{(2,1)} & a_{(n,2)}b_{(2,2)} & \cdots & a_{(n,2)}b_{(2,m)} & \cdots & a_{(n,n)}b_{(2,1)} & a_{(n,n)}b_{(2,2)} & \cdots & a_{(n,n)}b_{(2,m)} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & & \vdots & \ddots & & \vdots \\
a_{(n,1)}b_{(m,1)} & a_{(n,1)}b_{(m,2)} & \cdots & a_{(n,1)}b_{(m,m)} & a_{(n,2)}b_{(m,1)} & a_{(n,2)}b_{(m,2)} & \cdots & a_{(n,2)}b_{(m,m)} & \cdots & a_{(n,n)}b_{(m,1)} & a_{(n,n)}b_{(m,2)} & \cdots & a_{(n,n)}b_{(m,m)}
\end{bmatrix}
\tag{4.1.2}
$$

Figure 4.1.1. The expanded Kronecker product $M_{\rho_i} \otimes M_{\sigma_j}$.

By Theorem 4.1.1, $\rho_1 \otimes \sigma_1$, $\rho_1 \otimes \sigma_2$, ..., $\rho_1 \otimes \sigma_m$, $\rho_2 \otimes \sigma_1$, ..., $\rho_2 \otimes \sigma_m$, ..., $\rho_n \otimes \sigma_1$, ..., $\rho_n \otimes \sigma_m$ are irreducible representations of $G_1 \times G_2$. Call these τ_1, ..., τ_{nm} and choose some τ_α, τ_β. Then $\tau_\alpha = \rho_i \otimes \sigma_j$ and $\tau_\beta = \rho_s \otimes \sigma_t$ for $1 \leq i, s \leq n$, $1 \leq j, t \leq m$. Notice that $\alpha = im + j$ and $\beta = sm + t$. Consider M_{τ_α} and the tensor product $\tau_\alpha \otimes \tau_\beta = \sum_{\lambda=1}^{nm} c_{(\beta,\lambda)} \tau_\lambda$. Then the βth row of M_{τ_α} will be the horizontal vector $[c_{(sm+t,1)}, c_{(sm+t,2)}, ..., c_{(sm+t,nm)}]$, so M_{τ_α} is of the following form:

$$M_{\tau_\alpha} = \begin{bmatrix} c_{(1,1)} & c_{(1,2)} & \cdots & c_{(1,nm)} \\ c_{(2,1)} & c_{(2,2)} & \cdots & c_{(2,nm)} \\ \vdots & \ddots & & \vdots \\ c_{(nm,1)} & c_{(nm,2)} & \cdots & c_{(nm,nm)} \end{bmatrix}.$$

Note that $\lambda = qm + u$ for $1 \leq q \leq n$ and $1 \leq u \leq m$. Then $c_{(\beta,\lambda)} = c_{(sm+t,qm+u)}$. Our goal is to show $c_{(sm+t,qm+u)} = a_{(s,q)} b_{(t,u)}$.

Observe that

$$\begin{aligned}
\tau_\alpha \otimes \tau_\beta &= \tau_{im+j} \otimes \tau_{sm+t} \\
&= (\rho_i \otimes \sigma_j) \otimes (\rho_s \otimes \sigma_t) \\
&= (\rho_i \otimes \rho_s) \otimes (\sigma_j \otimes \sigma_t) \\
&= \Big(\sum_{k=1}^{n} a_{(s,k)} \rho_k\Big) \otimes \Big(\sum_{r=1}^{m} b_{(t,r)} \sigma_r\Big) \\
&= (a_{(s,1)}\rho_1 \oplus a_{(s,2)}\rho_2 \oplus ...a_{(s,n)}\rho_n) \otimes (b_{(t,1)}\sigma_1 \oplus b_{(t,2)}\sigma_2 \oplus ...b_{(t,m)}\sigma_m) \\
&= a_{(s,1)}\rho_1 \otimes (b_{(t,1)}\sigma_1 \oplus b_{(t,2)}\sigma_2 \oplus ...b_{(t,m)}\sigma_m) \\
&\quad \oplus a_{(s,2)}\rho_2 \otimes (b_{(t,1)}\sigma_1 \oplus b_{(t,2)}\sigma_2 \oplus ...b_{(t,m)}\sigma_m) \\
&\quad \vdots \\
&\quad \oplus a_{(s,n)}\rho_n \otimes (b_{(t,1)}\sigma_1 \oplus b_{(t,2)}\sigma_2 \oplus ...b_{(t,m)}\sigma_m)
\end{aligned}$$

$$\begin{aligned}
&= a_{(s,1)}b_{(t,1)}(\rho_1 \otimes \sigma_1) \oplus a_{(s,1)}b_{(t,2)}(\rho_1 \otimes \sigma_2) \oplus ... \oplus a_{(s,1)}b_{(t,m)}(\rho_1 \otimes \sigma_m) \\
&\quad \oplus a_{(s,2)}b_{(t,1)}(\rho_2 \otimes \sigma_1) \oplus a_{(s,2)}b_{(t,2)}(\rho_2 \otimes \sigma_2) \oplus ... \oplus a_{(s,2)}b_{(t,m)}(\rho_2 \otimes \sigma_m) \\
&\qquad \vdots \\
&\quad \oplus a_{(s,n)}b_{(t,1)}(\rho_n \otimes \sigma_1) \oplus a_{(s,n)}b_{(t,2)}(\rho_n \otimes \sigma_2) \oplus ... \oplus a_{(s,n)}b_{(t,m)}(\rho_n \otimes \sigma_m) \\
&= a_{(s,1)}b_{(t,1)}\tau_1 \oplus a_{(s,1)}b_{(t,2)}\tau_2 \oplus ... \oplus a_{(s,1)}b_{(t,m)}\tau_m \\
&\quad \oplus a_{(s,2)}b_{(t,1)}\tau_{m+1} \oplus a_{(s,2)}b_{(t,2)}\tau_{m+2} \oplus ...a_{(s,2)}b_{(t,2)}\tau_{2m} \\
&\qquad \vdots \\
&\quad \oplus a_{(s,n)}b_{(t,1)}\tau_{(n-1)m+1} \oplus a_{(s,n)}b_{(t,2)}\tau_{(n-1)m+2}... \oplus a_{(s,n)}b_{(t,m)}\tau_{nm}.
\end{aligned}$$

Then the $\beta = (sm+t)$th row of M_{τ_α} is the horizontal vector $[a_{(s,1)}b_{(t,1)}, a_{(s,1)}b_{(t,2)}, ..., a_{(s,1)}b_{(t,m)}, a_{(s,2)}b_{(t,1)}, a_{(s,2)}b_{(t,2)}, ..., a_{(s,2)}b_{(t,m)}, ..., a_{(s,n)}b_{(t,1)}, a_{(s,n)}b_{(t,2)}, ..., a_{(s,n)}b_{(t,m)})]$. Since the βth row of M_{τ_α} was defined as $[c_{(sm+t,1)}, c_{(sm+t,2)}, ..., c_{(sm+t,nm)}]$, it must be the case that $c_{(sm+t,qm+u)} = a_{(s,q)}b_{(t,u)}$. This completes the proof. □

Definition 4.1.4. Let G, H be two graphs. Then their **tensor product** $G \times' H$ is defined as follows:

1. $V(G \times H) = V(G) \times V(H)$ where $\times$ is the Cartesian product and

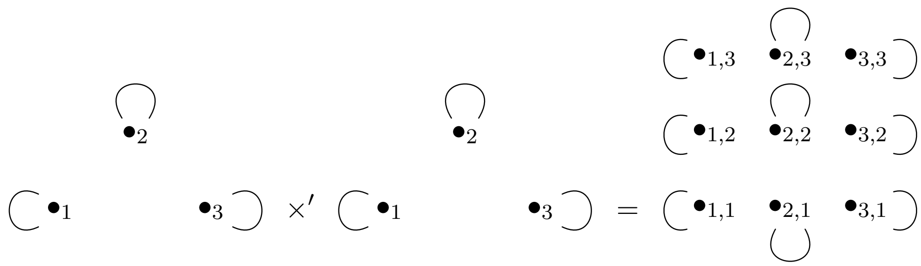
2. There exists an edge between $(u, u'), (v, v') \in V(G \times H)$ if
   (a) u is adjacent to v in G and

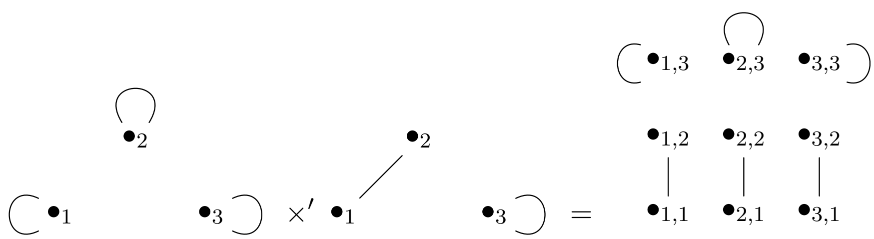
   (b) u' is adjacent to v' in H.

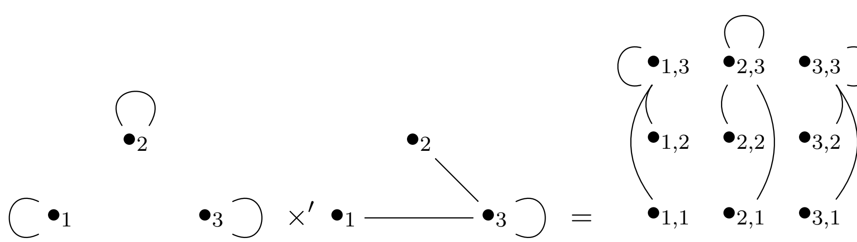
△

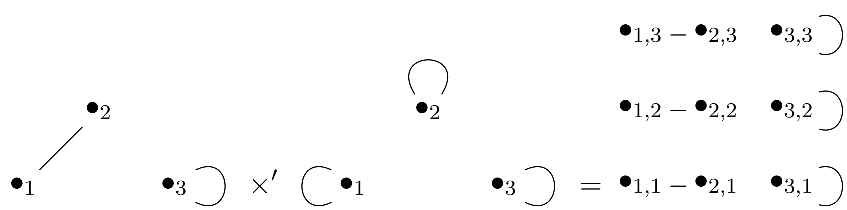
Corollary 4.1.5. *If Γ_1, Γ_2 are the McKay graphs of representations ρ, σ of G_1, G_2 respectively, then $\Gamma_1 \times' \Gamma_2$ is the McKay graph of $\rho \otimes \sigma$.*

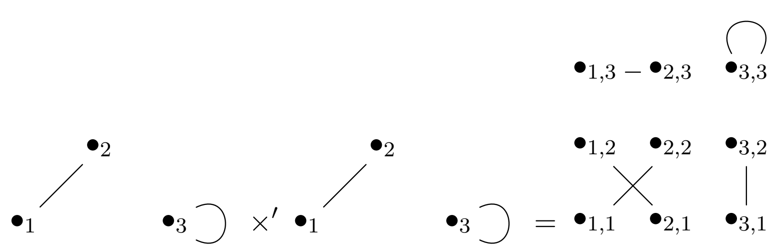
Example 4.1.6. We demonstrate the above using $S_3 \times S_3$ once more. Let us compute the nine $\Gamma_{\rho_i} \times' \Gamma_{\rho_j}$ for the McKay graphs of S_3.

$\Gamma_{\rho_1} \times' \Gamma_{\rho_1} =$ 

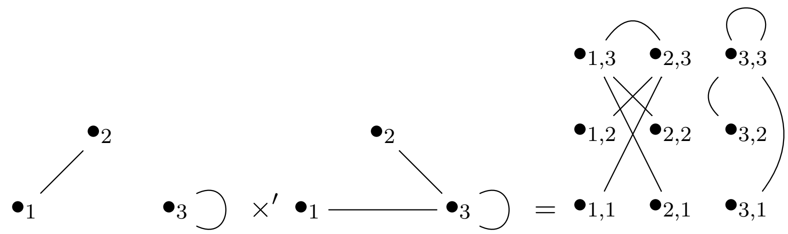
$\Gamma_{\rho_1} \times' \Gamma_{\rho_2} =$ 

$\Gamma_{\rho_1} \times' \Gamma_{\rho_3} =$ 

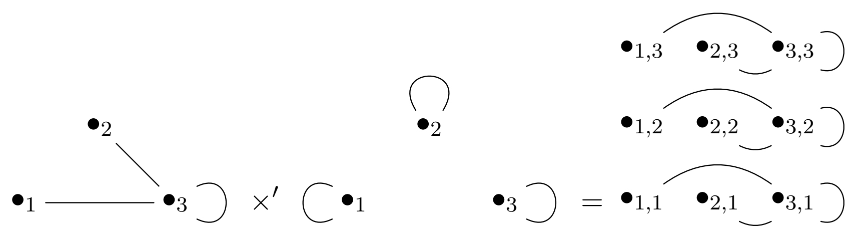
$\Gamma_{\rho_2} \times' \Gamma_{\rho_1} =$ 

$\Gamma_{\rho_2} \times' \Gamma_{\rho_2} =$ 

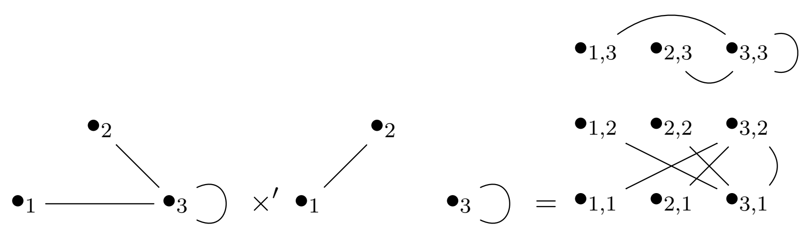
$\Gamma_{\rho_2} \times' \Gamma_{\rho_3} =$



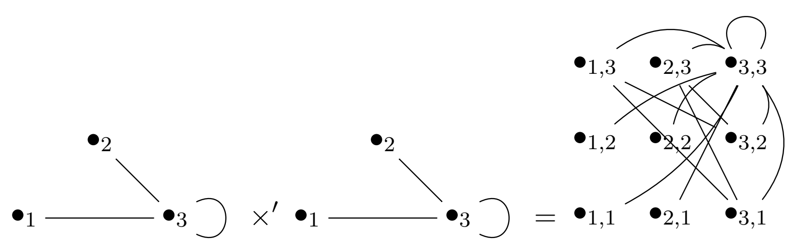
$\Gamma_{\rho_3} \times' \Gamma_{\rho_1} =$



$\Gamma_{\rho_3} \times' \Gamma_{\rho_2} =$

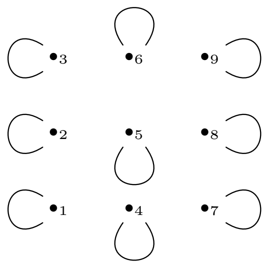


$\Gamma_{\rho_3} \times' \Gamma_{\rho_3} =$



We will compare these graph tensor products to Γ_{σ_m}, constructed from M_{σ_m} below.

$M_{\sigma_1} = I_9 \rightarrow$

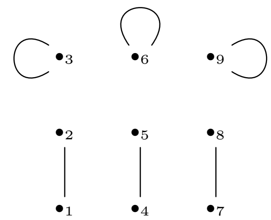


Γ_1

,

$$\begin{bmatrix}
0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1
\end{bmatrix} \rightarrow$$

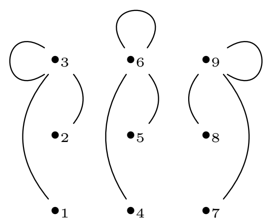
M_{σ_2}



Γ_{σ_2}

$$\begin{bmatrix}
0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&1
\end{bmatrix} \rightarrow$$

M_{σ_3}

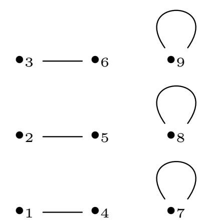


Γ_{σ_3}

,

$$\begin{bmatrix}
0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1
\end{bmatrix} \rightarrow$$

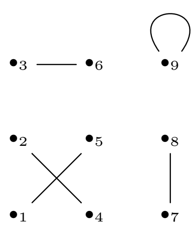
M_{σ_4}



Γ_{σ_4}

$$\begin{bmatrix}
0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1
\end{bmatrix} \rightarrow$$

M_{σ_5}

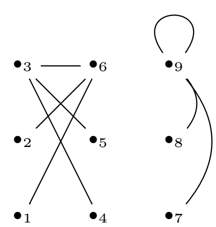


Γ_{σ_5}

,

$$\begin{bmatrix}
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&1&1&1&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&1
\end{bmatrix} \rightarrow$$

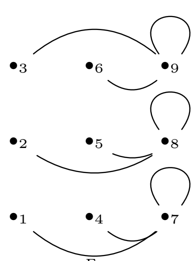
M_{σ_6}



Γ_{σ_6}

$$\begin{bmatrix}
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1
\end{bmatrix} \rightarrow$$

M_{σ_7}

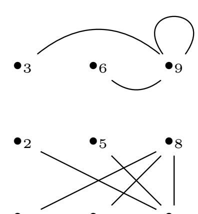


Γ_{σ_7}

,

$$\begin{bmatrix}
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1\\
0&1&0&0&1&0&0&1&0\\
1&0&0&1&0&0&1&0&0\\
0&0&1&0&0&1&0&0&1
\end{bmatrix} \rightarrow$$

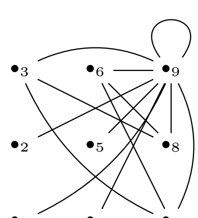
M_{σ_8}



Γ_{σ_8}

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&1\\
0&0&1&0&0&1&0&0&1\\
0&0&1&0&0&1&0&0&1\\
1&1&1&1&1&1&1&1&1
\end{bmatrix} \rightarrow$$

M_{σ_9}



Γ_{σ_9}

As expected, $\Gamma_{\sigma_m} = \Gamma_{\rho_i} \times' \Gamma_{\rho_j}$ for each m. In fact, we have $m = 3(i-1) + j$.

4.2 Dihedral Groups

A family of groups is unified by some underlying group structure. Since representations of a group rely on its group structure, it is implied that families of groups will have some underlying representation structure as well.

In this section, we will focus on the family of dihedral groups, D_n. There exist formulas for their character tables, which will be presented over the next few pages. Our goal is to extend these in a way that allows us to generate the McKay graphs of D_n. First, let us discuss its conjugacy classes and representations.

For D_n where n is odd, there are $\frac{n-1}{2} + 2$ conjugacy classes: $\{1\}, \{s, rs, r^2s, ..., r^{n-1}s\}$, $\{r, r^{n-1}\}, \{r^2, r^{n-2}\}, \ldots,$ and $\{r^{\frac{n-1}{2}}, r^{\frac{n-1}{2}+2}\}$. Then D_n has $\frac{n-1}{2} + 2$ irreducible representations. In particular, there are two irreducible representations of degree 1 and $\frac{n-1}{2}$ irreducible representations of degree 2. We discuss the group's $\frac{n-1}{2} + 2$ representations in terms of where each sends $g \in Cl_i$, a representative of a given conjugacy class. We call the one dimensional representations of D_{odd} χ_1 and χ_2, where χ_1 is trivial. We have $\chi_2(s) = -1$ and for $g \neq s$, $\chi_2(g) = 1$. From this, one can see that $\Im(\chi_2) \simeq \{\pm 1\} \subset GL_1(\mathbf{C})$. Let us denote the irreducible representations of degree 2 as $R_1, R_2, ..., R_{\frac{n-1}{2}}$. We have seen one such R in Example 2.2.15, from which we know $\chi_{R_j}(1) = 2, \chi_{R_j}(s) = 0$. These R_j are uniquely determined by where each sends the conjugacy class of r, so we say R_j the representation for which $\mathrm{Tr}(R_j(r)) = \omega^j + \omega^{-j}$ with $\omega = e^{\frac{2\pi i}{n}}$. Using the above notation, we present the character table for D_{odd}.

	1	s	r	r^2	$\dots$	$r^{\frac{n-1}{2}}$
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1
χ_{R_1}	2	0	$\omega+\omega^{-1}$	$\omega^2+\omega^{-2}$	$\dots$	$\omega^{\frac{n-1}{2}}+\omega^{-(\frac{n-1}{2})}$
χ_{R_2}	2	0	$\omega^2+\omega^{-2}$	$\omega^4+\omega^{-4}$	$\dots$	$\omega^{n-1}+\omega^{-(n-1)}$
χ_{R_3}	2	0	$\omega^3+\omega^{-3}$	$\omega^6+\omega^{-6}$	$\dots$	$\omega^{3(\frac{n-1}{2})}+\omega^{-3(\frac{n-1}{2})}$
$\vdots$				$\ddots$		
$\chi_{R_{\frac{n-1}{2}}}$	2	0	$\omega^{\frac{n-1}{2}}+\omega^{-(\frac{n-1}{2})}$	$\omega^{2(\frac{n-1}{2})}+\omega^{-2(\frac{n-1}{2})}$	$\dots$	$\omega^{\frac{n-1}{2}(\frac{n-1}{2})}+\omega^{-(\frac{n-1}{2})(\frac{n-1}{2})}$

It is important to note how D_n for n of different parities are related. If m, n are integers such that $m|n$, then there exists a subgroup $H \subseteq D_n$ such that $H \simeq D_m$. This becomes clear when we envision an inscribed $m-$gon within an n-gon. Consider D_{15}, for example. We can inscribe a regular 3-gon and a regular 5-gon inside a regular 15-gon. For $n = 2m$, it will always be the case that $D_m \subset D_n$. In fact, one can find the character table of D_m within that of D_n. However, the tables will not be the same since $Z(D_n)$ is nontrivial. This alters its conjugacy classes and thus its representations. Let us go into more detail.

When n is even, D_n has $\frac{n}{2}+3$ conjugacy classes. These are $\{1\}$, $\{s, r^2s, r^4s, ...\}$, $\{rs, r^3s, ...\}$, $\{r, r^{n-1}\}$, $\{r^2, r^{n-2}\}$, ..., $\{r^{\frac{n}{2}-1}, r^{\frac{n}{2}+1}\}$, and $\{r^{\frac{n}{2}}\}$. Then there are $\frac{n}{2}+3$ irreducible representations, four of which are 1-dimensional. Call them $\chi_1, \chi_2, \chi_3, \chi_4$ where χ_1 is the trivial representation and χ_2 is defined by $\chi_2(s) = \chi_2(rs) = -1$. The remaining irreducibles of degree 1 function similarly to each other. Namely, we have $\chi_3(g) = \chi_4(g)$ for $g \neq s, rs$ and $\chi_3(g) = -\chi_4(g)$ otherwise. Then their labeling is arbitrary. For the sake of consistency, we will say $\chi_3(s) = 1$ and $\chi_3(rs) = -1$, thus fixing $\chi_4(s) = -1$ and $\chi_4(rs) = 1$. Then $\chi_3(r) = \chi_4(r) = -1$, which means $\chi_{\{3,4\}}(r^{\frac{n}{2}})$ is determined by whether or not $n \equiv 0 \pmod 4$. If so, we have $\chi_{\{3,4\}}(r^{\frac{n}{2}}) = \chi_{\{3,4\}}(r^{even}) = (\chi_{\{3,4\}}(r))^{even} = (-1)^{even} = 1$. On the other hand, $n \equiv 2 \pmod 4$ implies $\frac{n}{2}$ is odd, so $\chi_{\{3,4\}}(r^{\frac{n}{2}}) = -1$ by similar reasoning. As with D_{odd}, we have $\Im(\chi_{\{i\neq 1\}}) \simeq \{\pm 1\} \in GL_1(\mathbf{C})$ for D_{even} as well.

We now move on to the remaining $\frac{n}{2}-1$ representations, all of which have degree 2. As with D_n for odd n, let us call them $R_1, R_2, ..., R_{\frac{n}{2}-1}$. Once again, we define R_j such that

$\text{Tr}(R_j(r)) = \omega^j + \omega^{-j}$ where $\omega = e^{\frac{2\pi i}{n}}$. The general character table of D_{even} is presented below.

	1	s	rs	r	r^2	$\dots$	$r^{\frac{n}{2}-1}$	$r^{\frac{n}{2}}$
χ_1	1	1	1	1	1	$\dots$	1	1
χ_2	1	-1	-1	1	1	$\dots$	1	1
χ_3	1	1	-1	-1	1	$\dots$	$-\chi_3(r^{\frac{n}{2}})$	± 1
χ_4	1	-1	1	-1	1	$\dots$	$-\chi_3(r^{\frac{n}{2}})$	$\chi_3(r^{\frac{n}{2}})$
χ_{R_1}	2	0	0	$\omega + \omega^{-1}$	$\omega^2 + \omega^{-2}$	$\dots$	$\omega^{\frac{n}{2}-1} + \omega^{-(\frac{n}{2}-1)}$	$\omega^{\frac{n}{2}} + \omega^{-\frac{n}{2}} = -2$
χ_{R_2}	2	0	0	$\omega^2 + \omega^{-2}$	$\omega^4 + \omega^{-4}$	$\dots$	$\omega^{2(\frac{n}{2}-1)} + \omega^{-2(\frac{n}{2}-1)}$	$\omega^{2(\frac{n}{2})} + \omega^{-2(\frac{n}{2})} = 2$
χ_{R_3}	2	0	0	$\omega^3 + \omega^{-3}$	$\omega^6 + \omega^{-6}$	$\dots$	$\omega^{3(\frac{n}{2}-1)} + \omega^{-3(\frac{n}{2}-1)}$	$\omega^{3(\frac{n}{2})} + \omega^{-3(\frac{n}{2})} = -2$
$\vdots$	$\vdots$					$\ddots$		
$\chi_{R_{\frac{n}{2}-1}}$	2	0	0	$\omega^{\frac{n}{2}-1} + \omega^{-(\frac{n}{2}-1)}$	$\omega^{n-2} + \omega^{-(n-2)}$	$\dots$	$\omega^{(\frac{n}{2}-1)^2} + \omega^{-[(\frac{n}{2}-1)^2]}$	$\omega^{\frac{n}{2}\cdot(\frac{n}{2}-1)} + \omega^{-\frac{n}{2}\cdot(\frac{n}{2}-1)} = \pm 2$

With regards to the last representation of degree 2, we have $\chi_{R_{\frac{n}{2}-1}}(r^{\frac{n}{2}}) = -2, 2$ for $n \equiv 0, 2$ (mod 4), respectively.

Now that the reader is familiar with the general character tables of D_n, we can present our derived tensor product decompositions. From these, we will generate the adjacency matrices of each group's McKay graphs. There is significant overlap in the decompositions for odd and even n, so we present the two together. One will be able to distinguish which cases apply to which parity by following the subscripts of the characters. For the more complicated decompositions, we show that each holds for the group generators s and r rather than iterate over each conjugacy class.

1. (trivial) $\chi_1 \cdot \chi_k = \chi_k$ and $\chi_1 \cdot \chi_{R_j} = \chi_{R_j}$ by Theorem 2.1.16.

2. $\chi_k \cdot \chi_k = \chi_1$ since $(\pm 1)^2 = 1$.

3. $\chi_2 \cdot \chi_{R_j} = \chi_{R_j}$. One can see from either character table that for nontrivial $\chi_2(g)$, we have $\chi_{R_j}(g) = 0$. More explicitly, either $\chi_2(g) \cdot \chi_{R_j}(g) = (1) \cdot \chi_{R_j}(g) = \chi_{R_j}(g)$, or $\chi_2(g) \cdot \chi_{R_j}(g) = (-1) \cdot 0 = \chi_{R_j}(g)$.

4. $\chi_2 \cdot \chi_3 = \chi_4$. As with the following two decompositions, this is fairly obvious from looking at the character table. We show this for $s, r \in D_n$. See that $\chi_2(s) \cdot \chi_3(s) = -1 \cdot 1 = -1 = \chi_4(s)$ and $\chi_2(r) \cdot \chi_3(r) = 1 \cdot (-1) = -1 = \chi_4(r)$.

5. $\chi_2 \cdot \chi_4 = \chi_3$. Similarly, $\chi_2(s) \cdot \chi_4(s) = -1 \cdot -1 = 1 = \chi_3(s)$ and $\chi_2(r) \cdot (\chi_4(r) = 1 \cdot -1 = -1 = \chi_3(r)$.

6. $\chi_3 \cdot \chi_4 = \chi_2$. Again, $\chi_3(s) \cdot \chi_4(s) = 1 \cdot -1 = -1 = \chi_2(s)$ and $\chi_3(r) \cdot \chi_4(r) = -1 \cdot -1 = 1 \chi_2(r)$.

7. $\chi_3 \cdot \chi_{R_j} = \chi_4 \cdot \chi_{R_j} = \chi_{R_{\frac{n}{2}-j}}$. First note that n is even since we discuss χ_3, χ_4. Then $-1 = \omega^{\frac{n}{2}}$, so we can rewrite $\chi_{\{3,4\}}(r) \cdot (\chi_{R_j}(r)) = (-1) \cdot (\omega^j + \omega^{-j}) = \omega^{(\frac{n}{2}+j)} + \omega^{\frac{n}{2}-j} = \omega^{-(\frac{n}{2}-j)} + \omega^{\frac{n}{2}-j} = \chi_{R_{\frac{n}{2}-j}}(r)$. For $g = s, rs$ we have $\chi_{R_j}(g) = 0$ and $\pm 1 \cdot 0 = 0$.

8. For $\ell = \frac{n}{4}$, $\chi_{R_\ell} \cdot \chi_{R_\ell} = \chi_1 + \chi_2 + \chi_3 + \chi_4$. The requirement on ℓ implies $n \equiv 0 \pmod 4$, which means its roots of unity are symmetric about both the real and imaginary axes. In other words, $\omega^{\frac{n}{4}} = -\omega^{-\frac{n}{4}}$. See that $\chi_{R_\ell} = [2, 0, 0, \omega^{\frac{n}{4}} + \omega^{-\frac{n}{4}} = 0, \omega^{2(\frac{n}{4})} + \omega^{-2(\frac{n}{4})} = -2, \omega^{3(\frac{n}{4})} + \omega^{-3(\frac{n}{4})} = 0, ..., \omega^{(\frac{n}{2})\frac{n}{4}} + \omega^{-(\frac{n}{2})\frac{n}{4}} = 2] = [2, 0, 0, 0, -2, 0, 2, ..., 2]$. Then $\chi_{R_\ell} \cdot \chi_{R_\ell} = [4, 0, 0, 0, 4, 0, 4, ..., 4] = \chi_1 + \chi_2 + \chi_3 + \chi_4$.

9. For $\ell \neq \lfloor \frac{n}{4} \rfloor$, $\chi_{R_\ell} \cdot \chi_{R_\ell} = \chi_1 + \chi_2 + \chi_{R_k}$ where $k = \begin{cases} 2\ell & \ell < \frac{n}{4} \\ n - 2\ell & \ell > \frac{n}{4} \end{cases}$.
Consider $\chi_{R_\ell}(r) = \omega^\ell + \omega^{-\ell}$. See that $\chi_{R_\ell}(r) \cdot \chi_{R_\ell} = (\omega^\ell + \omega^{-\ell})^2 = \omega^{2\ell} + \omega^0 + \omega^0 + \omega^{-2\ell} = 2 + \omega^{2\ell} + \omega^{-2\ell}$ which by definition equals $\chi_1(r) + \chi_2(r) + \chi_{R_{2\ell}}(r)$ for $\ell < \frac{n}{4}$. For larger ℓ, this follows from the fact that we can rewrite

$$\begin{aligned} \omega^{2\ell} + \omega^{-2\ell} &= \omega^{2\ell-n+n} + \omega^{-2\ell+n-n} \\ &= \omega^{2\ell-n} \cdot \omega^n + \omega^{n-2l} \cdot \omega^{-n} \\ &= \omega^{2\ell-n}(1) + \omega^{n-2\ell}(1) \\ &= \omega^{-(n-2\ell)} + \omega^{n-2\ell}, \text{ and so we have} \\ &= \chi_{R_{n-2\ell}}(r) \end{aligned}$$

as desired. For $g = s, rs$, see that $(\chi_{R_\ell}(g))^2 = 0^2 = 0 = 1 - 1 + 0 = \chi_1(g) + \chi_2(g) + \chi_{R_k}(g)$.

10. Without loss of generality, let $\ell > j$. Then for $\ell + j = \frac{n}{2}$, $\chi_{R_j} \cdot \chi_{R_\ell} = \chi_3 + \chi_4 + \chi_{R_{\ell-j}}$. See that $\chi_{R_j}(r) \cdot \chi_{R_\ell}(r) =$

$$\begin{aligned}(\omega^\ell + \omega^\ell)(\omega^j + \omega^{-j}) &= \omega^{\ell+j} + \omega^{-\ell+j} + \omega^{\ell-j} + \omega^{-\ell-j} \\ &= \omega^{\frac{n}{2}} + \omega^{-(\ell-j)} + \omega^{\ell-j} + \omega^{-\frac{n}{2}} \\ &= -1 + \omega^{\ell-j} + \omega^{-(\ell-j)} + (-1) \\ &= \chi_3(r) + \chi_{R_{\ell-j}}(r) + \chi_4(r).\end{aligned}$$

For $g = s, rs$, the reasoning from (9) applies since $\chi_3(g) = -\chi_4(g)$.

11. Otherwise, $\chi_{R_j} \cdot \chi_{R_\ell} = \chi_{R_{\ell-j}} + \chi_{R_k}$ where $k = \begin{cases} \ell + j & \ell + j < \frac{n}{2} \\ n - (\ell + j) & \ell + j > \frac{n}{2} \end{cases}$. Continuing from line 1 of the equations in (10), we can see that $\omega^{\ell+j} + \omega^{-\ell+j} + \omega^{\ell-j} + \omega^{-\ell-j} = \omega^{\ell+j} + \omega^{-(\ell+j)} + \omega^{\ell-j} + \omega^{-(\ell-j)}$, which suffices for $\ell + j < \frac{n}{2}$. For $\ell + j > \frac{n}{2}$, this simplifies further:

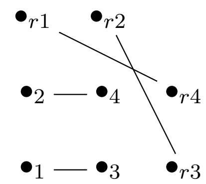
$$\begin{aligned}\omega^{\ell+j} + \omega^{-(\ell+j)} + \omega^{\ell-j} + \omega^{-(\ell-j)} &= \omega^{\ell+j-n+n} + \omega^{-(\ell+j)+n-n} + \chi_{R_{\ell-j}}(r) \\ &= \omega^{-(n-(\ell+j))}\omega^n + \omega^{n-(\ell+j)}\omega^{-n} + \chi_{R_{\ell-j}}(r) \\ &= \omega^{-(n-(\ell+j))} + \omega^{n-(\ell+j)} + \chi_{R_{\ell-j}}(r) \\ &= \chi_{R_{n-(\ell+j)}}(r) + \chi_{R_{\ell-j}}(r)\end{aligned}$$

as needed. For $g = s, rs$ we have $\chi_{R_j}(g) \cdot \chi_{R_\ell}(g) = 0^2 = 0 + 0 = \chi_{R_k}(g) + \chi_{R_{\ell-j}}(g)$.

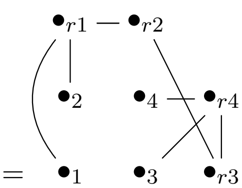
This lets us generate M_ρ of D_n for any reprsentation ρ. From this, we can create Γ_ρ. Below are several examples for varying n and ρ.

Example 4.2.1. Let $G = D_{10}$. We create M_{χ_3} and M_{R_1} using our formulas and construct $\Gamma_{\chi_3}, \Gamma_{R_1}$. The vertices corresponding to χ_i, χ_{R_j} will be labeled i, rj.

$$M_{\chi_3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \Gamma_{\chi_3} =$$



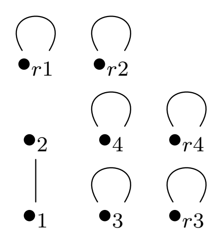
$$M_{R_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \Gamma_{\chi_3} =$$



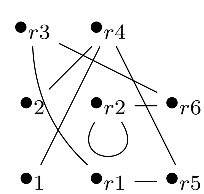
Let us now look at D_n where n is odd.

Example 4.2.2. Consider $G = D_{13}$ and let us compute M_{χ_2}, M_{R_4} with their respective graphs. Notation is similar to the graphs of D_{10}.

$$M_{\chi_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \Gamma_{\chi_2} =$$



$$M_{R_4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \Gamma_{R_4} =$$

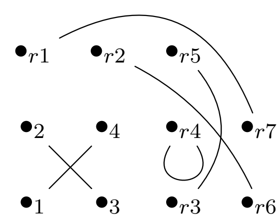


As one can see, Γ_{χ_2} is fairly simple. This is because it acts as the identity on $\rho \neq \chi_{\{1,2\}}$, which will be true for all D_n.

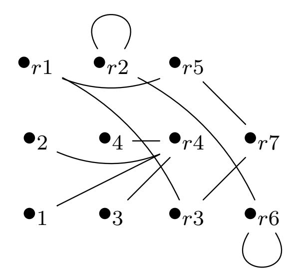
We now provide another example of D_{even}, this time for $n \equiv 0 \pmod 4$.

Example 4.2.3. For D_{16}, we create $M_{\chi_4}, M_{R_4}, M_{R_7}$ with their corresponding graphs.

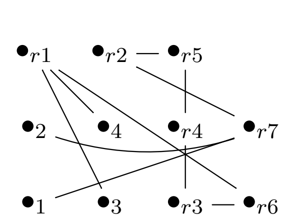
$$M_{\chi_4} = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0 \end{bmatrix} \to \Gamma_{\chi_4} =$$



$$M_{R_4} = \begin{bmatrix} 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 1&1&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0 \end{bmatrix} \to \Gamma_{\chi_4} =$$



$$M_{R_7} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 1&1&0&0&0&1&0&0&0&0&0 \end{bmatrix} \to \Gamma_{\chi_4} =$$



These examples show several patterns in the McKay graphs of general D_n. More specifically,

1. Γ_{χ_2} will have a single nontrivial edge between χ_1, χ_2 and self-loops on all other vertices.

2. For even n: if $n \equiv 2 \pmod 4$, $\Gamma_{\chi_{\{3,4\}}}$ will have $\frac{\frac{n}{2}+3}{2}$ pieces, each consisting of two vertices connected by an edge. Otherwise, we have $n \equiv 0 \pmod 4$, which yields $\Gamma_{\chi_{\{3,4\}}}$ with $\frac{\frac{n-1}{2}+3}{2}+1$ pieces. More specifically, $\frac{\frac{n-1}{2}+3}{2}$ of them will have two vertices each, while the last piece corresponds to $R_{\frac{n}{4}}$ with a self-loop.

3. Γ_{R_j} is connected for all R_j with the following edge patterns for a given vertex v:

   (a) If v corresponds to χ_i, v has exactly one edge.

   (b) Else for $v = R_k$, v has m edges where $2 \leq m \leq 4$.

In actuality, we know which vertices are connected for every Γ_ρ. Recall that there exists an edge between v_1, v_2 if v_1 appears in the decomposition of $\chi_\rho \cdot v_2$. Since we have formulas for the decompositions of all ρ of any D_n, we can produce identical ones for the group's McKay graphs. In order to avoid redundancy, we chose to highlight structural patterns of the graphs instead.

4.3 The Steinberg Representation of $\mathbf{SL}_2(p)$

In addition to D_n, we discuss another family of groups, $\mathbf{SL}_2(p)$. These groups not only have a well-known character table of ordinary representations, but also have a good theory of reduction $\pmod p$ which will be discussed in the next chapter. By definition, $\mathbf{SL}_2(p)$ is the special linear group of 2×2 matrices of determinant 1 with entries from $\mathbf{F}_p$.

We now discuss the $p+4$ conjugacy classes of $\mathbf{SL}_2(p)$ where 4 is interpreted as $|Z(\mathbf{SL}_2(p))|^2$. These are $\{1\}, \{z\}, \{a\}, \{a^2\}, ..., \{a^\ell\}, \{b\}, \{b^2\}, ..., \{b^m\}, \{c\}, \{d, d^2, ..., d^{(p-1)}\}$,

$\{zc\}, \{zd, zd^2, ..., zd^{(p-1)}\}$ for $\ell = {}^{(p-3)}/_2, m = {}^{(p-1)}/_2$ (or $m = 1$ for $p = 2$), $1 = I_2, z = -I_2, a = \left[\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right]$, b is some element of order $p+1$ that is undiagonalizable over $\mathbf{F}_p, c = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, and $d = \left[\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right]$ where v generates the multiplicative group of $\mathbf{F}_p$.[3, 28]

The character table for general $\mathbf{SL}_2(p)$ is given below, adhering to the original notation from [3, 30]. We omit the conjugacy classes zc, zd for the sake of brevity. For any representation ρ, it is the case that $\chi_\rho(zc) = \chi_\rho(z)(\chi_\rho(1))^{-1}\chi_\rho(c)$ - this holds for zd as well.

	1	z	a^ℓ	b^m	c	d
χ_1	1	1	1	1	1	1
χ_ψ	p	p	1	-1	0	0
χ_{ζ_i}	$p+1$	$(-1)^i(p+1)$	$\tau^{i\ell}+\tau^{-i\ell}$	0	1	1
χ_{ξ_1}	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^\ell$	0	$\frac{1}{2}(1+\sqrt{\varepsilon p})$	$\frac{1}{2}(1-\sqrt{\varepsilon p})$
χ_{ξ_2}	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^\ell$	0	$\frac{1}{2}(1-\sqrt{\varepsilon p})$	$\frac{1}{2}(1+\sqrt{\varepsilon p})$
χ_{θ_j}	$p-1$	$(-1)^j(p-1)$	0	$-(\sigma^{jm}+\sigma^{-jm})$	-1	-1
χ_{η_1}	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$
χ_{η_2}	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$

Figure 4.3.1. General Character table of $\mathbf{SL}_2(p)$

Here, $1 \leq i \leq \frac{(p-3)}{2}, 1 \leq j \leq \frac{(p-1)}{2}, \varepsilon = (-1)^{(p-1)/2}, \tau$ is a primitive $(p-1)$th root of unity, and σ is a primitive $(p+1)$th root of unity.

Example 4.3.1. We will illustrate the above formulas through the character table of $G = \mathbf{SL}_2(2) \simeq S_3$. We replicate the familiar character table below.

	()	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Notice that $\dim(\chi_2) = 1 = p-1$ and $\dim(\chi_3) = 2 = p$. Since we are in $\mathbf{F}_2$, $z = 1$ and $c = d$. Because of how small our p is, a^ℓ does not arise. The conjugacy class of (123) corresponds to b since $|(123)| = 3 = p+1$, and the conjugacy class of (12) is that of c. Then clearly $\chi_2((12)) = -1 = \chi_{\theta_j}(c)$. See that σ is a 3rd root of unity, which means $\chi_{\theta_j}(b) = -(\sigma + \sigma^{-1}) = -(-1) = 1 = \chi_2((123))$. Similarly, $\chi_3((12)) = \chi_\psi(c)$ and $\chi_3((123)) = -1 = \chi_\psi(b)$. Thus the general character table holds for $\mathbf{SL}_2(2)$.

For any p, the group $\mathbf{SL}_2(p)$ has a particular representation called the **Steinberg representation**. It is one of the most important representations in Lie theory and is fundamental in the theory of reductive groups. For details, see [4]. It can be identified as the only representation of dimension p and corresponds to ψ in the general character table of $\mathbf{SL}_2(p)$. If one wants to construct it by hand, it is $\mathbf{Sym}^{p-1}(\rho)$; the $(p-1)$th symmetric power representation where ρ is the standard representation defined by $\rho(g) = g$ for all $g \in \mathbf{SL}_2(p)$. The former are representations such that $\mathbf{Sym}^k(\rho) : G \longrightarrow GL_{k+1}(V)$. We construct this for $p = 2$ and $p = 3$, highlighting the special case of $\mathbf{SL}_2(p)$.

Example 4.3.2. *Note: This example will not generalize to other p and does not showcase the subtlety involved in constructing the Steinberg representation. We include it for its uniqueness.*

Let $G = \mathbf{SL}_2(2)$ once more. Constructing $\psi : G \longrightarrow GL_2(\mathbf{F}_p)$, the Steinberg representation of this group, means taking $\mathbf{Sym}^1(\rho)$, the 1st symmetric power representation. Let $g \in G$. Then $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ where $a, b, c, d \in \mathbf{F}_p$. We now take the basis elements $u = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], v = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ of $GL_2(\mathbf{F}_p)$ and take the dot product of each with g to yield $\psi(g) = [g \cdot u, g \cdot v]$. This results in

$$g \cdot u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = au + cv$$
$$g \cdot v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = bu + dv.$$

Perhaps somewhat surprisingly, this means $\psi(g) = g$. This will only happen for $p = 2$. The next example illustrates a more general case.

Example 4.3.3. Let $G = \mathbf{SL}_2(3)$ with $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for $a, b, c, d \in \mathbf{F}_3$ now. Then ψ of G is $\mathbf{Sym}^2(\rho)$, the 2nd symmetric power representation, which sends elements of G to $GL_3(\mathbf{F}_p)$.

Once again, we take the basis elements of the codomain and take their respective dot products with g to obtain $\psi(g) = \left[g \cdot u^2, g \cdot uv, g \cdot v^2\right]$. The former are $u^2 = \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$, $uv = \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$, and $v^2 = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$. Then

$$g \cdot u^2 = gugu = (au + cv)(au + cv) = a^2u^2 + 2acuv + c^2v^2$$

$$g \cdot uv = gugv = (au + cv)(bu + cd) = abu^2 + (ad + bc)uv + cdv^2$$

$$g \cdot v^2 = gvgv = (bu + cd)(bu + cd) = b^2u^2 + 2bduv + d^2v^2$$

yield $\psi(g) =$

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}.$$

The previous example shows what occurs when we construct the Steinberg representation of $\mathbf{SL}_2(p)$. Naturally, complexity increases with larger p. Because of the importance of the Steinberg representation and the fact that there is a well-known formula for it, we direct our focus to one of its decompositions. Specifically, we want to know how $\psi \otimes \psi$ decomposes as a direct sum. We find that the decomposition varies slightly depending on whether $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$. Using this notation discounts $p = 2$; this is fine since we know $\psi \otimes \psi = 1_G \oplus \theta_1 \oplus \psi$, as demonstrated in Example 3.2.4.

Using the notation from the general character table of $\mathbf{SL}_2(p)$ in Figure 4.3.1, we present the following:

Theorem 4.3.4. *1. For $p \equiv 1 \pmod 4$, the tensor product $\psi \otimes \psi$ decomposes into the direct sum*

$$\psi \otimes \psi = 1_G \oplus \xi_1 \oplus \xi_2 \oplus 2\psi \oplus 2(\theta_2 \oplus \theta_4 \oplus \ldots \oplus \theta_{\frac{(p-1)}{2}}) \oplus 2(\zeta_2 \oplus \zeta_4 \oplus \ldots \oplus \zeta_{\frac{(p-5)}{2}}).$$

2. For $p \equiv 3 \pmod 4$, the tensor product $\psi \otimes \psi$ decomposes into the direct sum

$$\psi \otimes \psi = 1_G \oplus \eta_1 \oplus \eta_2 \oplus 2\psi \oplus 2(\theta_2 \oplus \theta_4 \oplus \ldots \oplus \theta_{\frac{(p-3)}{2}}) \oplus 2(\zeta_2 \oplus \zeta_4 \oplus \ldots \oplus \zeta_{\frac{(p-3)}{2}}).$$

Our knowledge of character theory tells us that we are able to prove the above theorems by showing

$$(\chi_\psi(\alpha))^2 = \chi_1(\alpha) + \chi_{\xi_1}(\alpha) + \chi_{\xi_2}(\alpha) + 2\chi_\psi(\alpha) + 2(\chi_{\theta_2}(\alpha) + \chi_{\theta_4}(\alpha) + \ldots + \chi_{\theta_{\frac{(p-1)}{2}}}(\alpha)) \\ + 2(\chi_{\zeta_2}(\alpha) + \chi_{\zeta_4}(\alpha) + \ldots + \chi_{\zeta_{\frac{(p-5)}{2}}}(\alpha))$$

and

$$(\chi_\psi(\alpha))^2 = \chi_1(\alpha) + \chi_{\eta_1}(\alpha) + \chi_{\eta_2}(\alpha) + 2\chi_\psi(\alpha) + 2\big(\chi_{\theta_2}(\alpha) + \chi_{\theta_4}(\alpha) \ldots \chi_{\theta_{\frac{(p-3)}{2}}}(\alpha)\big) \\ + 2\big(\chi_{\zeta_2}(\alpha) + \chi_{\zeta_4}(\alpha) + \cdots + \chi_{\zeta_{\frac{(p-3)}{2}}}(\alpha)\big)$$

hold for each conjugacy class α of $\mathbf{SL}_2(p)$ where $p \equiv 1, 3 \pmod 4$, respectively. We proceed to show this for $\alpha = 1, z, a^\ell$ for $1 \leq \ell \leq \frac{p-3}{2}$, $b^m$for $1 \leq m \leq \frac{p-1}{2}, c, d, zc, zd$. We now prove this for $p \equiv 1 \pmod 4$.

Proof. Let $\alpha = 1$. Then $(\chi_\psi(1))^2 = p^2$. See that

$$\chi_1(1) + \chi_{\xi_1}(1) + \chi_{\xi_2}(1) + 2\chi_\psi(1) + 2(\chi_{\theta_2}(1) + \chi_{\theta_4}(1) + \ldots + \chi_{\theta_{\frac{(p-1)}{2}}}(1)) \\ + 2(\chi_{\zeta_2}(1) + \chi_{\zeta_4}(1) + \ldots + \chi_{\zeta_{\frac{(p-5)}{2}}}(1)) \\ = 1 + \tfrac{1}{2}(p+1) + \tfrac{1}{2}(p+1) + 2p + 2((p-1)(\tfrac{(p-1)}{4})) + 2((p+1)(\tfrac{(p-5)}{4})) \\ = 1 + p + 1 + 2p + (p-1)(\tfrac{p-1}{2}) + (p+1)(\tfrac{p-5}{2}) \\ = 3p + 2 + \tfrac{p^2-2p+1}{2} + \tfrac{p^2-4p-5}{2} \\ = 3p + 2 + \tfrac{2p^2+-6p-4}{2} \\ = 3p + 2 + p^2 - 3p - 2 \\ = p^2.$$

We move on to $\alpha = z$. Naturally, $\chi_1(z) = \chi_1(1)$ by definition. However, we also have $\chi_\psi(1) = \chi_\psi(z) = p$. Then ζ_i, ξ_1, ξ_2, and θ_j are the only representations for which the character $\chi(z)$ seems to differ from $\chi(1)$. We will show that $\chi(z) = \chi(1)$ for these representations as well, implying that no separate proof is needed for $\alpha = z$.

First, recall that $\chi_{\zeta_i}(z) = (-1)^i(p+1)$, $\chi_{\eta_{\{1,2\}}}(z) = \frac{1}{2}\varepsilon(p+1)$, and $\chi_{\theta_j}(z) = (-1)^j(p-1)$ where $\varepsilon = (-1)^{\frac{(p-1)}{2}}$. Notice that we only use even i, j in our decomposition formula and $\varepsilon = (-1)^{even}$ since $p \equiv 1 \pmod 4$. This means $\chi_{\zeta_i}(z) = (1)(p+1) = \chi_{\zeta_i}(1)$, $\chi_{\eta_{\{1,2\}}}(z) = \frac{1}{2}(p+1) = \chi_{\eta_{\{1,2\}}}(1)$, and $\chi_{\theta_j}(z) = (1)(p-1) = \chi_{\theta_j}(1)$. Thus we move on to $\alpha = a^\ell$.

For general ℓ, we have $(\chi_\psi(a^\ell))^2 = 1$, so we need to show equality to

$$\begin{aligned}
&\chi_1(a^\ell) + \chi_{\xi_1}(a^\ell) + \chi_{\xi_2}(a^\ell) + 2\chi_\psi(a^\ell) + 2(\chi_{\theta_2}(a^\ell) + \chi_{\theta_4}(a^\ell) + \ldots + \chi_{\theta_{\frac{(p-1)}{2}}}(a^\ell)) \\
&\qquad + 2(\chi_{\zeta_2}(a^\ell) + \chi_{\zeta_4}(a^\ell) + \ldots + \chi_{\zeta_{\frac{(p-5)}{2}}}(a^\ell)) \\
&= 1 + (-1)^\ell + (-1)^\ell + 2 + 2((0)(\tfrac{(p-1)}{4})) + 2((\tau^{2\ell} + \tau^{-2\ell}) + (\tau^{4\ell} + \tau^{-4\ell}) \\
&\qquad\qquad + \ldots + (\tau^{(\frac{(p-5)}{2})\ell} + \tau^{-(\frac{(p-5)}{2})\ell})) \\
&= 1 + 2(\sum_{i=2}^{(p-5)/2} \chi_{\zeta_i}(a^\ell)) \text{ for odd } \ell \text{ and} \\
&= 5 + 2(\sum_{i=2}^{(p-5)/2} \chi_{\zeta_i}(a^\ell)) \text{ for even } \ell.
\end{aligned}$$

where τ is a primitive $(p-1)$th root of unity. Then we need to show $\sum_{i=2}^{(p-5)/2} \chi_{\zeta_i}(a^\ell) = 0$ when ℓ is odd and $\sum_{i=2}^{(p-5)/2} \chi_{\zeta_i}(a^\ell) = -2$ when ℓ is even.

Let $\beta = \tau^{2\ell}$. Since τ is a primitive $(p-1)$th root of unity, then τ^2 is a primitive $\frac{(p-1)}{2}$th root of unity. On the other hand, $\tau^{2\ell}$ is not necessarily primitive. Regardless, we can now rewrite

$$\begin{aligned}
\sum_{i=2}^{(p-5)/2} \chi_{\zeta_i}(a^\ell) &= (\tau^{2\ell} + \tau^{-2\ell}) + (\tau^{4\ell} + \tau^{-4\ell}) + \ldots + (\tau^{(\frac{(p-5)}{2})\ell} + \tau^{-(\frac{(p-5)}{2})\ell}) \\
&= (\beta + \beta) + (\beta^2 + \beta^{-2}) + \ldots + (\beta^{\frac{(p-5)}{4}} + \beta^{-\frac{(p-5)}{4}}).
\end{aligned}$$

Set $N = \frac{(p-5)}{4}$. Then $\frac{(p-1)}{2} = 2N+2$, $N+1 = \frac{(p-1)}{4}$ and $-N = -\frac{(p-5)}{4} = N+2$. We incorporate this into our notation to obtain

$$\begin{aligned}
\beta + \beta^{-1} + \beta^{2} + \beta^{-2} + ... + \beta^{\frac{(p-5)}{4}} + \beta^{-\frac{(p-5)}{4}} &= \beta + \beta^{-1} + \beta^{2} + \beta^{-2} + ... + \beta^{N} + \beta^{-N} \\
&= \beta + \beta^{2N+1} + \beta^{2} + \beta^{2N} + ... + \beta^{N} + \beta^{N+2} \\
&= \beta + \beta^{2} + ... + \beta^{N} + \beta^{N+2} + ... + \beta^{2N+1}.
\end{aligned}$$

This is similar to the expanded geometric series $\frac{1-\beta^{2N+2}}{1-\beta}$, without a select few terms. Then we can rewrite our last equation

$$\begin{aligned}
\beta + \beta^{2} + ... + \beta^{N} + \beta^{N+2} + ... + \beta^{2N+1} &= \tfrac{1-\beta^{2N+2}}{1-\beta} - 1 - \beta^{N+1} \text{ and since } \beta^{2N+2} = 1, \\
&= 0 - 1 - \beta^{N+1}.
\end{aligned}$$

Then we must show $-1-\beta^{N+1} = 0$, or $\beta^{N+1} = -1$ for odd ℓ, and $-1-\beta^{N+1} = -2$, or $\beta^{N+1} = 1$ for even ℓ. Now let ℓ be odd and see that

$$\begin{aligned}
\beta^{N+1} &= (\tau^{2\ell})^{\frac{(p-1)}{4}} \\
&= (\tau^{\ell})^{\frac{(p-1)}{2}} \\
&= \tau^{(\ell-1)(\frac{(p-1)}{2}) + \frac{(p-1)}{2}} \\
&= (\tau^{\frac{(p-1)}{2}})^{\ell-1}\tau^{\frac{(p-1)}{2}} \text{ but since } \tau \text{ is primitive} \\
&= (-1)^{\ell-1}(-1) \text{ and } \ell-1 \text{ is even since } \ell \text{ is odd,} \\
&= (1)(-1) \\
&= -1.
\end{aligned}$$

This completes the proof for odd ℓ. For even ℓ, β^{N+1} becomes

$$\begin{aligned}
(\tau^{2\ell})^{\frac{(p-1)}{4}} &= (\tau^2)^{(\frac{\ell}{2})(\frac{(p-1)}{2})} \\
&= (\tau^{2(\frac{(p-1)}{2})})^{\frac{\ell}{2}} \\
&= (\tau^{p-1})^{\frac{\ell}{2}} \\
&= (1)^{\frac{\ell}{2}} \text{ since } \tau \text{ is a primitive } p-1\text{th root of unity} \\
&= 1.
\end{aligned}$$

Thus we can move on to the next set of conjugacy classes. For $\alpha = b^m$, $(\chi_\psi(b^m))^2 = (-1)^2 = 1$. Then we must show that $1 =$

$$\begin{aligned}
&\chi_1(b^m) + \chi_{\xi_1}(b^m) + \chi_{\xi_2}(b^m) + 2\chi_\psi(b^m) + 2(\chi_{\zeta_2}(b^m) + \chi_{\zeta_4}(b^m) + \ldots + \chi_{\zeta_{\frac{(p-5)}{2}}}(b^m)) \\
&\qquad + 2(\chi_{\theta_2}(b^m) + \chi_{\theta_4}(b^m) + \ldots + \chi_{\theta_{\frac{(p-1)}{2}}}(b^m)) \\
&= 1 + 0 + 0 - 2 + 2((0)\tfrac{(p-5)}{4}) + 2(-(\sigma^{2m} + \sigma^{-2m}) - (\sigma^{4m} + \sigma^{-4m}) \\
&\qquad\quad - \ldots - (\sigma^{\frac{(p-1)}{2}m} + \sigma^{-\frac{(p-1)}{2}m})) \\
&= -1 + 2(-(\sigma^{2m} + \sigma^{-2m}) - (\sigma^{4m} + \sigma^{-4m}) - \ldots - (\sigma^{\frac{(p-1)}{2}m} + \sigma^{-\frac{(p-1)}{2}m})) \\
&= -1 - 2(\sigma^{2m} + \sigma^{-2m} + \sigma^{4m} + \sigma^{-4m} + \ldots + \sigma^{\frac{(p-1)}{2}m} + \sigma^{-\frac{(p-1)}{2}m})
\end{aligned}$$

where σ is a primitive $(p+1)$th root of unity. Then it suffices to show that

$$\sigma^{2m} + \sigma^{-2m} + \sigma^{4m} + \sigma^{-4m} + \ldots + \sigma^{\frac{(p-1)}{2}m} + \sigma^{-\frac{(p-1)}{2}m} = -1.$$

We do this by setting $\gamma = \sigma^{2m}$ and $M = \frac{(p+1)}{2}$. Then γ is a $(p+1)$th root of unity as well, though it is not necessarily primitive, and $\frac{(p-1)}{4} = \frac{M-1}{2}, -\frac{(p-1)}{4} = \frac{M+1}{2}$. Substituting γ

and M into the above equation yields

$$\begin{aligned}\gamma+\gamma^{-1}+\gamma^{2}+\gamma^{-2}+\ldots+\gamma^{\frac{(p-1)}{4}}+\gamma^{-\frac{(p-1)}{4}} &= \gamma+\gamma^{M-1}+\gamma^{2}+\gamma^{M-2} \\ &\quad +\ldots+\gamma^{\frac{M-1}{2}}+\gamma^{\frac{M+1}{2}} \\ &= \gamma+\gamma^{2}+\ldots+\gamma^{\frac{M-1}{2}}+\gamma^{\frac{M+1}{2}} \\ &\quad +\ldots+\gamma^{M-1}.\end{aligned}$$

Notice that there are no integers between $\frac{M-1}{2}$ and $\frac{M+1}{2}$. Keeping this in mind, we once again, we put our equation in terms of a geometric series. This results in

$$\gamma+\gamma^{2}+\ldots+\gamma^{\frac{M-1}{2}}+\gamma^{\frac{M+1}{2}}+\ldots+\gamma^{M-1}=\tfrac{1-\gamma^{M}}{1-\gamma}-1$$

and since γ is a $(p+1)$th root of unity,

$$\tfrac{1-\gamma^{M}}{1-\gamma}-1=0-1=-1$$

as needed. This completes the proof for $\alpha=b^{m}$, so we move on to $\alpha=c$. For this conjugacy class we have $(\chi_{\psi}(c))^{2}=0^{2}=0$. Now recall that $\varepsilon=(-1)^{\frac{(p-1)}{2}}=(-1)^{even}=1$ since $p\equiv 1 \pmod 4$. Then we need to show the following equals 0 :

$$\begin{aligned}&\chi_{1}(c)+\chi_{\xi_{1}}(c)+\chi_{\xi_{2}}(c)+2\chi_{\psi}(c)+2(\chi_{\zeta_{2}}(c)+\chi_{\zeta_{4}}(c)+\ldots+\chi_{\zeta_{\frac{(p-5)}{2}}}(c)) \\ &\qquad +2(\chi_{\theta_{2}}(c)+\chi_{\theta_{4}}(c)+\ldots+\chi_{\theta_{\frac{(p-1)}{2}}}(c)) \\ &=1+\tfrac{1}{2}(1+\sqrt{\varepsilon p})+\tfrac{1}{2}(1-\sqrt{\varepsilon p})+2(0+2((1)(\tfrac{(p-5)}{4})))+2((-1)(\tfrac{(p-1)}{4})) \\ &=1+1+2(\tfrac{(p-5)}{4})-2(\tfrac{(p-1)}{4}) \\ &=2+2(\tfrac{(p-5)}{4})-2(\tfrac{(p-1)}{4}) \\ &=2+\tfrac{(p-5)}{2}-\tfrac{(p-1)}{2} \\ &=2+\tfrac{p-p-5+1}{2} \\ &=2+\tfrac{-4}{2} \\ &=0\end{aligned}$$

which completes the proof for $\alpha = c$.

Now let $\alpha = d$. Notice that for all $\rho \neq \xi_i$, one can see from the character table that $\chi_\rho(c) = \chi_\rho(d)$. For $\rho = \xi_i$, we have $\chi_{\xi_1}(c) = \chi_{\xi_2}(d)$ and $\chi_{\xi_2}(c) = \chi_{\xi_1}(d)$. Then the sum of the characters in our tensor product formula is identical for $\alpha = c$ and $\alpha = d$, which means we do not need a separate proof for $\alpha = d$ now that we know the formula holds for $\alpha = c$. This lets us move on to $\alpha = zc$, a conjugacy class that was omitted from the character table.

Recall that for any representation ρ, it is the case that $\chi_\rho(zc) = \chi_\rho(z)\chi_\rho(1)^{-1}\chi_\rho(c)$. Then $\chi_\psi(zc) = \chi_\psi(z)\chi_\psi(1)^{-1}\chi_\psi(c) = p \cdot \frac{1}{p} \cdot 0 = 0$ so $(\chi_\psi(zc))^2 = 0$ as well. Then we must show $0 =$

$$
\begin{aligned}
&\chi_1(z)\chi_1(1)^{-1}\chi_1(c) + \chi_{\xi_1}(z)\chi_{\xi_1}(1)^{-1}\chi_{\xi_1}(c) + \chi_{\xi_2}(z)\chi_{\xi_2}(1)^{-1}\chi_{\xi_2}(c) + 2(\chi_\psi(z)\chi_\psi(1)^{-1}\chi_\psi(c)) \\
&\qquad + 2((\chi_{\zeta_2}(z)\chi_{\zeta_2}(1)^{-1}\chi_{\zeta_2}(c)) + (\chi_{\zeta_4}(z)\chi_{\zeta_4}(1)^{-1}\chi_{\zeta_4}(c)) \\
&\qquad + \ldots + (\chi_{\zeta_{\frac{(p-5)}{2}}}(z)\chi_{\zeta_{\frac{(p-5)}{2}}}(1)^{-1}\chi_{\zeta_{\frac{(p-5)}{2}}}(c))) \\
&\qquad + 2((\chi_{\theta_2}(z)\chi_{\theta_2}(1)^{-1}\chi_{\theta_2}(c)) + (\chi_{\theta_4}(z)\chi_{\theta_4}(1)^{-1}\chi_{\theta_4}(c)) \\
&\qquad + \ldots + (\chi_{\theta_{\frac{(p-1)}{2}}}(z)\chi_{\theta_{\frac{(p-1)}{2}}}(1)^{-1}\chi_{\theta_{\frac{(p-1)}{2}}}(c))) \\
&= 1 + (\tfrac{1}{2}\varepsilon(p+1)(\tfrac{1}{2}(p+1))^{-1}\tfrac{1}{2}(1+\sqrt{\varepsilon p})) + (\tfrac{1}{2}\varepsilon(p+1)(\tfrac{1}{2}(p+1))^{-1}\tfrac{1}{2}(1-\sqrt{\varepsilon p})) + 2(0) \\
&\qquad + 2(((p+1)\tfrac{1}{(p+1)}(1))\tfrac{(p-5)}{4}) + 2(((p-1)(\tfrac{1}{(p-1)})(-1))\tfrac{(p-1)}{4}).
\end{aligned}
$$

While the value of $\chi_{\zeta_i}(zc)$ depends on the parity of i according to the character table, we use even i and so $\chi_{\zeta_i}(zc)$ is the same for all i, letting us group them together. The same goes for θ_j. We then use the fact that $\varepsilon = 1$ to turn the above equation into

$$
\begin{aligned}
1 + \tfrac{1}{2}(1+\sqrt{\varepsilon p}) + \tfrac{1}{2}(1-\sqrt{\varepsilon p}) + 2((1)(\tfrac{(p-5)}{4})) + 2((-1)\tfrac{(p-1)}{4}) &= 2 + 2((1)(\tfrac{(p-5)}{4})) + 2((-1)\tfrac{(p-1)}{4}) \\
&= 2 + \tfrac{p-5-p+1}{2} \\
&= 2 + \tfrac{-4}{2} \\
&= 0
\end{aligned}
$$

as desired. By the same reasoning that $\alpha = d$ was proved alongside $\alpha = c$, the proof for $\alpha = zc$ holds for $\alpha = zd$ as well. Thus we have proved our decomposition formula for all conjugacy classes of $\mathbf{SL}_2(p)$ where $p \equiv 1 \pmod 4$.

We now prove the decomposition formula for $p \equiv 3 \pmod 4$.

Let $\alpha = 1$. This gives us $(\chi_\psi(1))^2 = p^2$. See that

$$\begin{aligned}
&\chi_1(1) + \chi_{\eta_1}(1)+\chi_{\eta_2}(1) + 2\chi_\psi(1) + 2\big(\chi_{\theta_2}(1) + \chi_{\theta_4}(1)\dots\chi_{\theta_{\frac{(p-3)}{2}}}(1)\big)+\\
&\qquad 2\big(\chi_{\zeta_2}(1) + \chi_{\zeta_4}(1) + \cdots + \chi_{\zeta_{\frac{(p-3)}{2}}}(1)\big)\\
&\qquad\quad = 1 + \tfrac{1}{2}(p-1) + \tfrac{1}{2}(p-1) + 2(p) + 2\big((p-1)(\tfrac{p-3}{2})\big) + 2\big((p+1)(\tfrac{p-3}{2})\big)\\
&\qquad\quad = 1 + p - 1 + 2(p) + (p-1)(p-3) + (p+1)(p-3)\\
&\qquad\quad = 3p + p(p-3)\\
&\qquad\quad = 3p + p^2 - 3p\\
&\qquad\quad = p^2
\end{aligned}$$

We now move on to $\alpha = z$. Observe that for all representations except η_1, η_2, it is explicitly clear that $\chi_\rho(1) = \chi_\rho(z)$. For ζ_i, θ_j this holds since we only use even i, j in the decomposition formula). For η_1 and η_2, $\chi_{\eta_{\{1,2\}}}(z) = \frac{-1}{2}\varepsilon(p-1)$. However ε is defined as $(-1)^{\frac{p-1}{2}} = -1$ since $p \equiv 3 \pmod 4$. Then $\frac{-1}{2}\varepsilon(p-1) = p - 1$, showing that η_1, η_2 have the same character values for conjugacy classes 1 and z as well. Then in showing that our formula worked for $\alpha = 1$, we have shown the same for $\alpha = z$. Thus we can move on to $\alpha = a^\ell$.

For this conjugacy class we have $(\chi_\psi(a^\ell))^2 = 1$ and

$$\begin{aligned}
&\chi_1(a^\ell) + \chi_{\eta_1}(a^\ell) + \chi_{\eta_2}(a^\ell) + 2\chi_\psi(a^\ell) + 2\big(\chi_{\theta_2}(a^\ell) + \chi_{\theta_4}(a^\ell)\dots\chi_{\theta_{\frac{(p-3)}{2}}}(a^\ell)\big)\\
&\qquad + 2\big(\chi_{\zeta_2}(a^\ell) + \chi_{\zeta_4}(a^\ell) + \cdots + \chi_{\zeta_{\frac{(p-3)}{2}}}(a^\ell)\big)\\
&= 1 + 0 + 0 + 2(1) + 2\big((0(\tfrac{p-3}{2})\big) + 2\big((\tau^{2\ell} + \tau^{-2\ell}) + (\tau^{4\ell} + \tau^{-4\ell}) + ... + (\tau^{\frac{p-3}{2}\ell} + \tau^{-\frac{p-3}{2}\ell})\big)\\
&= 3 + 2\big((\tau^{2\ell} + \tau^{-2\ell}) + (\tau^{4\ell} + \tau^{-4\ell}) + ... + (\tau^{\frac{p-3}{2}\ell} + \tau^{-\frac{p-3}{2}\ell})\big)
\end{aligned}$$

where τ is a primitive $(p-1)$th root of unity. Then our goal is to show

$$(\tau^{2\ell}+\tau^{-2\ell})+(\tau^{4\ell}+\tau^{-4\ell})+...+(\tau^{\frac{p-3}{2}\ell}+\tau^{-\frac{p-3}{2}\ell})=-1.$$

Let $\beta=\tau^{2\ell}$. Then β is a $\frac{p-1}{2}=Nth$ root of unity, not necessarily primitive, and $\frac{p-3}{4}=\frac{N-1}{2}$. Substituting these terms gives us

$$\begin{aligned}&(\beta+\beta^{-1})+(\beta^2+\beta{-}2)+...+(\beta^{\frac{N-1}{2}}+\beta^{-\frac{N-1}{2}})\\&=\beta+\beta^{N-1}+\beta^2+\beta^{N-2}+...+\beta^{\frac{N-1}{2}}+\beta^{\frac{N+1}{2}} \qquad =\beta+\beta^2+\beta^3+...+\beta^{N-1}\\&=\tfrac{1-\beta^N}{1-\beta}-1\end{aligned}$$

since it is a geometric series. Because β is an Nth root of unity, this simplifies to

$$\tfrac{1-1}{1-\beta}-1=0-1=-1$$

as desired. We move on to our next group of conjugacy classes, for which a similar procedure is needed. We have $(\chi_\psi(b^m))^2=(-1)^2=1$ and

$$\begin{aligned}\chi_1(b^m)+&\chi_{\eta_1}(b^m)+\chi_{\eta_2}(b^m)+2\chi_\psi(b^m)+2\big(\chi_{\theta_2}(b^m)+\chi_{\theta_4}(b^m)+\ldots+\chi_{\theta_{\frac{(p-3)}{2}}}(b^m)\big)+\\&\qquad 2\big(\chi_{\zeta_2}(b^m)+\chi_{\zeta_4}(b^m)+\ldots+\chi_{\zeta_{\frac{(p-3)}{2}}}(b^m)\big)\\&=1+(-1)^{m+1}+(-1)^{m+1}+2(-1)+\\&\quad 2\big(-(\sigma^{2m}+\sigma^{-2m})-(\sigma^{4m}+\sigma^{-4m})-\ldots-(\sigma^{\frac{p-3}{2}m}+\sigma^{-\frac{p-3}{2}m})\big)+2\big(0(\tfrac{p-3}{4})\big)\end{aligned}$$

where σ is a primitive $(p+1)th$ root of unity. Notice that the desired value of the main parenthetical depends on m's parity, and thus results in two cases. When m is odd, we have

$$1-2\big((\sigma^{2m}+\sigma^{-2m})+(\sigma^{4m}+\sigma^{-4m})+\cdots+(\sigma^{\frac{p-3}{2}m}+\sigma^{-\frac{p-3}{2}m})\big)$$

so we want

$$(\sigma^{2m}+\sigma^{-2m})+(\sigma^{4m}+\sigma^{-4m})+\cdots+(\sigma^{\frac{p-3}{2}m}+\sigma^{-\frac{p-3}{2}m})=0$$

in order for the equation to have the value $(\chi_\psi(b^m))^2 = 1$. Similarly, even m yields

$$-3 - 2\big((\sigma^{2m} + \sigma^{-2m}) + (\sigma^{4m} + \sigma^{-4m}) + \cdots + (\sigma^{\frac{p-3}{2}m} + \sigma^{-\frac{p-3}{2}m})\big)$$

signifying that the following should be the case:

$$(\sigma^{2m} + \sigma^{-2m}) + (\sigma^{4m} + \sigma^{-4m}) + \cdots + (\sigma^{\frac{p-3}{2}m} + \sigma^{-\frac{p-3}{2}m}) = -2.$$

We proceed to work solely with $\sum\limits_{j=2}^{(p-3)/2} \chi_{\theta_j}(b^m)$ for general m. Using a similar method of substitution as when $\alpha = a^\ell$, we let $\gamma = \sigma^{2m}$. Note that this is a $\frac{p+1}{2}th$ root of unity, not necessarily primitive, and let $M = \frac{p-3}{4}$. Then $2M + 2 = \frac{p+1}{2}$ and $-\frac{p-3}{4} = -M = M + 2$. This gives us

$$\begin{aligned}
(\sigma^{2m} + \sigma^{-2m}) + (\sigma^{4m} + \sigma^{-4m}) + \ldots + (\sigma^{\frac{p-3}{2}m} + \sigma^{-\frac{p-3}{2}m}) \\
&= (\gamma + \gamma^{-1}) + (\gamma^2 + \gamma^{-2}) + \ldots + (\gamma^M + \gamma^{-M}) \\
&= (\gamma + \gamma^{2M-1}) + (\gamma^2 + \gamma^{2M-2}) + \ldots + (\gamma^M + \gamma^{M+2}) \\
&= \gamma + \gamma^2 + \gamma^3 + \ldots + \gamma^M + \gamma^{M+2} + \ldots + \gamma^{2M+1}.
\end{aligned}$$

This can be rewritten as the geometric series

$$\tfrac{1-\gamma^{2M+2}}{1-\gamma} - 1 - \gamma^{M+1}.$$

Recall that γ is a $\frac{p+1}{2} = (2M+2)th$ root of unity. Then this yields

$$\tfrac{1-1}{1-\gamma} - 1 - \gamma^{M+1} = 0 - 1 - \gamma^{M+1} = -1 - \gamma^{M+1} = -1 - (\sigma^{2m})^{\frac{p+1}{4}}$$

Now let m be odd. Then $m-1$ is even and $\frac{m-1}{2} \in \mathbf{Z}$. This gives us

$$\begin{aligned}
-1 - (\sigma^{2m})^{\frac{p+1}{4}} &= -1 - (\sigma^2)^{m\frac{p+1}{4}} \\
&= -1 - (\sigma^2)^{(m-1)\frac{p+1}{4} + \frac{p+1}{4}} \\
&= -1 - (\sigma^2)^{(\frac{m-1}{2})(\frac{p+1}{2}) + \frac{p+1}{4}} \\
&= -1 - \big((\sigma^2)^{\frac{p+1}{2}}\big)^{\frac{m-1}{2}} (\sigma^2)^{\frac{p+1}{4}}.
\end{aligned}$$

Recall that σ^2 is a primitive $\frac{p+1}{2}$ root of unity. Then this simplifies into

$$\begin{aligned}-1-(1)^{\frac{m-1}{2}}(\sigma^2)^{\frac{p+1}{4}} &= -1-(\sigma^2)^{\frac{p+1}{4}}\\ &= -1-(-1)\\ &= -1+1\\ &= 0\end{aligned}$$

which concludes the proof for b^{odd}.

When m is even, $\frac{m}{2}\in\mathbf{Z}$. Then

$$\begin{aligned}-1-(\sigma^{2m})^{\frac{p+1}{4}} &= -1-(\sigma^2)^{m\frac{p+1}{4}}\\ &= -1-(\sigma^2)^{(\frac{m}{2})(\frac{p+1}{2})}\\ &= -1-\left((\sigma^2)^{\frac{p+1}{2}}\right)^{\frac{m}{2}}\\ &= -1-(1)^{\frac{m}{2}} \text{ since } \sigma^2 \text{ is primitive}\\ &= -1-1\\ &= -2\end{aligned}$$

as desired for b^{even}. This allows us to move on to $\alpha = c$.

Note that as in the case for $p\equiv 1 \pmod 4$, $\chi_\rho(c) = \chi_\rho(d)$ unless $\rho = \eta_1, \eta_2$. For the latter representations, it is the case that $\chi_{\eta_1}(c) = \chi_{\eta_2}(d)$ and $\chi_{\eta_1}(d) = \chi_{\eta_2}(c)$. Then $\sum a_k\chi_\rho(c) = \sum a_k\chi_\rho(d)$ for a_k, ρ as in our decomposition formula, so it suffices to show that this holds for the conjugacy class $\alpha = c$ only. We have $(\chi_\psi(c))^2 = 0^2 = 0$, so we must show the following equals 0:

$$\begin{aligned}&\chi_1(c)+\chi_{\eta_1}(c)+\chi_{\eta_2}(c)+2\chi_\psi(c)+2\big(\chi_{\zeta_2}(c)+\chi_{\zeta_4}(c)+\ldots+\chi_{\zeta_{\frac{(p-3)}{2}}}(c)\big)\\ &\quad+2\big(\chi_{\theta_2}(c)+\chi_{\theta_4}(c)+\ldots+\chi_{\theta_{\frac{(p-3)}{2}}}(c)\big)\end{aligned}$$

$$= 1 + \tfrac{1}{2}(-1 + \sqrt{\varepsilon p}) + \tfrac{1}{2}(-1 - \sqrt{\varepsilon p}) + 2(0) + 2((1)(\tfrac{(p-3)}{4}) + 2((-1)(\tfrac{(p-3)}{4}))$$

$$= 1 - 1 + 2(\tfrac{(p-3)}{4}) - 2(\tfrac{(p-3)}{4})$$

$$= 0.$$

Finally, we reach the last pair of conjugacy classes, zc and zd. Recall that $\chi_\rho(zc) = \chi_\rho(z)\chi_\rho(1)^{-1}\chi_\rho(c)$ for any ρ. Because of the fact that $\chi_\rho(c) = \chi_\rho(d)$ for $\rho \neq \eta_1, \eta_2$, we have $\chi_\rho(zc) = \chi_\rho(zd)$. However, we have $\chi_{\eta_1}(c) = \chi_{\eta_2}(d)$ and $\chi_{\eta_2}(c) = \chi_{\eta_1}(d)$, so it is the case that $\sum a_k\chi_\rho(zc) = \sum a_k\chi_\rho(zd)$ for a_k, ρ as in our decomposition formula. Then we need only show that said formula holds for zc.

We have $(\chi_\psi(zc))^2 = (\chi_\psi(z)\chi_\psi(1)^{-1}\chi_\psi(c))^2 = (p \cdot \frac{1}{p} \cdot 0)^2 = 0^2 = 0$. It remains to show $0 =$

$$\chi_1(z)\chi_1(1)^{-1}\chi_1(c) + \chi_{\eta_1}(z)\chi_{\eta_1}(1)^{-1}\chi_{\eta_1}(c) + \chi_{\eta_2}(z)\chi_{\eta_2}(1)^{-1}\chi_{\eta_2}(c) + 2(\chi_\psi(z)\chi_\psi(1)^{-1}\chi_\psi(c))$$
$$+ 2((\chi_{\zeta_2}(z)\chi_{\zeta_2}(1)^{-1}\chi_{\zeta_2}(c)) + (\chi_{\zeta_4}(z)\chi_{\zeta_4}(1)^{-1}\chi_{\zeta_4}(c))$$
$$+ \ldots + (\chi_{\zeta_{\frac{(p-5)}{2}}}(z)\chi_{\zeta_{\frac{(p-5)}{2}}}(1)^{-1}\chi_{\zeta_{\frac{(p-5)}{2}}}(c)))$$
$$= 1 + (-\tfrac{1}{2}\varepsilon(p-1) \cdot (\tfrac{1}{2}(p-1))^{-1} \cdot \tfrac{1}{2}(-1 + \sqrt{\varepsilon p}))$$
$$+ (-\tfrac{1}{2}\varepsilon(p-1) \cdot (\tfrac{1}{2}(p-1))^{-1} \cdot \tfrac{1}{2}(-1 - \sqrt{\varepsilon p})) + 2(0)$$
$$+ 2(((p+1)\tfrac{1}{(p+1)}(1))(\tfrac{(p-3)}{4})) + 2(((p-1)(\tfrac{1}{(p-1)})(-1))(\tfrac{(p-3)}{4})).$$

As in the proof for $p \equiv 1 \pmod 4$, we group χ_{ζ_i} which each other since i is even, and similarly for χ_{θ_j}. Recall that $\varepsilon = -1$. Then the above simplifies to

$$1 - 1 + 0 + 2(\tfrac{(p-3)}{4}) - 2(\tfrac{(p-3)}{4}) = 0.$$

Thus the decomposition formula holds for all conjugacy classes of $p \equiv 3 \pmod 4$, concluding the proof. □

5

Conjectures in Modular Representation Theory

5.1 Introduction

Recall that an ordinary representation sends G to $GL(V)$ where V is a vector space over some field K of characteristic 0. A modular representation sends G to $GL(V')$ where V' is over K', a field of characteristic p for some prime p dividing the order of G. Among many other consequences, modular representations are not completely reducible, negating Maschke's theorem (2.1.17). We illustrate this in the following example.

Example 5.1.1. Let $G = C_2$, the cyclic group of order 2 and consider $\rho : G \longrightarrow GL_2(2)$. We define ρ such that

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(-1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

See that

$$(\rho(-1))^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \rho(1)$$

in characteristic 2. This shows that the multiplicity of G is preserved, so ρ is a faithful representation. Notice that ρ is not irreducible since $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is stable under $\rho(g)$. Furthermore, there is no complementary subspace W to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ such that $V = W \oplus \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. This means ρ is not completely reducible.

Thus we run into representations that are neither irreducible nor completely reducible, which means we cannot concern ourselves with tensor product decompositions. In turn, we are unable to create McKay graphs for such representations.

Further issues arise for groups G which contain elements of order p^m when we attempt to construct representations of G in characteristic p.

Example 5.1.2. Let $G = C_p$ and recall that there are p disctinct irreducible representations of C_p in characteristic 0. We will illustrate that this is not the case for $\rho : G \to GL_1(\mathbf{F}_p^*)$. See that since $|\mathbf{F}_p^*| = p-1$ and since it is cyclic, it has no nontrivial elements of order p. In contrast, all elements of C_p have order p, which means any representation of this group in characteristic p is trivial.

When given a group G, there will be two general ways to create its modular representations:

1. Take some ordinary representation ρ of G and reduce it $\pmod p$. We call this $\mathrm{Red}_p(\rho)$.

2. Find modular representations that arise solely from the characteristic of the field. These do not have analogues in characteristic $m \neq p$.

For reference, $\rho : C_2 \to GL_2(2)$ from Example 5.1.1 falls into the latter category. This is because $(\rho(-1))^2 \neq I_2$ unless we are in characteristic 2. With regards to representations from the first category, there exists a theorem which determines whether $\mathrm{Red}_p(\rho)$ is irreducible. Namely,

Theorem 5.1.3. [8, 7.3, Theorem 2.6] *Let ρ be a $\overline{\boldsymbol{Q}}$-representation of G of degree d. If p is prime to $\frac{n}{d}$ where $n = |G|$, then $Red_p(\rho)$ is irreducible.*

It is important to note that the converse of this statement is not necessarily true. More explicitly, there are groups G with ordinary irreducibles ρ of degree d such that $(p, \frac{|G|}{d}) \neq 1$, yet $\mathrm{Red}_p(\rho)$ is irreducible. We will see an example of this in the next section.

There exists a theory of modular characters, but it is beyond the scope of this project. Moreover, it is difficult to find the modular characters of a given group G. For a basic introduction, we refer the reader to [8].

We now proceed to illustrate that there is a significant difference between the ordinary and modular irreducible characters of a given group.

Example 5.1.4. Let $G = A_5$. Below are its ordinary and modular character tables, which we obtained from the Magma Calculator via the code CharacterTable(AlternatingGroup(5)); and [5] respectively. The latter table is of characteristic 2.

	()	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

	Cl_{1^*}	Cl_{2^*}	Cl_{3^*}	Cl_{4^*}
X_1	1	1	1	1
X_2	2	-1	$\frac{-1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
X_3	2	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
X_4	4	1	-1	-1

Observe that χ_2, χ_3, and χ_5 are not present in characteristic 2. The fact that X_2 and X_3 look similar to χ_2 and χ_3 is purely coincidental, for X_2 and X_3 correspond to representations of A_5 unique to characteristic 2. On the other hand, we can apply Theorem 5.1.3 to χ_4 and obtain the following: since $|G| = 60$, it is clear that 2 is prime to $60/4 = 15$. Then ρ_4 is irreducible in characteristic 2. More explicitly, $X_4 = \mathrm{Red}_2(\rho_4)$. This means we can use $X(g)$ to determine which conjugacy class of A_5 each Cl_i represents. We find that $Cl_{1^*} = ()$, $Cl_{2^*} = (123)$, $Cl_{3^*} = (12345)$, and $Cl_{4^*} = (12354)$. See that for g that consists of two 2-cycles, $|g| = 2$. Then $g = ()$ in characteristic 2. This explains why the conjugacy class of (12)(34) is not explicitly present in the modular character table of our group.

The next two sections will explore the irreducible modular representations of D_p and $\mathbf{SL}_2(p)$ in a similar fashion. Additionally, we will present several conjectures of tensor product decompositions for each of the two groups.

5.2 Dihedral Groups: D_p (mod 2) and D_p (mod p)

Having created formulas for the tensor product decompositions of the irreducible ordinary representations of D_n, we were curious to see if similar formulas existed for modular representations. As general D_n has fairly complicated modular representations, we instead look at D_p. More specifically, we map this family of groups to the algebraic closures $\overline{\mathbf{F}}_2$ and $\overline{\mathbf{F}}_p$. We replicate the ordinary character table of D_p below, with $\omega = e^{2\pi i/p}$, as it will be used to construct the relevant modular character tables.

	1	s	r	r^2	...	$r^{\frac{p-1}{2}}$
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1
χ_{R_1}	2	0	$\omega + \omega^{-1}$	$\omega^2 + \omega^{-2}$	...	$\omega^{\frac{p-1}{2}} + \omega^{-(\frac{p-1}{2})}$
χ_{R_2}	2	0	$\omega^2 + \omega^{-2}$	$\omega^4 + \omega^{-4}$	...	$\omega^{p-1} + \omega^{-(n-1)}$
χ_{R_3}	2	0	$\omega^3 + \omega^{-3}$	$\omega^6 + \omega^{-6}$	...	$\omega^{3(\frac{p-1}{2})} + \omega^{-3(\frac{p-1}{2})}$
$\vdots$				$\ddots$		
$\chi_{R_{\frac{p-1}{2}}}$	2	0	$\omega^{\frac{p-1}{2}} + \omega^{-(\frac{p-1}{2})}$	$\omega^{2(\frac{p-1}{2})} + \omega^{-2(\frac{p-1}{2})}$	...	$\omega^{\frac{p-1}{2}(\frac{p-1}{2})} + \omega^{-(\frac{p-1}{2})(\frac{p-1}{2})}$

Remark 5.2.1. [3, 8.1] Let q be prime and let $Cl_1, ..., Cl_A$ be the conjugacy classes of D_p for which $(|g|, q) = 1$ where $g \in Cl_k$ and $1 \leq k \leq i$. Then $A = \#$ of irreducible representations D_p has in characteristic q.

Let us use the above remark to determine how many irreducible representations D_p has in characteristic p. As discussed in Section 4.2, D_p has $(p-1)/2+2$ conjugacy classes. These are $\{1\}, \{s, rs, r^2s, ..., r^{p-1}s\}, \{r, r^{p-1}\}, \{r^2, r^{p-2}\}, \ldots,$ and $\{r^{\frac{p-1}{2}}, r^{\frac{p-1}{2}+2}\}$, with elements of order $1, 2, p, p, \ldots,$ and p respectively. Then there are two conjugacy classes with elements of order prime to p, which means D_p has two irreducible representations in characteristic p. Alternatively, there are $(p-1)/2 + 1$ conjugacy classes with elements of order prime to 2, so D_p has $(p-1)/2$ irreducible representations in characteristic 2. We now present the character table of D_p in characteristic p.

	1	s
χ_1	1	1
χ_2	1	-1

See that ρ_2 has degree 1, but $(p, \frac{2p}{1}) \neq 1$. Then ρ_2 is a modular representation that is irreducible despite not having satisfied Theorem 5.1.3. This is to show that the converse of said theorem does not necessarily hold. Since the above is a small character table, we can easily see that $\chi_2 \cdot \chi_2 = \chi_1$ corresponds to our only nontrivial tensor product decomposition. Then the formulas (1) and (2) for the ordinary tensor product decompositions of D_p hold in characteristic p as well. We will see that something similar occurs in characteristic 2.

Take any ordinary irreducible representation R_j of D_p which has degree 2. Since $(2, \frac{2p}{2}) = 1$, we know that $\mathrm{Red}_2(R_j)$ is irreducible by Theorem 5.1.3. Note that there are $\frac{(p-1)}{2}$ such R_j, which will account for $\frac{(p-1)}{2}$ irreducible representations in characteristic 2. Along with the trivial representation, this gives us the $\frac{(p-1)}{2}+1$ irreducible representations of D_p. Since the representations of degree 2 remain irreducible after reduction, it suffices to use the ordinary character table and interpret the complex characters as elements in the algebraic closure of F2. We can do this because p is an odd prime, so the algebraic closure $\bar{F}_2$ contains all pth roots of unity. Let ω be a fixed nontrivial pth root of unity in characteristic 2. Then the modular character table for D_p (mod 2) is given by the following:

	1	r	r^2	$\dots$	$r^{\frac{p-1}{2}}$
χ_1	1	1	1	1	1
$\chi_{\mathrm{Red}_2(R_1)}$	2	$\omega + \omega^{-1}$	$\omega^2 + \omega^{-2}$	$\dots$	$\omega^{\frac{p-1}{2}} + \omega^{-(\frac{p-1}{2})}$
$\chi_{\mathrm{Red}_2(R_2)}$	2	$\omega^2 + \omega^{-2}$	$\omega^4 + \omega^{-4}$	$\dots$	$\omega^{p-1} + \omega^{-(p-1)}$
$\chi_{\mathrm{Red}_2(R_3)}$	2	$\omega^3 + \omega^{-3}$	$\omega^6 + \omega^{-6}$	$\dots$	$\omega^{3(\frac{p-1}{2})} + \omega^{-3(\frac{p-1}{2})}$
$\vdots$			$\ddots$		
$\chi_{\mathrm{Red}_2(R_{\frac{(p-1)}{2}})}$	2	$\omega^{\frac{p-1}{2}} + \omega^{-(\frac{p-1}{2})}$	$\omega^{2(\frac{p-1}{2})} + \omega^{-2(\frac{p-1}{2})}$	$\dots$	$\omega^{\frac{p-1}{2}(\frac{p-1}{2})} + \omega^{-(\frac{p-1}{2})(\frac{p-1}{2})}$

Without loss of generality, let $\ell > j$ and let us see what happens to the general character dot product $\chi_{R_j} \cdot \chi_{R_\ell}$. Clearly $\chi_{R_j}(1) \cdot \chi_{R_\ell}(1) = 4$. We proceed to focus on the conjugacy class of r, since this determines the product for all other conjugacy classes as well. Then

$$\begin{aligned}
(\omega^\ell + \omega^{-\ell})(\omega^j + \omega^{-j}) &= \omega^{\ell+j} + \omega^{j-\ell} + \omega^{\ell-j} + \omega^{-\ell-j} \\
&= \omega^{\ell-j} + \omega^{-(\ell-j)} + \omega^{\ell+j} + \omega^{-(\ell+j)} \\
&= \chi_{R_{\ell-j}}(r) + \chi_{R_k}(r)
\end{aligned}$$

where $k = \begin{cases} \ell+j & \ell+j < \lfloor \frac{p}{2} \rfloor \\ p-(\ell+j) & \ell+j > \lfloor \frac{p}{2} \rfloor \end{cases}$, as in (11) of the tensor product decomposition formulas for the ordinary representations of D_n. The proof of this is identical to that of (11), so we do not replicate it here.

As the reader may have predicted, our result for the decomposition of $\chi_{\mathrm{Red}_2(R_j)} \cdot \chi_{\mathrm{Red}_2(R_j)}$ is similar. We focus on the conjugacy class of r once more. See that

$$\begin{aligned}
(\omega^j + \omega^{-j})^2 &= \omega^{2j} + \omega^0 + \omega^0 + \omega^{-2j} \\
&= \omega^{2j} + 2 + \omega^{-2j} \\
&= 2\chi_1(r) + \chi_{\mathrm{Red}_2(R_k)}(r)
\end{aligned}$$

where $k = \begin{cases} 2\ell & \ell < \lfloor \frac{p}{4} \rfloor \\ n-2\ell & \ell > \lfloor \frac{p}{4} \rfloor \end{cases}$ as in (9) of the original tensor product decomposition formulas. Once again, the proof of the above is identical to that of (9), with χ_1 replacing χ_2 in the decomposition since no such χ_2 exists in characteristic 2.

From this we gather that the tensor product decomposition formulas for the ordinary irreducible representations of D_n hold for the modular irreducible representations of D_p. Then we can create the McKay graphs of D_p for any p in characteristic 2 and characteristic p. Since the tensor product decompositions of the group's modular characters are essentially the same as that of its ordinary characters, the McKay graphs of D_p in characteristic 2 and p will be very similar to its ordinary McKay graphs.

5.3 The Steinberg Representation: $\mathbf{SL}_2(p)$ (mod p)

Let us move on to our other family of groups, $\mathbf{SL}_2(p)$. We look at its irreducible modular representations in characteristic p. As in the previous section, we present its ordinary character table for reference in constructing its modular characters. The Steinberg representation is denoted by ψ, as before.

	1	z	a^ℓ	b^m	c	d
χ_1	1	1	1	1	1	1
χ_ψ	p	p	1	-1	0	0
χ_{ζ_i}	$p+1$	$(-1)^i(p+1)$	$\tau^{i\ell}+\tau^{-i\ell}$	0	1	1
χ_{ξ_1}	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^\ell$	0	$\frac{1}{2}(1+\sqrt{\varepsilon p})$	$\frac{1}{2}(1-\sqrt{\varepsilon p})$
χ_{ξ_2}	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^\ell$	0	$\frac{1}{2}(1-\sqrt{\varepsilon p})$	$\frac{1}{2}(1+\sqrt{\varepsilon p})$
χ_{θ_j}	$p-1$	$(-1)^j(p-1)$	0	$-(\sigma^{jm}+\sigma^{-jm})$	-1	-1
χ_{η_1}	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$
χ_{η_2}	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$

It is known that $|\mathbf{SL}_2(p)| = p^3 - p = p(p+1)(p-1)$ [3, 5.1]. Then by Theorem 5.1.3, we know $\mathrm{Red}_p(\psi)$ is irreducible since $(p, \frac{p(p+1)(p-1)}{p}) = 1$. This reinforces the possibility of there existing a formula for the tensor product decomposition of $\mathrm{Red}_p(\psi) \otimes \mathrm{Red}_p(\psi)$ for any given p. First, let us discuss all modular irreducible representations of the group for a given p.

Recall that $\mathbf{SL}_2(p)$ has $p+4$ conjugacy classes, which are $\{1\}, \{z\}, \{a\}, \{a^2\}, ..., \{a^{\frac{(p-3)}{2}}\}$, $\{b\}, \{b^2\}, ..., \{b^{\frac{(p-1)}{2}}\}, \{c\}, \{d, d^2, ..., d^{(p-1)}\}, \{zc\}, \{zd, zd^2, ..., zd^{(p-1)}\}$ for $1 = I_2, z = -I_2, a = \left[\begin{smallmatrix} v & 0 \\ 0 & v^{-1}\end{smallmatrix}\right]$, b is some element of order $p+1$,$c = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]$, and $d = \left[\begin{smallmatrix} 1 & v \\ 0 & 1\end{smallmatrix}\right]$ for v such that $\langle v \rangle = \mathbf{F}_p^*$. Then by definition of v, $|a^k| = p-1$ and $|d| = p^j$ for some integer j. This gives us enough information to know which congjugacy classes of $\mathbf{SL}_2(p)$ have elements of order prime to p. These are $\{1\}, \{z\}, \{a\}, \{a^2\}, ..., \{a^\ell\}, \{b\}, \{b^2\}, ..., \{b^m\}$ with elements of orders $1, 2, p-1, ..., p-1, p+1, ..., p+1$ respectively. Altogether, this yields $2+\frac{p-3}{2}+\frac{p-1}{2} = p$ such conjugacy classes. Then by Remark 5.2.1, $\mathbf{SL}_2(p)$ has p irreducible representations in characteristic p.

It is known that the full list of irreducible modular representations of $\mathbf{SL}_2(p)$ are $\mathrm{Sym}^k(\rho)$ for $k = 0, ..., p-1$. Recall that for representations of this form, ρ is the standard representation and $k \in \mathbf{Z}_+$. See that since $\mathrm{Sym}^k(\rho)$ is defined over the ground field, we do not need to consider the algebraic closure $\overline{\mathbf{F}}_p$ in order to obtain modular representations of the group as was needed for D_p. For ease of notation, we will denote $\mathrm{Sym}^k(\rho)$ by ϕ_k. Then ϕ_0 is the trivial representation and $\phi_{p-1} = \mathrm{Red}_p(\psi)$.

Conjecture 5.3.1. *The tensor product $Red_p(\psi) \otimes Red_p(\psi)$ decomposes into the direct sum*

$$3\phi_0 \oplus 4\phi_2 \oplus 4\phi_4 \oplus ... \oplus 4\phi_{p-3} \oplus 2\phi_{p-1}i.$$

Let us show that this decomposition holds for conjugacy classes 1 and z. It is the case that the character values $\phi_k(1) = \phi_k(z)$ for any ϕ_k used in the above decomposition. Then it suffices to show that the decomposition holds for the identity conjugacy class. See that

$$\begin{aligned}
3\phi_0(1) + 4\phi_2(1) + 4\phi_4(1) + ... + 4\phi_{p-3}(1) + 2\phi_{p-1}(1) &= 3 + 4(3) + 4(5) + ... + 4(p-2) + 2p \\
&= 3 + 4\sum_{k=1}^{(p-3)/2}(2k+1) + 2p \\
&= 3 + 4(2\sum_{k=1}^{(p-3)/2} k) + 4\sum_{k=1}^{(p-3)/2} 1 + 2p \\
&= 3 + 8\frac{(\frac{(p-3)}{2}(\frac{(p-3)}{2}+1))}{2} + 4(\frac{(p-3)}{2}) + 2p \\
&= 3 + 4(\frac{(p-3)}{2}(\frac{(p-1)}{2})) + 2p - 6 + 2p \\
&= 3 + (p-3)(p-1) + 4p - 6 \\
&= 3 + p^2 - 4p + 3 + 4p - 6 \\
&= p^2
\end{aligned}$$

as desired. We have checked that the above holds for other conjugacy classes on a case by case basis. This provides good evidence for our conjecture, but is not a proof.

Appendix A

Preliminaries

Below is an exercise from a Bard course on Representation Theory taught a few years ago to illustrate the nuances of stable subspaces. This is referenced in 2.1.6.

Exercise: Write out the alternate proof of Mashke's Theorem using an inner product when $k = \mathbf{R}$ or $\mathbf{C}$.

Solution. Let $\langle\ ,\ \rangle$ be an inner-product on V. Define a new inner-product $\langle\ ,\ \rangle_{\text{new}}$ by

$$\langle x, y\rangle_{\text{new}} = \sum_{g\in G} \langle \rho(g)x, \rho(g)y\rangle.$$

Start by checking that is indeed an inner-product. First, $\langle\ ,\ \rangle_{\text{new}}$ is linear in both variables since matrix multiplication is linear, and it's a finite sum of such. Explicitly, if $\{e_1, e_2, \ldots, e_n\}$ is a basis for V, and $x = \sum \alpha_i e_i$, $y = \sum \beta_j e_j$, then we need to check that $\langle x, y\rangle_{\text{new}} = \sum_{i,j} \alpha_i \beta_j \langle e_i, e_j\rangle_{\text{new}}$:

$$\begin{aligned}
\langle x, y\rangle_{\text{new}} &= \sum_{g\in G} \langle \rho(g)x, \rho(g)y\rangle \\
&= \sum_{g\in G} \langle \rho(g) \sum \alpha_i e_i, \rho(g) \sum \beta_j e_j\rangle \\
&= \sum_{g\in G} \langle \sum \alpha_i \rho(g) e_i, \sum \beta_j \rho(g) e_j\rangle \\
&= \sum_{g\in G} \sum_{i,j} \alpha_i \beta_j \langle \rho(g)e_i, \rho(g)e_j\rangle \text{ (since } \langle\ ,\ \rangle \text{ is linear)} \\
&= \sum_{i,j} \alpha_i \beta_j \sum_{g\in G} \langle \rho(g)e_i, \rho(g)e_j\rangle \text{ (since the sums are finite)} \\
&= \sum_{i,j} \alpha_i \beta_j \langle e_i, e_j\rangle_{\text{new}}.
\end{aligned}$$

Moreover, it's easy to check that $\langle x, x\rangle_{\text{new}} \geq 0$ with equality precisely when $x = 0$.

Next, we should check that the new inner-product is invariant under the action of G:

$$\begin{aligned}\langle \rho(h)x, \rho(h)y\rangle_{\text{new}} &= \sum_{g\in G}\langle \rho(g)\rho(h)x, \rho(g)\rho(h)y\rangle \\ &= \sum_{g\in G}\langle \rho(gh)x, \rho(gh)y\rangle \\ &= \sum_{\sigma\in G}\langle \rho(\sigma)x, \rho(\sigma)y\rangle \\ &= \langle x, y\rangle_{\text{new}}.\end{aligned}$$

Now let $W \subset V$ be G-stable, and let $W^\perp$ be the orthogonal complement to W under the new inner-product, *i.e.*

$$W^\perp = \{v \in V \ : \ \langle w, v\rangle_{\text{new}} = 0 \text{ for all } w \in W\}.$$

Let $x \in W^\perp$ so that $\langle w, x\rangle_{\text{new}} = 0$ for all $w \in W$. We need to show that $\rho(h)x \in W^\perp$ for all $h \in G$. But for any $w \in W$ we have

$$\begin{aligned}\langle w, \rho(h)x\rangle_{\text{new}} &= \sum_{g\in G}\langle \rho(g)w, \rho(g)\rho(h)x\rangle \\ &= \sum_{g\in G}\langle \rho(g)\rho(h)\rho(h)^{-1}w, \rho(g)\rho(h)x\rangle \\ &= \sum_{g\in G}\langle \rho(g)\rho(h)w', \rho(g)\rho(h)x\rangle \text{ (for some } w' \in W \text{ since } W \text{ is } G\text{-stable)} \\ &= \sum_{g\in G}\langle \rho(gh)w', \rho(gh)x\rangle \\ &= \sum_{\sigma\in G}\langle \rho(\sigma)w', \rho(\sigma)x\rangle \\ &= \langle w', x\rangle_{\text{new}} = 0.\end{aligned}$$

Therefore $W^\perp$ is G-stable. This finishes the proof since $\dim W + \dim W^\perp = \dim V$.

Appendix B

Sample Code

B.1 Magma

Character tables are difficult to produce by hand, so we turn to an online resource. More specifically, we use the Magma Calculator at http://magma.maths.usyd.edu.au/calc/. In general, the command

$$\text{CharacterTable}(G');$$

yields the character table of the group G where G' is the notation for G used by Magma. We go through the calculations needed to obtain the character table of D_8 as an example. The code CharacterTable(DihedralGroup(8)); generates

```
------------------------------
Class |   1  2  3  4  5   6   7
Size  |   1  1  4  4  2   2   2
Order |   1  2  2  2  4   8   8
------------------------------
p  =  2   1  1  1  1  2   5   5
------------------------------
X.1   +   1  1  1  1  1   1   1
X.2   +   1  1 -1  1  1  -1  -1
X.3   +   1  1  1 -1  1  -1  -1
X.4   +   1  1 -1 -1  1   1   1
X.5   +   2  2  0  0 -2   0   0
X.6   +   2 -2  0  0  0  Z1 -Z1
X.7   +   2 -2  0  0  0 -Z1  Z1
```

Assuming no previous knowledge of the characters of D_8, this we can also find in Magma. The console includes an explanation of character value symbols alongside each generated character table. Here, it defines Z1 via the following:

```
Z1     = (CyclotomicField(8: Sparse := true)) ! [ RationalField() | 0, 1, 0, -1 ]
```

Then we can obtain the minimal polynomial of Z1 and solve for the variable. The code

MinimalPolynomial((CyclotomicField(8: Sparse := true)) ! [RationalField() — 0, 1, 0, -1]);

yields the solution

```
$.1^2 - 2
```

which, by adopting the notation used, is

$$x^2 - 2.$$

Then

$$(Z1)^2 - 2 = 0 \Rightarrow Z1 = \pm\sqrt{2}.$$

Since $Z1$ and $-Z1$ are featured interchangeably in the character table, we allow sign ambiguity. Reflecting on our knowledge of the characters of general D_n, we know $Z1 = e^{2\pi i/8} + e^{-2\pi i/8} = \sqrt{2}$ which matches the output from Magma.

We see that obtaining character tables in Magma is fairly straight forward. For the two infinite families of groups this project focuses on, the codes for the character tables ofre CharacterTable(DihedralGroup(n)); for that of D_n, and CharacterTable(SpecialLinearGroup(2,p)); for the that of $\mathbf{SL}_2(p)$ for any positive n, p. The Magma Calculator was an important resource for this project and we thank the University of Sydney for providing public access.

B.2 Mathematica

We now demonstrate how to decompose a tensor product in Mathematica by using $G = \mathbf{SL}_2(11) \pmod{11}$ as an example. More specifically, we are looking for the decomposition of the Steinberg representation in characteristic 11 with itself, $\psi \otimes \psi$.

The first step is to replicate the group's character table in matrix form. Here we can afford to take a shortcut, for we know that the decomposition of $\psi \otimes \psi$ only involved representations of the form $\chi_i(1) = \chi_i(z)$ since $(\chi_\psi(1))^2 = (\chi_\psi(z))^2$.. Thus we eliminate χ_{even}. We begin by replicating the character table such that it only includes representations that may appear in the decomposition at hand. The following encodes this:

```
In[3]:= A = {{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
          {3, 3, -1, 0, 0, -b5, -b5, -b5s, -b5s, 2, 2},
        {5, 5, 1, -1, -1, 0, 0, 0, 0, 1, 1},
        {7, 7, -1, 1, 1, b5, b5, b5s, b5s, -1, -1},
        {9, 9, 1, 0, 0, -1, -1, -1, -1, -2, -2},
        {11, 11, -1, -1, -1, 1, 1, 1, 1, -1, -1}}
```

Here, $b5 = \frac{1-\sqrt{5}}{2}$ and $b5s = \frac{-1-\sqrt{5}}{2}$. One can see these characters clearer in matrix form, easily identifying which row corresponds to which representation. Conveniently, row i corresponds to χ_i for us.

```
In[7]:= MatrixForm[A]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & 0 & 0 & \frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{1}{2}+\frac{\sqrt{5}}{2} & \frac{1}{2}+\frac{\sqrt{5}}{2} & 2 & 2 \\ 5 & 5 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 7 & 7 & -1 & 1 & 1 & -\frac{1}{2}+\frac{\sqrt{5}}{2} & -\frac{1}{2}+\frac{\sqrt{5}}{2} & -\frac{1}{2}-\frac{\sqrt{5}}{2} & -\frac{1}{2}-\frac{\sqrt{5}}{2} & -1 & -1 \\ 9 & 9 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & -2 & -2 \\ 11 & 11 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix}$$

Since there are six representations, we create a vector containing six variables. Each serves as the coefficient of a representation. Multiplying it by A yields the system of equations we solve to obtain our answer. The code for these steps is below.

```
In[8]:= Sol = {a, b, c, d, e, f}

Out[8]= {a, b, c, d, e, f}

In[9]:= Sol.A
```

Out[9]=

$$\Big\{a+3b+5c+7d+9e+11f,\ a+3b+5c+7d+9e+11f,\ a-b+c-d+e-f,\ a-c+d-f,\ a-c+d-f,$$
$$a+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)d-e+f,\ a+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)d-e+f,\ a+\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)d-e+f,$$
$$a+\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)d-e+f,\ a+2b+c-d-2e-f,\ a+2b+c-d-2e-f\Big\}$$

$$\begin{aligned}
&\texttt{Solve}\Big[121 == a+3b+5c+7d+9e+11f\ \&\& \\
&\quad 121 == a+3b+5c+7d+9e+11f\ \&\& \\
&\quad 1 == a-b+c-d+e-f\ \&\& \\
&\quad 1 == a-c+d-f\ \&\& \\
&\quad 1 == a-c+d-f\ \&\& \\
&\quad 1 == a+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)d-e+f\ \&\& \\
&\quad 1 == a+\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)b+\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)d-e+f\ \&\& \\
&\quad 1 == a+2b+c-d-2e-f\ \&\& \\
&\quad \texttt{Element[a, Integers] \&\& Element[b, Integers] \&\& Element[c, Integers] \&\& Element[d, Integers] \&\&} \\
&\quad \texttt{Element[e, Integers] \&\& Element[f, Integers] \&\&} \\
&\quad a \geq 0\ \&\&\ b \geq 0\ \&\&\ c \geq 0\ \&\&\ d \geq 0\ \&\&\ e \geq 0\ \&\&\ f \geq 0,\ \{a, b, c, d, e, f\},\ \texttt{Complexes}\Big]
\end{aligned}$$

$$\{\{a \to 3,\ b \to 4,\ c \to 4,\ d \to 4,\ e \to 4,\ f \to 2\}\}$$

We obtain the solution

$$\begin{aligned}\psi \otimes \psi &= a\rho_1 \oplus b\rho_3 \oplus c\rho_5 \oplus d\rho_7 \oplus e\rho_9 \oplus f\psi \\ &= 3\rho_1 \oplus 4\rho_3 \oplus 4\rho_5 \oplus 4\rho_7 \oplus 4\rho_9 \oplus 2\psi\end{aligned}$$

which supports Conjecture 5.3.1.

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