Centroidal Voronoi Tessellations with Few Generator Points

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Abstract

A Voronoi tessellation with n generator points is the partitioning of a bounded region in $\mathbb{R}^2$ into polygons such that every point in a given polygon is closer to its generator point than to any other generator point. A centroidal Voronoi tessellation (CVT) is a Voronoi tessellation where each polygons generator point is also its center of mass. In this project I will demonstrate what kinds of CVTs can exists within specific parameters, such as a square or rectangular region, and a set number generator points. I will also prove that the examples I present are the only CVTs that can possibly exist within their given parameters.

Contents

Acknowledgments

I would like to acknowledge the help I received from Ethan Bloch during this project, without whom I wouldn't have gotten nearly as far in exploring this topic.

1
Background

1.1 Voronoi Tessellations

A Voronoi tessellation with n generator points is the partitioning of a bounded region in $\mathbb{R}^2$ into convex polygons such that every point in a given polygon is closer to its generator point than to any other.

Figure 1.1.1, taken from [1], is an example of a Vornoi tessellation where $n = 20$.

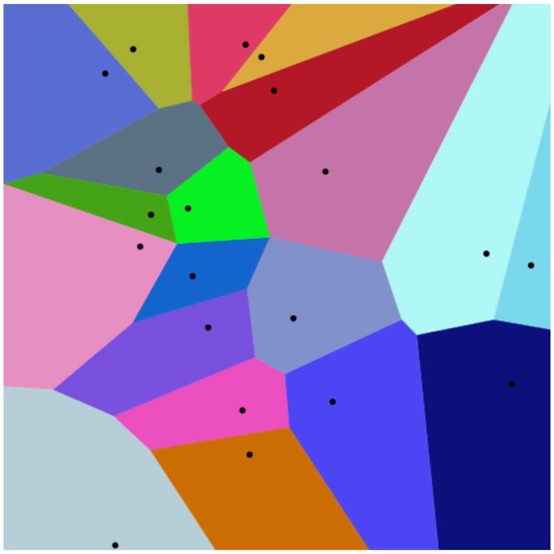

Figure 1.1.1.

For more details on how Voronoi tessellations are formed, please refer to chapter 2 in [2].

Definition 1.1.1. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points in $\mathbb{R}^2$. The distance between A and B is given by

$$D(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

△

Proof of the following Lemma can be found in [3].

Lemma 1.1.2. *Let P_1, P_2, A, B be points in $\mathbb{R}^2$, as can be seen in Figure 1.1.2. The line $\overline{AB}$ is a perpendicular bisector of the line $\overline{P_1P_2}$ if and only if $D(A, P_1) = D(A, P_2)$ and $D(B, P_1) = D(B, P_2)$.*

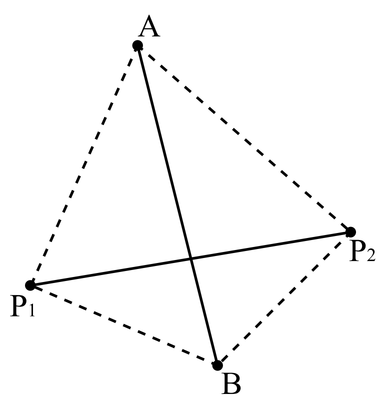

Figure 1.1.2.

1.2 Centroids

Definition 1.2.1. The centroid of a polygon is the mean position of all the points in the polygon.

△

Proof of the following Lemma can be found in [4].

Lemma 1.2.2. *Let* $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})$ *be the vertices of a polygon in* $\mathbb{R}^2$*. Let*

$$A = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1} - x_{i+1} y_i).$$

Then the centroid of the polygon is the point (C_x, C_y)*, where*

$$C_x = \frac{1}{6A}\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$
$$C_y = \frac{1}{6A}\sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i).$$

In Lemma 1.2.2, the number A is the area of the polygon and the vertices are assumed to be numbered in order of their occurrence along the polygon's perimeter.

1.3 Centroidal Voronoi Tessellations

A centroidal Voronoi tessellation (CVT) of a bounded region in $\mathbb{R}^2$ is a Voronoi tessellation where each polygon's generator point is also its' centroid. Figure 1.3.1 demonstrates 3 different centroidal Voronoi tessellation for 5 generator points in a square region in $\mathbb{R}^2$. While Voronoi tessellation can be formed with any given points, CVTs are much more rare as will be shown in the next chapter.

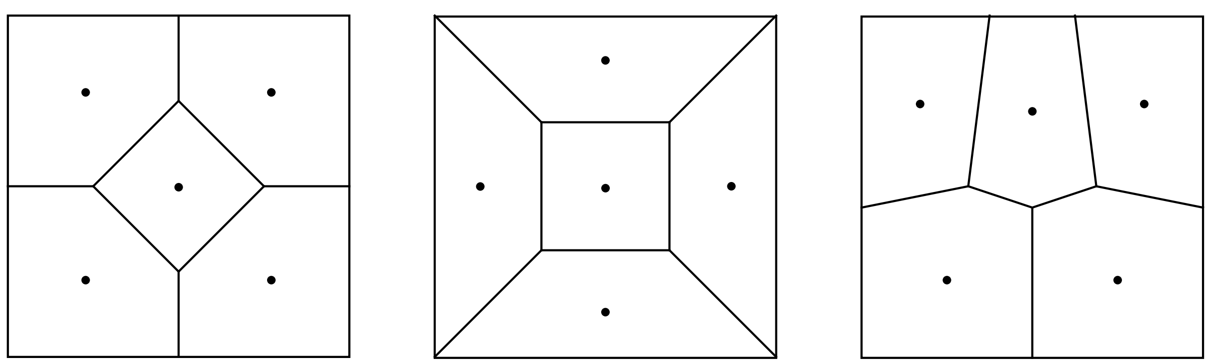

Figure 1.3.1.

The most commonly used method for building centroidal Voronoi tessellations is Lloyd's algorithm.

Using randomly distributed generator points, the algorithm works as follows

1. Create a Voronoi diagram with the current generator points.

2. Find the centroids of each region.

3. Move the generator points to the centroids of their respective regions.

4. Repeat steps 1-3 until the centroid of each region has the same coordinates as its respective generator point.

A demonstration of Lloyd's algorithm can be found at [5].

However, this method involves constant relocation of the generator points, while we are more interested in a fixed set of generator points.

Lastly, for the purposes of some of the proofs, We will be assuming that CVTs are equivalent if one CVT can be transformed into other CVT via a symmetry of the region. Since we will be mostly dealing with square shaped regions, this means that, for example, a CVT is equivalent to its' reflection about a horizontal line through the center of the square. Additional information on symmetry groups can be found at [6].

2

CVTs with 2 Generator Points

2.1 Lemmas

First we need to prove some of the properties of a centroid.

Lemma 2.1.1. *Let R be the interior of a bounded rectangle in $\mathbb{R}^2$. Let the point G be the centroid of R. Then any line that passes through G divides the area of R into two parts with equal area.*

Proof. Suppose the rectangle has height h and length s, as can be seen in Figure 2.1.1.

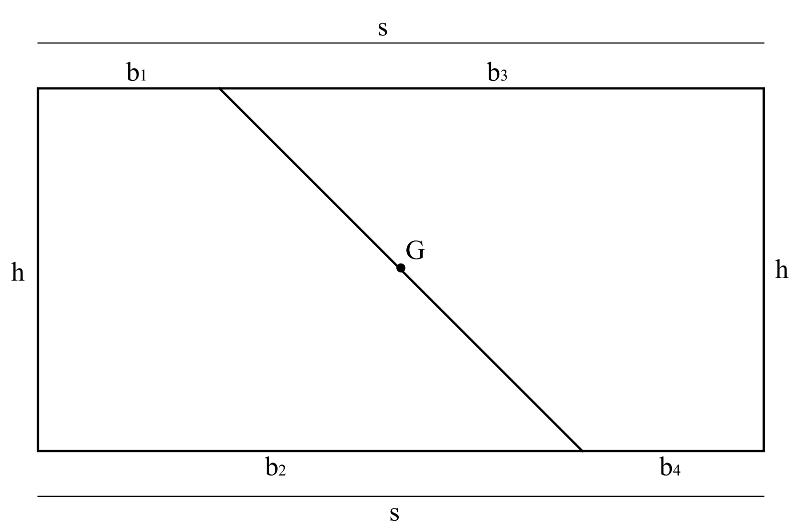

Figure 2.1.1.

A line through G divides the rectangle into trapezoid 1 with hight h and lengths b_1, b_2, and trapezoid 2 with hight h and lengths b_3, b_4, where $b_1 + b_3 = s$ and $b_2 + b_4 = s$. The area of trapezoid 1 is given by $A_1 = \frac{b_1+b_2}{2}h$, and the area of trapezoid 2 is given by $A_2 = \frac{b_3+b_4}{2}h$. Assuming that the lower left hand corner has coordinates $(0, 0)$, we know that the coordinates of G are $(\frac{s}{2}, \frac{h}{2})$, we can therefore say that $b_1 = b_4$ and $b_2 = b_3$. It follows from there that $b_1 + b_2 = b_3 + b_4$, and therefore $\frac{b_1+b_2}{2}h = \frac{b_3+b_4}{2}h$, and finally $A_1 = A_2$. □

Lemma 2.1.2. *Let R be the interior of a bounded rectangle in $\mathbb{R}^2$. Let the point G be the centroid of R. If a line divides R into two regions with equal areas, then the line passes through G.*

Proof. Suppose the rectangle has height h and length s, as can be seen in Figure 2.1.1. Let that the line that divides the rectangle into two regions with equal areas intersects the rectangle at points B_1 and B_2.

Suppose the line $\overline{B_1B_2}$ is parallel to the line through G. Since we know from Lemma 2.1.1 that a line through G divides the area of R into two parts with equal area, we can say that a parallel line would divide the area of R into two parts with different areas. Therefore, the line $\overline{P_1P_2}$ has to go through G. □

Lemma 2.1.3. *Let R be the interior of a bounded polygon in $\mathbb{R}^2$. Suppose $R = N_1 \cup N_2$, where N_1 and N_2 are polygons whose interiors do not intersect. Let P_1 and P_2 be the centroids of N_1 and N_2 respectively. Then the line $\overline{P_1P_2}$ contains the centroid of R.*

Proof. In Figure 2.1.2 let P_1 be the centroid of N_1, and let A_1 and A_2 be the areas of the 2 regions of N_1 created by a line through P_1 and P_2. Also, let P_2 be the centroid of N_2, and let A_3 and A_4 be the areas of the 2 regions of N_2 created by a line through P_1 and P_2.

We can see by Lemma 2.1.1 that $A_1 = A_2$ and $A_3 = A_4$. It follows that $A_1 + A_3 = A_2 + A_4$. Hence the line $\overline{P_1P_2}$ divides R into two pieces with equal area. Therefore, by Lemma2.1.2 , the line $\overline{P_1P_2}$ passes through the centroid of R. □

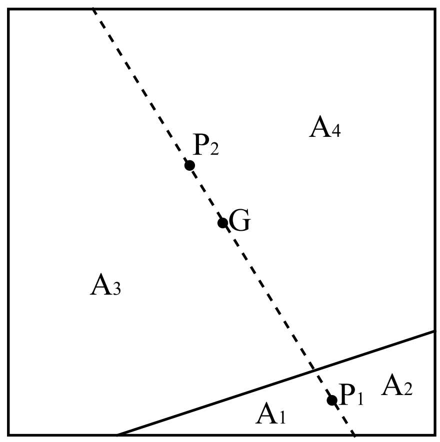

Figure 2.1.2.

2.2 Two Generator Points in a Square

Theorem 2.2.1. *For 2 generator points in a square region, there exist precisely two centroidal Voronoi tessellations.*

Proof. Let P_1 and P_2 be generator points in a square region. Let $B = (b, 0)$ and $A = (a, 1)$ be the points at which the perpendicular bisector of the line $\overline{P_1P_2}$ intersect with the edge of the region, as can be seen in Figure 2.2.1.

We know by Lemma 2.1.3 that the line $\overline{AB}$ passes through G, and therefore we can say that $a = 1 - b$.

Using Lemma 1.2.2 we can find the coordinates of P_1 and P_2, which are

$$P_1 = (\frac{b^2 - b + 1}{3}, \frac{2 - b}{3})$$
$$P_2 = (\frac{-b^2 + b + 2}{3}, \frac{b + 1}{3}).$$

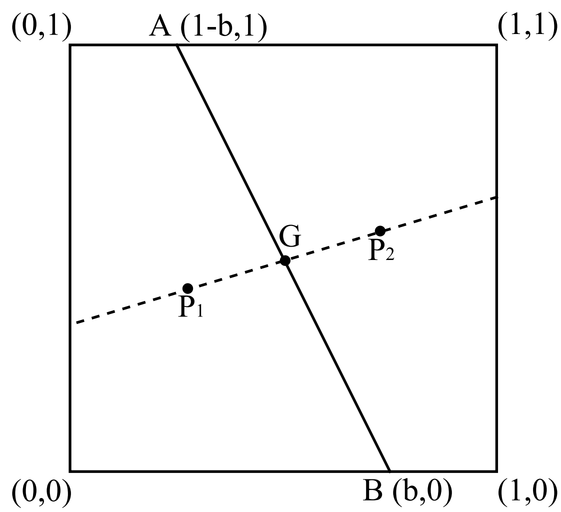

Figure 2.2.1.

By Lemma 1.1.2 we can say that the following has to be true

$$D(A, P_1) = D(A, P_2)$$
$$D(B, P_1) = D(B, P_2).$$

Since we know the coordinates of all the points, we can use Code A.1 to express $D(A, P_1) = D(A, P_2)$ and $D(B, P_1) = D(B, P_2)$ as

$$\frac{((b-1)^2 + 4b - 3)^2 + (b+1)^2}{9} = \frac{((b-2)b - 2b + 1)^2 + (b-2)^2}{9}$$
$$\frac{((b-1)^2 - 2b)^2 + (b-2)^2}{9} = \frac{((b-2)b + 4b - 2)^2 + (b+1)^2}{9}$$

respectively.

We can now use Code A.1 to solve this equation for b, which gives us the following solutions

$$b = 0, b = 1, b = \frac{1}{2}.$$

Since $r = 0$ is a reflection of $r = 1$ about the line $x = \frac{1}{2}$, we can conclude that $r = 1$ and $r = \frac{1}{2}$, which can be seen in Figure 2.2.2, are the only possible centroidal Vornoi tessellations for 2 generator points in a square region. □

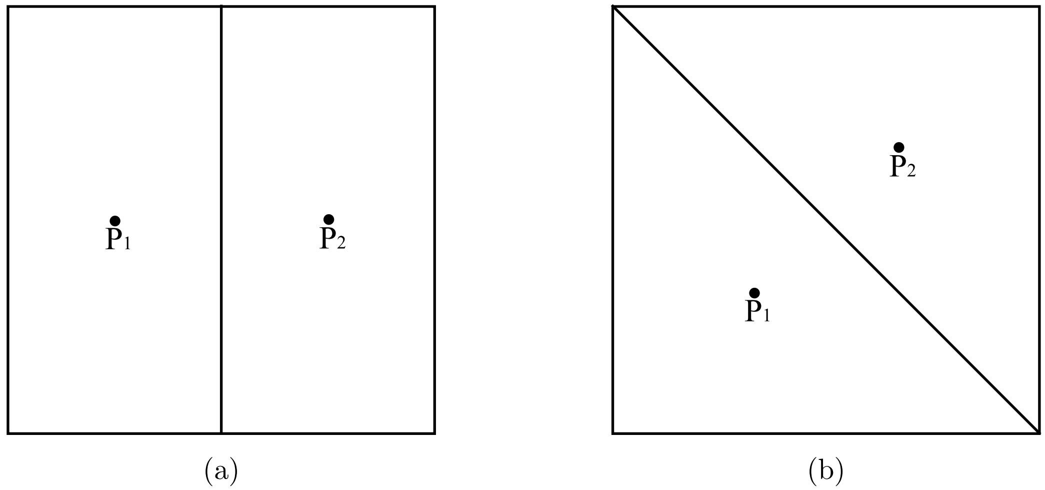

Figure 2.2.2.

2.3 Two Generator Points in a Rectangle

Theorem 2.3.1. *Ler R be a rectangle with length r and hight 1. If $\sqrt{\frac{2}{3}} < r \leq 1$, then for 2 generator points in R, there exist precisely three centroidal Voronoi tessellations. For all other values of r there exists precisely one centroidal Voronoi tessellation.*

Proof. Let P_1 and P_2 be generator points in a rectangular region with length r and hight 1. Let $B = (b, 0)$ and $A = (a, 1)$ be the points at which the perpendicular bisector of the line $\overline{P_1 P_2}$ intersect with the edge of the region, as can be seen in Figure 2.3.1.

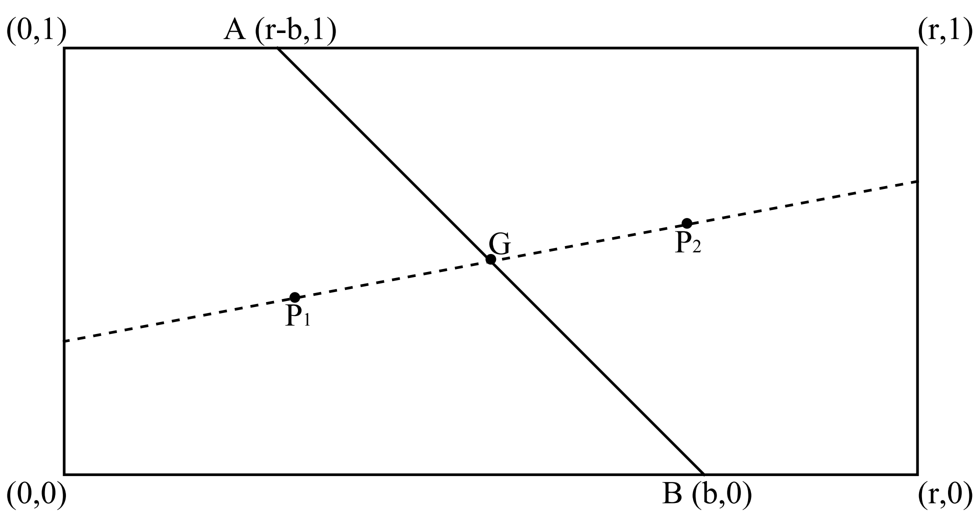

Figure 2.3.1.

We know by Lemma 2.1.3 that the line $\overline{AB}$ passes through G, and therefore we can say that $a = r - b$.

Using Lemma 1.2.2 we can find the coordinates of P_1 and P_2, which are

$$P_1 = (\frac{(b-r)^2 + br}{3r}, \frac{b-2r}{r3})$$
$$P_2 = (-\frac{(b-2r)b + br - 2r^2}{3r}, \frac{b+r}{3r}).$$

By Lemma 1.1.2 we can say that the following has to be true

$$D(A, P_1) = D(A, P_2)$$
$$D(B, P_1) = D(B, P_2).$$

Since we know the coordinates of all the points, we can use Code A.2 to express $D(A, P_1) = D(A, P_2)$ and $D(B, P_1) = D(B, P_2)$ as

$$\frac{1}{9}((3b - 3r + \frac{(b-r)^2 + br}{r})^2 + (\frac{b-2r}{r} + 3)^2) = \frac{1}{9}((3b - 3r - \frac{(b-2r)b + br - 2r^2}{r})^2 + (\frac{b+r}{r} - 3)^2)$$
$$\frac{1}{9}((3b - \frac{(b-r)^2 + br}{r})^2 + (\frac{b-2r}{r})^2) = \frac{1}{9}((3b + \frac{(b-2r)b + br - 2r^2}{r})^2 + (\frac{b+r}{r})^2)$$

respectively.

We can now use Code A.2 to solve these equations for b, which gives us the following solutions

$$b = \frac{r + \sqrt{3r^2 - 2}}{2}, b = \frac{r - \sqrt{3r^2 - 2}}{2}, b = \frac{r}{2}.$$

If $r \leq \sqrt{\frac{2}{3}}$, then $3r^2 - 2 < 0$, and $\sqrt{3r^2 - 2}$ becomes an imaginary number. Therefore $\frac{r+\sqrt{3r^2-2}}{2}$ and $\frac{r-\sqrt{3r^2-2}}{2}$ do not exists in $\mathbb{R}$ when $r \leq \sqrt{\frac{2}{3}}$.

If $r > 1$, then $r < \sqrt{3r^2 - 2}$. It follows from there that $r < \frac{r+\sqrt{3r^2-2}}{2}$. Therefore, when $r > 1$, then $r < b$, which puts the point B outside the rectangle. It also follows from there that $0 > \frac{r-\sqrt{3r^2-2}}{2}$. Therefore, when $r > 1$, then $o > b$, which puts the point B outside the rectangle.

Therefore, we can conclude that $b = \frac{r+\sqrt{3r^2-2}}{2}$ and $b = \frac{r-\sqrt{3r^2-2}}{2}$ for $\sqrt{\frac{2}{3}} < r \leq 1$, and $b = \frac{r}{2}$ for all values of r, are the only possible centroidal Vornoi tessellations for 2 generator points in a rectangular region. □

2.4 Two Generator Points in a Trapezoid

Before we look at regions with 3 generator points, we will first consider a trapezoid region as can be seen in Figure 2.4.1

Theorem 2.4.1. *For 2 generator points in a right trapezoid region, there is no centroidal Voronoi tessellations.*

Proof. Let P_1 and P_2 be generator points in a trapezoid region with hight 1, and lengths c and r. Let $B = (b, 0)$ and $A = (a, 1)$ be the points at which the perpendicular bisector of the line $\overline{P_1P_2}$ intersect with the edge of the region, and let $G = (m, n)$ be centroid of the region, as can be seen in Figure 2.4.1.

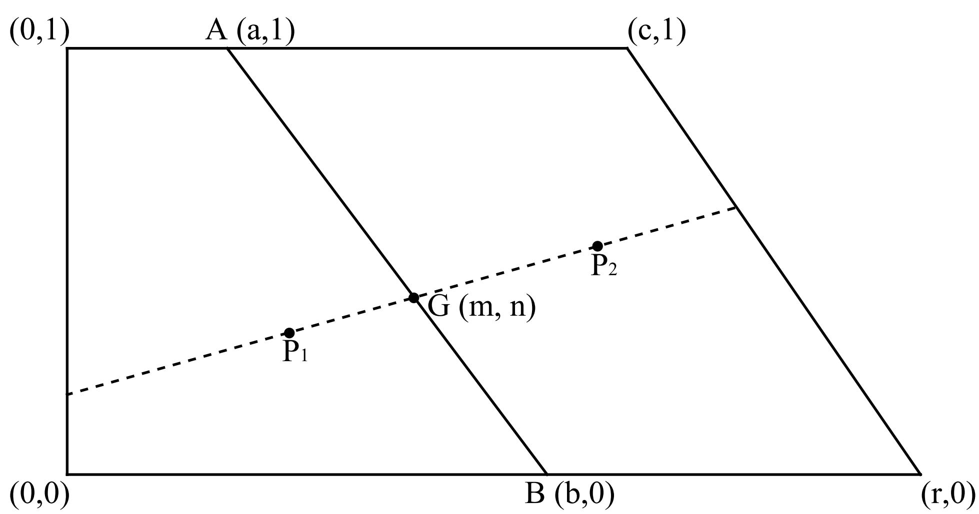

Figure 2.4.1.

We know by Lemma 2.1.3 that the line $\overline{AB}$ passes through G. Since A, B, and G are all part of the same line, we can express a in terms of b, m, and n.

Suppose that the equation of the line $\overline{AB}$ has the form $x = ry + k$.

Substituting the coordinates of point B, this equation becomes $b = r \cdot 0 + k$. Therefore $k = b$.

Substituting the coordinates of point G, this equation becomes $m = rn + b$. Therefore $r = \frac{m-b}{n}$, which gives us

$$x = \frac{m-b}{n}y + b.$$

Substituting the coordinates of point A, the equation becomes $a = \frac{m-b}{n} + b$.

Furthermore, by Lemma 1.2.2, we can say that $m = \frac{c^2+r(c+r)}{3(c+r)}$, and $n = \frac{2c+r}{3(c+r)}$. Therefore, $a = \frac{c^2+r^2+cr-bc-2br}{2c+r}$.

Using Lemma 1.2.2 we can find the coordinates of P_1 and P_2, which are

$$P_1 = (\frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b + \frac{(bc-c^2+2br-cr-r^2)^2}{(2c+r)^2}}{3(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})}, \frac{b - \frac{2(bc-c^2+2br-cr-r^2)}{2c+r}}{3(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})})$$

$$P_2 = (\frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b - (c - \frac{bc-c^2+2br-cr-r^2}{2c+r})(c + \frac{bc-c^2+2br-cr-r^2}{2c+r}) - (c+r)r}{3(b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r})},$$
$$\frac{b - 2c - r - \frac{2(bc-c^2+2br-cr-r^2)}{2c+r}}{3(b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r})}).$$

By Lemma 1.1.2 we can say that the following has to be true

$$D(A, P_1) = D(A, P_2)$$
$$D(B, P_1) = D(B, P_3).$$

Since we know the coordinates of all the points, we can use Code A.3 to express $D(A, P_1) = D(A, P_2)$ as

$$\frac{1}{9}(\frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b + \frac{(bc-c^2+2br-cr-r^2)^2}{(2c+r)^2}}{b - \frac{bc-c^2+2br-cr-r^2}{2c+r}} + \frac{3(bc - c^2 + 2br - cr - r^2)}{2c+r})^2$$
$$+ \frac{1}{9}(\frac{b - \frac{2(bc-c^2+2br-cr-r^2)}{2c+r}}{b - \frac{bc-c^2+2br-cr-r^2}{2c+r}} - 3)^2 =$$
$$= \frac{1}{9}(\frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b - (c - \frac{bc-c^2+2br-cr-r^2}{2c+r})(c + \frac{bc-c^2+2br-cr-r^2}{2c+r}) - (c+r)r}{b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r}}$$
$$+ \frac{3(bc - c^2 + 2br - cr - r^2)}{2c+r})^2 + \frac{1}{9}(\frac{b - 2c - r - \frac{2(bc-c^2+2br-cr-r^2)}{2c+r}}{b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r}} - 3)^2.$$

And $D(B,P_1) = D(B,P_3)$ can be expressed as

$$\frac{1}{9}(3b - \frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b + \frac{(bc-c^2+2br-cr-r^2)^2}{(2c+r)^2}}{b - \frac{bc-c^2+2br-cr-r^2}{2c+r}} + \frac{1}{9}(\frac{b - \frac{2(bc-c^2+2br-cr-r^2)}{2c+}}{b - \frac{bc-c^2+2br-cr-r^2}{2c+r}})^2 =$$

$$=\frac{1}{9}(3b - \frac{(b - \frac{bc-c^2+2br-cr-r^2}{2c+r})b - (c - \frac{bc-c^2+2br-cr-r^2}{2c+r})(c + \frac{bc-c^2+2br-cr-r^2}{2c+r}) - (c+r)r}{b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r}})^2$$

$$+ \frac{1}{9}(\frac{b - 2c - r - \frac{2(bc-c^2+2br-cr-r^2)}{2c+r}}{b - c - r - \frac{bc-c^2+2br-cr-r^2}{2c+r}})^2.$$

We can now use Code A.3 to solve these equations for b, with parameter $c \neq r$, which shows us that this case has no solution. These equations only have a solution when the $c = r$, at which point the region just becomes a rectangle.

Therefore, we can conclude that 2 generator points in a right trapezoid region have no centroidal Voronoi tessellations. □

3

CVTs with 3 Generator Points

Lemma 3.0.1. *Let P_1, P_2 and P_3 be points of a triangle. The perpendicular bisectors of the lines $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_3P_1}$ meet at a single point.*

Proof. Let the perpendicular bisectors of $\overline{P_1P_2}$ and $\overline{P_1P_3}$ intersect at a point O. Since any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, we can say that $D(O, P_1) = D(O, P_2)$, and $D(O, P_1) = D(O, P_3)$. Then, by the transitive property, $D(O, P_2) = D(O, P_3)$. We can therefore say that the point O is on the perpendicular bisectors of $\overline{P_2P_3}$. Since $D(O, P_1) = D(O, P_2) = D(O, P_3)$, then the point O is equidistant from P_1, P_2 and P_3. □

Voronoi tessellations for 3 generator points in a square region can be separated into two groups.

The first group consists of tessellations in which the perpendicular bisectors do not intersect inside the square, as can be seen in Figure 3.0.1a.

The second group consists of tessellations in which the perpendicular bisectors do intersect inside the square, as can be seen in Figure 3.0.1b. Furthermore, by Lemma 3.0.1 we know that that the only possibility when the perpendicular bisectors do intersect inside the square is when all three intersect in a common point.

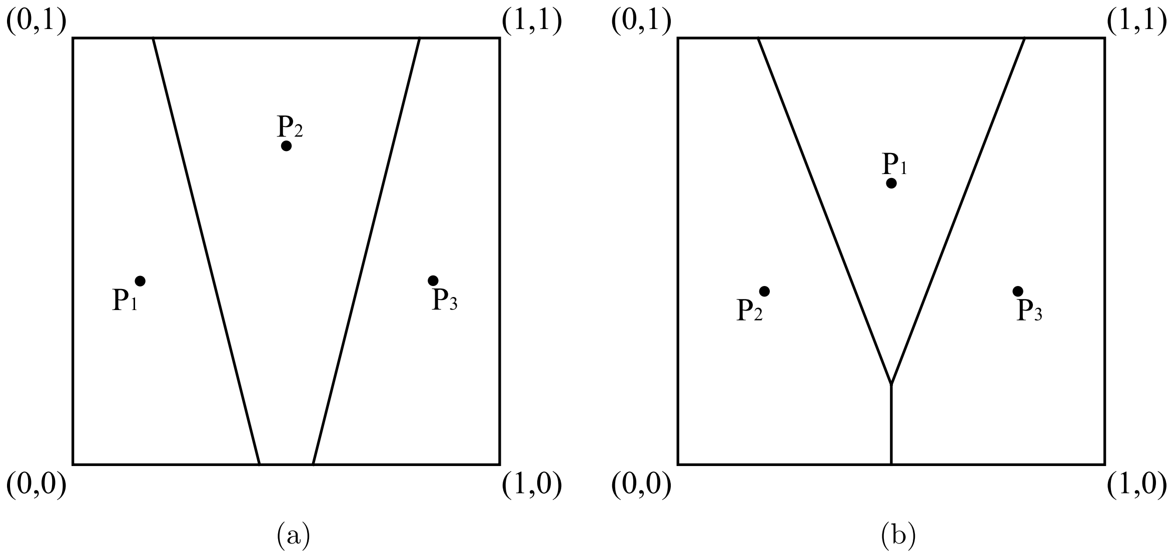

Figure 3.0.1.

3.1 Three Generator Points in a Square - Group 1

Theorem 3.1.1. *For* 3 *generator points in a square region where the perpendicular bisectors of the lines between the generator points do not intersect inside the square, there exists precisely one centroidal Voronoi tessellation.*

Proof. Let P_1, P_2 and P_3 be generator points in a square region. Let $A = (a, 1)$, $B = (b, 0)$, $C = (c, 1)$ and $R = (r, 0)$ be the points at which the perpendicular bisectors of the lines $\overline{P_1P_2}$ and $\overline{P_2P_3}$ intersect with the edge of the region, as can be seen in Figure 3.1.1,

Since we know for from Theorem 2.4.1 that a right trapezoid has no CVTs, we can say that the polygon made by the points $(0, 0)$, $(0, 1)$, C, R can not be a trapezoid, and thus can only be a rectangle. Therefore it must be true that $c = r$.

Furthermore, since the polygon made by the points $(0, 0)$, $(0, 1)$, C, R is a rectangle we can say that $a = r - b$.

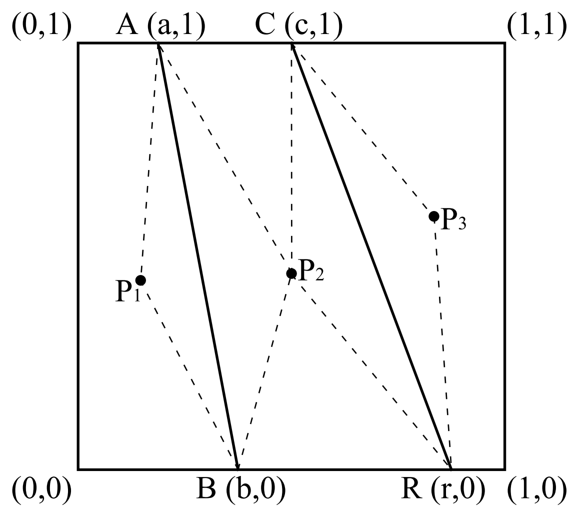

Figure 3.1.1.

Using Lemma 1.2.2 we can find the coordinates of P_1, P_2 and P_3, which are

$$P_1 = (\frac{(b-r)^2+br}{3r}r, -\frac{b-2r}{3r})$$
$$P_2 = (-\frac{(b-2r)b+br-2r^2}{3r}r, -\frac{b+r}{3r})$$
$$P_3 = (\frac{(r+1)(r-1)+2r^2-2}{6(r-1)}, \frac{1}{2}).$$

By Lemma 1.1.2 we can say that the following has to be true

$$D(A,P_1) = D(B,P_2)$$
$$D(A,P_1) = D(B,P_2)$$
$$D(C,P_2) = D(C,P_3)$$
$$D(R,P_2) = D(R,P_3).$$

Since we know the coordinates of all the points, we can use Code A.4 to express both $D(C,P_2) = D(C,P_3)$, $D(R,P_2) = D(R,P_3)$ $D(C,P_2) = D(C,P_3)$ and $D(R,P_2) =$

$D(R, P_3)$ as

$$\frac{1}{9}((3b-3r+\frac{(b-r)^2+br}{r})^2+(\frac{b-2r}{r}+3)^2)=\frac{1}{9}((3b-3r-\frac{(b-2r)b+br-2r^2}{r})^2+(\frac{b+r}{r}-3)^2),$$
$$\frac{1}{9}((3b-\frac{(b-r)^2+br}{r})^2+(\frac{b-2r}{r})^2)=\frac{1}{9}((3b+\frac{(b-2r)b+br-2r^2}{r})^2+(\frac{b+r}{r})^2),$$
$$\frac{1}{9}((3r+\frac{(b-2r)b+br-2r^2}{r})^2+(\frac{b+r}{r}-3)^2)=\frac{1}{36}(6r-\frac{(r+1)(r-1)+2r^2-2}{r-1})^2+\frac{1}{4},$$
$$\frac{1}{9}((3r+\frac{(b-2r)b+br-2r^2}{r})^2+(\frac{b+r}{r})^2)=\frac{1}{36}(6r-\frac{(r+1)(r-1)+2r^2-2}{r-1})^2+\frac{1}{4}$$

respectively.

We can now use Code A.4 to solve this equation for b and r, which gives us the following solutions

$$b=1, r=2$$
$$b=0, r=0$$
$$b=\frac{1}{3}, r=\frac{2}{3}.$$

The solution $b=1$, $r=2$ is invalid because it puts the point R outside the square. The solution $b=0$, $r=0$ is invalid because it puts the centroids P_1 and P_2 on the edges of the square, and we are only interested in situations where P_1, P_2 and P_3 are in the interior. Therefore, $b=\frac{1}{3}$, $r=\frac{2}{3}$ is the only valid solution. So $b=\frac{1}{3}$, $r=\frac{2}{3}$, $c=\frac{2}{3}$ and $a=\frac{1}{3}$, as illustrated in Figure 3.1.2, is the only centroidal Voronoi tessellation for 3 generator points in a square region where the perpendicular bisectors do not intersect inside the square.

□

3.2 Three Generator Points in a Square - Group 2

Theorem 3.2.1. *For* 3 *generator points in a square region where the perpendicular bisectors of the lines between the generator points DO intersect inside the square, and where the line* $x=\frac{1}{2}$ *is assumed to be one of the perpendicular bisectors, there exists precisely one centroidal Voronoi tessellation.*

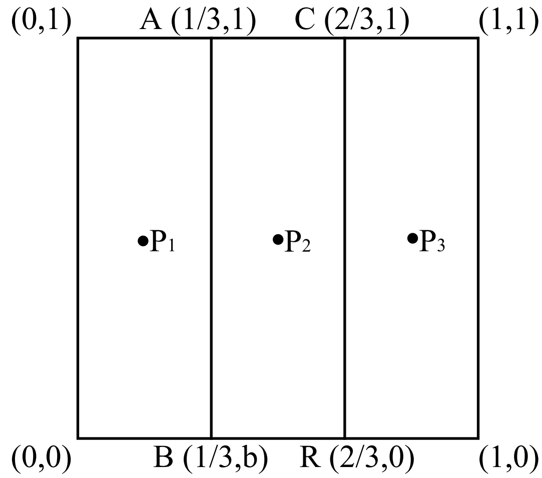

Figure 3.1.2.

Proof. Let P_1, P_2 and P_3 be generator points in a square region. Let $A = (0, a)$, $B = (1, b)$ and $C = (c, 0)$ be the points at which the perpendicular bisectors of the lines $\overline{P_1P_2}$, $\overline{P_1P_3}$ and $\overline{P_2P_3}$ intersect with the edge of the region, respectively. Since we know by Lemma 3.0.1 that all three of the perpendicular bisectors meet at a single point, let $G = (m, n)$ be the point where all three of the perpendicular bisectors meet, as can be seen in Figure 3.2.1.

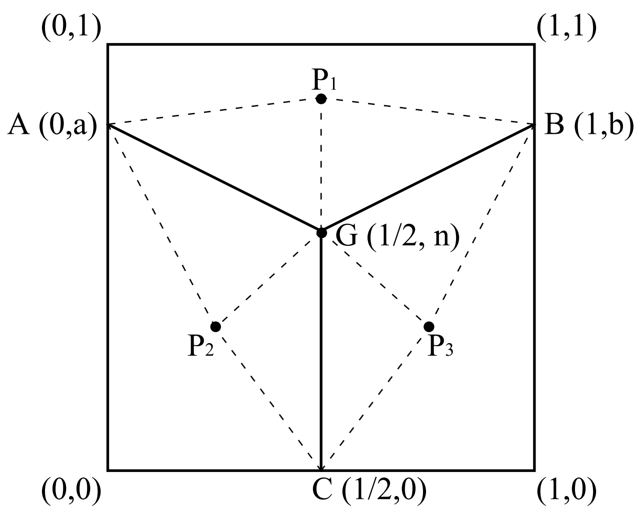

Figure 3.2.1.

Since we are assuming that the line $x = \frac{1}{2}$ is one of the perpendicular bisectors, we can say that the points C and G are on that line, and therefore $c = \frac{1}{2}$ and $m = \frac{1}{2}$.

Using Lemma 1.2.2 we can find the coordinates of P_1, P_2 and P_3, which are

$$P_1 = (\frac{a+5b+6n-12}{6(a+b+2n-4)}, \frac{(a+n)a-(b+n)(b-2n)+2(b+1)(b-1)-4}{a+b+2n-4})$$
$$P_2 = (\frac{a+2n}{6(a+n)}, \frac{(a+n)a+n^2}{3(a+n)})$$
$$P_3 = (\frac{5b+4n}{6(b+n)}, -\frac{(b+n)(b-2n)-2b^2+n^2}{3(b+n)}).$$

By Lemma 1.1.2 we can say that the following has to be true

$$D(A,P_1) = D(A,P_2)$$
$$D(G,P_1) = D(G,P_2)$$
$$D(B,P_1) = D(B,P_3)$$
$$D(G,P_1) = D(G,P_3)$$
$$D(C,P_2) = D(C,P_3)$$
$$D(G,P_2) = D(G,P_3).$$

Since we know the coordinates of all the points, we can use Code A.5 to express $D(A,P_1) = D(A,P_2)$ as

$$\frac{1}{9}(3a - \frac{(a+n)a-(b+n)(b-2n)+2(b+1)(b-1)-4}{a+b+2n-4})^2 + \frac{1}{36}(\frac{a+5b+6n-12}{a+b+2n-4})^2 =$$
$$= \frac{1}{9}(3a - \frac{(a+n)a+n^2}{a+n})^2 + \frac{1}{36}(\frac{a+2n}{a+n})^2,$$

$D(G,P_1) = D(G,P_2)$ as

$$\frac{1}{9}(3n - \frac{(a+n)a-(b+n)(b-2n)+2(b+1)(b-1)-4}{a+b+2n-4})^2 + \frac{1}{36}(\frac{a+5b+6n-12}{a+b+2n-4} - 3)^2 =$$
$$= \frac{1}{9}(3n - \frac{(a+n)a+n^2}{a+n})^2 + \frac{1}{36}(\frac{a+2n}{a+n} - 3)^2,$$

$D(B,P_1) = D(B,P_3)$ as

$$\frac{1}{9}(3b - \frac{(a+n)a-(b+n)(b-2n)+2(b+1)(b-1)-4}{a+b+2n-4})^2 + \frac{1}{36}(\frac{a+5b+6n-12}{a+b+2n-4} - 6)^2 =$$
$$= \frac{1}{9}(3b + \frac{(b+n)(b-2n)-2b^2+n^2}{b+n})^2 + \frac{1}{36}(\frac{5b+4n}{b+n} - 6)^2,$$

$D(G, P_1) = D(G, P_3)$ as

$$\frac{1}{9}(3n - \frac{(a+n)a-(b+n)(b-2n)+2(b+1)(b-1)-4}{a+b+2n-4})^2 + \frac{1}{36}(\frac{a+5b+6n-12}{a+b+2n-4} - 3)^2 =$$
$$= \frac{1}{9}(3n + \frac{(b+n)(b-2n)-2b^2+n^2}{b+n})^2 + \frac{1}{36}(\frac{5b+4n}{b+n} - 3)^2,$$

$D(C, P_2) = D(C, P_3)$ as

$$\frac{1}{9}(\frac{(a+n)a+n}{a+n})^2 + \frac{1}{36}(\frac{a+2n}{a+n} - 3)^2 = \frac{1}{9}(\frac{(b+n)(b-2n)-2b^2+n^2}{b+n})^2 + \frac{1}{36}(\frac{5b+4n}{b+n} - 3)^2,$$

and $D(G, P_2) = D(G, P_3)$ as

$$\frac{1}{9}(3n - \frac{(a+n)a+n}{a+n})^2 + \frac{1}{36}(\frac{a+2n}{a+n} - 3)^2 = \frac{1}{9}(3n + \frac{(b+n)(b-2n)-2b^2+n^2}{b+n})^2 + \frac{1}{36}(\frac{5b+4n}{b+n} - 3)^2.$$

We can now use Code A.5 to solve these equations for a, b and n, which gives us the following solutions

1. $a = \frac{1}{2}I$, $b = \frac{1}{2}I$, $n = -\frac{1}{2}I$
2. $a = -\frac{1}{2}I$, $b = -\frac{1}{2}I$, $n = \frac{1}{2}I$
3. $a = 1.00974195868289e - 28 - \frac{1}{2}I$, $b = -\frac{1}{2}I$, $n = \frac{1}{2}I$
4. $a = \frac{1}{2}I$, $b = \frac{1}{2}I$, $n = -3.82610927270291e - 16 - 0.4999999999999999 \cdot I$
5. $a = -\frac{1}{2}I$, $b = -\frac{1}{2}I$, $n = -7.86299641775921e - 16 + 0.4999999999999999 \cdot I$
6. $a = 0.1030571046735446 - 0.5475362769479363 \cdot I$, $b = 0.1030571046735446 - 0.5475362769479363 \cdot I$, $n = -0.1557575148939911 + 0.452543468512838 \cdot I$
7. $a = 0.1030571046735446 + 0.5475362769479363 \cdot I$, $b = 0.1030571046735446 + 0.5475362769479363 \cdot I$, $n = -0.1557575148939908 - 0.4525434685128378 \cdot I$

8. $a = 0$, $b = 0$, $n = 0$

9. $a = 1$, $b = 1$, $n = 1$

10. $a = -\frac{1}{2}$, $b = -\frac{1}{2}$, $n = 1$

11. $a = 1.400583090379009$, $b = 1.400583090379009$, $n = 0.1253255465235577$

12. $a = 1.070293398533007$, $b = 1.070293398533007$, $n = 0.9333089043605717$

13. $a = -0.1893382352941176$, $b = -0.1893382352941176$, $n = 0.2664015627180131$

14. $a = 0.7623474723997675$, $b = 0.7623474723997675$, $n = 0.4864789517702816$.

Solutions 1 - 7 are invalid because they contain imaginary numbers.

Solutions 8 and 9 are invalid because they put the centroids P_1, P_2 and P_3 on the edges of the square, and we are only interested in situations where P_1, P_2 and P_3 are in the interior.

Solutions 10, 11, 12 and 13 are invalid because the values of a and b are larger than 1 or smaller then 0, thus putting them outside the square.

Therefore, $a = 0.7623474723997675$, $b = 0.7623474723997675$, $n = 0.4864789517702816$ is the only solution. □

The above proof is not entirely conclusive, as it does not account for the possibility that the points A and B cross the edges of the region along a different line segment.

We will need to check for the pairs $A = (0, a)$ and $B = (b, 1)$ as seen in Figure 3.2.2a, $A = (a, 1)$ and $B = (b, 0)$ as seen in Figure 3.2.2b, $A = (a, 1)$ and $B = (b, 1)$ as seen in Figure 3.2.2c.

The pairs $A = (a, 1), B = (1, b)$ and $A = (a, 0), B = (b, 1)$ do not need to be checked because they are reflections of $A = (0, a), B = (b, 1)$ and $A = (a, 1), B = (b, 0)$ about the line $x = \frac{1}{2}$, respectively.

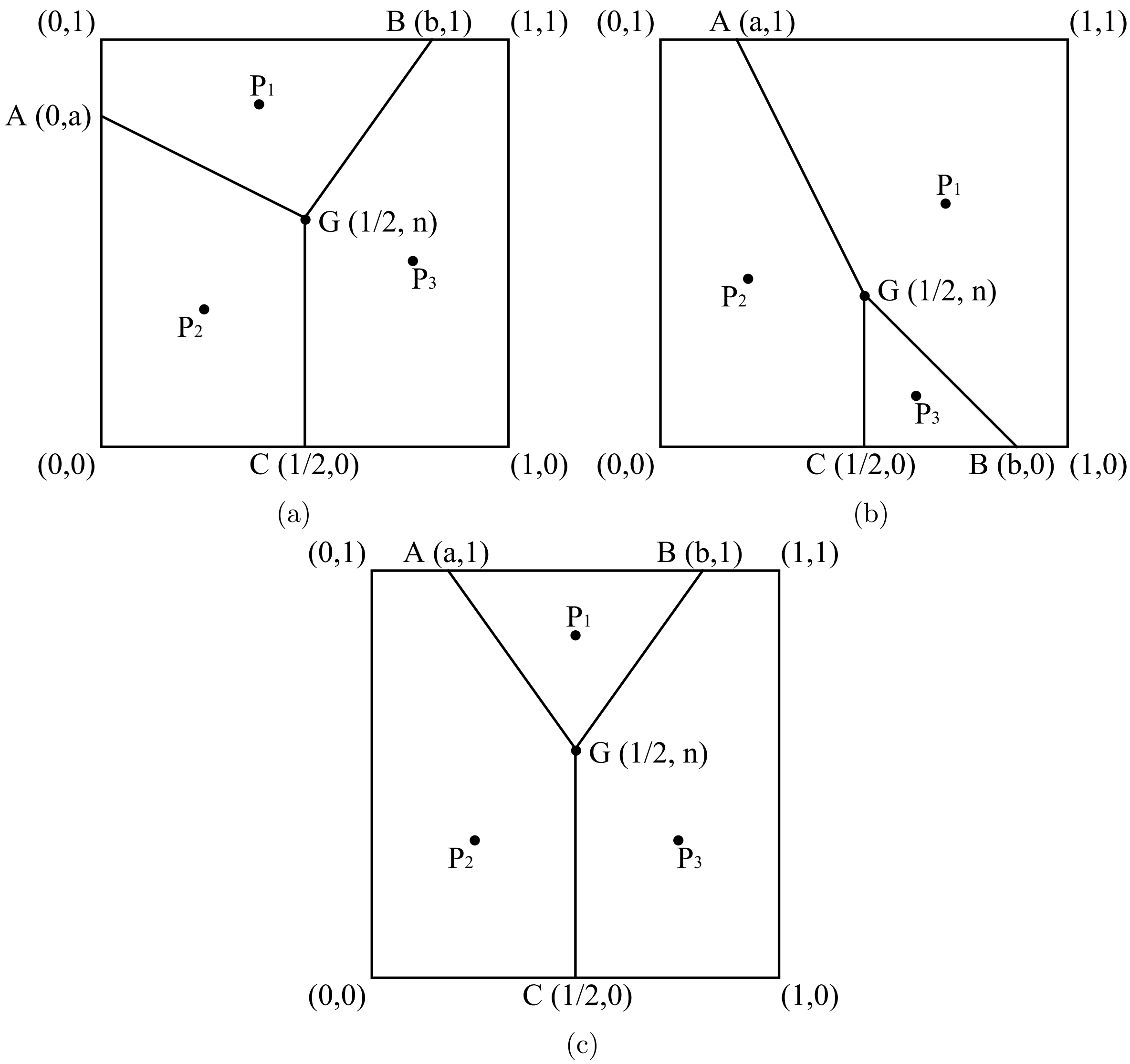
(0,1)
B (b,1)
(1,1)
A (0,a)
P1
G (1/2, n)
P3
P2
(0,0)
C (1/2,0)
(1,0)
(a)
(0,1)
A (a,1)
(1,1)
P1
P2
G (1/2, n)
P3
(0,0)
C (1/2,0)
B (b,0)
(1,0)
(b)
(0,1)
A (a,1)
B (b,1)
(1,1)
P1
G (1/2, n)
P2
P3
(0,0)
C (1/2,0)
(1,0)
(c)

Figure 3.2.2.

The pairs $A = (a, 0), B = (1, b)$ and $A = (0, a), B = (b, 0)$ do not need to be checked because they are reflections of $A = (a, 1), B = (1, b)$ and $A = (0, a), B = (b, 1)$ about the line $y = \frac{1}{2}$, respectively.

The pair $A = (a, 0), B = (b, 0)$, does not need to be checked because in it the polygon formed by generator point P_1 would be concave, and CVTs can only have convex polygons.

Repeating the process from the proof of Theorem 3.2.1 for $A = (0, a), B = (b, 1)$ gives us the following solutions

$$a = 1, b = (1/2), n = 1$$

$$a = 0.9999983117974968, b = 1.497287326388889, n = 0.9999979278862989$$

$$a = 1.0, b = 0.7506799637352675, n = 1.0$$

$$a = 1.0, b = 0.4915223987149741, n = 1.0$$

$$a = 0, b = (1/2), n = 0.$$

All of these solutions are invalid because when $a = n = 1, 0$, it puts the centroids P_2 and P_3 on the edges of the square, and we are only interested in situations where P_2 and P_3 are in the interior.

Repeating the process from the proof of Theorem 3.2.1 for $A = (a, 1), B = (b, 0)$ gives us the following solution

$$a = \frac{1}{2}, b = \frac{5}{4}, n = \frac{3}{2}.$$

This solution is invalid because it puts points B and G outside the square.

Repeating the process from the proof of Theorem 3.2.1 for $A = (a, 1), B = (b, 1)$ gives us the following solutions

$$a = \frac{1}{2}, b = \frac{5}{4}, n = -\frac{1}{2}$$
$$a = -\frac{1}{4}, b = \frac{1}{2}, n = -\frac{1}{2}$$
$$a = -\frac{1}{2}\sqrt{2} + \frac{1}{2}, b = \frac{1}{2}\sqrt{2} + \frac{1}{2}, n = -\frac{1}{2}$$
$$a = \frac{1}{2}\sqrt{2} + \frac{1}{2}, b = -\frac{1}{2}\sqrt{2} + \frac{1}{2}, n = -\frac{1}{2}$$
$$a = 1.851430158289364, b = -0.8514301582893641, n = 2.601207590569293$$
$$a = 2.499010257333093, b = -1.499010257333093, n = 0.7766141097939924$$
$$a = -0.1507406988578537, b = 1.150740740740741, n = -0.4355982612674445$$
$$a = 0.2328028055031022, b = 0.7671971944968977, n = -0.3731344371247012$$
$$a = 2.499010257333093, b = -1.499010257333093, n = 0.7766141097939924$$
$$a = -0.1507406988578537, b = 1.150740740740741, n = -0.4355982612674445$$
$$a = 0.2328028055031022, b = 0.7671971944968977, n = -0.3731344371247012$$
$$a = 1.851430158289364, b = -0.8514301582893641, n = 2.601207590569293$$
$$a = (0.01173174360500562 + 1.146401586303696 \cdot I), b = (0.9882682563949989 - 1.146401586303694 \cdot I), n = (0.6944841956433466 - 1.162959733078031 \cdot)$$
$$a = (0.01173174360500235 - 1.146401586303706 \cdot I), b == (0.9882682563949955 + 1.146401586303684 \cdot I), n = (0.6944841956433466 + 1.162959733078031 \cdot I)$$
$$a = (-0.2279829885120222 - 0.3357976917092346 \cdot I), b = (1.227982988512035 + 0.3357976917092294 \cdot I), n = (0.5209712936710452 - 0.02090145223396259 \cdot I)$$
$$a = (-0.2279829885120221 + 0.3357976917092346 \cdot I), b = (1.227982988512035 - 0.3357976917092294 \cdot I), n = (0.5209712936710452 + 0.02090145223396259 \cdot I)$$
$$a = (0.4298731892826677 + 1.467511436524454 \cdot I), b = (0.5701268107173323 - 1.467511436524454 \cdot I), n = (0.9675114365244545 - 0.6798731892826693 \cdot I)$$

$a =(0.4298731892826694 - 1.467511436524453 \cdot I), b = (0.5701268107173305+ 1.467511436524453 \cdot I), n = (0.9675114365244545 + 0.6798731892826693 \cdot I)$

$a =(-0.1798731892826694 + 0.03248856347554584 \cdot I), b = (1.179873189282669- 0.03248856347554584 \cdot I), n = (-0.4675114365244545 - 0.07012681071733087 \cdot I)$

$a =(-0.1798731892826694 - 0.0324885634755458 \cdot I), b = (1.179873189282669+ 0.0324885634755458 \cdot I), n = (-0.4675114365244545 + 0.07012681071733087 \cdot I)$

$a =(0.01173174360500562 + 1.146401586303696 \cdot I), b = (0.9882682563949944- 1.146401586303696 \cdot I), n = (0.6944841956433466 - 1.162959733078031 \cdot I)$

$a =(0.01173174360500235 - 1.146401586303706 \cdot I), b = (0.9882682563949976+ 1.146401586303706 \cdot I), n = (0.6944841956433466 + 1.162959733078031 \cdot I)$

$a =(-0.2279829885120222 - 0.3357976917092346 \cdot I), b = (1.227982988512022+ 0.3357976917092346 \cdot I), n = (0.5209712936710452 - 0.02090145223396259 \cdot I)$

$a =(-0.2279829885120221 + 0.3357976917092346 \cdot I), b = (1.227982988512022- 0.3357976917092346 \cdot I), n = (0.5209712936710452 + 0.02090145223396259 \cdot I).$

The first eleven of these solutions are invalid because the values of a, b and n are higher than 1 or lower then 0, thus putting them outside the square. The rest of the solutions are invalid because they contain imaginary numbers.

Therefore, we can conclude that $A = (0, a)$ and $B = (1, b)$, as shown in Figure 3.2.1, is the only situation where a centroidal Voronoi tessellation exists.

Appendix A
Sage Code

A.1 Two Generator Points in a Square

```
a, b = var('a, b')


def Area(L):
    v = (1/2) * sum(L[i][0] * L[(i+1)%(len(L))][1]
                   - L[(i+1)%(len(L))][0] * L[i][1] for i in range(len(L)))
    return v


def Cx(L):
    v = (1/(6 * Area(L))) * sum((L[i][0]+L[(i+1)%(len(L))][0])
                   * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                   * L[i][1]) for i in range(len(L)))
    return v


def Cy(L):
    v = (1/(6 * Area(L))) * sum((L[i][1]+L[(i+1)%(len(L))][1])
                   * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
```

```
                    * L[i][1]) for i in range(len(L)))
    return v


def Cent(L):
    v = [Cx(L), Cy(L)]
    return v


def D(A, B):
    v = (A[0] - B[0])^2 + (A[1] - B[1])^2
    return v


L = [[0,0], [0,1], [1,1], [1,0]]
A = [1-b,1]
B = [b,0]
P1 = [L[0], L[1], A, B]
P2 = [B, A, L[2], L[3]]

solve ([D(A, Cent(P1)) == D(A, Cent(P2)), D(B, Cent(P1)) == D(B, Cent(P2))], b)
```

A.2 Two Generator Points in a Rectangle

```
r, a, b = var('r, a, b')


def Area(L):
    v = (1/2) * sum(L[i][0] * L[(i+1)%(len(L))][1]
                    - L[(i+1)%(len(L))][0] * L[i][1] for i in range(len(L)))
    return v
```

```
def Cx(L):
    v = (1/(6 * Area(L))) * sum((L[i][0]+L[(i+1)%(len(L))][0])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cy(L):
    v = (1/(6 * Area(L))) * sum((L[i][1]+L[(i+1)%(len(L))][1])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cent(L):
    v = [Cx(L), Cy(L)]
    return v


def D(A, B):
    v = (A[0] - B[0])^2 + (A[1] - B[1])^2
    return v


L = [[0,0], [0,1], [r,1], [r,0]]
A = [r-b,1]
B = [b,0]
G = Cent(L)
P1 = [L[0], L[1], A, B]
P2 = [B, A, L[2], L[3]]
```

```
solve ([D(A, Cent(P1)) == D(A, Cent(P2)), D(B, Cent(P1)) == D(B, Cent(P2))], b)
```

A.3 Two Generator Points in a Trapezoid

```
r, a, b, c, m, n = var('r, a, b, c, m, n')


def Area(L):
    v = (1/2) * sum(L[i][0] * L[(i+1)%(len(L))][1]
                    - L[(i+1)%(len(L))][0] * L[i][1] for i in range(len(L)))
    return v


def Cx(L):
    v = (1/(6 * Area(L))) * sum((L[i][0]+L[(i+1)%(len(L))][0])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cy(L):
    v = (1/(6 * Area(L))) * sum((L[i][1]+L[(i+1)%(len(L))][1])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cent(L):
    v = [Cx(L), Cy(L)]
    return v


def D(A, B):
```

```
    v = (A[0] - B[0])^2 + (A[1] - B[1])^2
    return v


L = [[0,0], [0,1], [c,1], [r,0]]
a = (c^2 + r^2 + c*r - b*c - 2*b*r)/(2*c + r)
#or a = (Cent(L)[0] - b)/Cent(L)[1] + b
A = [a,1]
B = [b,0]
P1 = [L[0], L[1], A, B]
P2 = [B, A, L[2], L[3]]

solve ([D(A, Cent(P1)) == D(A, Cent(P2)), D(B, Cent(P1)) == D(B, Cent(P2))], b)
```

A.4 Three Generator Points in a Square - 1

```
r, a, b, c = var('r, a, b, c')


def Area(L):
    v = (1/2) * sum(L[i][0] * L[(i+1)%(len(L))][1]
                       - L[(i+1)%(len(L))][0] * L[i][1] for i in range(len(L)))
    return v


def Cx(L):
    v = (1/(6 * Area(L))) * sum((L[i][0]+L[(i+1)%(len(L))][0])
                       * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                       * L[i][1]) for i in range(len(L)))
    return v
```

```
def Cy(L):
    v = (1/(6 * Area(L))) * sum((L[i][1]+L[(i+1)%(len(L))][1])
                   * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                   * L[i][1]) for i in range(len(L)))
    return v


def Cent(L):
    v = [Cx(L), Cy(L)]
    return v


def D(A, B):
    v = (A[0] - B[0])^2 + (A[1] - B[1])^2
    return v


L = [[0,0], [0,1], [1,1], [1,0]]
A = [r-b,1]
B = [b,0]
C = [r,1]
R = [r,0]
P1 = [L[0], L[1], A, B]
P2 = [B, A, C, R]
P3 = [R, C, L[2], L[3]]

solve ([D(A, Cent(P1)) == D(A, Cent(P2)), D(B, Cent(P1)) == D(B, Cent(P2)),
             D(C, Cent(P2)) == D(C, Cent(P3)), D(R, Cent(P2)) == D(R, Cent(P3))], r)
```

A.5 Three Generator Points in a Square - 2

```
a, b, c, m, n = var('a, b, c, m, n')


def Area(L):
    v = (1/2) * sum(L[i][0] * L[(i+1)%(len(L))][1]
                    - L[(i+1)%(len(L))][0] * L[i][1] for i in range(len(L)))
    return v


def Cx(L):
    v = (1/(6 * Area(L))) * sum((L[i][0]+L[(i+1)%(len(L))][0])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cy(L):
    v = (1/(6 * Area(L))) * sum((L[i][1]+L[(i+1)%(len(L))][1])
                    * (L[i][0] * L[(i+1)%(len(L))][1] - L[(i+1)%(len(L))][0]
                    * L[i][1]) for i in range(len(L)))
    return v


def Cent(L):
    v = [Cx(L), Cy(L)]
    return v


def D(A, B):
    v = (A[0] - B[0])^2 + (A[1] - B[1])^2
    return v
```

```
L = [[0,0], [0,1], [1,1], [1,0]]

A = [0,a]

B = [1,b]

C = [1/2,0]

G = [1/2,n]

P1 = [L[1], L[2], B, G, A]

P2 = [A, G, C, L[0]]

P3 = [B, G, C, L[3]]


solve ([D(A, Cent(P1)) == D(A, Cent(P2)), D(G, Cent(P1)) == D(G, Cent(P2)),
         D(B, Cent(P1)) == D(B, Cent(P3)), D(G, Cent(P1)) == D(G, Cent(P3)),
         D(C, Cent(P2)) == D(C, Cent(P3)), D(G, Cent(P2)) == D(G, Cent(P3))], a, b, n)
```

Bibliography

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