Connectedness in Cayley Graphs and P/NP Dichotomy for Quay Algebras

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Abstract

This senior thesis attempts to determine the extent to which the P/NP dichotomy of finite algebras (as proven by Bulatov, et.al in 2017) can be cast in terms of connectedness in Cayley graphs. This research is motivated by Prof. Robert McGrail's work "CSPs and Connectedness: P/NP-Complete Dichotomy for Idempotent, Right Quasigroups" published in 2014 in which he demonstrates the strong correspondence between tractability and total path-connectivity in Cayley graphs for right, idempotent quasigroups. In particular, we will introduce the notion of total V-connectedness and show how it could be potentially used to phrase the dichotomy in terms of connectivity for another class of algebras, namely for Quay algebras.

Contents

Dedication

To my parents.

Acknowledgments

First, I would like to thank my adviser Prof. Robert McGrail for his constant guidance, commitment, support and patience throughout this project as well as the past four years. Without his help and encouragement this thesis would not be possible.

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Last but not least, I would like to thank my friends and everyone else who made my experience at Bard a precious and unforgettable one.

1
Introduction

Constraint Satisfaction Problems (CSPs) [1] are mathematical problems expressed as a set of objects: a variable set, a value domain and a set of constraints. Their goal is to produce variable assignments which satisfy given limitations. CSPs are widely used in the field of artificial intelligence as well as operations research. In 2017, Andrei A. Bulatov [3] proved that for every constraint language Γ, the problem $CSP(\Gamma)$ is either solvable in polynomial (P) or non-deterministic polynomial (NP) time. In Prof. Robert McGrail's paper "CSPs and Connectedness: P/NP-Complete Dichotomy for Idempotent, Right Quasigroups" published in 2014 [2], the authors proved that an idempotent, right quasigroup Q is path-connected in its right Cayley graph if and only if it has a Merling term. They reduced the problem of P/NP classification of such quasigroups to the search for Merling terms in Q. The goal of this project is to continue the authors' research and attempt to express the P/NP dichotomy of finite algebras in terms of connectedness of their Cayley graphs. In order to do so, we restrict our study to binary algebras due to the fact that Cayley graphs are generally constructed from algebras with binary operations. We start our research by examining Cayley graphs of algebras with a weak near-unanimity term, an operation whose existence implies tractability.

1.1 Summary

First, we discuss the background knowledge behind our research. In Chapter 2, we describe Constraint Satisfaction Problems and in the next chapter introduce the idea of a CSP over an algebra. In the fourth chapter we proceed to summarize the results of the aforementioned publication [3] regarding the dichotomy of finite CSPs. Chapter 5 presents our results on connectivity of a Cayley graph with regard to the existence of a near-unanimity and weak near-unanimity term. We also introduce notions of weak-connectedness and V-connectivity. In the next chapter we conclude the results from Prof. McGrail's paper "CSPs and Connectedness: P/NP-Complete Dichotomy for Idempotent, Right Quasigroups" [2] as well as phrase the dichotomy for this group of algebras using the notion of V-connectedness. Lastly, Chapter 7 describes our results on V-connectedness for Quay algebras.

2
Constraint Satisfaction Problems (CSPs)

Definition 2.0.1. *A Constraint Satisfaction Problem (CSP)* [1] is formally defined as a triple $\langle X, D, C \rangle$ where:

- $X = \{x_1, \ldots, x_n\}$ is a variable set,
- $D = \{d_1, \ldots, d_n\}$ is a nonempty domain of values for each variable,
- $C = \{c_1, \ldots, c_k\}$ is set of constraints.

Each constraint $c_i \in C$ is a pair (t_j, ρ_j) where t_j is a tuple of variables of length m_j, called the *constraint scope*, and ρ_j is an m_j-ary relation on the corresponding domains, called the *constraint relation*. A *state* is an assignment of values to some (incomplete assignment) or all (complete assignment) variables. An assignment is *consistent* if it does not violate any constraints. The *solution* to a CSP is a *complete* and *consistent* assignment that satisfies all the constraints. Some CSPs also require a solution to maximize an objective function. △

There are CSPs with *discrete* or *continuous* variables. Discrete CSPs can have finite or infinite domains. Boolean CSPs are examples of the former group where the variables can either take *true* or *false* values. They include instances which are known to be NP-complete. Many scheduling problems are instances of the latter type of CSPs. With infinite domains, a *constraint language* must be employed in order to express the variable constraints. CSPs with continuous variables are commonly studied in operations research, including the problem of scheduling experiments on the Hubble Space Telescope [10].

We can also differentiate constraint satisfaction problems based on the type of their constraints. The varieties comprise of *unary*, *binary* and *higher-order* constraints depending on whether they relate one, two or more than two variables, respectively.

Example 2.0.1. 3-satisfiability (called 3-SAT or 3CNFSAT) is an example of a boolean constraint satisfaction problem that is known to be NP-complete [9]. The goal of this problem is to determine the satisfiability of a formula in conjunctive normal form (CNF) where each clause has at most three literals. For example, $(x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_5) \wedge (x_4 \vee \neg x_6)$ is a 3-SAT expression. It has 3 clauses, each with no more than 3 literals. We can rewrite this 3-SAT instance into constraints of form $\langle (x_{i_1}, x_{i_2}, x_{i_n}), R\rangle$ where R is a relation over the given expression. Thus we obtain a set C containing the following constraints:

$$c_1 = \langle (x_1, x_2, x_4), \{(1,0,0), (1,0,1), (1,1,1), (1,1,0), (0,1,0), (0,0,0), (0,0,1)\}\rangle$$
$$c_2 = \langle (x_2, x_3, x_5), \{(1,0,0), (1,0,1), (1,1,1), (1,1,0), (0,1,0), (0,1,1), (0,0,1)\}\rangle$$
$$c_3 = \langle (x_4, x_6), \{(1,0), (0,0), (1,1)\}\rangle.$$

Notice, each of these constraints contain all possible variable assignments except for the ones that make a given clause false. A solution to this problem is an assignment of truth values from $D = \{0 \text{ (false)}, 1 \text{ (true)}\}$ to our variable set $X = \{x_1, \ldots, x_6\}$ so that all the constraints above are satisfied and thus force the Boolean expression to evaluate to true. $\lozenge$

3
CSPs and Algebras

3.1 Definitions

Definition 3.1.1. In universal algebra, an *algebra* is a structure which consists of a set of elements A together with a collection of operations on A. An *k-ary operation* on A is a function that takes k elements from A and returns one element of A. Algebras with one basic operation are usually expressed by their Cayley tables. $\triangle$

Example. The projection algebra U_n is a binary algebra $(A, *)$ on the underlying set $A = \{0, 1, \ldots, n\}$ defined by $x * y = x$ for all $x, y \in A$. The Cayley table for the algebra U_2 looks as following: $\Diamond$

*	0	1
0	0	0
1	1	1

Figure 3.1.1. Cayley Table for U_2.

Definition 3.1.2. A binary algebra A with an operator $*$ is *idempotent* if for all $x \in A$ we have $x * x = x$. $\triangle$

Definition 3.1.3. Let A be an algebra. Then B is a *subalgebra* of A, denoted by $B \leq A$ if $B \subseteq A$ and every fundamental operation of B is the restriction of the corresponding operation of A, i.e., for each function symbol f, f^B is f^A restricted to B. △

Definition 3.1.4. Let A be an algebra. Then a *subpower* of A is a subalgebra of A^n. By $Sub(A)$ we denote the class of subpowers of A. △

Definition 3.1.5. Let A be a finite algebra and let the set of all subsets of A^n (relations) for some n be denoted as R_A. Then all the constraint relations can be expressed as relations from R_A. For a set of relations $\Gamma \in R_A$, $CSP(\Gamma)$ is a problem in which all constraint relations come from Γ. This problem is referred to as a *nonuniform CSP* and the set Γ is called a *constraint language*. Notice, in $CSP(\Gamma)$ an instance is a pair (V, C) where V is the set of variables and C the set of constraints. Each constraint $c_i \in C$ is a pair $\langle s_i, R_i \rangle$, where $s_i = (v_1, \ldots, v_n)$ is a variable tuple and $R_i \in \Gamma$ is an n-ary constraint relation. The solution of $CSP(\Gamma)$ is a mapping $\varphi : V \to A$ s.t. for every $c_i \in C$, $\varphi(s_i) \in R_i$. △

Definition 3.1.6. Let F denote the set of operations on an algebra A. Then $CSP(A)$ can be expressed as a class of nonuniform CSPs as follows $CSP(A) = \{CSP(\Gamma) \mid \Gamma \leq Sub(A)\}$. △

Example. The *3-coloring* problem is an example of a CSP over the projection algebra U_3, where U_3 consists of the underlying set $\{0, 1, 2\}$. 3 coloring is known to be NP-complete. Given a graph $G(V, E)$, where V is a set of vertices (or variables in the language of CSPs) and E a set of edges, the coloring problem asks whether there exists an assignment of 3 colors to the vertices $c : V \to \{0, 1, 2\}$ such that no two adjacent vertices have the same color: $\forall (u, v) \in E$, $c(u) \neq c(v)$.

I.e. consider graph G with $V = \{v_1, \ldots, v_5\}$ and $E = \{e_1, \ldots, e_5\}$ shown in Figure 3.1.2:

For each edge we will have a constraint of form $c_e = \langle (v_n, v_m), R_k \rangle$, where $v_n, v_m \in V$ and R_k represents a constraint relation consisting of tuples $t_{ij} = (x_i, x_j)$ where $x_i, x_j \in \{0, 1, 2\}$. Consider the following set of constraints:

$C =$

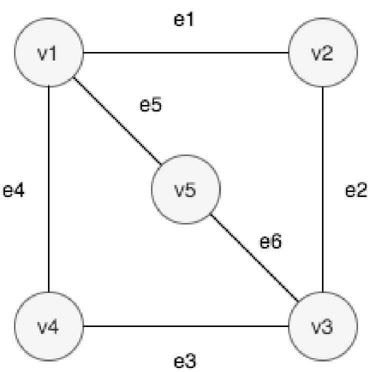

Figure 3.1.2. Graph G with five vertices and 6 edges.

$\{\langle(v_1, v_2), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle,$

$\langle(v_1, v_5), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle,$

$\langle(v_1, v_4), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle,$

$\langle(v_2, v_3), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle,$

$\langle(v_3, v_4), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle,$

$\langle(v_3, v_5), \{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle\}.$

The solution to this problem is an assignment of $\{v_1, v_2, \ldots, v_5\}$ to $\{0, 1, 2\}$ that is complete and does not violate any constraint c in C.

Consider an assignment f such that $f(v_1) = 0$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 2$ and $f(v_5) = 1$. Notice, f does not violate any of the constraints. Thus G is tri-colorable. The 3-coloring of G yielded by this assignment is visualized below in Figure 3.1.3. We assign the values 0,1,2 to colors green, red and blue, respectively.

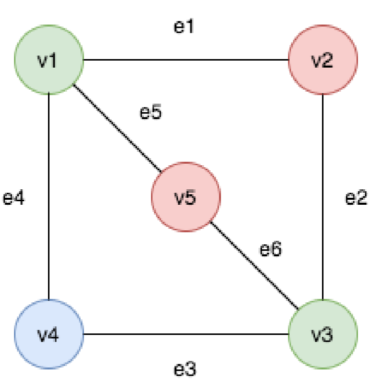

Figure 3.1.3. Instance of 3-coloring for Graph G.

$\Diamond$

3.2 Background

In 1999, Feder and Vardi conjectured that the class of all CSPs exhibits P/NP-complete dichotomy [5]. If a CSP is not NP-complete, then it is tractable. It is still unknown whether the class of NP-complete problems is distinct from the class of P problems.

4

Proof of The Dichotomy Conjecture

There have been many advances to prove the P/NP-complete dichotomy theorem for all algebras. Peter E. Jeavons et al. observed that polymorphisms (higher order symmetries of constraint languages) are of great importance to the study of CSPs.

4.1 Definitions

Definition 4.1.1. A *polymorphism* of a relation R over an algebra A is an operation $f(x_1, \ldots, x_n)$ on A such that for any $a_1, \ldots, a_k \in R$, $f(a_1, \ldots, a_n) \in R$. In this case, we say that R is invariant with respect to f. Thus, a polymorphism of a constraint language Γ is an operation which is a polymorphism of every $R \in \Gamma$. △

Definition 4.1.2. A *k-ary weak near-unanimity operation (WNU)* is a polymorphism $w(x_1, \ldots, x_k)$ over A such that for any $x, y \in A$ we have $w(y, x, \ldots, x) = \cdots = w(x, \ldots, x, y) = z$. △

Definition 4.1.3. A *k-ary near-unanimity operation (NU)* is a polymorphism $u(x_1, \ldots, x_k)$ over A such that for any $x, y \in A$ we have $u(y, x, \ldots, x) = \cdots = u(x, \ldots, x, y) = x$. △

Notice, a near-unanimity term is an instance of a weak near-unanimity operation, however with a stronger condition.

4.2 Background

In his paper *"A dichotomy theorem for nonuniform CSPs" (2017)*, Bulatov states that by Definition (3.1), tractability of $CSP(A)$ can be understood as the existence of a polynomial time algorithm for each $CSP(\Gamma)$ from this class or a uniform polynomial time algorithm for all of the problems. He shows that these notions of tractability are equivalent. He also proves that for any constraint language Γ over a finite set of problems, $CSP(\Gamma)$ is either solvable in polynomial time or is NP-complete.

Ultimately, he states this result in a stronger form and shows that for an idempotent, finite algebra $\mathbb{A}$ the following are equivalent:

1. $CSP(\mathbb{A})$ is solvable in polynomial time,
2. $\mathbb{A}$ has a weak near-unanimity term,
3. every algebra from $HS(\mathbb{A})$ (which is a set of all quotient algebras of all subalgebras of $\mathbb{A}$) has a nontrivial term operation of form $f(x_1, \ldots, x_n = x_i)$ for some $n, k \in \mathbb{N}$, which is not a projection.

Otherwise, $CSP(\mathbb{A})$ is NP-complete. Bulatov presents a polynomial time algorithm for solving CSPs which returns a solution if one exists.

In 2017, the same result was obtained by Dimitriy Zhuk in his independent work *"A Proof of CSP Dichotomy Conjecture"* [4]. The author introduces an algorithm that solves a CSP in polynomial time for constraint languages with a weak near unanimity polymorphism and, using this result, proceeds to confirm the dichotomy conjecture.

5
Cayley Graphs and Connectivity

Definition 5.0.0. In graph theory a *directed path* is a path in which the edges are all oriented in the same direction. For example, a path $x \to y \to z$ is a directed path. △

Definition 5.0.1. A *Cayley graph* is a diagram which encodes the structure of an algebra. We differentiate between a right and a left Cayley graph. Let A be an algebra with binary operations $\{*, /\}$.

The *right* $Cayley(A)$ is a graph where each element of A is assigned a vertex and there is an edge between x and y in A labeled a if $x * a = y$ or $x/a = y$.

Similarly, the *left* $Cayley(A)$ is a graph where each element of A is assigned a vertex and there is an edge between x and y in A such that $a * x = y$ or $a/x = y$. △

Each $Cayley(A)$ is *path connected* if there exists a directed path from each vertex to another vertex. If both *right* $Cayley(A)$ and *left* $Cayley(A)$ are path connected, then we say that A is path connected. $Cayley(A)$ is *weakly connected* if there exists an undirected path between any pair of vertices. Analogously, if both *right* and *left* $Cayley(A)$ are weakly-connected, then we describe A as weakly-connected. Otherwise, A is said to be *disconnected.*

Example. The following Figure 5.0.1 presents the Cayley table and Cayley graph for an algebra, which we will later refer to as the Tait quandle. Observe, *right* $Cayley(A)$ is exactly same as *left* $Cayley(A)$, therefore we speak of one $Cayley(Tait)$ graph. Notice, $Cayley(Tait)$ is path-connected, and therefore $Tait$ is path-connected. ◊

*, /	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

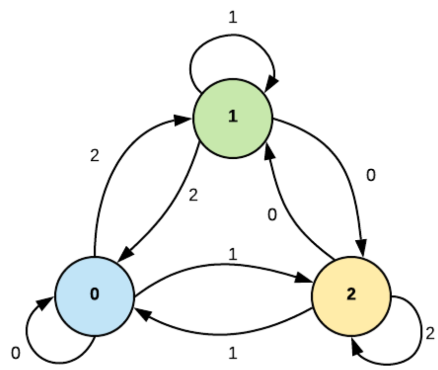

Figure 5.0.1. Cayley table and graph for the Tait quandle.

In his paper on the dichotomy result for idempotent right quasigroups, Prof. McGrail demonstrates that if an algebra A has a U_2 in its variety, then it is NP-complete (Lemma 13). He proves this by arguing that due to U_2 being NP-complete by its unary factor, there exists an NP-complete CSP over A. Hence A is also NP-complete.

Lemma 5.0.1. *If an algebra A has a projection algebra U_2 or T_2 which is a transpose (twisted version) of U_2 in its variety, then A is NP-complete* [2].

Proof. McGrail proves this by showing that there is a subpower $R \leq A^n$ such that there exists a homomorphism $h : R \to U_2$. Since U_2 is known to be NP-complete by its unary factor, then there exists an NP-complete CSP over R which is also a CSP over A. Hence CSP(A) is NP-compete and therefore A is also NP-complete. This is certainly true for T_2 as well, since the 3-SAT problem is also a problem over the transpose of U_2. □

5.1 Weak Near-Unanimity Terms

Definition 5.1.1. Let A be an algebra with binary operations and let $t(x_1, \ldots, x_k)$ be a k-nary operation of A. Let '$\circ$' denote any operator in A. Using recursion on the number of instances of operators in the term t, denoted as n, we define the notion of the *leftmost variable* in t as follows:

Base case: $n = 0$. We have $t(x_1, \ldots, x_k) = x_i$ for some $i \in \langle 1, k \rangle$. Then x_i is the *leftmost variable* in t.

For any $n \geq 1$ we have:

$t(x_1, \ldots, x_k) = r(x_1, x_2, \ldots, x_k) \circ_i s(x_1, x_2, \ldots, x_k)$ where r and s are some k-nary operations of A. Let the leftmost variable of t be the leftmost variable of r. $\triangle$

Definition 5.1.2. Let A be an algebra with binary operations and let $t(x_1, \ldots, x_k)$ be a k-nary operation of A. Let '$\circ$' denote any operator in A. Using recursion on the number of instances of operators in the term t, denoted as n, we define the notion of the *rightmost variable* in t as follows:

Base case: $n = 0$. We have $t(x_1, \ldots, x_k) = x_i$ for some $i \in \langle 1, k \rangle$. Then x_i is the *rightmost variable* in t.

For any $n \geq 1$ we have:

$t(x_1, \ldots, x_k) = r(x_1, x_2, \ldots, x_k) \circ_i s(x_1, x_2, \ldots, x_k)$ where r and s are some k-nary operations of A. Let the rightmost variable of t be the rightmost variable of s. $\triangle$

These two definitions will be used in Theorem (5.2.1) and Theorem (5.3.1) in the next two sections.

5.2 Existence of a Near-Unanimity Term Implies Path-connectedness

Theorem 5.2.1. *Let $u(x_1, \ldots, x_k)$ be a k-ary near-unanimity operation over an algebra A with binary operations. Then both left and right $Cayley(A)$ are path-connected. Thus A is path-connected.*

Proof. We will prove the path-connectedness of the *right Cayley(A)* and then *left Cayley(A)* respectively.

Case 1: Let x_i denote the *leftmost variable* in the near-unanimity operation u. We know that $\forall x, y \in A$, $u(x, x, \ldots, y) = u(x, x, \ldots, y, x) = \cdots = u(y, x, x, \ldots, x) = x$. We consider the instance in which we place y at the position of the i-th variable: $t(\ldots, y, \ldots) = x$. Notice, this term starts with y and ends with x. Thus, for every pair $(x, y) \in A$ there exists a path from y to x and by analogy from x to y in right $Cayley(A)$. We conclude that *right Cayley(A)* is path-connected.

Case 2: Let x_i denote the *rightmost variable* in the near-unanimity operation u. We know that $\forall x, y \in A$, $u(x, x, \ldots, y) = u(x, x, \ldots, y, x) = \cdots = u(y, x, x, \ldots, x) = x$. We consider the instance in which we place y at the position of the i-th variable: $t(\ldots, y, \ldots) = x$. Notice, this term starts with y and ends with x. Thus, for every pair $(x, y) \in A$ there exists a path from y to x and analogically from x to y in *left Cayley(A)*. We conclude that *left Cayley(A)* is path-connected.

Thus we have shown that A is path-connected. $\square$

We observe that if an algebra A has a near-unanimity term, then by Bulatov [3] and Zhuk [4] it is tractable and its Cayley graph is path-connected. It is worth mentioning that we are not sure whether this is true in the opposite direction. Given an algebra A with path-connected Cayley graphs, we do not know whether A is tractable or if it has a near-unanimity term.

The existence of a near-unanimity term is a very strong condition for an algebra and therefore, in the next chapter, we consider a broader group of polymorphisms that contains near-unanimity terms. Thus, we proceed to explore what the existence of a weak near-unanimity term implies about the structure of an algebra's Cayley graph.

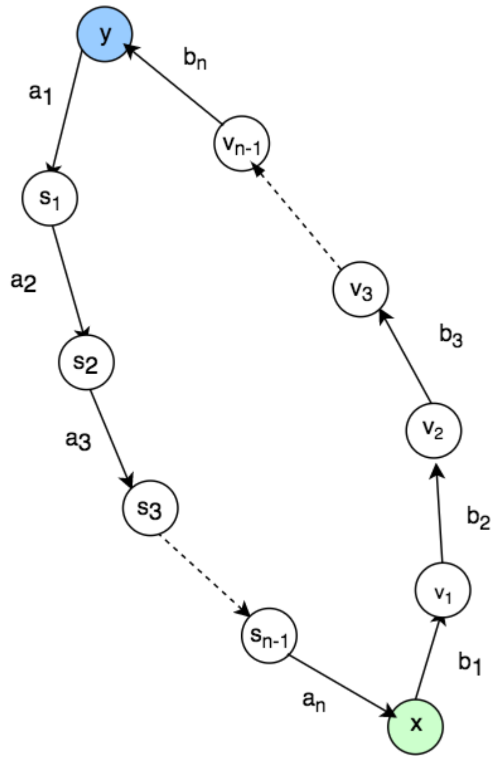

Figure 5.2.1. Path connectedness with paths of length n from y to x and x to y where $a_1, \ldots, a_n, b_1, \ldots b_n \in \{x, y\}$ and $s_1, \ldots, s_{n-1}, v_1, \ldots, v_{n-1} \in A$.

5.3 Weak connectedness for Weak Near-Unanimity terms

Theorem 5.3.1. *Let $w(x_1, \ldots, x_k)$ be a k-ary weak near-unanimity operation over an algebra A with binary operations. Then both right and left Cayley(A) are weakly-connected. Thus A is weakly-connected.*

Proof. We will prove the weak-connectedness of the *right Cayley(A)* and then *left Cayley(A)* respectively.

Case 1: Let x_i denote the *leftmost variable* in the weak near-unanimity operation u. From definition of a WNU we have $\forall x, y \in A,\ w(x, x, ..., y) = w(x, x, \ldots, y, x) = \cdots = w(y, x, \ldots, x) = z$ for some $z \in A$. We consider instances of the term in which we substitute x_i with y and x respectively. We have $t(\ldots, y, \ldots) = t(\ldots, x, \ldots) = z$. Notice, the former is a path from y to z and the latter is a path from x to z. Note, both paths are of equal length. Since there is an undirected path for each pair of elements in A, *right Cayley(A)* is weakly-connected.

Case 2: Let x_i denote the *rightmost variable* in the weak near-unanimity operation u. From definition of a WNU we have $\forall x, y \in A,\ w(x,x,...,y) = w(x,x,\ldots,y,x) = \cdots = w(y,x,\ldots,x) = z$ for some $z \in A$. We consider instances of the term in which we substitute x_i with y and x respectively. We have $t(\ldots,y,\ldots) = t(\ldots,x,\ldots) = z$. The former is a path from y to z and the latter is a path from x to z. Note, both paths are of equal length. Since there is an undirected path for each pair of elements in A, *left Cayley(A)* is weakly-connected.

Thus we have shown that A is weakly-connected. $\square$

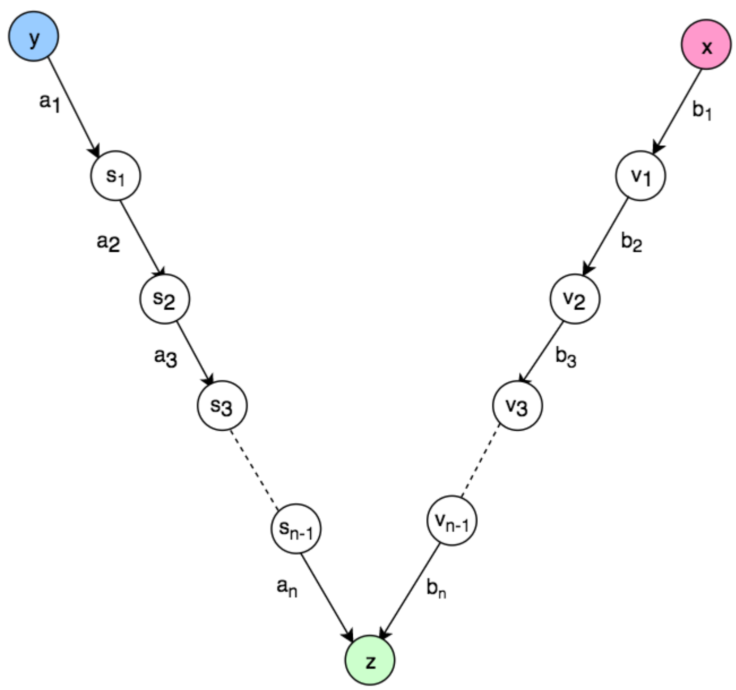

Figure 5.3.1. Weak connectedness with paths of length n from x and y to a common z where $a_1,\ldots,a_n,b_1,\ldots,b_n \in \{x,y\}$ and $z, s_1,\ldots,s_{n-1}, v_1,\ldots,v_{n-1} \in A$.

We observe that if an algebra A has a weak near-unanimity term, then it is tractable and both its Cayley graphs are weakly-connected. On top of the weak-connectedness, we also get a stronger restriction on the structure of the Cayley graphs. We know that for any pair of vertices there are paths of equal length from those vertices to some common vertex.

Example. Consider a binary algebra B defined by the Cayley table in *5.3.2*.

*	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

Figure 5.3.2. Algebra B with 3 elements and a binary operation $*$.

In *Figure 5.3.3* we build a Cayley graph for B and notice that both left and right $Cayley(B)$ are exactly the same. We observe that B is weakly-connected and that for each pair $x, y \in B$ there exists a path of length 1 to some common vertex z. In this example, $(0, 1)$ both go to 2, $(1, 2)$ both have a path to 2, and $(0, 2)$ also go to a common vertex 2. In the trivial cases $(0, 0)$ both go to 0, $(1, 1)$ go to 1 and $(2, 2)$ go to 2. We do not know whether this algebra is tractable, but since it does not contain the projection algebra U_2 we suspect it is solvable in polynomial time. $\Diamond$

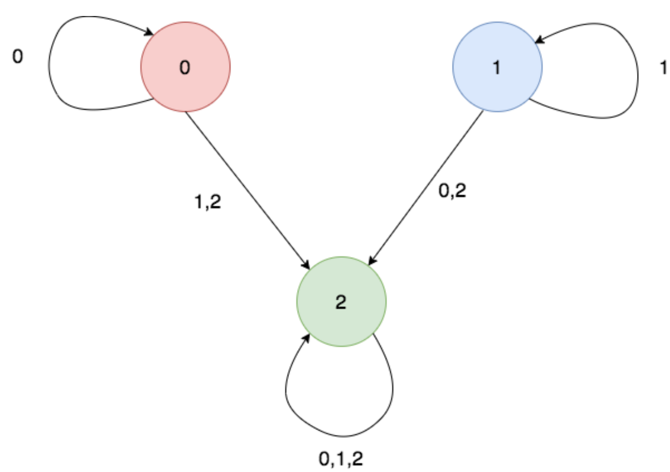

Figure 5.3.3. Right and left Cayley(B).

5.4 Definition of V-connectedness

In the last section we observe that the existence of a WNU polymorphism in an algebra puts some strong restrictions on the type of connectedness in the algebra's Cayley graph. We try to relax some of these restrictions and introduce a broader notion of ”V-connectedness”.

Definition 5.4.1. For an algebra A, we say Cayley(A) is *V-connected* if for all x and for all y in A there exists z in A such that there are paths from x to z and y to z. These paths need not be of the same length. △

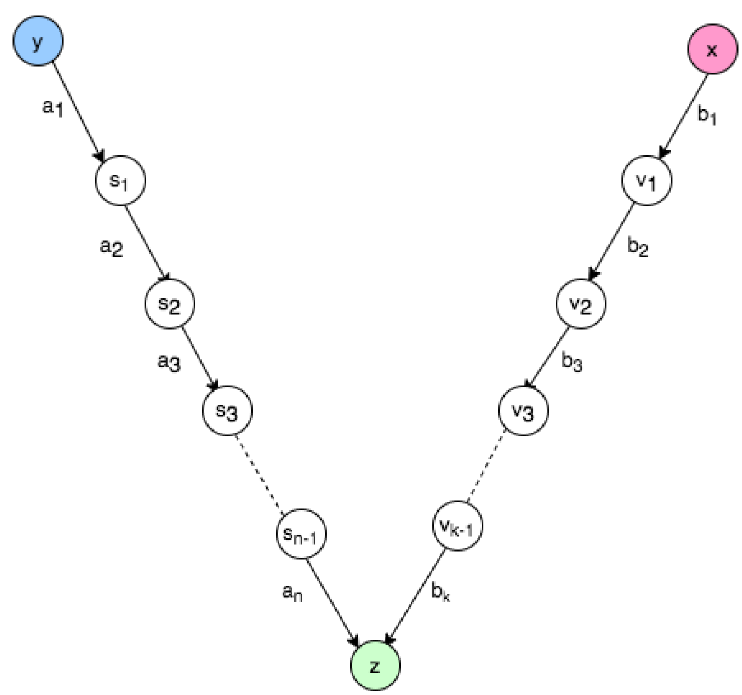

Figure 5.4.1. V-connectivity: path of length n from x to z and path of length k from y to z where $a_1, \ldots, a_n, b_1, \ldots, b_k \in \{x, y\}$ and $z, s_1, \ldots, s_{n-1}, v_1, \ldots, v_{n-1} \in A$.

Example. Consider a binary algebra C defined by the following Cayley table:

In *Figure 5.4.3* we build Cayley graphs for C. We observe that both *right* and *left Cayley(C)*

*	0	1	2	3	4	5
0	0	3	3	3	3	3
1	1	1	4	3	4	3
2	3	3	2	3	3	3
3	5	5	5	3	5	5
4	5	5	5	5	4	5
5	5	5	5	5	5	5

Figure 5.4.2. Algebra C with 6 elements and a binary operation $*$.

are V-connected (but not path-connected). In both Cayley graphs for every $x, y \in C$ there is a path from x and from y to some common vertex $z \in C$. The lengths of these paths can be different. For example, in *right Cayley(C)*, for the pair $(2, 3)$, there is path of length two from the vertex 2 to 5 and there is a path of length one from 3 to 5.

$\Diamond$

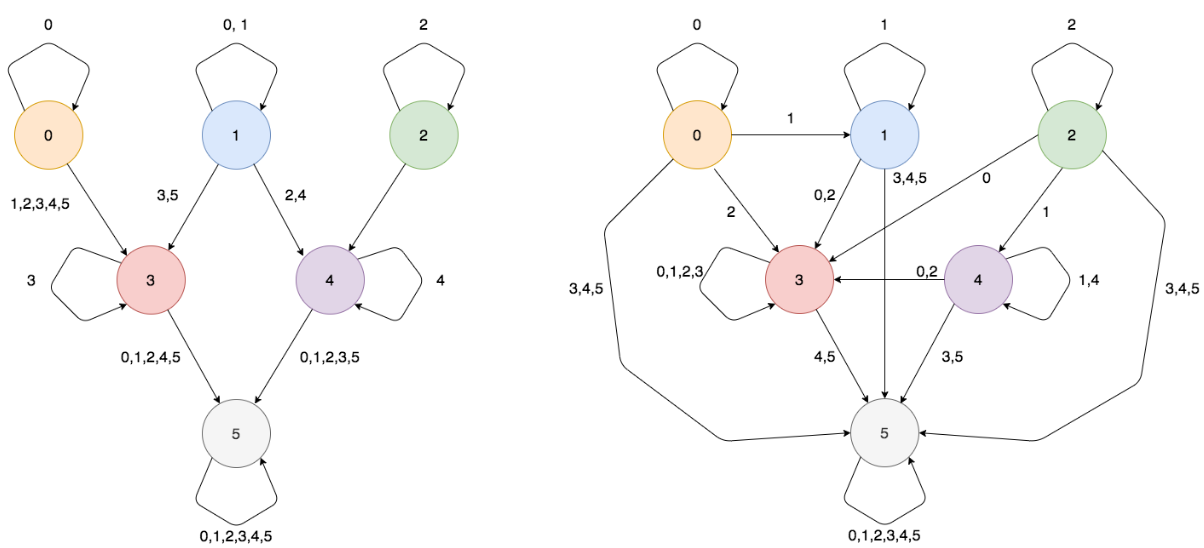

Figure 5.4.3. From left to right: Right and Left Cayley(C).

6

P/NP-Complete Dichotomy for Idempotent, Right Quasigroups

According to Prof. McGrail's work "CSPs and Connectedness: P/NP-Complete Dichotomy for Idempotent, Right Quasigroups" [2] it follows that we can fully characterize these algebras, along with their tractability, in terms of connectedness in their right Cayley graphs. This is due to the fact that they possess right-cancellation properties. Thanks to this result we are able to comprehend these algebras better as connectedness provides us with a visualization of the problem.

It has been shown that the existence of a ternary *Malcev term* [6] in an algebra implies that the algebra is tractable. This paper [2] introduces another term-based test for tractability for right, idempotent quasigroups, namely the binary *Merling term*. This new term determines the P/NP dichotomy for this group of algebras. Moreover, it turns out that the existence of a *Merling term* determines a strong notion of connectedness in the right Cayley graph of these algebras. In the end, the authors provide the readers with a notion of a *Merling condition*, a first order formula, that axiomatizes the P/NP dichotomy for *involuntory quandles*, a subclass of right, idempotent quasigroups.

6.1 Definitions

Definition 6.1.1. An *idempotent, right quasigroup* Q [2] consists of a set Q along with two binary basic operations $*, / : Q^2 \to Q$ that satisfy the following axioms:

1. Idempotence: $\forall x \in Q,\ x * x = x$,
2. Right Cancellation I: $\forall x, y \in Q,\ (x * y)/y = x$,
3. Right Cancellation II: $\forall x, y \in Q,\ (x/y) * y = x$.

Notice, U_n is an idempotent, right quasigroup where the operations $*, /$ are identical. Also, U_n is known to be NP-complete for $n \geq 2$ since it admits the NP-complete 3-SAT problem. △

Definition 6.1.2. Let A be an algebra. A *Mal'cev term* for A is a ternary term $p(x, y, z)$ such that $p(x, y, y) = x$ and $p(x, x, y) = y$ in A. It has been proven by Andrei Bulatov and Victor Dalmau [6] that if A has a Mal'cev term, then A is tractable. △

Definition 6.1.3. Let Q be an idempotent, right quasigroup. Then a binary term m(x,y) in Q is a *Merling term* [2] if for all x, y in Q, $m(x, y) = y$ but $m(x, y) = x$ in U_2 which means that $m(x, y)$ begins with x. △

Definition 6.1.4. Let Q be an idempotent, right quasigroup.

(i) Q is *locally path-connected* if every subalgebra $Q' \leq Q$ is path-connected.

(ii) Q is *totally path-connected* if for all $n \in \mathbb{N}$ and $Q' \leq Q^n$, Q' is locally path-connected.

(iii) Q is *uniformly path-connected* if it has a Merling term. [2]

△

Definition 6.1.5. Let A be an algebra and let F be a set of binary operations. Then A with generators X is free over a class of algebras with binary operations F if for every every algebra B in that class and any function $f : X \to B$ there exists a unique F-homomorphism $g : A \to B$ such that g agrees with f on X. △

Definition 6.1.6. A quandle Q consists of a set Q together with two binary operations $*, / : Q^2 \to Q$ that satisfy the following axioms:

1. Idempotence: $\forall x \in Q,\ x * x = x$,
2. Right Cancellation I: $\forall x, y \in Q,\ (x * y)/y = x$,
3. Right Cancellation II: $\forall x, y \in Q,\ (x/y) * y = x$,
4. Right Self-Distributivity: $\forall x, y, z \in Q,\ ((x * y) * z = (x * z) * (y * z))$.

The first-order theory of quandles was first introduced by David Joyce in his work *"A classifying invariant of knots; the knot quandle"* [7]. It is based on the "crossover algebra" of three-dimensional knots. Quandles form a subclass of right, idempotent quasigroups so all the results that apply to the larger group of algebras also apply to quandles. △

Definition 6.1.7. An quandle Q with binary operations $\{*, /\}$ satisfies the *Merling condition* if: $Q \models \forall xy(x * y = x \implies x = y)$.

It was shown that if Q satisfies the *Merling Condition*, then Q^n also satisfies it for any $n \in \mathbb{N}$. △

Definition 6.1.8. An *involuntary quandle* Q consists of a set Q together with a binary operation $* : Q^2 \to Q$ that satisfies the following axioms:

1. Idempotence: $\forall x \in Q,\ x * x = x$,
2. Right Cancellation: $\forall x, y \in Q,\ (x * y) * y = x$,
3. Right Self-Distributivity: $\forall x, y, z \in Q,\ ((x * y) * z = (x * z) * (y * z))$.

Notice, an *involuntary quandle* is a quandle where the operation $*$ is equivalent to $/$. △

6.2 Background

In the paper *"CSPs and Connectedness: P/NP-Complete Dichotomy for Idempotent, Right Quasigroups"* [2], the authors showed that an idempotent right quasigroup Q has a Merling term if and only if it has a Malcev term. Notice, $m(x, y) = y$ is a path from x to y in Q. Thus, if Q is uniformly connected, then it is tractable and $Cayley(Q)$ is path-connected.

They also demonstrated that Q is totally path-connected if and only if it has a Merling term. They let $F(2, Q)$ be the free term algebra on two generators over Q. It follows that $F(2, Q)$ is a subpower of Q. First they let Q be totally path-connected. This means that its subpowers also have to be path-connected. Hence there is a path in the right Cayley graph from x to y for the generators $x, y \in F(2, Q)$. This path corresponds to a sequence of translations by binary terms in $F(2, Q)$ from which the authors build the Merling term $m(x, y)$ which is also a Merling term in Q. In the next step, they let Q be an idempotent, right quasigroup with a Merling term $m(x, y)$. Since all the algebras in the variety of Q must satisfy $m(x, y) = y$ it follows that for all $n \in \mathbb{N}$ $R \leq Q^n$ has to inherit this Merling term. Thus each subpower R of Q is locally path-connected and thus Q is totally path-connected.

For the converse, the authors proved that if Q is not totally path-connected, then U_2 is in its variety. They extended the result to all algebras and showed that if any algebra A has U_2 in its variety, then it is NP-complete.

Finally, they demonstrated the Dichotomy Theorem for right, idempotent quasigroups which states that if an idempotent, right quasigroup Q is not NP-complete, then it must be tractable. Equivalently, Q is NP-complete if and only if Q is not totally path-connected.

Then the authors proceeded to demonstrate that if a finite quandle Q does not satisfy the *Merling condition*, then U_2 must be a subalgebra of Q. In the following step, they prove that the *Merling condition* axiomatizes the P/NP dichotomy for finite, involuntary quandles. Thus, if a finite, involuntary quandle does not satisfy the *Merling condition* then U_2 is a subalgebra of Q thus Q is NP-complete. Otherwise, they show that Q must be totally path-connected and therefore tractable.

6.3 Tractability Phrased in Terms of V-connectedness for Right, Idempotent Quasigroups

In this section we present our results regarding tractability of right, idempotent quasigroups with regard to the notion of V-connectedness that was introduced in Chapter 5.

Definition 6.3.1. Let Q be an idempotent, right quasigroup. Q is *totally V-connected* if for all $n \in \mathbb{N}$ and $Q' \leq Q^n$, Q' is V-connected. △

Theorem 6.3.1. *A right, idempotent quasigroup Q is totally V-connected iff it is tractable.*

Lemma 6.3.2. *If Q is V-connected, then it is path-connected in its Cayley graph.*

Proof. Suppose Q is V-connected. This means that $\forall x \forall y \in Q\ \exists z \in Q$ such that there are paths from x to z and y to z. Thus there is a path of length n from y to z and so we have a sequence of operations starting from y and ending at z that has form: $((\ldots(y \circ_1 v_1) \circ_2 v_2) \circ_3 \ldots) \circ_n v_n = z$ where $v_1, \ldots v_n \in Q$. Given right-cancellation, there must exist a sequence of term operations from z to y: $((\ldots(z \circ {}_n{}' v_n) \circ'_{n-1} v_{n-1}) \circ'_{n-2} \ldots) \circ'_1 v_1 = y$ where by $\circ'_k$ we denote the operation '$*$' if $\circ_k = /$, and the operation '$/$' if $\circ_k = *$. Since there exists a path from x to z and z to y, we conclude that there exists a path from x to y for any $x, y \in Q$. Hence Q is path-connected in its Cayley graph. □

Proof. Let Q be totally V-connected right, idempotent quasigroup. Since Q is totally V-connected, so is $F(2,Q)$ (since as mentioned in *Lemma 6.3.3*, it is a subpower of Q). Then let A be a subpower of Q and let $a, b \in A$. Consider the function on $\{x, y\}$ that sends x to a and y to b. Notice, since $F(2,Q)$ is a subpower of Q then all the axioms in Q also hold for the free term algebra and therefore $F(2,Q)$ must also be an idempotent, right quasigroup. So, since $F(2,Q)$ is V-connected, then by *Lemma 6.3.2* it has to be path-connected. Hence, there exists $z_1, z_2, \ldots, z_n \in F(2,Q)$ where $((x * z_1) * z_2) * \cdots * z_n = y$. Recall, freeness says there is a unique homomorphism $H : F(2, Q) \to A$ that sends x to a and y to b. Consider $b = H(y) = H(((x*z_1)*z_2)*\cdots*z_n) = ((H(x)*H(z_1))*H(z_2))*\cdots*H(z_n) = ((a*H(z_1))*H(z_2))*\cdots*H(z_n)$ which is a path in right Cayley(A) from a to b. Thus A is (right) path-connected. With regard to the dichotomy result from the previous chapter, since all subpowers of Q are path-connected, then we conclude that Q is totally path-connected. Thus Q has to be tractable.

Now let Q be tractable. Then, by *Chapter 5*, we know it has a Merling term and therefore it is totally path-connected. Since it is totally path-connected then it follows that it is totally V-connected. $\square$

7
Quay Algebras and Results

In the last chapter we found a more relaxed notion of connectivity, as compared to path-connectivity, which was still be able to determine the tractability of some algebras. In particular, with V-connectedness we were able to phrase tractability for right, idempotent quasigroups. In the next step, we want to investigate whether there exists a larger universe of algebras for which we could translate the P/NP dichotomy into the notion of V-connectivity. That being said, we will examine *Quay algebras* which contain all *finite, involuntary quandles* but do not necessarily have *right-cancellation* properties.

7.1 Definitions

Definition 7.0.1. A *Quay algebra* Q [2] consists of a set Q along with a binary operation $* : Q^2 \to Q$ that satisfies the following axioms:

1. $\forall x \in Q,\ x * x = x$,
2. $\forall x, y, z \in Q,\ (((x * z) * y) * z) = x * (y * z)$.

All involuntary quandles and semilattices (see definition below) belong to Quay algebras. △

Definition 7.0.2. A semilattice S is an algebraic structure with a set S and a binary operation $*$, which satisfies the following axioms:

1. $\forall x \in S,\ x * x = x$,
2. $\forall x, y \in S,\ (x * y = y * x)$,
3. $\forall x, y, z \in S,\ ((x * y) * z) = x * (y * z)$.

It has been shown by Jeavons, Cohen and Gyssens in their work *"Closure properties of constraints"* that all semilattices are tractable. $\triangle$

7.2 Results

7.2.1 Quay algebras and V-connectedness

Theorem 7.2.1. *Let Q be a Quay algebra and let $F(2, Q)$ denote the the free term algebra on two generators over Q. If $F(2, Q)$ is left (right) V-connected, then Q is totally left (right) V-connected.*

Proof. Let Q be a Quay algebra. Suppose that $F(2, Q)$ is V-connected. Let A be a subalgebra of Q and let $a, b \in Q$. Consider the function on $\{x, y\}$ that sends x to a and y to b. Since $F(2, Q)$ is V-connected then there must exists some $z \in F(2, Q)$ such that there is a path from x to z and y to z. Hence for some $n, k \in \mathbb{N}$ we have $v_1, v_2, \ldots, v_n \in F(2, Q)$ and $w_1, w_2, \ldots, w_k \in F(2, Q)$ such that $(((x * v_1) * v_2) * \ldots) * v_n = z$ and $(((y * w_1) * w_2) * \ldots) * w_k = z$. Recall, freeness says there is a unique homomorphism $H : F(2, Q) \to A$ that sends x to a and y to b. Consider $H(z) = H((((x * v_1) * v_2) * \ldots) * v_n) = ((H(x) * H(v_1)) * H(v_2)) * \cdots * H(v_n) = ((a * H(v_1)) * H(v_2)) * \cdots * H(v_n)$ which is a path in Cayley(A) from a to some $H(z) \in A$. Similarly, we have $H(z) = H((((y * w_1) * w_2) * \ldots) * w_k) = ((H(y) * H(w_1)) * H(w_2)) * \cdots * H(w_k) = ((b * H(w_1)) * H(w_2)) * \cdots * H(w_k)$ which is a path in Cayley(A) from b to $H(z) \in A$. Since there is a path from a to $H(z)$ and a path from b to $H(z)$ in Cayley(A), then we conclude that A is V-connected. Thus we have shown that all subpowers of Q are V-connected, so Q is totally V-connected. $\square$

Conjecture 7.2.2. *Let Q be a Quay algebra. If $F(2, Q)$ is V-connected in both its left and right Cayley graph then Q is tractable. Otherwise it is NP-complete.*

We believe that the sufficient condition to determine the tractability of a Quay algebra Q is its total V-connectedness in both right and left Cayley($F(2,Q)$). We have computed free term algebras on two generators over many examples of Quay algebras. We found that as long they did not have the unary algebra U_2 or T_2 as a subalgebra, both their left and right Cayley($F(2,Q)$) were V-connected. We present some of the examples below and explain the process of deriving $F(2,A)$ for a finite, binary algebra A.

7.2.2 Procedure of deriving the free term algebra on two generators over a finite, binary algebra

Let A be a finite, binary algebra. We let x, y be our two generators. In order to compute $F(2,A)$ we have to first derive all the identities on two generators in A. We check all possible, unique terms containing x and y, taking the placement of parentheses into account. We start with expressions of length 2 and keep increasing their length until we find no new identities - we stop when all the larger terms can be simplified to the one of the previously found identities.

In the next step, we gradually create a Cayley table for $F(2,A)$. We keep the head row and head column the same and we fill them out with unique terms consisting of x and y which increase in length as we progress. We start by placing x and then y in the head row and column. Then we fill the empty grids in the table by computing $(x*x)$, $(x*y)$, $(y*x)$ and $(y*y)$ using the identities that we derived in the previous step. If any of the above evaluate to terms that are not yet present in our head row or head column, then we proceed to add them to the head row and column (we add $x*y$ and $y*x$ in case both are not defined yet) and compute the resulted new, empty grids in the table. This process continues until we obtain no new terms inside our table (we reach closure).

Example. The *Lopsided quandle* L is a quandle with a binary operations $*, /$ (where $* = /$) defined by the following Cayley table in Figure 7.2.1:

We start by listing all the identities on two generators that exist in L. We have:

*	0	1	2
0	0	0	1
1	1	1	0
2	2	2	2

Figure 7.2.1. Cayley Table for the Lopsided quandle

1. $\forall x \in L$, $x * x = x$
2. $\forall x, y \in L$, $x * (x * y) = x$
3. $\forall x, y \in L$, $x * (y * x) = x * y$
4. $\forall x, y \in L$, $(x * y) * x = x * y$
5. $\forall x, y \in L$, $(x * y) * y = x$
6. $\forall x, y \in L$, $(x * y) * (y * x) = x$

One can verify that these hold by assigning any elements of L to x and y.

Then we proceed to build a Cayley table for $F(2, L)$.

Step 1: We add x and y to the head column and head row and compute the empty grids inside the table. We check our list of identities and since $x * y$ and $y * x$ cannot be simplified, we fill the blanks with those terms. See Figure 7.2.2

Step 2: Since $x*y$ nor $y*x$ are not present in the current head row and column, then we expand our rows and columns by those terms. Then, analogically, we evaluate the newly formed empty grids. We use the following identities to simplify the new terms generated by multiplying the columns by the rows (starting from first column to last):

- $(x * y) * x = x * y$ by Identity #4

*	x	y
x		
y		

*	x	y
x	x	$x*y$
y	$y*x$	y

Figure 7.2.2. Step 1 of building Cayley Table for $F(2, L)$.

- $(y * x) * x = y$ by Identity #5
- $(x * y) * y = x$ by Identity #5
- $(y * x) * y = y * x$ by Identity #4
- $x * (x * y) = x$ using Identity #2
- $y * (x * y) = x$ using Identity #3
- $(x * y) * (x * y) = x * y$ using Identity #1
- $(y * x) * (x * y) = x * y$ using Identity #6
- $x * (y * x) = x * y$ using Identity #3
- $y * (y * x) = y$ using Identity #2
- $(x * y) * (y * x) = x$ using Identity #6
- $(y * x) * (y * x) = y * x$ using Identity #1

This step is shown in Figure 7.2.3.

Step 3: We observe that there are no new terms in the table. This means that we have reached closure and therefore obtained the final Cayley table for $F(2, L)$.

Now we can build a Cayley graph for $F(2, L)$ using the generated Cayley table. In order to

*	x	y	$x*y$	$y*x$
x	x	$x*y$		
y	$y*x$	y		
$x*y$				
$y*x$				

*	x	y	$x*y$	$y*x$
x	x	$x*y$	$x*(x*y)=x$	$x*(y*x)=x*y$
y	$y*x$	y	$y*(x*y)=y*x$	$y*(y*x)=y$
$x*y$	$(x*y)*x=x*y$	$(x*y)*y=x$	$(x*y)*(x*y)=x*y$	$(x*y)*(y*x)=x$
$y*x$	$(y*x)*x=y$	$(y*x)*y=y*x$	$(y*x)*(x*y)=y$	$(y*x)*(y*x)=(y*x)$

Figure 7.2.3. Step 2 of building Cayley Table for $F(2,L)$.

better visualize the graph, we will assign integers $0,1,2,3$ to the terms x, y, $(x*y)$ and $(y*x)$, respectively. The simplified Cayley table for $F(2,L)$ is shown in Figure 7.2.4 and its left and right Cayley graphs can be seen in Figure 7.2.5 and Figure 7.2.6, respectively. Observe, the initial Lopsided quandle is disconnected in its right Cayley graph since the vertex 2 forms a disconnected component. Thus, L has to be NP-complete. Notice, $F(2,Q)$ is path-connected (which means it is also V-connected) in its left Cayley graph but disconnected in its right Cayley graph.

$\Diamond$

$*$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

Figure 7.2.4. Simplified Cayley Table for $F(2, L)$.

7.2.3 Examples for Quay algebras

In order to obtain examples of Quay algebras we used the program *Mace4* [15] which searches for finite models conforming to user-provided formulas and assumptions. First, we decided to restrict our search to Quay algebras of size 3 which are neither involuntary quandles nor semilattices. To do so, we inputted the following constraints into the program:

$$\text{all } x\ (x * x = x).$$

$$\text{all } x \text{ all y all z } (((x * z) * y) * z = x * (y * z)).$$

$$\text{exists } x \text{ exists } y\ ((x * y) * y \neq x).$$

$$\text{exists } x \text{ exists } y\ (x * y \neq y * x).$$

The first two assumptions are axioms for a Quay algebra. The next two make sure that we do not get quandles and semilattices by removing right-cancellability and commutativity, respectively. We included 2 examples below along with their free algebras over two generators.

Example. In Figure 7.2.7 we have the algebra T_3 which is the transpose of projection algebra U_3. We generate $F(2, T_3)$ in Figure 7.2.8. We observe that it is equivalent to T_2. We have shown the right and left Cayley graph of $F(2, T_3)$ in Figure 7.2.9. Notice, $F(2, T_3)$ is path-connected in the right Cayley graph, but disconnected in the left Cayley graph.

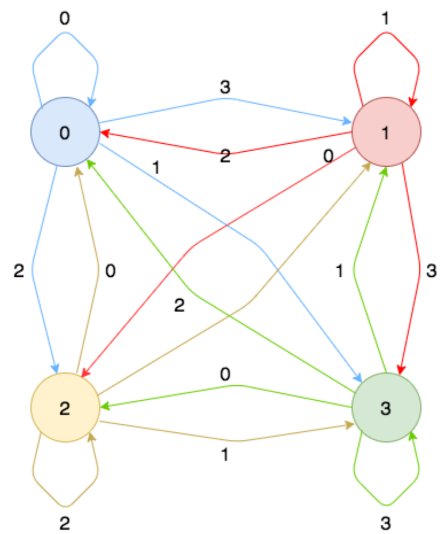

Figure 7.2.5. Left Cayley($F(2, L)$).

$\Diamond$

Example. In Figure 7.2.10 we have an algebra Q_1 which contains U_2 as subalgebra. We generate $F(2, Q_1)$ in Figure 7.2.11. We have shown right and left Cayley graphs of $F(2, Q_1)$ in Figure 7.2.12. Notice, $F(2, Q_1)$ is V-connected in its right, but disconnected in its left Cayley graph.

$\Diamond$

Then, we searched for an example of a semilattice of size 3 that has no right-cancellation. We inserted the following assumptions into the *Mace4* program:

$$\text{all } x \ (x * x = x).$$

$$\text{all } x \text{ all } y \ (x * y = y * x).$$

$$\text{all } x \text{ all y all z } ((x * y) * z = x * (y * z)).$$

$$\text{exists } x \text{ exists } y \ ((x * y) * y! = x).$$

Example. We show one of the found algebras, a semilattice S in Figure 7.2.13. It turns out that $F(2, S)$ is identical as S itself. In Figure 7.2.14 we show left and right Cayley graphs of S which happen to be identical to each other. Notice, Cayley($F(2, S)$) which is same as Cayley(S) is not path-connected but it is V-connected.

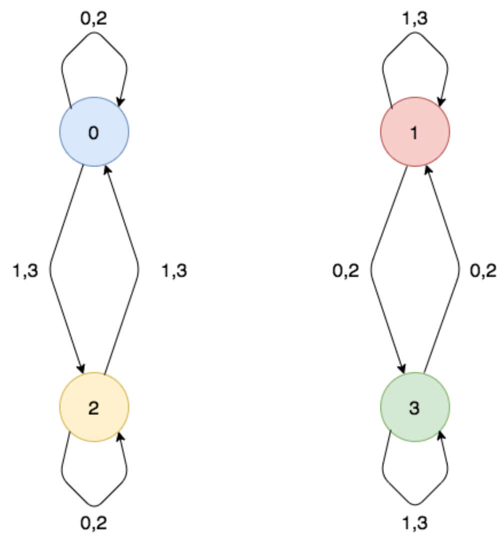

Figure 7.2.6. Right Cayley($F(2, L)$).

*	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

Figure 7.2.7. Quay algebra T_3.

$\Diamond$

From these examples we can observe that when a Quay algebra Q contains U_n or T_n, at least one (left or right) Cayley($F(2, Q)$) is disconnected. We also noticed that S, which is tractable due to being a semilattice, has a V-connected and not path-connected Cayley($F(2, S)$). Thus we conclude that *total path-connectedness* does not determine the P/NP dichotomy for Quay algebras since it is too strong a requirement. However, perhaps the weaker notion of *total V-connectedness* is able to determine tractability for this group of algebras.

*	0	1
0	0	1
1	0	1

Figure 7.2.8. $F(2, T_3) = T_2$.

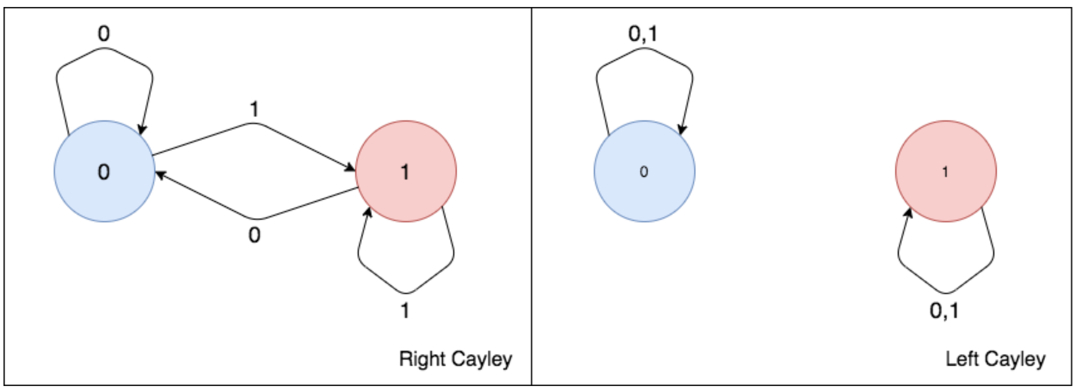

Figure 7.2.9. Right and Left Cayley($F(2, T_3)) = Cayley(T_2)$.

*	0	1	2
0	0	1	2
1	1	1	1
2	0	1	2

Figure 7.2.10. Quay algebra Q_1.

*	0	1	2	3
0	0	2	2	3
1	3	1	2	3
2	3	2	2	3
3	3	2	2	3

Figure 7.2.11. $F(2, Q_1)$.

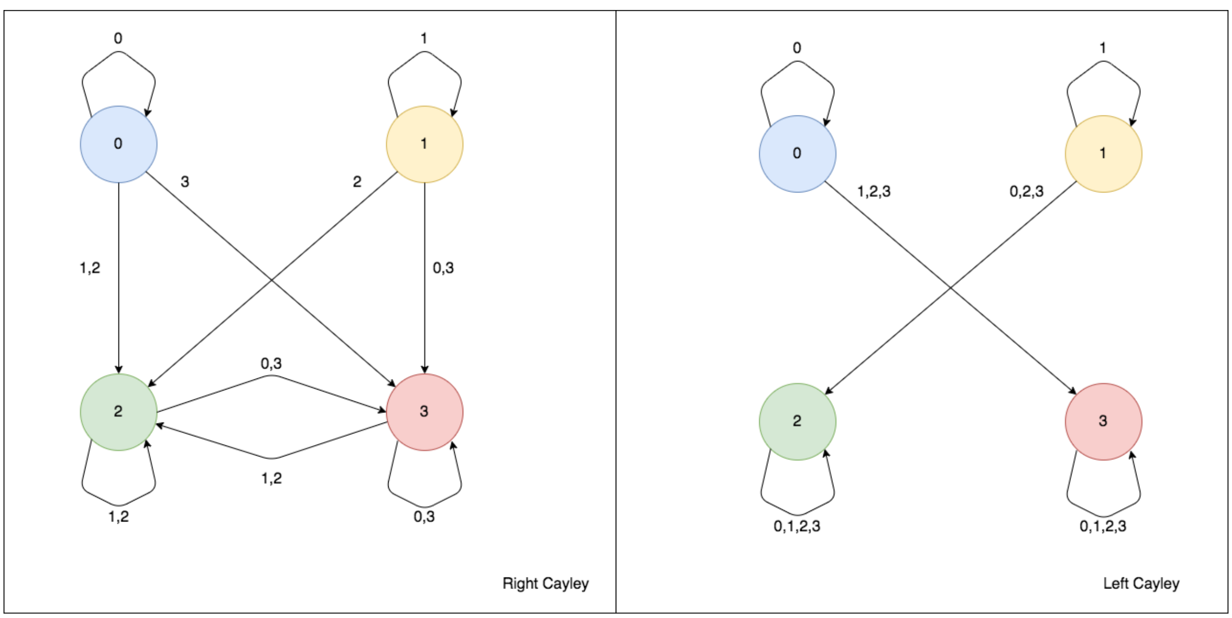

Figure 7.2.12. Right and Left Cayley($F(2, Q_1)$).

*	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

Figure 7.2.13. Semilattice S.

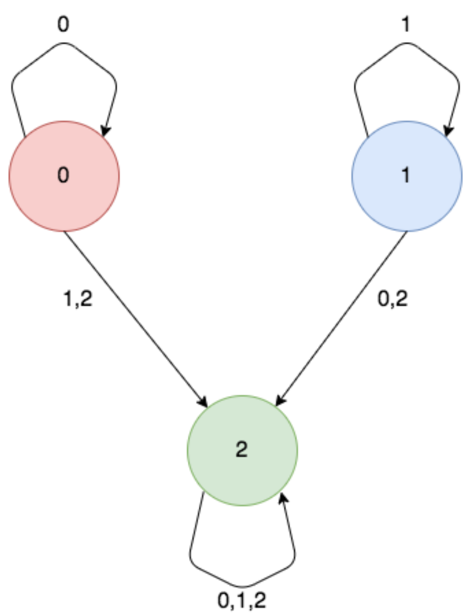

Figure 7.2.14. Right and Left Cayley(S) (identical).

8
Conclusion

8.1 Results

In Prof. McGrail's work [2] we saw that the P/NP dichotomy of finite CSPs proved by Bulatov [3] and Zhuk [4] can be phrased in terms of connectedness in the right Cayley graph for right, idempotent quasigroups. Prof. McGrail has shown that *total path-connectivity* is equivalent to tractability in this group of algebras. As part of this senior thesis we examine Quay algebras in order to answer the question as to whether there exists a larger universe of algebras for which the dichotomy could be expressed by some notion of connectedness. As we investigate weak near-unanimity terms and their implication on the structure of Cayley graphs for algebras, we introduce the idea of V-connectedness. While much weaker than path-connectedness, this notion of connectivity still provides a Cayley graph with enough restriction on its structure so that the algebra could be potentially tractable. In fact, in Chapter 6., we demonstrate that *total V-connectedness* in the right Cayley graph can determine the P/NP dichotomy for right, idempotent quasigroups. In the following chapter we prove that for a Quay algebra Q it is sufficient to show that its free term algebra over 2 generators is V-connected to demonstrate that Q is totally V-connected. We then introduce a conjecture in which we propose that *total V-connectivity* in both the left and right Cayley graph of a Quay algebra A implies its tractability. Otherwise, we believe A is NP-complete. The examples of Quay algebras that we provide at the end of this chapter strengthen our assumption. What is also worth mentioning, is that the structure of the Cayley graph of the free term algebra over the semilattice S shows that *total path-connectedness* fails to determine the dichotomy for this group of algebras as it is too strong.

8.2 Future work

As for future research, it would be valuable to examine other algebras and investigate the relationship between connectedness and tractability for them. For example, one could look into binary algebras with a single operator along with axioms of idempotence and right self-distributivity.

Another way to extend the project would perhaps be to create a program that would be able to produce a weak near-unanimity term for a tractable algebra. Bulatov [3] has demonstrated that there exists a weak near-unanimity isomorphism for every tractable algebra, however, the problem of generating one given such an algebra is not trivial.

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