Module Basis of Mixed Splines over R[x]

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Abstract

A mixed spline is a piecewise polynomial with varying degrees of smoothness. In this project, we characterize a basis for mixed splines over subdivisions of the reals based on a characterization for integer spline bases. We use our new characterization to find bases for modules of splines with boundary conditions with particular differentiability requirements on their boundaries and compare various aspects of the two.

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Dedication

To my family, for their unending creativity and support.
To my friends, for the bonds we share.
To Talia, for all the rest.

Acknowledgments

I am thankful to the entirety of the Bard Math Department, in particular I want to thank Steve Simon, Stefan Méndez-Diez, Matt Junge, and Lauren Rose for serving as my primary guides through my undergraduate journey.

1

Introduction

A polynomial spline is a continuous piece wise polynomial function defined over an interval in $\mathbb{R}$. Splines have been studied extensively due to their applications in data interpolation and computer graphics.

In this project we study modules of splines with boundary conditions and different differentiability requirements over $\mathbb{R}[x]$ on intervals in $\mathbb{R}$. Our main goal is to find a basis for these modules. Upon determining the module basis, we prove that the elements exist using results from [3]. We then give a corollary giving a module basis for sets of $C^{r,p}$ splines on intervals in $\mathbb{R}$. We consider this newfound module basis for $C^{r,p}$ splines and compare it to a known basis for $C^{r,0}$ splines. We also prove results for the leading terms of the elements of our module basis.

In chapter 2, we remind the reader of some basic number theory and abstract algebra concepts.

In chapter 3, we define and introduce polynomial splines, divided sub-intervals, $C^{r,p}$ splines and mixed splines. We study the set of all mixed splines, $C^M(I)$, and determine it is a module over $\mathbb{R}[x]$. We then relay a known basis for $C^{r,0}$ splines which we compare to later. We end the chapter by introducing a polynomial Q and showing that it can be used to determine when a basis for $C^M(I)$ has been found.

In chapter 4, we introduce and explore the concept of splines on graphs and prove the case where a spline on a graph corresponds to a C^M spline.

In chapter 5, we define and introduce integer splines. We study integer splines on $n-$cycles before discussing flow-up classes. These flow-up classes are sets of splines on graphs which have some number of leading zeroes. We discuss a basis for integer splines which uses particular elements of these flow-up classes. This basis inspires our own basis for the module of C^M splines. We finish out the chapter with key results proving the existence of particular elements of the flow-up classes on $n-$cycle graphs.

In chapter 6, we give an explicit basis for the module of C^M splines and prove the existence of its basis elements. We discuss the form of this basis and its relationship to previously known bases for modules of integer splines. We then give a corollary for modules of $C^{r,p}$ splines and lastly we prove the degrees of the leading terms of the basis elements.

In chapter 7, we compare our basis found in chapter 6 to the known basis for modules of $C^{r,0}$ splines discussed in chapter 3. We compare the two bases as well as the matrices formed from the basis elements. We prove a relationships between these two matrices. We compare the sum of the degrees of the basis elements found using the method in chapter 3 and the leading terms of the elements found using the method in chapter 6, proving that they are equal.

2
Preliminaries

The purpose of this chapter is to remind the reader to some concepts from abstract algebra. These will be important in understanding the spaces we work over. We also define two key concepts from number theory, those being GCD and LCM. Lastly, we include the generalized Chinese remainder Theorem which will later be used to show the existence of key splines.

2.1 Abstract Algebra

Definition 2.1.1. Let k be a field. Then $k[x]$ is the set of all polynomials in x with coefficients in k.

Definition 2.1.2. The **leading term of a polynomial** f in x is the highest degree term with a non-zero coefficient.

Definition 2.1.3. A **commutative ring** R, is a ring in which the binary operation of multiplication in R is commutative.

Definition 2.1.4. A **zero-divisor** is a nonzero element a of a commutative ring R such that there is a nonzero element $b \in R$ such that $ab = 0$.

Definition 2.1.5. A **Ring with unity** is a Ring R, which contains an element, usually denoted 1 such that for any element $r \in R$ $r1 = 1r = r$.

Definition 2.1.6. An **integral domain** is a commutative ring with unity and no zero-divisors.

Theorem 2.1.7. *Let R be a commutative ring, then if R is an integral domain, $R[x]$ is an integral domain.*

Proof. By construction $R[x]$ is a ring, however we must also show that $R[x]$ is commutative with unity and contains no zero-divisors. R is commutative by construction and therefore so is $R[x]$. Let an element $1 \in R$ be the unity element of R. Then let there exist $f \in R[x]$ such that $f(x) = 1$ is the unity element of $R[x]$ satisfying our first two requirements. next we suppose

$$f(x) = a_1 x^{b_1} + ...a_{n-1}x^{b_{n-1}} + a_n x_n^b + a_0 \tag{2.1.1}$$

$$g(x) = c_1 x^{d_1} + ...c_{m-1}x^{d_{m-1}} + c_m x_m^d + c_0 \tag{2.1.2}$$

where $c_m a_n$ are non-zero, $c_i, b_j \in R$, and $a_n X^{b_n}$ and $c_m X^{d_m}$ are the leading terms of f and g respectively. Then $f \cdot g$ has leading coefficient $a_n c_m$. R is an integral domain and so $a_n c_m \neq 0$. Therefore for all $f, g \in R[x]$, $f \cdot g \neq 0$. Thus $R[x]$ has no zero divisors and must be an integral domain. $\square$

Definition 2.1.8. A **module over a ring** R, also called an **R-Module** is a set M with a binary operation, written as *addition*, and an operator of R on M, written as *multiplication*. In addition R and M must satisfy the following:

M is closed under addition and scalar multiplication.

M is an abelian group under addition.

For all $a \in R$ and all $f, g \in M, a(f + g) = af + ag$.

For all $a, b \in R$ and all $f \in M, (a + b)f = af + bf$.

For all $a, b \in R$ and all $f \in M, (ab)f = a(bf)$.

If 1 is the multiplicative identity in $R, 1f = f$ for all $f \in M$.

Definition 2.1.9. A module M is **finitely generated** if there exists a finite number of elements $g_1, g_2, ..., g_n$ in M such that $g_1, g_2, ..., g_n$ span M.

Definition 2.1.10. A subset $G \subset M$ is called a **module basis** for M if M is finitely generated by G and each $g \in G$ is linearly independent.

We next outline a few concepts from real algebraic geometry that we will rely on in later chapters. The following definitions for greatest common divisor and least common multiple come from [6].

Definition 2.1.11. Let p and q be polynomials with integer coefficients. The **greatest common divisor** of p and q is a polynomial $g \in \mathbb{R}[x]$ such that g is a divisor of both p and q, and any divisor of both p and q is a divisor of g.

Definition 2.1.12. Let p and q be polynomials with integer coefficients. The **least common multiple** of p and q is a polynomial such that g is a multiple of both p and q, and any multiple of both p and q is a multiple of g.

The following theorem is the Generalized Chinese Remainder theorem.

Theorem 2.1.13. *The system of congruences*

$$\begin{aligned} x &\equiv a_1 \mod m_1 \\ x &\equiv a_1 \mod m_2 \\ &\vdots \\ x &\equiv a_1 \mod m_n \end{aligned}$$

has a solution if and only if $(m_i, m_j)|a_i - a_j$ *for all pairs of integers* (i, j)*, where* $1 \leq i < j \leq n$*. If a solution exists, then it is unique module* $[m_1, m_2, ..., m_n]$*.*

The proof of this theorem can be found in chapter 2 of [1].

3
Polynomial Splines

This chapter will focus on polynomial splines in one variable with boundary conditions.These splines serve as our main object of study for which we develop results for in later chapters. We begin by defining polynomial splines and what it means for them to have boundary conditions. We then explore two types of polynomial splines with boundary conditions, $C^{r,p}$ splines and the more general C^M splines. We prove that the set of all C^M splines forms a finitely generated module, the basis for which we prove later. Lastly, we end the chapter by defining some better notation for ease of use.

We begin with the definition of a polynomial spline.

Definition 3.0.1. Let $I = I_1 \cup I_2 \cup ... \cup I_n \subset \mathbb{R}$ where
$I_1 = (-\infty, a_1], I_2 = [a_1, a_2], ..., I_{n-1} = [a_{n-2}, a_{n-1}], I_n = [a_{n-1}, \infty)$ and $a_1 < a_2 < ... < a_{n-1}$. A **polynomial spline over** I is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F|_{I_j} = f_j$$

where $f_j \in \mathbb{R}[x]$ for all $j \in \{1, ..., n\}$.

We write the spline F as the $n - tuple$ $F = (f_1, f_2, ..., f_n)$ where $f_j \in \mathbb{R}[x]$ is defined on the sub-interval I_j.

For ease of notation we define the intervals we will be working over.

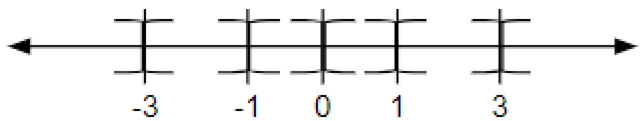

Figure 3.0.1. The subdivided interval $I_{(-3,-1,0,1,3)}$.

Definition 3.0.2. $I_{(a_0,a_1,...,a_n)} = (-\infty, a_0] \cup [a_1, a_2] \cup [a_2, a_3] \cup ... \cup [a_{n-1}, a_n] \cup [a_n, \infty)$, a subdivided interval of $\mathbb{R}$.

Figure 3.0.1 serves to illustrate this definition.

3.1 C^r splines

When considering polynomial splines we often care about how "smoothly" the polynomials meet at the $a_i's$. To this end define C^r polynomial splines to be polynomial splines which are r times differentiable. At the end of this section we show that sets of such splines form a module.

Definition 3.1.1. Let I be a subdivided interval. Then, the set of all splines which are C^r over I is $C^r(I)$.

The following theorem tells us that $C^r(I)$ is a finitely generated module. This will be used to prove the same for $C^M(I)$.

Theorem 3.1.2. *theorem 3.0.20 from* [5]. *Let* $I = I_1 \cup I_2 \cup ... \cup I_n$ *where* $I_1 = (-\infty, a_1], I_2 = [a_1, a_2], ..., I_{n-1} = [a_{n-2}, a_{n-1}], I_n = [a_{n-1}, \infty)$. *Then* $C^r(I)$ *is a finitely generated module over* $\mathbb{R}[x]$

3.2 Splines with boundary conditions

In this project we focus on polynomial splines with boundary conditions. These are polynomial splines defined over $I_{(a_0,\ldots,a_n)}$ which the spline is equal to the function 0 on the intervals $(-\infty, a_0]$ and $[a_n, \infty)$.

Definition 3.2.1. A **polynomial spline with boundary conditions** F is a polynomial spline defined over some $I_{(a_0,\ldots,a_n)}$ and on which $F(x) = 0$ for all $x \in (-\infty, a_0] \cup [a_n, \infty)$. We write the polynomial spline with boundary conditions F as $F = (f_1, f_2, ..., f_n)$ where $F = f_i$ on the sub-interval $[a_{i-1}, a_i]$ where $i \in \{1, ..., n\}$.

Example 3.2.2. Consider the spline with boundary conditions
$F = (x(x+4)(x+2)^3, (x-2)^3(x+2)^3, (x-2)^3(x+1)(x-4))$ defined over
$I = (-\infty, -4] \cup [-4, -2] \cup [-2, 2] \cup [2, 4] \cup [4, \infty)$. We can see the graph of F in the $xy-$plane in 3.3.1.

3.3 $C^{r,p}$ splines

Next we will introduce the set of all splines which are C^r splines on the interior of a subdivided interval I and C^p on it's boundaries. A $C^{r,p}$ spline is a spline,F defined over the subdivided interval $I = I_0, ..., I_{n+1}$ where F is r times differentiable on $I_1 \cup ... \cup I_n$ and p times differentiable on I_0 and I_{n+1}.

Definition 3.3.1. A $C^{r,p}$ polynomial spline over I, where $I = I_0, ..., I_{n+1}$ is a polynomial spline with boundary conditions with p continuous derivatives over I_0 and I_{n+1} and r continuous derivatives everywhere else in I.

A spline is $C^{r,p}$ according to the following theorem

Theorem 3.3.2. *Let I be defined on n intervals and let $F = (f_1, ..., f_n)$ be a spline over I. Then F is C^r if and only if*

$$f_1 \equiv 0 \mod (x - a_0)^{p+1}$$

$$f_n \equiv 0 \mod (x - a_n)^{p+1}$$

$$f_i \equiv f_{i+1} \mod (x - a_i)^{r+1}$$

for all $1 < i < n$. Where $x - a_i$ is a linear polynomial defining the boundary $x - a_i = 0$ between I_i and I_{i+1}.

Example 3.3.3. Consider the previous spline with boundary conditions $//F = (x(x+4)(x+2)^3, (x-2)^3(x+2)^3, (x-2)^3(x+1)(x-4))$ on $//I = (-\infty, -4] \cup [-4, -2] \cup [-2, 2] \cup [2, 4] \cup [4, \infty)$. We can see the following equations hold

$$x(x+4)(x+2)^3 \equiv 0 \mod (x+4)$$

$$(x-2)^3(x+1)(x-4)) \equiv 0 \mod (x-4)$$

$$x(x+4)(x+2)^3 \equiv (x-2)^3(x+2)^3 \mod (x+2)^3$$

$$(x-2)^3(x+2)^3 \equiv (x-2)^3(x+1)(x-4) \mod (x-2)^3$$

Thus by 3.3.2 F is $C^{2,0}$.

3.4 Mixed splines

In the previous section we considered splines with two degrees of differentiability, however we find it can be useful to consider varying degrees of differentiabilty over the subdivided interval. To that end we explore and develop results for mixed splines which have such varying degrees of differentiabilty. We note that these mixed splines will also be $C^{r,p}$ for particular values of r and p.

Definition 3.4.1. Let $I = I_{(a_1,...,a_n)}$. A **mixed spline** or C^M **spline** over I is a polynomial spline with boundary conditions which has r_i continuous derivatives at a_i for $0 \leq i \leq n$. Where $M = (r_1, ..., r_n)$ and the $r_i \in \mathbb{N} \cup \{0\}$.

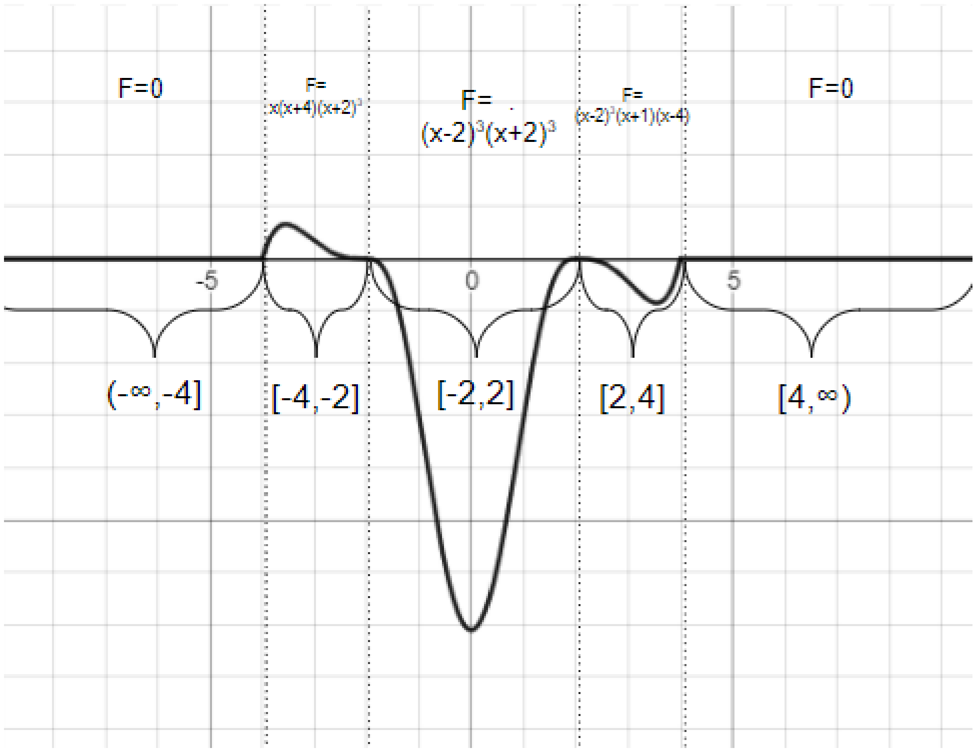

Figure 3.3.1. The graph of a polynomial spline with boundary conditions which is $C^{2,0}$

We next give our defining equations. These are the set of equations which a spline must satisfy in order to be a C^M spline.

Definition 3.4.2. Let $M = (r_0, ..., r_n)$ and let $I = I_{(a_0,...,a_n)}$. Then let a spline $F = (f_1, f_2, ..., f_n)$. The **defining equations** are as follows:

$$f_i \equiv f_{i+1} \mod (x - a_i)^{r_i+1}$$

$$f_0 \equiv 0 \mod (x - a_0)^{r_0+1}$$

$$f_n \equiv 0 \mod (x - a_0)^{r_n+1}$$

for all $1 \leq i \leq n-1$.

We now show that a spline satisfying these equations fits the definition for being a mixed spline.

Theorem 3.4.3. *Let $F = (f_1, ..., f_n)$ be a spline. Let $M = (r_0, ..., r_n)$ and let $I = I_{(a_0,...,a_n)}$. The spline F is C^M over I if and only if the defining equations are true.*

Proof. Assume F is C^M over I. Then the following comes from the definition of C^M.

$$
\begin{aligned}
f_i(a_i) &= f_{i+1}(a_i) \\
f_i'(a_i) &= f_{i+1}'(a_i) \\
&\vdots \\
f_i^{(r_i)}(a_i) &= f_{i+1}^{(r_i)}(a_i) \\
f_1(a_0) &= 0 \\
f_1'(a_0) &= 0 \\
&\vdots \\
f_1^{(r_0)} &= 0 \\
f_n(a_n) &= 0 \\
f_n'(a_n) &= 0 \\
&\vdots \\
f_n^{(r_n)} &= 0
\end{aligned}
$$

for all $1 \leq i \leq n-1$. This is equivalent to saying

$$(x-a_i)|f_i - f_{i+1}$$

$$(x-a_i)|f_i' - f_{i+1}'$$

$$\vdots$$

$$(x-a_i)|f_i^{(r_i)} - f_{i+1}^{(r_i)}$$

$$(x-a_0)|f_1$$

$$(x-a_0)|f_1'$$

$$\vdots$$

$$(x-a_0)|f_1^{r_0}$$

$$(x-a_n)|f_n$$

$$(x-a_n)|f_n'$$

$$\vdots$$

$$(x-a_n)|f_n^{r_n}$$

and that can be rewritten as simply

$$f_i \equiv f_{i+1} \mod (x-a_i)^{r_i+1}$$

$$f_0 \equiv 0 \mod (x-a_0)^{r_0+1}$$

$$f_n \equiv 0 \mod (x-a_0)^{r_n+1}$$

. So the defining equations are true. Otherwise if we assume that the defining equations are true then F is C^M over I because it is r_i times differentiable at the point a_i for all $0 \leq i \leq n$. □

In order to prove our main result of a basis for $C^M(I)$ we show that the set of all C^M polynomial splines over a given I is a finitely generated module over $\mathbb{R}[x]$.

Definition 3.4.4. Let I be a subdivided interval. Then the set of all polynomials splines that are C^M over I is $C^M(I)$.

Proposition 3.4.5. *$C^M(I)$ is a finitely generated module over $\mathbb{R}[x]$.*

Proof. We will show $C^M(I)$ is a finitely generated module by showing that it is a submodule of $\mathbb{R}[x]^n$.

Let $F = (f_0, ..., f_{n+1}), B = (b_0, ..., B_{n+1})$ be elements of $C^M(I)$ and let $p \in \mathbb{R}[x]$. For ease of notation throughout this proof let $\ell_i = (x - a_i)^{r_i+1}$.

First it is clear that $C^M(I)$ is non-empty because $(0, 0, ..., 0) \in C^M(I)$.

Let r' be the smallest element of r such that $r' \leq r_i$ for all $r_i \in M$. We observe that $C^M(I) \subset C^{r'}(I)$. Suppose that $F = (f_0, ..., f_{n+1}$ is in $C^M(I)$. Then by the definition of C^r

$$f_i^{(j)}(a_i) = f_{i+1}^{(j)}(a_i)$$

for all $i = 0, 1, ..., n + 1$, $j = 0, 1, ..., r', ...r_i$. Since $r' \leq r_i$ by definition, $f \in C^{r'}(I)$. Thus $C^M(I) \subset C^{r}\prime(I)$. By 3.1.2 $C^{r'} \subset \mathbb{R}[x]^n$. Thus $C^M(I) \subset \mathbb{R}[x]^n$.

Next we show $C^M(I)$ is closed under subtraction. We recall $F, B \in C^M(I)$, we know that $f_0, f_{n+1}, b_0, b_{n+1} = 0$ and that

$$f_i \equiv f_{i+1} \mod (x - a_i)^{r_i}$$

$$b_i \equiv b_{i+1} \mod (x - a_i)^{r_i}$$

for all $i \in \{0, ..., n\}$.

We can see that $F - B = (f_0 - b_0, ..., f_{n+1} - b_{n+1})$. It is then easy to see that

$$f_i - b_i \equiv f_{i+1} - b_{i+1} \mod (x - a_i)^{r_i}$$

and thus $F - B$ is C^M and thus $F - B \in C^M(I)$.

Lastly we show that $C^M(I)$ is closed under scalar multiplication. $F \in C^M(I)$ and so we know that

$$f_i \equiv f_{i+1} \mod \ell_i^{r_i}$$

for all $i \in \{0, ..., n\}$.

Let $c \in \mathbb{R}$. We can see $cF = (cf_0, ..., cf_{n+1})$. Then consider that

$$cf_i \equiv cf_{i+1} \mod c(x - a_i)^{r_i} = (x - a_i)^{r_i} \mod (x - a_i)^{r_i}$$

for all $i \in \{0, ..., n\}$. Thus cF is C^M on I and so $cF \in C^M(I)$. Therefore $C^M(I)$ is a submodule of $\mathbb{R}[x]^n$ and so is a finitely generated module over $\mathbb{R}[x]$ by Proposition 2.2.12 in [5]. □

The next few definitions serve to ease our discussion of these splines. For simplicity we define some cleaner notation for the linear polynomials defining the boundaries of our sub-intervals.

Definition 3.4.6. Let $I = I_{(a_0,\dots,a_n)}$. Then ℓ_i represents the linear polynomial $x - a_i = 0$.

We also define the degree of the leading term of a spline to be the LT degree.

Definition 3.4.7. Let $F = (f_1, ..., f_n)$ be a polynomial spline. Then the **LT degree of F** or $LT(F)$ is the degree of the polynomial f_1.

3.5 Previous Results

In this section we list a few important results from previous work on Polynomial Splines. First, a module basis for $C^{r,0}$ polynomial splines which we will use for comparison later. We then adapt some useful results for determining if a set of polynomial splines form a basis.

Theorem 3.5.1. *Let*

$$g_0(a_i) = (x - a_0)((x - a_n)x^r - (x - a_i)^{r+1})$$

and

$$g_n(a_i) = (x - a_n)((x - a_0)x^r - (x - a_i)^{r+1}).$$

The vectors

$$B_1 = \{g_0(a_1), g_n(a_1), ..., g_n(a_1)\}$$

$$B_2 = \{g_0(a_2), g_0(a_2), g_n(a_2), ..., g_n(a_2)\}$$

$$\vdots$$

$$B_2 = \{g_0(a_i), ..., g_0(a_i), g_n(a_i), ..., g_n(a_i)\}$$

$$\vdots$$

$$B_{n-1} = \{g_0(a_{n-1}), ..., g_0(a_{n-1}), g_n(a_{n-1})\}$$

$$B_n = \{(x-a_0)(x-a_n), (x-a_0)(x-a_n), ..., (x-a_0)(x-a_n)\}$$

form a basis $B = (B_1, B_2, ..., B_n)$ *for* $C^{r,0}(I)$.

Here it becomes important to define a polynomial Q which will be integral in proving whether or not we have found a basis for $C^M(I)$.

Definition 3.5.2. Let $I = I_{(a_0,...,a_n)}$. Then the polynomial Q is defined as

$$Q = (\prod_{1\leq i\leq n-1} \ell_i^{r_i+1}) * (\prod_{i\in\{0,n\}} \ell_i^{r_i+1})$$

We write Q this way to easily draw a distinction between in the interior and exterior edges of I. This will be particularly useful when considering a basis for $C^{r,p}$ splines.

We include an important result regarding C^r splines from [4]. A more general form of this theorem will be used to prove our results.

Theorem 3.5.3. *Theorem 2.3 from* [4] *The set of splines* $F = \{B_1, ..., B_n\}$ *in* $C^r(I)$ *form a basis over* $\mathbb{R}$ *if and only if* $\det[B_1, ..., B_n] = cQ$, *for some nonzero real constant* c.

This was proved by [4] over more general regions. We prove it below for the C^M case, and to that end we also adapt proposition 2.2 from [4] for the C^M case which will be used to show the desired res of theorem 3.5.5.

Proposition 3.5.4. *Modification of Proposition 2.2 from* [4]. *Let* $\{B_1, ..., B_n\} \in C^M(I)$. *Then* Q *divides* $\det[B_1, ..., B_n]$.

Proof. Let $B_i = (b_{1i}, ..., b_{ni})$. Then
$\det[B_1, ..., B_n] = \begin{vmatrix} f_{1,i} & \dots & f_{1n} \\ f_{2,i} & \dots & f_{2n} \\ \vdots & \dots & \vdots \\ f_{n1} & \dots & f_{nn} \end{vmatrix} = \begin{vmatrix} f_{1,i} - f_{2i} & \dots & f_{1n} - f_{2i} \\ f_{2,i} & \dots & f_{2n} \\ \vdots & \dots & \vdots \\ f_{n1} & \dots & f_{nn} \end{vmatrix}$. For each i, $\ell_i^{r_i+1}$ divides $f_{1i} - f_{2i}$ by definition of $C^M(I)$, and so $\ell_i^{r_i+1}$ divides $\det[B_1, ..., B_n]$. The $\ell_i' s$ are distinct by construction and so are pairwise relatively prime. Therefore, Q must divide $\det[B_1, ..., B_n]$. □

Proposition 3.5.5. *Modification of Theorem 2.3 from* [4]. *Let* $\det[B_1, ..., B_n] = cQ$ *for some nonzero real constant* c *and let* $\{B_1, ..., B_n\}$ *in* $C^M(I)$. $\{B_1, ..., B_n\}$ *in* $C^M(I)$ *form a basis over* $\mathbb{R}[x]$.

Proof. Suppose $\det[B_1, ..., B_n] = Q$. Then $\{B_1, ..., B_n\}$ must be linearly independent over $\mathbb{R}[x]$. For any $v \equiv Q[b_1, b_2, ..., b_n]^T \in Q\mathbb{R}[x]^n$, for all $i \leq n$, let

$$\begin{aligned} x_i &= \frac{\det[B_1, .., B_{i-1}, Q[b_1, b_2, ..., b_n]^T, B_{i+1}, ..., B_n]}{\det[B_1, ..., B_n]} \\ &= \frac{Q \det[B_1, .., B_{i-1}, [b_1, b_2, ..., b_n]^T, B_{i+1}, ..., B_n]}{Q} \\ &= \det[B_1, .., B_{i-1}, [b_1, b_2, ..., b_n]^T, B_{i+1}, ..., B_n] \end{aligned}$$

Thus, $x_i \mathbb{R}[x]$ for all i. Let $x = [x_1, ..., x_n]^T$. By Cramer's rule, $[B_1, ..., B_n]x = v$, so $v \in span(B_1, ..., B_n)$, the module generated by $B_1, ..., B_n$. Since our choice of $v \in Q\mathbb{R}[x]^n$ was arbitrary, $Q\mathbb{R}[x]^n \subset span(B_1, ..., B_n)$. Let $B \in C^M(I) - \{0\}$. Then $QB \in span(B_1, ..., B_n)$, so $QB = \sum_{i=1}^n b_i B_i$ for some $\{b_i\}$ in $\mathbb{R}[x]$. Then

$$\begin{aligned} b_i Q &= b_i(\det[B_1 ... B_n]) \\ &= \det[B_1 ... B_{i-1} b_i B_i B_{i+1} ... B_n] \\ &= \det[B_1 ... B_{i-1} \sum b_j B_j B_{i+1} ... B_n] \\ &= \det[B_1 ... B_{i-1} QB B_{i+1} ... B_n] \\ &= Q \det[B_1 ... B_{i-1} B B_{i+1} ... B_n]. \end{aligned}$$

By proposition 3.5.4, $Q | \det[B_1 ... B_{i-1} B B_{i+1} ... B_n]$. Thus, we can write $\det[B_1 ... B_{i-1} B B_{i+1}] = QP$ for some $P \in \mathbb{R}[x]$, and thus,

$$Q\det[B_1 ... B_{i-1} B B_{i+1} ... B_n] = QQP.$$

Thus, we can see

$$b_i Q = QQP.$$

Therefore $Q|b_i$. Then $B = \sum(b_i/Q)B_i \in span(B_1, ..., B_n)$. Therefore $\{B_1, ..., B_n\}$ spans $C^M(I)$. Therefore $[B_1, ..., B_n]$ forms a basis for $C^M(I)$. $\square$

4

Mixed Splines on Cycle Graphs

The following chapter explores and introduces the concept of a spline on a graph. We then give a method for obtaining a C^M spline by first finding a spline on a specially constructed cycle graph. Being able to do so will allow us to access key results proving the existence of particular splines in chapter 5 which will serve as basis elements in chapter 6.

Example 4.0.1. Consider the following polynomial spline:

$$F = \begin{cases} f_1 = -x : x \leq 0 \\ f_2 = x^2 : x \geq 0 \end{cases}$$

Then clearly $F = (-x, x^2)$ is a spline over $I = (-\infty, 0] \cup [0, \infty)$ as $f_1(0) = f_2(0)$. Figures 4.0.1 and 4.0.2 depict two visual representations of F.

We will now discuss how we can visualize a mixed spline as a vertex labeling. We do this so we can use some previous results about splines on cycle graphs.

Definition 4.0.2. Let G be a graph with edge set $E = \{e_1, e_2, ..., e_k\}$ and vertex set $V = \{v_1, v_2, ..., v_n\}$. Let ℓ_i be a polynomial label on edge e_i and $A = \{\ell_1, \ell_2, ..., \ell_k\}$ be the set of edge labels. then (G, A) is an edge labelled graph.

Definition 4.0.3. A **polynomial spline on** (G, A) is an vertex labelling $(f_1, f_2, ..., f_n) \in \mathbb{R}[x]$, such that for any two vertices $i, j \in V$ connected by edge e_k, we have $f_i \equiv f_j \mod \ell_k$.

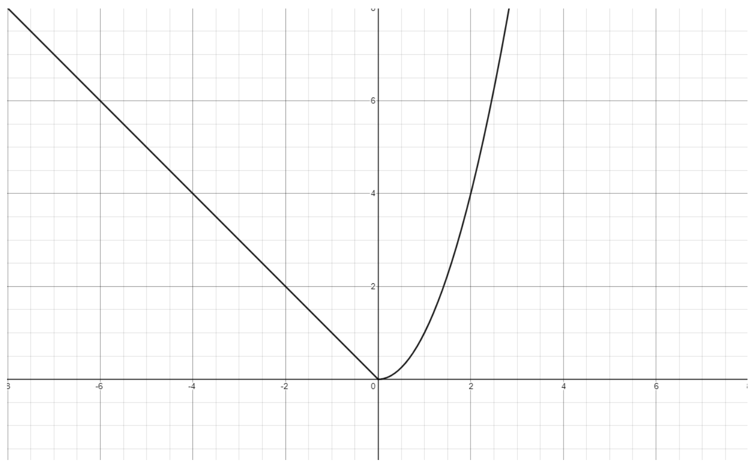

Figure 4.0.1. The polynomial spline $F = (-x, x^2)$

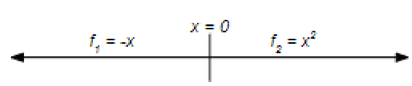

Figure 4.0.2. A second representation of the polynomial spline $F = (-x, x^2)$

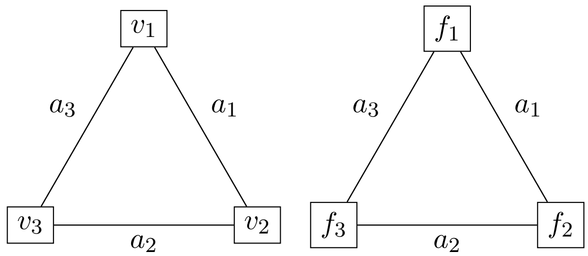

Figure 4.0.3. An edge-labeled graph (left) and a graphical representation of a generalized polynomial spline on the edge-labeled graph (right).

Figure 4.0.3 illustrates this definition.

These graphical interpretations allow us to find splines over given intervals. By considering particular splines on cycle graphs we are able to get mixed splines. We do this by only considering splines on cycle graphs where the first vertex is labeled 0.

Theorem 4.0.4. *Let (G, A) be an $n+1-$cycle graph with $A = \{\ell_0^{r_0+1}, \ell_1^{r_1+1}, ..., \ell_n^{r_n+1}\}$. Then let $B = (0, b_1, b_2, ..., b_n)$ be a polynomial spline on (G, A). $B' = (b_1, b_2, ..., b_n)$ is in $C^M(I_{(a_0,a_1,...,a_n)})$ spline where $M = (r_0, r_1, ..., r_n)$.*

Proof. $B = (0, b_1, b_2, ..., b_n)$ is a polynomial spline on (G, A) and so by definition the following is true:

$$\begin{aligned} 0 &\equiv b_1 \mod \ell_0^{r_0+1} \\ b_1 &\equiv b_2 \mod \ell_1^{r_1+1} \\ &\vdots \\ b_{n-1} &\equiv b_n \mod \ell_{n-1}^{r_{n-1}+1} \\ b_n &\equiv 0 \mod \ell_n^{r_n+1} \end{aligned}$$

then by Theorem 3.4.3 these defining equations tell us $B' \in C^M(I_{(a_0,a_1,...,a_n)})$. $\square$

We outline an example of a $C^{(1,2,1)}$ spline over $I_{(-1,0,1)}$ found on a 3-cycle graph.

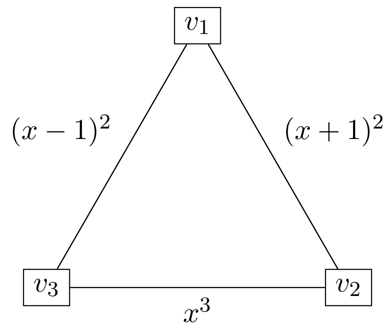

Figure 4.0.4. An edge weighted 3−cycle graph.

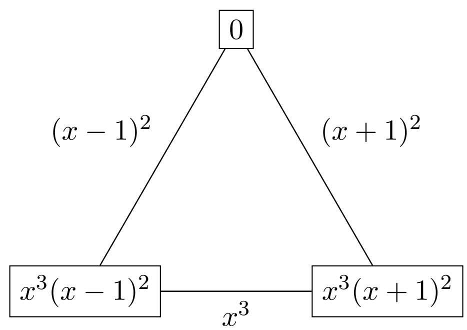

Figure 4.0.5. A polynomial spline on G

Example 4.0.5. Let (G, A) be the graph in figure 4.0.4 with $A = \{(x+1)^2, x^3, (x-1)^2\}$. Then $F = (0, f_2, f_3)$ is a polynomial spline on (G, A) provided the following equations hold.

$$0 \equiv f_2 \mod (x+1)^2$$

$$f_2 \equiv f_3 \mod x^3$$

$$f_3 \equiv 0 \mod (x-1)^2$$

One solution is $(0, f_2, f_3) = (0, x^3(x+1)^2, x^3(x-1)^2)$ shown in figure 4.0.5. Then by Theorem 4.0.4 we find that $B' = (x^3(x+1)^2, x^3(x-1)^2)$ is a $C^{(1,2,1)}$ spline over $I_{(-1,0,1)}$. The graph of which is seen in figure 4.0.6.

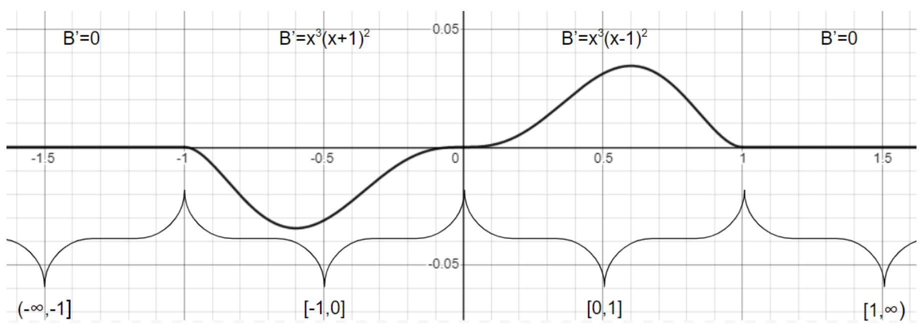

Figure 4.0.6. The $C^{(1,2,1)}$ spline $B' = (x^3(x+1)^2, x^3(x-1)^2)$ over $I_{(-1,0,1)}$.

5
Integer Splines

In the following section we explore generalized integer splines. These splines exist on graphs with integer edge labels. We begin by laying out some basic definitions before discussing flow-up classes. The elements of these flow-up classes are used in a basis for integer splines. Lastly, we recount some important results proving the existence of particular elements of flow-up classes on graphs, and use this to prove the existence of C^M splines that will be used to form our basis in chapter 6.

Definition 5.0.1. Let G be an edge-weighted graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, edge set $E = \{e_1, e_2, ..., e_m\}$ and polynomial edge weights $A = \{x - a_1, x - a_2, ..., x - a_m\}$. A **generalized integer spline** is a vertex labelling $(f_1, ..., f_n) \in \mathbb{Z}^n$ such that if v_i and v_j are connected by an edge e_k, then $f_i \equiv f_j \mod a_k$.

The following figure illustrates definition 5.0.1. Note: We limit the labelling of vertices to integers and elements of the edge weight set A to the natural numbers.

The results we seek to extend for polynomial splines has been proven for $n-$cycle splines as defined below.

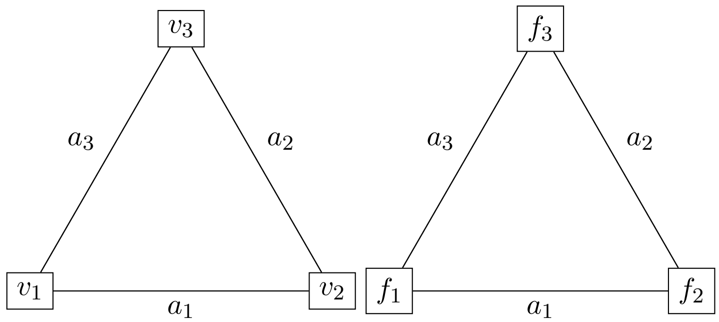

Figure 5.0.1. An edge-labeled graph (left) and a graphical representation of a generalized integer spline (right).

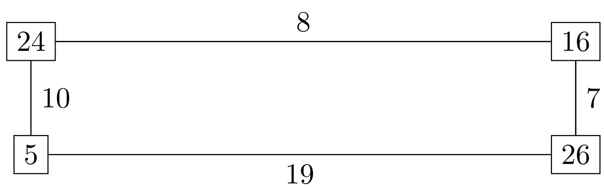

Figure 5.0.2. A $4 - cycle$ spline.

Definition 5.0.2. Let $A = \{a_1, .., a_n\}$ be the ordered set of edge-labels on an $n-$cycle graph with ordered vertices $\{v_1, ..., v_n\}$. If the following conditions are satisfied

$$f_1 \equiv f_2 \mod a_1 \tag{5.0.1}$$

$$f_2 \equiv f_3 \mod a_2 \tag{5.0.2}$$

$$\vdots \tag{5.0.3}$$

$$f_{n-1} \equiv f_n \mod a_{n-1} \tag{5.0.4}$$

$$f_n \equiv f_1 \mod a_n. \tag{5.0.5}$$

then $F = (f_1, f_2, ..., f_n)$ is an **n-cycle spline**.

Example 5.0.3. Figure 5.0.2 represents a $4 - cycle$ spline, $F = (24, 16, 26, 5)$.

5.1 Flow-Up Class

We now discuss a method developed for integer splines on $n-$cycle graphs. We call this method the **Flow-up Class method** due to its relationship to the Flow-up classes. We begin by defining a flow-up class and giving an outline of the results for integer splines.

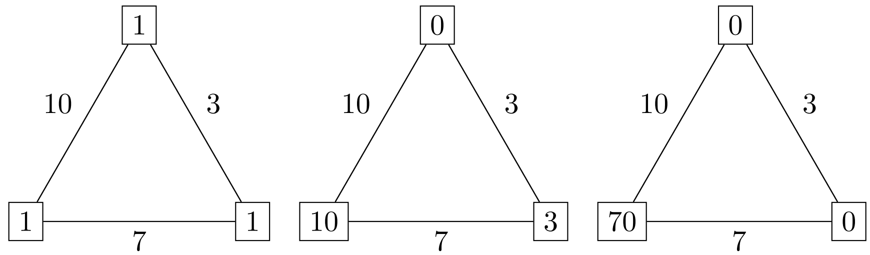

Figure 5.1.1. Elements of the Flow-up classes F_0, F_1 and F_2 on (G, A).

Definition 5.1.1. Fix the edge labels on (G, A). For $0 \leq i < n$, let the flow-up class F_i be the set of splines on (G, A) where the first i components are 0 and the $i+1$st component is non-zero. Note $F_n = \{0\}$.

Example 5.1.2. Fix the edges on the $3 - cycle$ (G, A), where $A = \{3, 7, 10\}$. Let the integer splines $B_0 = (1, 1, 1), B_1 = (0, 3, 70), B_2 = (0, 0, 70)$. Clearly B_0 is in the flow-up class F_0, B_1 is in F_1 and B_2 is in F_2. Figure 5.1.1 illustrates these elements of different flow-up classes.

We now define a minimal element of a flow-up class.

Definition 5.1.3. A **minimal element** of a flow-up class is an element such that the leading element is the smallest value possible.

The following theorem tells us when we have a minimal element.

Theorem 5.1.4. *([3], Theorem 4.5) Fix the edge labels on (G, A), where $A = \{\ell_1, \ell_2, ..., \ell_n\}$. Let $1in1$ and let $F = (0, ..., 0, f_{i+1}, f_{i+2}, ..., f_n)$ be an element in the flow-up class F_i on (G, A). Then, the leading element, f_{i+1}, is a multiple of $[\ell_i, (\ell_{i+1}, ..., \ell_n)]$ and $f_{i+1} = [\ell_i, (\ell_{i+1}, ..., \ell_n)]$ is the smallest value that satisfies the v_k edge conditions.*

An important result of [1] was that a careful selection of elements of these flow-up classes along with the all 1 vector resulted in a basis for splines on $n-$cycle graphs. Unfortunately the proof of which was omitted by [1].

This inspired us to consider if such a basis might exists for mixed splines. For that to be possible we much first have that such elements exist for polynomial splines.

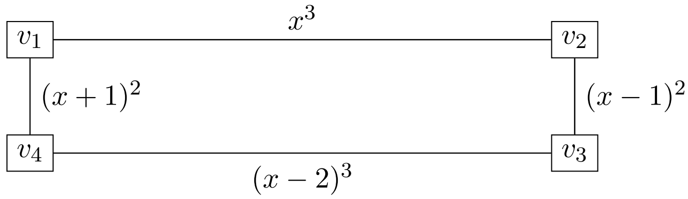

Figure 5.1.2. A 4−cycle graph on which we find minimal elements.

Their existence was proved by [3] on cycle graphs by the following theorems. This we will be useful as we have previously shown we can can get mixed splines from splines on cycle graphs, and we recall those splines on cycle graphs had a leading zero.

The first theorem guarantees that our flow-up classes are not empty.

Theorem 5.1.5. *Theorem 4.3 in* [3]*. Fix an* $n-$*cycle with edge labels* (G, A)*. Let* $n \geq 3$ *and* $1 \leq i < n$*. There exists an element of the flow-up class* F_i*,* B_i *on* (G, A)*.*

Lastly we show the minimal elements exist in the flow-up classes.

Theorem 5.1.6. *Theorem 4.6 in* [3]*. Fix an* $n-$*cycle with edge labels* (G, A)*. Fix* $n \geq 3$ *and* $2 \leq i < n$*. There exists a minimal element of the flow-up class* F_i*,* $B_i = (0, ..., 0, f_{i+1}, ..., f_n)$ *on* (G, A)*.*

The following example illustrates Theorem 5.1.6 for C^M splines on a 4−cycle graph.

Example 5.1.7. Let (G, A) be the edge labeled 4−cycle graph in figure 5.1.2. We find the minimal elements B_i of the form we expect on (G, A) for $1 \leq i < 4$.

$$B_1 = (0, x^3, x^3 + x^2 - 2x + 1, 2x^3 - 5x^2 + 10x - 7)$$

$$B_2 = (0, 0, (x-1)^2, x^3 - 5^2 + 10x - 7)$$

$$B_3 = (0, 0, 0, (x-2)^3(x+1)^2)$$

6
Current Research

In this chapter, we present the main result of our research of mixed splines. We prove that if we take a set of mixed splines from distinct flow-up classes with minimal leading terms, they will form a basis for $C^M(I)$. We give an example to illustrate the construction of such a basis. We show that for any $C^M(I)$ we can find such a basis. We then give corollaries about bases for $C^{r,p}(I)$, and finally prove a result relating $C^M(I)$ to the leading term degrees of its basis.

We now prove that given a vector $M \in (\mathbb{N} \cup \{0\})^n$ and an interval $I \in \mathbb{R}$ we can determine a basis for $C^M(I)$.

Theorem 6.0.1. *Let $I = I_{(a_0,\ldots,a_n,a_{n+1})}$. Let $\mathcal{B} = (B_0, B_1, ..., B_{n-1})$ be a set of C^M splines such that*

$$B_0 = (\ell_0^{r_0+1}, f_{0,2}, f_{0,3}, ..., f_{0,n})$$

$$B_i = (0, ..., \ell_i^{r_i+1}, f_{i,i+2}, f_{i,i+3}, ..., f_{i,n})$$

$$B_{n-1} = (0, 0, ..., 0, (\ell_{n-1}^{r_{n-1}})\ell_n^{r_n+1})$$

for some $f_{i,j}'s$ where each B_i has i leading zeros for $1 \leq i \leq n-2$. Then $\mathcal{B}$ is a basis for $C^M(I)$.

Proof. Consider the matrix $\mathcal{M} = [B_0, B_1, ..., B_{n-1}]$ then

$$M=\begin{bmatrix} [\ell_0^{r_0+1},(\ell_1^{r_1+1},..,\ell_n^{r_n+1})] & 0 & ... & 0 & 0 \\ f_{0,2} & [\ell_1^{r_1+1},(\ell_2^{r_2+1},..,\ell_n^{r_n+1})] & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{0,n-2} & f_{1,n-2} & ... & [\ell_{n-2}^{r_{n-2}+1},(\ell_{n-1}^{r_{n-1}+1},\ell_n^{r_n+1})] & 0 \\ f_{0,n-1} & f_{1,n-1} & ... & f_{n-2,n-1} & [\ell_{n-1}^{r_{n-1}+1},\ell_n^{r_n+1}] \end{bmatrix}$$

By definition each a_i is unique and so is each ℓ_i. Thus, $(\ell_i, \ell_j) = 1$ for any $0 \le i < j \le n$. Thus we can simplify the diagonals of $\mathcal{M}$ such that

$$M=\begin{bmatrix} [\ell_0^{r_0+1},1] & 0 & ... & 0 & 0 \\ f_{0,2} & [\ell_1^{r_1+1},1] & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{0,n-2} & f_{1,n-2} & ... & [\ell_{n-2}^{r_{n-2}+1},1] & 0 \\ f_{0,n-1} & f_{1,n-1} & ... & f_{n-2,n-1} & [\ell_{n-1}^{r_{n-1}+1},\ell_n^{r_n+1}] \end{bmatrix}.$$

$\mathcal{M}$ is upper triangular and so we can easily calculate the determinant to be the product of the diagonals.

$$\det \mathcal{M} = \prod_{i=0}^{n} \ell_i^{r_i+1}$$

Which is equal to the polynomial Q. By 3.5.4, $\mathcal{B}$ is a basis for $C^M(I)$. $\square$

We consider a small example of finding a basis this way.

Example 6.0.2. Let $M = (1, 2, 3)$ and let $I = I_{(-1,0,1)}$. Consider the set $\mathcal{B} = (B_0, B_1)$. where

$$B_0 = ((x+1)^2, (x+1)^2(x-1)^3)$$

$$B_1 = (0, x^3(x-1)^4)$$

We can see that B_0 is a minimal element of the flow-up class F_0 by theorem 5.1.4, which satisfies

$$(x+1)^2 \equiv 0 \mod (x+1)^2$$

$$(x+1)^2 \equiv (x+1)^2(x-1)^3 \mod x^3$$

$$(x+1)^2(x-1)^4 \equiv 0 \mod (x-1)^4$$

and is thus a C^M spline.

We can also see that B_1 is a minimal element of the flow-up class F_1 by theorem 5.1.4, which satisfies

$$\begin{aligned} 0 &\equiv 0 \mod (x+1)^2 \\ 0 &\equiv x^3(x-1)^4 \mod x^3 \\ x^3(x-1)^4 &\equiv 0 \mod (x-1)^4 \end{aligned}$$

by Theorem 6.0.1 $\mathcal{B}$ is a basis for $C^M(I)$.

We next show that such basis elements exist as to satisfy Theorem 6.0.1.

Theorem 6.0.3. *Let $I = I_{(a_0,...,a_n,a_{n+1})}$. There exists $\mathcal{B} = (B_0, B_1, ..., B_{n-1})$ C^M splines over I such that*

$$\begin{aligned} B_0 &= (\ell_0^{r_0+1}, f_{0,2}, f_{0,3}, ..., f_{0,n}) \\ B_i &= (0, ..., f_{i,i+1}, f_{i,i+2}, f_{i,i+3}, ..., f_{i,n}) \\ B_{n-1} &= (0, 0, ..., 0, (\ell_{n-1}^{r_{n-1}})\ell_n^{r_n+1}) \end{aligned}$$

where $f_{i,i+1} = \ell_i^{r_i+1}$ for $1 \leq i \leq n-2$. The elements of $\mathcal{B}$ exist.

Proof. Let (G, A) be an $n+1-$cycle where $A = \{\ell_0^{r_0+1}, ..., \ell_n^{r_n+1}\}$. Then by Theorem 5.1.6 there exists the set of splines $\mathcal{G} = (G_1, G_2, ..., G_n)$ on (G, A) such that

$$G_i = (0..., \ell_i^{r_i+1}, g_{i+2,2}, ..., g_{n,n})$$

where G_i has i leading zeros for $1 \leq i < n+1$. Let $\mathcal{B} = (B_0, B_1, ..., B_{n-1})$ where $B_i = G_{i+1}$ with the first element removed, by Theorem 4.0.4 these splines exist and are C^M over I. Thus we have found a set that satisfies the conditions of Theorem 6.0.1. $\square$

Thanks to Theorem 6.0.1 we can easily arrive at a basis for all $C^{r,p}(I)$ splines.

Corollary 6.0.4. *Let* $I = I_{(a_0,\ldots,a_n,a_{n+1})}$. *Let* $\mathcal{B} = (B_0, B_1, \ldots, B_{n-1})$ *be a set of* $C^{r,p}$ *splines such that*

$$B_0 = (\ell_0^{p+1}, f_{0,2}, f_{0,3}, \ldots, f_{0,n})$$

$$B_i = (0, \ldots, f_{i,i+1}, f_{i,i+2}, f_{i,i+3}, \ldots, f_{i,n})$$

$$B_{n-1} = (0, 0, \ldots, 0, (\ell_{n-1}^{r+1})\ell_n^{p+1})$$

where $f_{i,i} = \ell_i^{r+1}$ *for* $1 \leq i \leq n-2$. *Then* $\mathcal{B}$ *is a basis for* $C^{r,p}(I)$.

Proof. Let $M = (p, r, r \ldots, r, p)$, then by Theorem 6.0.1 $\mathcal{B}$ is a basis for $C^{r,p}(I)$. □

We also make note of the LT's of the basis.

Theorem 6.0.5. *Let* $I = I_{(a_0,\ldots,a_n)}$. *Then there exists a basis for* $C^M(I)$ *with LT degrees* $r_0 + 1, r_1 + 1, \ldots, r_{n-2} + 1, (r_{n-1} + 1) + (r_n + 1)$.

Proof. This follows directly from the basis found in Theorem 6.0.1. □

Using the basis found in example 6.0.2 we can see these LT's.

Example 6.0.6. Consider the basis $\mathcal{B} = (((x+1)^2, (x+1)^2(x-1)^3), (0, x^3(x-1)^4))$ for $C^{(1,2,3)}(I_{(-1,0,1)})$. We can easily see

$$LT(((x+1)^2, (x+1)^2(x-1)^3)) = 2 = 1 + 1$$

$$\textit{and}$$

$$LT((0, x^3(x-1)^4)) = 7 = (2+1) + (3+1)$$

as expected.

7
Comparison of Bases

This chapter will examine the relationship between a pre-existing basis for $C^{r,0}(I)$ and our new one. First we will consider an example of finding both bases for a set of $C^{3,0}$ splines. Then, we will show that the determinants of the matrices with the basis vectors as columns are equal, and that the sum of the degree's of the basis from Theorem 3.5.1 is equal to the sum of the LT degrees of the basis from Theorem 6.0.1.

Example 7.0.1. Let $I = I_{(-2,-1,0,1)}$ and let $r = 3$. We will now construct a basis using two different methods. By Theorem 3.5.1 we know $\mathcal{B} = (B_1, B_2, B_3)$ where

$$B_1 = ((x+2)((x-1)x^3 - (x+1)^4), (x-1)((x+2)x^3 - (x+1)^4, (x-1)((x+2)x^3 - (x+1)^4))$$

$$B_2 = ((x+2)((x-1)x^3 - (x)^4), (x+2)((x-1)x^3 - (x)^4), (x-1)((x+2)x^3 - (x)^4))$$

$$B_3 = ((x+2)(x-1), (x+2)(x-1), (x+2)(x-1))$$

forms a basis for $C^{3,0}(I)$.

We now find a second basis with our new method and get that $\mathcal{B}' = (B_0', B_1', B_2')$ where

$$B_0' = ((x+2), (x+1)^4 + x + 2, -19x^4 + (x+1)^4 + x + 2)$$
$$B_1' = (0, (x+1)^4, -16x^4 + (x+1)^4)$$
$$B_2' = (0, 0, x^4(x-1))$$

also forms a basis for $C^{3,0}(I)$ by Theorem 6.0.1.

Considering both of these basis we compare the derivatives of the matrices $[B_1, B_2, B_3]$ and $[B_0', B_1', B_2']$. We then determine a relationship between the two.

Example 7.0.2. Consider the bases found in example 7.0.1. We consider the derivatives of the matrices $[B_1, B_2, B_3]$ and $[B_0', B_1', B_2']$.

$\det[B_1, B_2, B_3] =$

$$\det \begin{bmatrix} (x+2)((x-1)x^3 - (x+1)^4) & (x+2)((x-1)x^3 - x^4) & (x+2)(x-1) \\ (x-1)((x+2)x^3 - (x+1)^4) & (x+2)((x-1)x^3 - x^4) & (x+2)(x-1) \\ (x-1)((x+2)x^3 - (x+1)^4) & (x-1)((x+2)x^3 - x^4) & (x+2)(x-1) \end{bmatrix}$$

$= 9x^{10} + 45x^9 + 72x^8 + 18x^7 - 63x^6 - 63x^5 - 18x^4$.

$\det[B_0', B_1', B_2'] =$

$$\det \begin{bmatrix} (x+2) & 0 & 0 \\ (x+1)^4 + x + 2 & (x+1)^4 & 0 \\ -19x^4 + (x+1)^4 + x + 2 & -16x^4 + (x+1)^4 & x^4(x-1) \end{bmatrix}$$

$= (x+2)(x+1)^4 x^4 (x-1) = x^{10} + 5x^9 + 8x^8 + 2x^7 - 7x^6 - 7x^5 - 2x^4$. We note that $\det[B_1, B_2, B_3] = 9 \det[B_0', B_1', B_2']$.

Theorem 7.0.3. *Let $I = I_{(a_0, \ldots, a_n)}$ and fix $r \in \mathbb{N}$. Then let $\mathcal{B}$ be a basis of $C^{r,0}(I)$ found using Theorem 3.5.1, and let $\mathcal{B}'$ be a basis of $C^{r,0}(I)$ using our new method. Then,* $\det[\mathcal{B}] = (a_0 - a_n)^{n-1} \det[\mathcal{B}']$.

Proof. We showed in the proof of Theorem 6.0.1 that $\det[\mathcal{B}'] = \prod_{i=0}^{n} \ell_i^{r_i+1}$ for the general case. Recalling that we are in the case where $M = (0, r, r, ..., r, 0)$ we obtain

$$\det[\mathcal{B}'] = (\ell_0)(\ell_n) \prod_{i=1}^{n-1} \ell_i^{r+1}. \tag{7.0.1}$$

We then obtain $\det[\mathcal{B}]$ from Lemma 3.3 in [2] which tells us

$$\det[\mathcal{B}] = (a_0 - a_n)^{n-1}(\ell_0)(\ell_n) \prod_{i=1}^{n-1} \ell_i^{r+1}. \tag{7.0.2}$$

Thus, we have

$$\det[\mathcal{B}] = (a_0 - a_n)^{n-1}(\ell_0)(\ell_n) \prod_{i=1}^{n-1} \ell_i^{r+1} = (a_0 - a_n)^{n-1} \det[\mathcal{B}']. \tag{7.0.3}$$

□

We are also able to come up with a result relating the LT's of our basis to the degree of the basis elements of the old basis.

Theorem 7.0.4. *Let $I = I_{(a_0,...,a_n)}$ and fix $r \in \mathbb{N}$. Then let $\mathcal{B} = (B_1, B_2, ..., B_n)$ be a basis of $C^{r,0}(I)$ found using Theorem 3.5.1, and let $\mathcal{B}' = (B_0', B_1', ..., B_{n-1}')$ be a basis of $C^{r,0}(I)$ using our new method. Then, $\sum_{i=1}^{n} \deg(B_i) = \sum_{i=0}^{n-1}(B_i')$.*

Proof. The first thing we note, is that by Theorem 4.1 from [2] we know

$$\sum_{i=1}^{n} \deg(B_i) = 2 + (n-1)(r+1).$$

We then calculate the sum of the degrees for our new basis. Our basis $\mathcal{B}'$ was carefully chosen so that $LT(B_i') = r + 1$ for $0 \leq i \leq n-2$ and $LT(B_{n-1}') = 2$ therefore

$$\sum_{i=0}^{n-1} LT(B_i') = \sum_{i=0}^{n-2} LT(B_i') + LT(B_{n-1}') = (n-1)(r+1) + 2.$$

Thus, $(n-1)(r+1) + 2 = \sum_{i=1}^{n} \deg(B_i) = \sum_{i=0}^{n-1}(B_i')$. □

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