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The Facilitation of Sound Waves Using Mathematical and Scientific Methods of Digital Signal Processing

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by
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Abstract

Mathematics and music have been lifelong partners since the beginning of time. Rhythm and time are two fundamental aspects of music which rely solely on counting, an under-appreciated skill in mathematics, yet recognized by all mathematically-minded people as the foundation of some of the most important mathematical findings; as John B. Fraleigh would say, "Never underestimate a theorem that counts something!" However, music recordings have evolved through the use of technology further than merely possessing the capabilities to quantify and archive the notes that were played in the recording. In the days before digital recordings, the only way to ensure better sound quality was by modifications to the recording equipment and acoustical adjustments to the setting of the recording. It is an indisputable fact that the overwhelming majority of recordings in our modern day are produced using digital technology. The discovery that sound is quantified by a sum of the disturbances the source of the perturbations created by changes in air pressure, known as sound waves, preceded the use of digital technology, but it was this amazing observation that first shaped how mathematicians and scientists utilized mathematical methods in audio engineering. This group of mathematicians and scientists, known as acousticians, aspire to utilize digital signal processing techniques to reproduce the sound created by a source and enhance the sound quality of the recording. There are many ways this can be accomplished. However, one of the most important methods available to acousticians is filtering. Filtering is the process of accentuating or attenuating certain frequencies in the content of the signal.

The focus of this paper is digital filtration techniques and the fundamental workings of the mathematical methods that allow these techniques to be possible, as well as the various kinds of software that utilize these techniques. In addition to detailed explanations of the processes by which these methods were born, there is also a case study comparing the effect of a piece of hardware used for filtration with the effect of a digital filter modeled to exhibit the same behavior and functionality as the analog filter.

This intensive and comprehensive study of digital signal processing and the mathematical methods acousticians used to create this field of science has been something I have been in-

terested in for many years since first learning of my passion for mathematics. I am excited to share my research and data with the science and math community.

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Dedication

To my Grandfather Rocco Ravaschiere and Aunt Antoinette Tesorerio

Acknowledgments

I would like to thank my high school Pre-calculus teacher Antonio Centeio for making me understand that mathematics is a subject that one must actively and independently study rigorously in order to understand and conceptualize information. I would like to thank Lauren Rose, my Number Theory teacher in Freshman year, whose topics in class influenced me to become more interested in mathematics. I am very grateful to have the privilege of learning from Amir Barghi, Matt Szczesny, Matt Kelly, Joseph Kirtland, Mary Krembs, John Cullinan, Jim Belk, Ethan Bloch, and Nicholas Lanzillo, all of whom taught me the subjects necessary for my mathematics career to prosper. I would also like to thank my senior project advisor Matthew Deady for supporting me through the process of senior project and teaching me the finer intricacies of signal processing. I would also like to acknowledge my parents Michael Moss and Claudia Ravaschiere who influenced me to pursue a career in the sciences, despite the fact that they are both artists. I believe a lot of areas of art and music have elements of mathematics, and to be mathematically-minded there is a certain kind of creativity that distinguishes itself in those individuals who are the most successful in their fields of study. For this reason I would also like to thank the intelligent and creative minds at Brown Innovations in South Boston for giving me the opportunity to experience signal processing first-hand, influencing me to study digital signal processing, and encouraging me to explore how mathematics and music merge into a interdisciplinary study that has fascinated some of the most brilliant minds throughout history.

1
Introduction

There are many scientists and mathematicians who have contributed to the field of acoustics and audio engineering. However, the man who changed how we understand the behavior of waveforms is one of the most critically acclaimed mathematicians, and whom acousticians should show the most gratitude. Jean Baptiste Fourier (1768-1830) was a French scientist and mathematician who proved that any continuous periodic waveform can be expressed as an infinite series of sinusoids. His work was the foundation for the analytical understanding of waveforms in a wide variety of fields in mathematics and science.

One part of this paper is to expound the fundamental methods of Fourier analysis and reveal how these techniques are incorporated in audio engineering. The majority of the formulations of the mathematical proofs that contribute to Fourier analysis are well-known; additionally, the real analysis proofs that greatly influenced our understanding of Fourier analysis are revered among mathematicians and physicists alike. Much of this paper is devoted to establishing the methods of signal processing; thus, a majority of the information included conveys how mathematics influenced the evolution of signal processing and the production of digital recordings.

The other part of this paper explores the case study of a digital filter which was created to model the functionality of an analog filter.

The topic of digital signal processing and the mathematical methods behind this field of study are enthralling, but like many fields of mathematics, our understanding of digital signal processing begins with the humblest of origins. The beginning of digital signal processing began with signal processing. Signal processing began with a disgruntled audience member in a theater muttering the all too familiar phrase: “I would like this instrument to sound louder, and this one to sound softer." In other words, a signal was processed aurally by a person actively judging which elements of the sound should be increased and which ones should be decreased. Two variables were important in this person’s observations: the magnitude of the sound and the direction of the sound. This is where we begin our study. We build upon the mathematical construction which has magnitude and direction: the vector.

2
The Construction of Signals

2.1 Core Definitions Regarding Vectors

Before the explanation of how waves are constructed, it is useful to first establish the definition of several devices used in Euclidean space. These principles are universal in how mathematicians and physicists view the world. Understanding the role of these fundamental concepts is crucial in beginning to grasp how acousticians formulated methods to qualitatively and quantitatively measure waves. We begin with the humblest of origins. The vector.

Definition 2.1.1. A **vector** is a quantity in mathematics which has a direction as well as a magnitude.

Vectors are used to determine the position of one point in space relative to another.

Example 2.1.2. Taking an example in 2-dimensional Euclidean space, let $\vec{v} = \langle x, y \rangle$. $\vec{v}$ has direction x on the x-axis and direction y on the y-axis and magnitude, denoted $|\vec{V}| = \sqrt{x^2 + y^2}$.

Vectors are not merely some tool mathematicians devised solely to measure linear systems. Vectors in linear systems are the foundation of important conceptual models which have been expanded upon to apply to many physical systems. But how can one measure give an accurate representation of a vastly dynamical system? The answer lies in the construction of a zone specially formatted for operating on vectors in order to design models of complex dynamical systems. The root of these systems are vector spaces.

Definition 2.1.3. A **vector space** over a **field** F is a set of vectors which satisfies vector addition and multiplication, scalar multiplication, and the 0 vector must be present.

The axioms of a field are expressed using the following mathematical notation. Let V be a vector space over the field F, let $\mathbf{X},\mathbf{Y},\mathbf{Z} \in V$ and let $r, s \in F$ be scalars.

1. Commutativity:

$\mathbf{X}+\mathbf{Y}=\mathbf{Y}+\mathbf{X}$.

2. Associativity of vector addition:

$(\mathbf{X}+\mathbf{Y})+\mathbf{Z}=\mathbf{X}+(\mathbf{Y}+\mathbf{Z})$.

3. Additive identity: For all $\mathbf{X}$,

$\mathbf{0}+\mathbf{X}=\mathbf{X}+\mathbf{0}=\mathbf{X}$.

4. Existence of additive inverse: For any $\mathbf{X}$, there exists a $\mathbf{-X}$ such that

$\mathbf{X}+(\mathbf{-X})=\mathbf{0}$.

5. Associativity of scalar multiplication:

$r(s\mathbf{X}) = (rs)\mathbf{X}$.

6. Distributivity of scalar sums:

$(r+s)\mathbf{X} = r\mathbf{X} + s\mathbf{X}$.

7. Distributivity of vector sums:

$r(\mathbf{X}+\mathbf{Y}) = r\mathbf{X} + r\mathbf{Y}$.

8. Scalar multiplication identity:

$1 \cdot \mathbf{X} = \mathbf{X}$.

The aforementioned vector $\vec{v} = \langle x, y \rangle$ was in component form, where x was the x component and y was the y component. This can also be written as a matrix defined

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x is the x component and y is the y component.

Mathematicians are interested in a special set of vectors. These vectors are ones which are not copies of the same few vectors, but are a particular set of vectors which can be gathered to create any vector in the vector space.

Definition 2.1.4. A **basis** (finite or infinite) is a set of vectors that spans the whole vector space and is linearly independent. These basis vectors are linearly independent if and only if they cannot be expressed as

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = 0$$

with $a_1, \ldots, a_n$ constants which are not all zero.

The significance of a basis can be demonstrated with the following example. Let $V = \mathbb{R}^2$ be a vector space, namely the entire $x - y$ plane. Take the vector with coordinates $(3,4)$. This vector can be written as a sum of its x component and y component:

$$(3,4) = (3,0) + (0,4)$$

it can be decomposed even further and written in terms of a“unit" x vector and a “unit" y vector:

$$(3,4) = 3 \cdot (1,0) + 4 \cdot (0,1).$$

The pair $\{(1,0),(0,1)\}$ of vectors span $\mathbb{R}^2$ because any vector can be decomposed this way:

$$(a,b) = a(1,0) + b(0,1).$$

The expressions of the form $a(1,0)+b(0,1)$ fill the space $\mathbb{R}^2$. This is also true of $(1,1)$ and $(0,1)$, where we can write our vector as

$$(3,4) = 3\cdot(1,1) + 1\cdot(0,1)$$

and more generally

$$(a,b) = a\cdot(1,1) + (b-a)\cdot(0,1).$$

This relates to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If these linearly independent vectors are in the subspace of a vector space they must also be in the vector space. Mathematicians have a definition for when these linearly independent vectors are members of all the same subspaces of a vector space.

Definition 2.1.5. The **span** of a set S vectors in a vector space V is defined to be the intersection W of all subspaces of V that contain the set S.

In this way any vector $\vec{v}$ in the span can be represented as a finite sum (linear combination) of basis elements

$$a_1\vec{b_{i1}} + a_2\vec{b_{i2}} + \cdots + a_n\vec{b_{in}} = \vec{v} \tag{2.1.1}$$

where $a_1,\ldots,a_n$ are constants and $\vec{b_{i1}},\ldots,\vec{b_{in}}$ are vectors that form the basis.

2.2 The Relationship Between the Dot Product and a Basis of Vectors

There are other operations on vectors in a field besides scalar multiplication, addition, and the axioms. A common operation on vectors is known as the **dot product**. The dot product, universal notation $\langle\cdot\rangle$, is particularly useful for our purposes in order to understand the link between vectors and waveforms.

Example 2.2.1. Let $\vec{v}$ and $\vec{w}$ be vectors defined by $\vec{v} = \langle 1,2,3\rangle$ and $\vec{w} = \langle 4,5,6\rangle$. The dot product of these vectors is

$$\begin{aligned}\vec{v}\cdot\vec{w} &= 1(4)+2(5)+3(6)\\ &= 4+10+18\\ &= 32.\end{aligned}$$

The dot product has many applications in physics and mathematics. Our interest in the dot product lies in its role in the definition of **orthogonality**.

Definition 2.2.2. Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if and only if their dot product is equal to 0. If these two vectors are **orthogonal** a widespread notation used to convey this fact is $\vec{v} \perp \vec{w}$. If two vectors are **orthogonal** they are positioned apart by an angle of 90° or $\frac{\pi}{2}$ radians.

Example 2.2.3. Let $\vec{v}$ and $\vec{w}$ be vectors defined by $\vec{v} = \langle x,y\rangle$ and $\vec{w} = \langle -y,x\rangle$. Showing these vectors are orthogonal is trivial, $\vec{v}\cdot\vec{w} = x(-y)+xy = -xy+xy = 0$.

Take two vectors $\vec{a} = \langle 1,0\rangle$ and $\vec{b} = \langle 0,1\rangle$. Clearly these vectors are orthogonal [1(0)+0(1)=0]. However, these vectors form a basis spanning $\mathbb{R}^2$, which is significant because any vector can be constructed with this basis. This means that at discrete points in time any level can be achieved by multiplying this basis by the desired scalars at that discrete point. This yields a figure which can represent any discrete sample of data, with time on the x-axis and the level on the y-axis. Only the orthogonality of these vectors that form the basis allow this construction to be achieved.

Orthogonality will be crucial in understanding how the frequency domain is formulated. However, if we are to impose the principles of orthogonality on functions in the time domain to create functions in the frequency domain, there are important criteria that these functions must satisfy for the transformation from the time domain to the frequency domain to be complete. The formulation of this criteria must be revealed, but it is not possible without the use of the fundamental properties of real numbers and real analysis. The following section is devoted to establishing the the role of real analysis in signal processing.

2.3 Foundations of Real Analysis and the Connection to Signal Processing

What is real analysis? Real analysis is the qualitative and quantitative means of analyzing how functions, series, and sequences on the real number line behave. Real analysis is important in understanding the construction of the frequency spectrum, the criteria for certain kinds of functions that can be used as input signals for filtering, the reason for the use of different windowing functions for displaying the graph of the discrete-time Fourier transform (DTFT), and many other areas in signal processing where acousticians' utilization of technology may inadvertently gloss over the finer details of the material provided. We begin with discussing the cardinal points of real analysis study concerning ordered fields, boundaries of intervals, limits and continuous functions, and the $\epsilon - \delta$ style proof.

Definition 2.3.1. An ordered field is a set F with elements $0, 1 \in F$, binary operations $+$ and $\cdot$, unary operations $-$, $^{-1}$, and relation $<$ on $F - \{0\}$, which satisfy the following properties. Let $x, y, z \in F$.

a. $(x + y) + z = x + (y + z)$ (Associative law for Addition).

b. $x + y = y + x$ (Commutative Law for Addition).

c. $x + 0 = x$ (Identity Law for Addition).

d. $x + (-x) = 0$ (Inverses Law For Addition).

e. $(xy)z = z(yz)$ (Associative Law for Multiplication).

f. $xy = yx$ (Commutative Law for Multiplication).

g. $x \cdot 1 = x$ (Identity Law for Multiplication).

h. If $x \neq 0$, then $xx^{-1} = 1$ (Inverses Law for Multiplication).

i. $x(y + z) = xy + xz$ (Distributive Law).

j. Precisely one of $x < y$ or $x = y$ or $x > y$ holds (Trichotomy Law).

k. If $x < y$ and $y < z$, then $x < z$ (Transitive Law).

l. If $x < y$ then $x + z < y + z$ (Addition Law for Order).

m. If $x < y$ and $z > 0$, then $xz < yz$ (Multiplication Law for Order).

n. $0 \neq 1$ (Non-Triviality).

These properties are necessary to discuss in order to begin understanding real analysis and the rules which apply to how we manipulate our functions, series, and sequences in order to arrive at sensical conclusions. The reason why we have focused on the ordered field is because the real numbers are an ordered field. In order to delve further into the properties of the real numbers, there are certain principles that apply to the real numbers that aid in our use of functions for analytical tools to understand fundamental laws in nature. The following definition relates to an ordered field with boundaries. In signal processing, when using discrete-time samples of a continuous sound wave, imposing a certain time limit of the sampled signal and choosing specifically sized increments of the discrete samples are two forms of figurative boundaries which affect the data drawn from the continuous signal. However, in an ordered field such as the real numbers, boundaries mark the ends of information which could extend ad infinitum.

However, since the real numbers are uncountable [2, Theorem 8.4.8], the information is still infinite within these boundaries. However, establishing boundaries allows for other mathematical conclusions to be drawn.

Definition 2.3.2. Let F be an ordered field, and let $A \subseteq F$ be a set.

1. The set A is **bounded above** if there is some $M \in F$ such that $x \leq M$ for all $x \in A$. The number M is called an **upper bound** of A.

2. The set A is **bounded below** if there is some $P \in F$ such that $x \geq M$ for all $x \in A$. The number P is called an **lower bound** of A.

3. The set A is **bounded** if it is bounded above and bounded below.

4. Let $M \in F$. The number M is a **least upper bound** (also known as a **supremum**) of A if M is an upper bound of A, and if $M \leq T$ for all upper bounds T of A.

5. Let $P \in F$. The number P is a **greatest lower bound** (also known as an **infimum**) of A if P is a lower bound of A, and if $P \geq V$ for all upper bounds V of A.

To quantify the smallest distance a value may approach from another point, a measure of “arbitrary closeness" must be created. To measure this distance, an arbitrary positive number is chosen, often denoted with a symbol ϵ or δ. The number known as the limit of a function, L, is the point which output values as these values approach a fixed number. Then, if for any $\epsilon > 0$ we choose, there exists a $\delta > 0$ such that for all choices of input within this distance δ of our fixed number, c, the value of our function will be within proximity ϵ of L. The conclusion is that the limit of the function as values in the domain approach the fixed number c is equal to L.

Definition 2.3.3. Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$, let $f\colon I - \{c\} \to \mathbb{R}$ be a function and let $L \in \mathbb{R}$. The number L is the **limit** of f as x goes to c, written

$$\lim_{x \to c} f(x) = L,$$

if for each $\epsilon > 0$, there is some $\delta > 0$ such that $x \in I - \{c\}$ and $|x - c| < \delta$ imply $|f(x) - L| < \epsilon$. If $\lim_{x \longrightarrow c} f(x) = L$, we also say that f **converges** to L as x goes to c. If f converges to some real number as x goes to c, we say that $\lim_{x \longrightarrow c} f(x)$ exists.

Continuous signals in the time domain are the subject of various analyses by acousticians. So what does it mean for a signal to be continuous, let alone a function? Using our $\epsilon - \delta$ proof writing style to define what makes a function continuous is our next step to understanding how functions can be harnessed to create any kind of curve in space. But there are only particular curves that can be created, specifically continuous curves. We must start with the definition of a continuous function.

Definition 2.3.4. Let $A \subseteq \mathbb{R}$ be a set, and let $f : A \longrightarrow \mathbb{R}$ be a function.

1. Let $c \in A$. The function f is **continuous at** c is for each $\epsilon > 0$, there is some $\delta > 0$ such that $x \in A$ and $|x - c| < \delta$ imply $|f(x) - f(c)| < \epsilon$. The function f is **discontinuous at** c if f is not continuous at c; in that case we also say that f has a **discontinuity at** c.

2. The function f is **continuous** if it is continuous at every number in A. The function f is **discontinuous** if it is not continuous.

If there is a gap in the curve of a function (such as a piece-wise function) or an asymptote (vertical or horizontal "barrier" that values approach but never reach) then the function is not continuous. There is a more stringent condition on continuous functions mathematicians seek in these functions that bears greater significance than if the function were merely continuous. The stronger condition is significant because instead of our choice of δ depending on c, ϵ and the function, now our choice of δ only depends on ϵ and the function. This is important because we know that for any point we can choose a point δ that will be within the same distance ϵ of our point.

Definition 2.3.5. Let $A \subseteq \mathbb{R}$ be a set, and let $f\colon A \longrightarrow \mathbb{R}$ be a function. The function f is **uniformly continuous** if for each $\epsilon > 0$, there is some $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

Observe that the notion of continuity is defined at distinct numbers in the domain of the function, but mathematicians do not have a concept of "uniformly continuous at a point." The reason for this is that the idea behind uniform continuity is the same δ works for a given ϵ, regardless of the specific point so long as that point is in the domain of the definition of the function. For our purposes we will be concerned with bounded, uniformly continuous functions; this fact necessitates stating the following corollary.

Corollary 2.3.6. *Let $C \subseteq \mathbb{R}$ be a closed bounded interval, and let $f\colon C \longrightarrow \mathbb{R}$ be a function. If f is continuous, then f is bounded.*

We have just discussed how a basis can be used to create a graph resembling to shape of a waveform by boosting the levels of each basis element at the given sample point. In signal processing it is crucial to know that the function being sampled is uniformly continuous in order to be able to pick arbitrary sample points of any desired increment; if points that are picked along the domain and the function is not continuous at these points there will be data missing in the discrete sample. This will lead to erroneous information. In this way it is crucial to know that the function is uniformly continuous over the domain is it defined.

Just as any discrete signal form can be created with orthogonal basis elements, and any sinusoid can be generated by the summation of sinusoids, any continuous function on a closed interval can be uniformly approximated on that interval by polynomials to any degree of accuracy. This idea is rooted in a theorem that we will use to show how all these facts are amalgamated for the purpose of creating the frequency spectrum. The theorem is called the **Weierstrass Approximation Theorem**, and it cannot be stated until a few results and definitions are given to provide context for the theorem.

2.4 Continuous Bounded Functions Created Using Polynomials

For a bounded uniformly continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ define for $h > 0$

$$S_h f(x) = \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) e^{-(\frac{u-x}{h})^2} du. \tag{2.4.1}$$

As a side note we define what it means for a sequence of functions to be **uniformly convergent** to another function. The sequence of functions can be thought of as approaching the function in all values of the domain of the family corresponding to that of the function these values are approaching. The proof style is similar to proving uniform continuity, except only the choice of N must depend on ϵ.

Definition 2.4.1. Let $A \subseteq \mathbb{R}$ be a non-empty set, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $A \longrightarrow \mathbb{R}$ and let $f : A \longrightarrow \mathbb{R}$ be a function. The sequence of functions $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** to f if for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n \geq N$ imply $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.

Theorem 2.4.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function. Then $S_h f$ converges uniformly to f as $h \longrightarrow 0$.*

Proof: Let $\epsilon > 0$. Then by definition of uniform continuity there exists $\delta > 0$ such that $x, y \in \mathbb{R}$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $M \in \mathbb{R}$ and suppose $|f(x)| \leq M$ for all $x \in \mathbb{R}$. We know M exists because f is bounded. We will show that the Gaussian integral can be computed by shell integration (double integration in polar coordinates) where we have

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r dr d\theta \\
&= 2\pi \int_{0}^{\infty} e^{-r^2} r dr \\
&= 2\pi \int_{-\infty}^{0} \frac{1}{2} e^{s} ds \{\text{Change of variables } [s = -r^2]\} \\
&= \pi \int_{-\infty}^{0} e^{s} ds \\
&= \pi e^{s}\Big|_{-\infty}^{0} \\
&= \pi (e^0 - e^{-\infty}) \\
&= \pi.
\end{aligned}
$$

Using the fact that $\int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi}$, it can be verified easily that

$$\frac{1}{h\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\frac{u-x}{h})^2} du = 1.$$

This implies that we can write

$$f(x) = \frac{1}{h\sqrt{\pi}}\int_{-\infty}^{\infty} f(x) e^{-(\frac{u-x}{h})^2} du.$$

Now let $h_0 > 0$ such that $h_0 < \frac{\epsilon\delta\sqrt{\pi}}{2M}$, then

$$
\begin{aligned}
|S_h f(x) - f(x)| &\le \frac{1}{h\sqrt{\pi}}\int_{-\infty}^{\infty} |f(u) - f(x)| e^{-(\frac{u-x}{h})^2} du \\
&\le \frac{\epsilon}{2} + \frac{1}{h\sqrt{\pi}}\int_{|x-u|\ge\delta} |f(u) - f(x)| e^{-(\frac{u-x}{h})^2} du \\
&\le \frac{\epsilon}{2} + \frac{2M}{h\sqrt{\pi}}\int_{|x-u|\ge\delta} e^{-(\frac{u-x}{h})^2} du \\
&= \frac{\epsilon}{2} + \frac{2M}{\sqrt{\pi}}\int_{|v|\ge\frac{\delta}{h}} e^{-v^2} dv \le \frac{2Mh}{\delta\sqrt{\pi}}\int_{|v|\ge\frac{\delta}{h}} |v| e^{-v^2} dv \\
&\le \frac{\epsilon}{2} + \frac{4Mh}{\delta\sqrt{\pi}}\int_{0}^{\infty} v e^{-v^2} dv = \frac{\epsilon}{2} + \frac{2hM}{\delta\sqrt{\pi}} < \epsilon
\end{aligned}
$$

for all $0 < h < h_0$ and all $x \in \mathbb{R}$. □

Theorem 2.4.3. *If f is a continuous real-valued function on $[a,b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial P on $[a,b]$ such that $|f(x) - P(x)| < \epsilon$ for all $x \in [a,b]$.*

Proof: We begin by extending f to a bounded uniformly continuous function on $\mathbb{R}$ by defining $f(x) = f(a)(x-a+1)$ on $[a-1,a), f(x) = -f(b)(x-b-1)$ on $(b,b+1]$, and $f(x) = 0$ on $\mathbb{R}|_{[a-1,b+1]}$. The notation used in the segment $\mathbb{R}|_{[a-1,b+1]}$ means that the values of $\mathbb{R}$ are **restricted** to only values including $a-1$, every number in between $a-1$ and $b+1$, and $b+1$. For the sake of the reader understanding more common notation, $[a,b]$ denotes that a and b are included in the interval along with numbers in between a and b, where an interval (c,d) indicates that c and d are not included in the interval but all numbers between c and d are included. We will proceed accordingly. In particular there exists an $R > 0$ such that $f(x) = 0$ for $|x| > R$. Let $\epsilon > 0$. Let $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all x. We know M exists because f is defined over a bounded interval and is continuous, so by our corollary we know f is bounded. By the theorem proved above we know there exists an $h_0 > 0$ such that for all $x \in \mathbb{R}$ we have $|S_h f(x) - f(x)| < \frac{\epsilon}{2}$. since $f(u) = 0$ for $|u| > R$, we can write

$$S_{h_0} f(x) = \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) e^{-(\frac{u-x}{h_0})^2}\, du.$$

On $[-\frac{2R}{h_0}, \frac{2R}{h_0}]$ the power series of e^{-v^2} converges uniformly, so there exists N such that

$$\left| \frac{1}{h_0\sqrt{\pi}} e^{-(\frac{u-x}{h_0})^2} - \frac{1}{h_0\sqrt{\pi}} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0}\right)^{2k} \right| < \frac{\epsilon}{4RM}$$

for all $|x| \leq R$ and all $|u| \leq R$, since in that case $|u-x| \leq 2R$. This implies that

$$\left| S_{h_0} f(x) - \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0}\right)^{2k} du \right| < \frac{\epsilon}{2}$$

for all $|x| \leq R$. If we put $P(x) = \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0}\right)^{2k} du$, then $P(x)$ is a polynomial in x of degree at most $2N$ such that $|S_{h_0} f(x) - P(x)| < \frac{\epsilon}{2}$ for all $|x| \leq R$. This implies that $|f(x) - P(x)| < \epsilon$ for all $x \in [a,b]$. □

The function $S_h f$ is the convolution of f with a Gaussian heat kernel. These heat kernels form an approximate identity. The topic of convolution will be discussed in the Chapter 5.

The significance of the Gaussian is that it corresponds to a random sample of data, where the majority of samples (representing the average values) of the data are centered around the peak of the parabolically-shaped curve, and the outliers of the data are located near the ends of curve and asymptotically approach zero. In our case study we will be analyzing samples of data sets, which is why establishing a connection between the Gaussian curves and how they are used in our analysis is important to our study. It is here that we find an overlap between these continuous function approximations and the summations of sinusoids used to create waveforms. Soon we will discuss how the convolution of uniformly continuous functions with Gaussians are infused with the concept of basis elements of vectors spaces and orthogonality of these basis elements, and how convolution of series of orthogonal waveforms may in turn be used to create the frequency spectrum. Before we reach that point, we must first reveal how orthogonality is applied to Fourier series.

2.5 Orthogonality of Fourier Series

We have discussed how a basis of vectors may be used to create different levels of a curve at discrete points depending on the value of the scalar which corresponds to the amplitude at that point. Now we will make the connection between these basis vectors and integrals to show how the orthogonality principle applies to continuous functions which we use in Fourier analysis.

Example 2.5.1. Let $\vec{v} = \sum_{i=1}^{i=k} v_i \vec{e}_i$ be a set of basis vectors. This means that the set $\vec{v}$ are linearly independent and every vector in the vector space is a linear combination of the set. By the orthogonality principle of basis vectors $\vec{e}_i \cdot \vec{e}_j = 0$ (if $i \neq j$).

This may be related to the integral of two functions which are orthogonal.

Example 2.5.2. Let $F\colon [-\pi,\pi] \longrightarrow \mathbb{R}$ be a continuous periodic function defined

$$F(x) = \sin(mx)\sin(nx)$$

where $m, n \in \mathbb{Z}$.

Suppose that $n \neq m$. Then we have $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx$. We may use one of the trigonometric addition formulas $-\frac{1}{2}[\cos((m+n)x) - \cos((m-n)x)] = \sin(mx)\sin(nx)$. Then we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = -\frac{1}{2}\int_{-\pi}^{\pi} [\cos((m+n)x) - \cos((m-n)x)]dx \\
&= -\frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n} - \frac{\sin((m-n)x)}{m-n}\right]\Big|_{-\pi}^{\pi} \\
&= -\frac{1}{2}\left[\frac{\sin((m+n)\pi)}{m+n} - \frac{\sin((m-n)\pi)}{m-n}\right] - \left(\left[\frac{\sin((m+n)(-\pi))}{m+n} - \frac{\sin((m-n)(-\pi))}{m-n}\right]\right) \\
&= 0.
\end{aligned}$$

$m - n \neq 0$ and $m + n = 0$ or another integer, therefore we know that $\sin(k\pi) = 0$ where $k \in \mathbb{Z}$. Therefore these two waveforms $\sin(mx)$ and $\sin(nx)$ are orthogonal to each other. Now take the case where $m = n$. Substitute m for n and we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(mx)dx &= \int_{-\pi}^{\pi} \sin^2(mx)dx \\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2mx)dx \\
&= \frac{1}{2}x - \frac{1}{4m}\sin(2mx)\Big|_{-\pi}^{\pi} \\
&= \frac{\pi}{2} - 0 - [-\frac{\pi}{2} - 0] \\
&= \pi.
\end{aligned}$$

A similar argument applies to cosines and exponentials.

The application of orthogonality extends to signal processing because when two waves are orthogonal their dot product will equal 0, and when two waves are equal they will add constructively. Intuitively this makes sense, because when the waves are orthogonal they are out of phase from one another by 90 degrees ($\frac{\pi}{2}$ radians), which means that they behave destructively and cancel one another. When the waves are of the same frequency they behave constructively and their amplitudes add. This is known as Fourier's trick. There are other forms of this method that allow us to see how a basis of vectors may have the same principle of orthogonality applied above.

Example 2.5.3. Consider $\vec{v} = \sum_{i=1}^{i=k} v_i \vec{e}_i$. Multiply both sides by $\vec{e}_j$ and we have

$$\begin{aligned}\vec{e}_j \cdot \vec{v} &= \vec{e}_j \cdot \sum_{i=1}^{i=k} v_i \vec{e}_i \\ &= \sum_{i=1}^{i=k} v_i \vec{e}_i \cdot \vec{e}_j \\ &= \sum_{i=1}^{i=k} v_i \delta_{i,j} \\ &= v_j\end{aligned}$$

where v_j is the amplitude in the $\vec{e}_j$ direction.

We have seen how orthogonality is used to attenuate certain functions and permit certain functions to be displayed. Before we establish the connection between orthogonality and sampling a signal over the time domain, a definition of Fourier series and Fourier transforms must be provided. The connection between discrete-time signals and continuous-time signals will be evident once we have shown how Fourier series and Fourier transforms relate to delta functions through the property of orthogonality.

3

Fourier Series and Fourier Transforms

3.1 Definition of Fourier Series and Fourier Transforms

In the context of digital signal processing, the purpose of a Fourier transform is to reconstruct a signal from the time domain to the frequency domain spectrum. The Fourier transform is a generalization of the complex Fourier series in the limit as $L \longrightarrow \infty$. The sum of the Fourier series is changed into an integral by the Fourier transform, which is useful in our study of acoustics because the sum of frequencies are changed into an integral that may be evaluated over the entire spectrum of our frequencies. We will introduce definitions of Fourier series and Fourier transform and discuss the variations of the Fourier transform that are applicable to our study of acoustics in digital signal processing.

Definition 3.1.1. A **Fourier series** of a continuous periodic function f is a discrete representation of **Fourier amplitudes** derived using the formula for **Fourier coefficients** of sines and cosines. Let $f\colon [-L, L] \longrightarrow \mathbb{R}$ be a continuous, periodic function. We will use the notation $\left\{\mathscr{F} \circ f(x)\right\}$ to denote the **Fourier series** of f for all $x \in [-L, L]$. We have

$$\left\{\mathscr{F} \circ f(x)\right\} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{3.1.1}$$

where $\frac{1}{2}a_0, a_n\cos(nx), b_n\sin(nx)$ are the **Fourier amplitudes** of the function f for all $n \in \mathbb{N}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{3.1.2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)dx \tag{3.1.3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(nx)dx \tag{3.1.4}$$

a_0, a_n, b_n are the **Fourier coefficients** of f for all $n \in \mathbb{N}$.

Definition 3.1.2. The **Fourier transform** converts a continuous time function into a frequency spectrum of **Fourier amplitudes**.

Example 3.1.3. Define frequency as $f = \frac{\omega}{2\pi}$ where ω = angular velocity and frequency is in units radians per second. Let $f, g\colon [-L, L] \longrightarrow \mathbb{R}$. In the limit as $L \longrightarrow \infty$, replace the discrete A_n with the continuous $F(k)dk$ while letting $\frac{n}{L} \rightarrow k$. Then change the sum to an integral and we have

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx. \tag{3.1.5}$$

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikx}dk \tag{3.1.6}$$

Equation 3.1.5 is the **Fourier transform** of f and equation 3.1.6 is the **inverse Fourier transform** of F. Notice that this form is written without ω, and the **Fourier transform** may be rewritten as

$$\begin{aligned} g(\omega) &= \left\{\mathscr{F}[f(t)]\right\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \end{aligned} \tag{3.1.7}$$

$$\begin{aligned} f(t) &= \left\{\mathscr{F}^{-1}[g(\omega)]\right\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega. \end{aligned} \tag{3.1.8}$$

Equation 3.1.7 is the **Fourier transform** expressed in terms of ω and equation 3.1.8 is the **inverse Fourier transform** expressed in terms of t.

3.2 Applications of the Fourier transform in Digital Signal processing

We will present an example of a transformation that yields a curve whose behavior is at the core of digital signal processing.

Example 3.2.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ Ae^{iw_0 t}, & \text{if } 0 < t < T \\ 0, & \text{if } T < t \end{cases} \tag{3.2.1}$$

where A, w_0, T are constants. A is the amplitude of the signal, w_0 is the frequency of the signal, and T is a value of time. We will take the Fourier transform of this function to find out information about the frequency content of the signal. The first step is to evaluate the integral.

$$\begin{aligned} g(\omega) &= F(f(t)) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^T f(t) e^{-i\omega t} dt \\ &= \frac{A}{\sqrt{2\pi}} \int_0^T e^{i\omega_0 t} e^{-i\omega t} dt \\ &= \frac{A}{\sqrt{2\pi}} \int_0^T e^{i(\omega_0 - \omega)t} dt \\ &= \left(\frac{A}{\sqrt{2\pi}}\right)\left(\frac{1}{i(\omega_0 - \omega)}\right) e^{i(\omega_0-\omega)t}\Big|_0^T \\ &= \left(\frac{A}{\sqrt{2\pi}}\right)\left(\frac{1}{i(\omega_0 - \omega)}\right) e^{i(\omega_0-\omega)T} - \left(\frac{A}{\sqrt{2\pi}}\right)\left(\frac{1}{i(\omega_0 - \omega)}\right) \\ &= \frac{A}{\sqrt{2\pi} \cdot i(\omega_0 - \omega)} \left(e^{i(\omega_0-\omega)T} - 1\right) \\ &= -\frac{Ai}{\sqrt{2\pi}(\omega_0 - \omega)} \left[e^{i(\omega_0-\omega)T} - 1\right] \end{aligned}$$

We will use Euler's formula, $e^{ix} = \cos(x) + i\sin(x)$, to rewrite the previous equation

$$\begin{aligned} g(\omega) &= -\frac{Ai}{\sqrt{2\pi}(\omega_0 - \omega)} \left[e^{i(\omega_0-\omega)T} - 1\right] \\ &= -\frac{A}{\sqrt{2\pi}(\omega_0 - \omega)} [\cos((\omega_0 - \omega)T) + i\sin((\omega_0 - \omega)T) - 1]i \\ &= \frac{A}{\sqrt{2\pi}(\omega_0 - \omega)} [\sin((\omega_0 - \omega)T) - i\cos((\omega_0 - \omega)T) + i] \end{aligned}$$

We will take the Real and Imaginary parts of this solution and separate them. We are only concerned in the real part of this equation so we may gain information about our frequency.

$$g(\omega) = \frac{A}{\sqrt{2\pi}(\omega_0 - \omega)}[\sin((\omega_0-\omega)T) - i\cos((\omega_0-\omega)T) + i] \tag{3.2.2a}$$

$$\Re\left\{g(\omega)\right\} = \frac{A}{\sqrt{2\pi}(\omega_0 - \omega)}\sin((\omega_0-\omega)T) \tag{3.2.2b}$$

$$\Im\left\{g(\omega)\right\} = \frac{A}{\sqrt{2\pi}(\omega_0 - \omega)}[1-\cos((\omega_0-\omega)T)] \tag{3.2.2c}$$

Observe that in the case where $\omega = \omega_0$ we have $\frac{0}{0}$, therefore we must use take the limit of our real solution as ω approaches ω_0 and use L'Hôpital's rule.

$$\lim_{\omega\to\omega_0}\left[\frac{A}{\sqrt{2\pi}}\right]\cdot\frac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)}$$

We will make a substitution of variables and define $x = (\omega - \omega_0)$, taking the limit as x approaches 0.

$$\begin{aligned}
\lim_{x\longrightarrow 0}\left[\frac{A}{\sqrt{2\pi}}\right]\cdot\frac{\sin(x)T}{x} \\
&= \left[\frac{AT}{\sqrt{2\pi}}\right]\lim_{x\longrightarrow 0}\frac{\sin(x)}{x} \\
&= \frac{AT}{\sqrt{2\pi}}\cdot 1 \\
&= \frac{AT}{\sqrt{2\pi}}.
\end{aligned}$$

Equation 3.2.2b is referred to as a sinc function, and at $\omega = \omega_0$ the value of the function is $\frac{AT}{\sqrt{2\pi}}$. ω_0 is known as the *center frequency*. In order to gain a visualization of this kind of function, the a graph of a different sinc function curve is provided. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined

$$f(x) = \frac{8000000\pi\sin(15x)^2}{(x^2 - 4000000\pi^2)^2} \tag{3.2.3}$$

where $\omega_0 = 2000\pi$ is the center frequency. Figure 3.2.1 is a graph of this function with the center frequency as the only point represented on the x-axis. This family of functions that exhibit

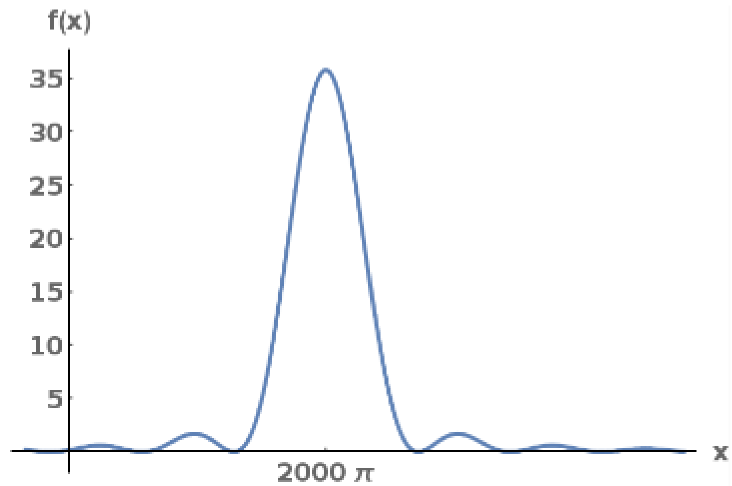


Figure 3.2.1. Graph of equation 3.2.3.

the same behavior as the curve in Figure 3.2.1 are known as sinc functions. Normalized sinc functions are of the form

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{3.2.4}$$

where $x \neq 0$. The value at $x = 0$ is defined to be the limiting value since $\operatorname{sinc}(0) = 1$. These functions are perfect models of how acousticians may gain information about the frequency of a signal. Recall that any wave signal, no matter how seemingly complex, may be expressed as an infinite sum of sine waves. Any number of sine waves of different periods may act destructively and constructively at various points to create this sinc curve. In equation 3.2.2b the central peak of the sinc curve gets taller and T is increased and the width of the central curve gets narrower as T is increased. Looking at a finite sample of a pure tone, $e^{i\omega_0 t}$, the sample time T of the signal changes the frequency information of signal in the following ways:

1. For a small T a broader frequency spectrum is seen around ω_0. In this way we have that f is not precisely defined. This is testament to a smaller amount of time spent sampling the signal which results in a less defined frequency spectrum.

2. For a large T a narrower frequency spectrum is seen around ω_0, which is well defined at $\omega \approx \omega_0$. In this way longer sampling always gives a better frequency definition.

3.3 Fourier Operations and Constructing the Frequency Domain

There are three different types of Fourier decompositions that are common in signal processing. We will discuss these Fourier forms in a way which accentuates their common traits, despite the fact that the conventions we are following may not be standard in the methodology of other mathematicians' derivations. Steps will be labeled and equations will be numbered to alleviate laborious book-keeping.

The first case we explore is when a function is periodic with period T. We want to express this in terms of a sum of periodic functions

$$f_T(t) = \sum_{n=-\infty}^{\infty} g_n e^{i\omega_n t}; \omega_n = \frac{2\pi}{T} \cdot n \tag{3.3.1}$$

Notice in equation 3.3.1 the frequency of the signal ranges over all values of n and $f_n = \frac{n}{T}$.

The second case is a general function $f(t)$, which we express as an integral over <u>all</u> possible oscillations

$$f(t) = \int_{\omega=-\infty}^{\infty} g(\omega) \cdot e^{i\omega t} d\omega \tag{3.3.2}$$

The third case is a finite set of measurements of f at equal intervals of time. We will express this as a finite sum of oscillatory terms evaluated at those times

$$\begin{aligned} f_k = f(t_k) &= \sum_{n=0}^{N-1} g_n e^{i\omega_n t_k} \\ &= \sum_{n=0}^{N-1} g_n e^{i\frac{2\pi nk}{N}} \end{aligned} \tag{3.3.3}$$

Equation 3.3.3 has either $\omega_n = \frac{2\pi}{T} \cdot n$ or $t_k = k \cdot \frac{T}{N}$.

In each case we must find the value of the g functions using the value of the f functions.

To accomplish this we will use a delta function depending on whether we are dealing with a continuous interval or discrete increments. The purpose of the delta function is to make all but one value of the expression under consideration go to 0. Recall that Fourier's Trick was derived in Chapter 2. The same principle may be extended to integrals of a complex exponential as well as discrete series of exponentials. This is the connection between orthogonality and the delta function. Our delta functions are defined as

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\frac{2\pi}{T}(n-m)t}dt = \delta_{n,m} \tag{3.3.4}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\alpha)t}dt = \delta(\omega-\alpha) \tag{3.3.5}$$

$$\frac{1}{N}\sum_{k=0}^{N-1}\left(e^{\frac{2\pi i}{N}k}\right)^{(n-m)} = \delta_{n,m} \tag{3.3.6}$$

Equation 3.3.4 was established in Chapter 2 and is commonly referred to as “Fourier’s Trick." Equation 3.3.5 is the Dirac delta Function, which is an extension to continuous functions over $(-\infty,\infty)$. Equation 3.3.6 is found by taking the summation of the geometric series for the n^{th} roots of 1.

We will begin with equation 3.3.1. We multiply both sides by equation 3.3.4 and we have

$$\begin{aligned} f_T(t) &= \sum_{n=-\infty}^{\infty} g_n e^{i\frac{2\pi}{T}nt} \\ \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-i\frac{2\pi}{T}mt}dt &= \sum_{n=-\infty}^{\infty} g_n\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\frac{2\pi}{T}(n-m)t}dt\right] \\ &= \sum_{n=-\infty}^{\infty} g_n\delta_{n,m} \\ &= g_m. \end{aligned} \tag{3.3.7}$$

By starting with equation 3.3.2, we may arrive at a similar result analogous to equation 3.3.7. Multiply equation 3.3.2 by equation 3.3.5 and we have

$$\begin{aligned} f(t) &= \int_{\omega=-\infty}^{\infty} g(\omega)\cdot e^{i\omega t}d\omega \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt &= \int_{\omega=-\infty}^{\infty} g(\omega)\cdot\left[\frac{1}{2\pi}\int_{t=-\infty}^{\infty} e^{i(\omega-\alpha)t}dt\right]d\omega \\ &= \int_{\omega=-\infty}^{\infty} g(\omega)\cdot\delta(\omega-\alpha)d\omega \\ &= g(\alpha). \end{aligned} \tag{3.3.8}$$

Now using the latter part of equation 3.3.3 and multiplying both sides by equation 3.3.6 we have

$$\begin{aligned} f_k &= \sum_{n=0}^{N-1} g_n e^{i\frac{2\pi}{N}nk} \\ \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-\frac{2\pi ik}{N}m} &= \sum_{n=0}^{N-1} g_n \left[\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi ik}{N}(n-m)}\right] \\ &= \sum_{n=0}^{N-1} g_n \delta_{n,m} \\ &= g_m. \end{aligned} \tag{3.3.9}$$

A summary is provided of the previous derivations.

$$f_T(t) = \sum_{n=-\infty}^{\infty} g_n e^{i\omega_n t} \Leftrightarrow g_n = \frac{1}{T}\int_{t=-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-i\omega_n t} dt \tag{3.3.10}$$

$$f(t) = \int_{\omega=-\infty}^{\infty} g(\omega) e^{i\omega t} dt \Leftrightarrow g(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{3.3.11}$$

$$f_k = \sum_{n=0}^{N-1} g_n e^{i\frac{2\pi}{N}nk} \Leftrightarrow g_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi}{N}nk} \tag{3.3.12}$$

The fact that equation 3.3.1 reveals that $f_T(t)$ may be expressed as an infinite but <u>countable</u> sum of functions tells us that these f_T functions are not the full class of functions, which is uncountable. In fact, the f_T functions which may be represented by a Fourier series are square-integrable function in an $\mathscr{L}^2$ space, a countably-dimensioned *Hilbert space*. A *Hilbert space* is an abstract vector space possessing the structure of an inner product (integral version of the dot product seen in Chapter 1) that allows length and angle to be measured. We will be concerned with finite Hilbert spaces for finite sampling of periodic functions.

Additionally, the Fourier transform of equation 3.3.2, being an integral over all frequencies, applies to the complete set of functions of a real variable, including unbounded ones.

The discrete Fourier transform, equation 3.3.3, involving only <u>finite</u> sums, has the least information of all. There it is evident that if one starts with N data points over a continuous function $\{f_k\}$, the result is ending up with N oscillation amplitudes $\{g_n\}$. Now that we have discussed how the frequency domain is derived we will move on to discussing how the fast Fourier transform works. The fast Fourier transform is the means by which acousticians analyze continuous-time signals and change the signal from the time domain to the frequency domain for further frequency analysis. Once we understand the fast Fourier transform we may discuss the methods used for creating a digital filter based on the functionality of an analog filter.

4

The fast Fourier Transform

4.1 The Process of the fast Fourier Transform

In the solution of a discrete transformation problem, (equation 3.3.9), we have time data sampled N times on an interval $t:[0,T]$. Let

$$\triangle \equiv \frac{T}{N}, t_k \equiv k \cdot \triangle, f_k \equiv f(t_k), k:[0,N]. \tag{4.1.1}$$

We want to extract frequency information out of this, but since we only have N data points, so we may only obtain N frequency points,

$$\omega_n = n \cdot \frac{2\pi}{T}, n:[0,N) \tag{4.1.2}$$

notice that

$$\omega_n \cdot t_k = (n \cdot \frac{2\pi}{T})(k \cdot \frac{T}{N}) = \frac{2\pi}{N} kn \tag{4.1.3}$$

So we may express f_k as a finite sum of oscillation terms:

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} g_n e^{i\omega_n t_k} = \frac{1}{N} \sum_{n=0}^{N-1} g_n e^{i\frac{2\pi}{N} kn} \tag{4.1.4}$$

The trick is to invert the sum, which is the same as multiplying by the inverse of the exponent or dividing by the exponent. Then we have

$$g_n = \sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi}{N}kn} \tag{4.1.5}$$

Initially, this form seems to resemble $\vec{v} = A\vec{w}$ matrix multiplication, with N^2 steps. The idea behind the fast Fourier transform is that an N-point transform may be split up into two almost identical $\frac{N}{2}$-point transforms, which is known as radix-2. This leaves $2 \cdot (\frac{N}{2})^2 = \frac{1}{2}N^2$ steps. Doing this repeatedly yields fewer steps (roughly $N\log_2 N$ for a large enough N). This is possible because the $e^{\frac{2\pi i}{N}nk}$ terms are really <u>not</u> N^2 separate terms. Many are repeated multiple times.

The splitting takes place along "even/odd" lines, or more generally mod(2^m) arithmetic. In general, letting $e^{\frac{2\pi i}{N}n} \equiv a$, we have

$$\begin{aligned} g_n &= f_0 + a \cdot f_1 + a^2 \cdot f_2 + \cdots + a^{N+2} \cdot f_{N-2} + a^{N-1} \cdot f_{N-1} \\ &= [f_0 + a^2 f_2 + a^4 f_4 + \cdots + a^{N-2} f_{N-2}] + a[f_1 + a^2 f_3 + a^4 f_5 + \cdots + a^{N-2} f_{N-1}] \\ &= [f_0 + b f_2 + b^2 f_4 + \cdots + b^{\frac{N}{2}-1} \cdot f_{N-2}] + a \cdot [f_1 + b \cdot f_3 + b^2 f_5 + \cdots + b^{\frac{N}{2}-1} f_{N-1}] \end{aligned} \tag{4.1.6}$$

Here we have assumed that N is even, and let $b \equiv a^2$ to accentuate how the last step is just two identical $\frac{N}{2}$-point transforms.

This process may be repeated, splitting each sum into two transforms, each of which is $\frac{N}{4}$ long. If N is originally a power of 2, this process may be repeated all the way down to 1. The full Fourier transform is then found by recursively summing pieces, using the same addition steps at each stage.

Example 4.1.1. Let $N = 16$. Let $a \equiv e^{\frac{2\pi i}{N}n}, b \equiv a^2, c \equiv b^2, d \equiv c^2$. Then we have

$$
\begin{aligned}
g_n &= f_0 + a \cdot f_1 + a^2 \cdot f_2 + \cdots + a^{14} \cdot f_{14} + a^{15} \cdot f_{15} \\
&= [f_0 + b \cdot f_2 + b^2 \cdot f_4 + b^3 \cdot f_6 + b^4 \cdot f_8 + b^5 \cdot f_{10} + b^6 \cdot f_{12} + b^7 \cdot f_{14}] \\
&+ a \cdot [f_1 + b \cdot f_3 + b^2 \cdot f_5 + b^3 \cdot f_7 + b^4 \cdot f_9 + b^5 \cdot f_{11} + b^6 \cdot f_{13} + b^7 \cdot f_{14}] \\
&= [f_0 + c \cdot f_4 + c^2 \cdot f_8 + c^3 \cdot f_{12}] + b \cdot [f_2 + c \cdot f_6 + c^2 \cdot f_{10} + c^3 \cdot f_{14}] \qquad (4.1.7) \\
&+ a \cdot [f_1 + c \cdot f_5 + c^2 \cdot f_9 + c^3 \cdot f_{13}] + b \cdot [f_3 + c \cdot f_7 + c^2 \cdot f_{11} + c^3 \cdot f_{15}] \\
&= [(f_0 + d \cdot f_8) + c \cdot (f_4 + d \cdot f_{12})] + b \cdot [(f_4 + d \cdot f_{12}) + c\dot{(}f_6 + d \cdot f_{14})] \\
&+ a \cdot [(f_1 + d \cdot f_9) + c \cdot (f_5 + d \cdot f_{13})] + b \cdot [(f_3 + d \cdot f_{11}) + c \cdot (f_7 + d \cdot f_{15})]
\end{aligned}
$$

We have seen how to write a recursive algorithm to add pairs, or pairs of pairs, and so on and so forth. The question remains: How do we get the numbers in the correct order? Looking at the binary representation of the indices will be helpful, since each level of division and grouping hinges on the congruences of numbers $\mathrm{mod} 2^m$. The first division is into evens and odds, i.e. $\{0 \mathrm{mod} 2\}, \{1 \mathrm{mod} 2\}$. Next we get $\{0 \mathrm{mod} 4\}, \{2 \mathrm{mod} 4\}, \{1 \mathrm{mod} 4\}, \{3 \mathrm{mod} 4\}$. These numbers are represented in binary and reverse binary in Figure 4.1.1.

k	Binary	Reversed	Number
0	0000	0000	0
8	1000	0001	1
4	0100	0010	2
12	1100	0011	3
2	0010	0100	4
10	1010	0101	5
6	0110	0110	6
14	1110	0111	7
1	0001	1000	8
9	1001	1001	9
5	0101	1010	10
13	1101	1011	11
3	0011	1100	12
11	1011	1101	13
7	0111	1110	14
15	1111	1111	15

Figure 4.1.1. Binary representation and reverse binary representation.

We will take the numbers in the order we end up adding them, then look at their binary representations, and finally that binary representation written in reverse order. A glance at Figure 4.1.1 shows that this is precisely the order in which we want to put the terms. The reason this works is because of the successive stages of Mod2^m decisions that were made in putting the terms in this order in the first place.

A fast Fourier transform routine generally has two stages. First the f_k terms are sorted into a new order determined by their reversed binary representation. Then, these are recursively added in pairs with the appropriate coefficients.

The fact that the FFT may be written recursively and will only use the coefficients $(e^{\frac{2\pi i}{N}\cdot n})^m$ is what makes the routine fast. Now we will discuss how the FFT works with convolution in digital signal processing and relate how the convolution of uniformly continuous functions with Gaussians are infused with the concept of basis elements of vector spaces and may be applied to create Fourier series.

5

An Important Property in Signal Processing

5.1 The Definition of Convolution

Convolution is a very important operation in signal processing. **Convolution** is the means by which continuous-time signals are combined with other continuous-time signals. The output of these combined signals may be viewed as an integral of the pointwise multiplication of the two functions as a function spanning the length over the domain of one of the original input functions (the bounds of the integration). Convolution is the operation used when a continuous-time signal is transformed with a *window function* to display the information from the time domain to the frequency domain. We will introduce window functions in Chapter 6 and discuss their important role in our case study in Chapter 8. In the next section we will show that this property applies to discrete-time signals as well.

Definition 5.1.1. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous-time functions. **Convolution** of f and g over a finite range $[0, t]$ is given by

$$[f * g](t) \equiv \int_0^t f(\tau)g(t-\tau)d\tau, \tag{5.1.1}$$

where the symbol $[f * g](t)$ denotes the convolution of f and g.

Convolution is often taken over an infinite range,

$$\begin{aligned}[f*g](t) &\equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau.\end{aligned} \tag{5.1.2}$$

5.2 Convolution property of FFT and DTFT

Convolution is a property which is used regularly in signal processing. We will discuss the properties of the FFT under convolution and the result of the theorem we state similarly applies to the DTFT (discrete-time Fourier transform).

Theorem 5.2.1. *Let $f(t)$ and $g(t)$ be arbitrary functions of time t with Fourier transforms. Take*

$$f(t) = \mathscr{F}_v^{-1}[F(v)](t) = \int_{-\infty}^{\infty} F(v)e^{2\pi ivt}dv \tag{5.2.1}$$

$$g(t) = \mathscr{F}_v^{-1}[G(v)](t) = \int_{-\infty}^{\infty} G(v)e^{2\pi ivt}dv, \tag{5.2.2}$$

where $\mathscr{F}_v$ denotes the Fourier transform and $\mathscr{F}_v^{-1}$ denotes the inverse Fourier transform. Then the convolution is

$$f*g \equiv \int_{-\infty}^{\infty} g(t')f(t-t')dt' \tag{5.2.3}$$

$$= \int_{-\infty}^{\infty} g(t')\left[\int_{-\infty}^{\infty} F(v)e^{2\pi iv(t-t')}dv\right]dt'. \tag{5.2.4}$$

Interchange the order of integration and we have

$$f*g = \int_{-\infty}^{\infty} F(v)\left[\int_{-\infty}^{\infty} g(t')e^{-2\pi ivt'}dt'\right]e^{2\pi ivt}dv \tag{5.2.5}$$

$$= \int_{-\infty}^{\infty} F(v)G(v)e^{2\pi ivt}dv \tag{5.2.6}$$

$$= \mathscr{F}_v^{-1}[F(v)G(v)](t). \tag{5.2.7}$$

Applying a Fourier transform to each side of the equation, we have

$$\mathscr{F}[f*g] = \mathscr{F}[f]\mathscr{F}[g]. \tag{5.2.8}$$

The convolution theorem also takes alternate forms

$$\mathscr{F}[fg] = \mathscr{F}[f] * \mathscr{F}[g] \tag{5.2.9}$$

$$\mathscr{F}^{-1}(\mathscr{F}[f]\mathscr{F}[g]) = f * g \tag{5.2.10}$$

$$\mathscr{F}^{-1}(\mathscr{F}[f] * \mathscr{F}[g]) = fg. \tag{5.2.11}$$

Theorem 5.2.1 may also be applied to the inverse Fourier transform. Theorem 5.2.1 indicates that multiplication of two continuous-time signals in the time domain is equivalent to convolution of two continuous signals in the frequency domain; likewise, convolution of two continuous-time signals in the time domain is equivalent to multiplication of two continuous signals in the frequency domain. In a similar manner it can be shown that Theorem 5.2.1 applies to functions of discrete variable sequences as well. Now that we have an understanding of the methods that form the basis of signal processing, we transition to discussing the finer details of sampling a continuous-time signal; specifically, we are interested in arriving at a solution for a major issue regarding the discrete-time sampling of a continuous, periodic signal. The following chapter discusses how sampling a continuous-time signal may go awry and the appropriate response to this issue that yields a desirable result.

6

Details of Periodic Sampling

6.1 A Conundrum Regarding Continuous-Time Sampling

This section discusses how periodic sampling may confuse frequencies of a continuous-time signal.

If a system is sampled on a time cycle $\triangle t$, then we are interested in cyclic behavior with periods of integer multiples to this value. Let T=period and let $N \in \mathbb{Z}_+$. Then we are interested in a period such that $T = N \cdot \triangle t$. Specifically, if we take N data points, each $\triangle t$ apart, the frequencies we observe are

$$f_n = \frac{n}{\triangle t} \tag{6.1.1}$$

where $n = 0, 1, \ldots, N-1$ and t is time.

In reality, all we know is that we observe N frequencies, each separated by $\frac{1}{\triangle t} = \triangle f$. It may be the case that $0 \longrightarrow N-1$, or $-\frac{N}{2} \longrightarrow 0 \longrightarrow \frac{N}{2} - 1$, or even possibly $N_0 \longrightarrow N_0 + N - 1$, the point being that sampling only N times does not permit us to distinguish between two cycles that differ by $\frac{N}{\triangle t}$ in frequency.

One way to observe this is with a graphical representation seen in Figure 6.1.1 depicts the curves of frequency 1 and 5.

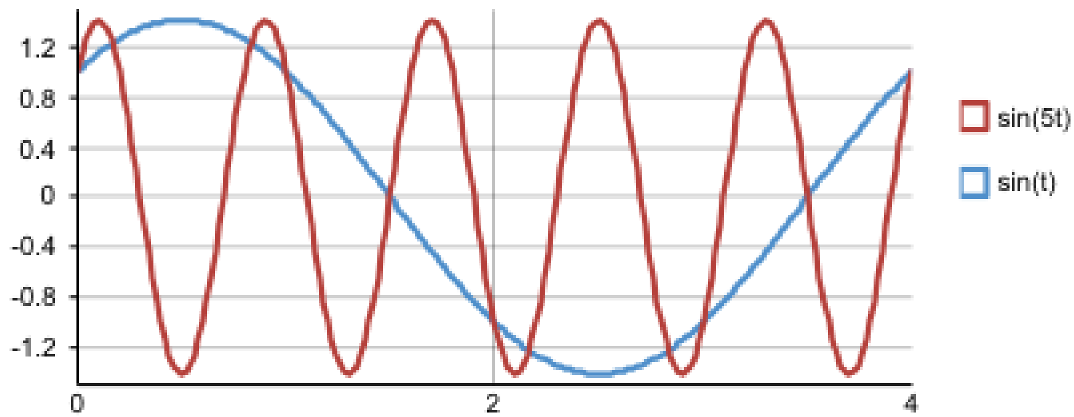


Figure 6.1.1. sin(t) and sin(5t)

In Figure 6.1.1 we see that sampling at $t = 0,1,2,3,4$ gives the same values for $\sin(t)$ and $\sin(5t)$. Therefore, at these discrete points in the frequency domain the sampling could not distinguish between these contributions.

Let us prove this fact algebraically. We may compare the "time arguments" of two oscillations that differ by N. The phase angle is adjusted to make this occur at $t = t_0$.

Let ϕ be the phase and let f_n be the frequency at a particular positive integer n.

$$f_n \cdot t_0 = f_{n+N} \cdot t_0 - \phi \tag{6.1.2}$$

$$\frac{n}{\triangle t} \cdot t_0 = \frac{(n+N)}{\triangle t} \cdot t_0 - \phi \tag{6.1.3}$$

$$\text{so } \phi = \frac{N}{\triangle t} \cdot t_0. \tag{6.1.4}$$

We sample these two oscillations again at $t_k = t_0 + k \cdot \triangle t$. Then we have

$$f_n \cdot t_k = \frac{n}{\triangle t} \cdot t_0 + nk \tag{6.1.5}$$

$$\text{and } f_{n+N} \cdot t_k - \phi = \frac{(n+N)}{\triangle t} \cdot t_0 + (n+N)k - \frac{N}{\triangle t} \cdot t_0 \tag{6.1.6}$$

$$= \frac{n}{\triangle t} \cdot t_0 + nk + Nk \tag{6.1.7}$$

$$= f_n \cdot t_k + Nk \tag{6.1.8}$$

so the phase angle would be $2\pi f t$, or

$$2\pi(f_{n+N} \cdot t_k - \phi) = 2\pi f_n \cdot t_k + 2\pi \cdot Nk \tag{6.1.9}$$

We see that the second oscillation has just gone through a number of complete oscillations, so it will assume the same values for sine or cosine. In context to these oscillations, the sampling will not be able to discern the difference between f_n and f_{n+N}.

It is simple to see that when the sampling frequency is exactly equal to the oscillating frequency the sample results are a facsimile of the sample results of a constant function ($f = 0$, in the most trivial case).

This means if $f = f_0 + \triangle f$, the information that will be displayed in the results of the sampling is the change in the signal relative to what would be observed if $f = f_0$. Another way of describing this is that it is only $\triangle f$ that matters, since f_0 always reverts values to the same value. This substantiates the reason why f and $f + f_0$ are indiscernible to a sampling at frequency f_0.

The solution to this problem is picking a sampling rate that is divisible by the number of increments chosen for frequency in the frequency spectrum, and making sure that the signals used have frequencies which are less than half the sampling rate. The frequency that is half of the sampling rate is known as the *Nyquist frequency*. It is crucial that we discuss the importance of remedying the problem that is unique to taking the FFT of a continuous-time signal. The issue we will discuss is one which necessitates the use of windowing functions.

6.2 A Problem with the FFT that Necessitates the use of Windowing Functions

When an FFT is used to measure the frequency component of a signal, the assumption is that the analysis is based on a finite set of data. The actual FFT posits that it is a finite data set, a continuous spectrum that is one period of a periodic signal. The FFT regards a signal in the time domain or frequency domain as a circular topologies, so the two endpoints of the time waveform are regarded as being joined to one another. When the measured signal is periodic and the sampled time interval is composed of an integer number of periods, the FFT does not encounter any issues. However, it is often that the measured signal is not an integer number of periods. Therefore, the finite aspect of the measured signal may yield a truncated waveform with different characteristics from the original continuous-time signal. In this way, the finite aspect of the signal may prompt abrupt transitions in the measured signal. These sharp transitions are discontinuities.

When the number of periods in the signal is not an integer, the endpoints are discontinuous. These discontinuities appear in the FFT as high-frequency components which would normally be absent in the original signal. These frequencies may be higher than the Nyquist frequency and are aliased between 0 and half of the sampling rate. This means that the spectrum generated by the FFT is not the actual spectrum of the original signal and is a jumbled imposter. It appears as if energy at one frequency bin seeps into other frequency bins. This phenomenon is known as *spectral leakage*, which causes the fine spectral lines to spread into wider signals. Acousticians knew that they had to devise a way to modify the output of the FFT so that these discontinuities were bypassed.

6.3 Windowing Functions

The negative effects of taking an FFT over a non-integer number of cycles may be resolved by used a technique called **windowing**. Windowing reduces the amplitude of the discontinuities at the boundaries of the frequency content of the signal. Windowing involves multiplying the continuous-time signal content by a finite-length window with an amplitude that varies, depending on the choice of window, but whose amplitude asymptotically approaches 0 along the edges of the signal. This causes the endpoints of the wave to connect and creates a continuous waveform free of sharp transitions.

There are a few different kinds of window functions that are more suitable than others depending on the signal. One must understand more about the frequency characteristics of window functions in order to possess the knowledge of how these functions may be applied to the frequency spectrum properly.

A realistic plot of a window conveys the fact that the frequency characteristic of a window is a continuous spectrum with a main lobe and two or more side lobes. The main lobe is centered at each frequency component of the time domain signal and the side lobes approach zero. The height of the side lobes is indicative of the kind of effect the windowing function has on frequencies near the main lobes. The side lobe response of a sinusoidal signal with large amplitude and/or short period may overpower the main lobe response of a nearby sinusoidal signal with small amplitude and/or large period. In general, the lower side lobes reduce spectral leakage in the measured FFT but increase the bandwidth of the major lobe. The side lobe slope steepness (known as the *roll-off rate*) is the asymptotic decay rate of the side lobe peaks. Thus, increasing the side lobe roll-off rate reduces spectral leakage.

Selecting the window function may appear to be a daunting task. Each window function has its own behavior whose compatibility with the signal in question depends on the goal of the user. Each type of problem necessitates the use of different windowing functions based on the way the input signal convolves with that particular window function in order to create a processed function that displays the desired frequency information and whose behavior is suitable for the conditions imposed by the acoustician. Initially it is advised to choose a window function based on an estimate of the frequency content in the signal. The following should be taken into consideration:

1. If the signal contains high amplitude and/or short period frequency components that interfere with the frequency components of a distant frequency of interest, a smoothing window with a high side lobe roll-off rate is advised.

2. If the signal contains high amplitude and/or short period frequency components that interfere with the frequency components of a close frequency of interest, a window function with a low maximum side lobe level is advised.

3. If the frequency of interest contains two or more signals of close proximity, spectral resolution is important. In this circumstance it is best to choose a smoothing window with a narrow main lobe.

4. If the amplitude exactness of a sole frequency component is more vital than the precise location of the component in a given frequency bin, a window with a wide main lobe is advised.

5. It the signal spectrum is broad in frequency content, a uniform window or no window at all is advised.

6. In general, the Hanning (also known as Hann) window is satisfactory in the vast majority of cases. It has exceptional frequency resolution is effective in reducing spectral leakage. If one is incapable estimate the frequency content of a signal but the need for a smoothing window is unavoidable, starting with the Hann window is highly advised.

Note that even if one uses no window, the signal is convolved with a rectangular-shaped window of a set height, by virtue of the interval in time of the input signal and working with a signal sampled at discrete points. This convolution has a sine function characteristic spectrum best known as sinc. For this reason, using no window is often called the uniform or rectangular window because there is still an effect on the signal reminiscent of applying a window to the signal.

The Hamming and Hanning window functions have curves which resemble the shape of sinc curves. Both windows yield a wide peak but low side lobes. However, the Hann window reaches zero at both endpoints of the signal and eliminates the discontinuities. The Hamming window only asymptotically approaches zero at these discontinuities, thus the discontinuities are not completely ameliorated. As a result, the Hamming window cancels the side lobes nearest to the main lobe more effectively than the Hann window, but it falls short of the mark when tasked with canceling other lobes. These window functions are helpful for noise measurements in the case when better frequency resolution than other windows present is desired but moderate side lobes do not pose an issue. The difference between these windowing functions can be seen in Figure 6.3.1 and Figure 6.3.2.

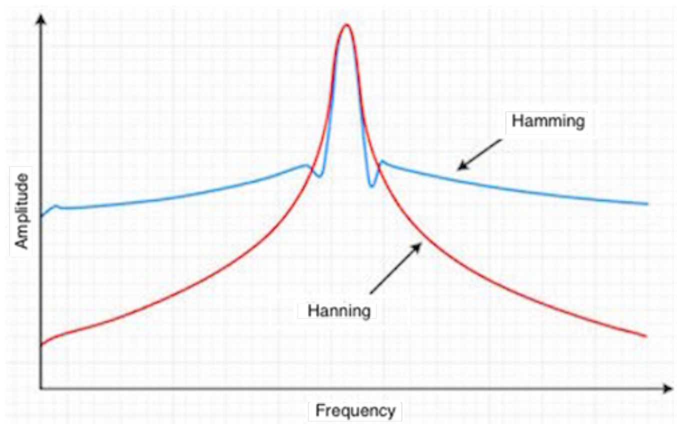


Figure 6.3.1. FFT of Hann and Hamming Window Functions

For the purposes of our case study we will use the Hann window to display our frequency information.

We will now move to the subject of filters which will provide us with the information necessary to begin detailing the case study of a digital filter modeled after an analog filter.

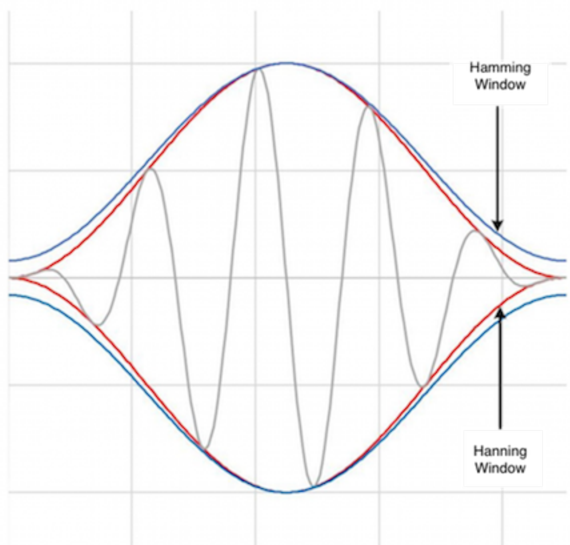


Figure 6.3.2. Comparison of Hann and Hamming Window Functionality

7
Analog Filters

7.1 Four General Filter Types

There are four general filter types that each have their perks in signal processing.

Definition 7.1.1. A **low-pass filter** is a filter that permits frequency content below a certain frequency to be displayed in the frequency spectrum and mitigates all frequencies higher than the designated frequency. The frequency that marks this divide is known as the cut-off frequency.

Definition 7.1.2. A **high-pass filter** is a filter that permits frequency content above the designated cut-off frequency and mitigates all frequencies lower than the cut-off frequency.

Definition 7.1.3. A **band-pass filter** is a filter that permits frequency content between two designated cut-off frequencies and mitigates all frequencies other than ones in the designated interval.

Definition 7.1.4. A **band-stop filter** is a filter that permits frequency content on either side of two designated cut-off frequencies and mitigates all frequencies in the designated interval.

For our purposes we will be interested in low-pass and high-pass filters.

7.2 Introduction to Analog Filters

Mathematicians fervently seek to establish connections between the causes and effects of natural phenomena in the world. In fact, the study of mathematics often involves input submitted to a device that returns that input modified slightly or drastically, depending on how the device operates. In signal processing, mathematicians and physicists have agreed on a commonly used term analogous to the concept of an input-output machine driven by empirical implications provided by teachers in the standard grade school mathematics classroom.

Definition 7.2.1. The **system function** (or *transfer function*) of a digital processor, applied to a continuous wave signal, is a representation of the means by which a mathematical formula is used to describe how the input of the system, in the form of the continuous-time signal, returns an output in the form of another new signal.

In the context of digital signal processing applied to the study of Acoustics, mathematicians and scientists are interested in the means with which they observe how a system function is modified to suit the standards of audio engineering. In order to optimize the system function, a sample of amplitude points of the continuous signal are taken which best represent the information of the analog wave.

Definition 7.2.2. The mathematical method which acousticians modify the display of the signal information gathered from a pre-existing system function to enhance certain properties of the signal is called a **window function**.

We will compare two different filter types which are the most basic and fundamental means of displaying certain aspects of signal information in signal processing. These two types are low-pass filters and high-pass filters. We will first derive these filters' formulas from various circuit schematics. From these formulas and relations we derive, we see a direct relationship between the filter-setup functionality in circuitry and signal processing.

7.3 Derivation of Low-pass and High-pass Filters

RC filter

Consider an RC circuit in series configuration. Let voltage input, V_{in}, be defined as $V_{in}(t) = V_0 e^{j\omega t}$ where V_0 is initial voltage, e is Euler's constant, $j = \sqrt{-1}$, ω = angular frequency, and t is time. The voltage across the capacitor is defined as $V_C = \frac{Q}{C}$, where Q is the charge over the plates of the capacitor and C is the capacitance. The voltage across the resistor is defined as $V_R = I(T)R = RI_0 e^{j(\omega t+\phi)}$, where I_0 is the initial current, R is the value of resistance of the resistor, and ϕ is the phase constant. $Q(t) = \frac{I_0}{j\omega} e^{j(\omega t+\phi)}$. Then we have that

$$
\begin{aligned}
V_{in} &= V_C + V_R \\
V_0 e^{j\omega t} &= \frac{I_0}{j\omega C} \cdot e^{j(\omega t+\phi)} + I_0 R \cdot e^{j(\omega t+\phi)} \\
V_0 &= I_0 \left[\frac{1}{j\omega C} + R \right] e^{j\phi} \text{ (We must find real solution)} \\
e^{-j\phi} &= \left(\frac{V_0}{I_0} \right) \left(\frac{-j}{\omega C} + R \right) \\
\frac{V_0}{I_0} \cdot e^{-j\phi} &= \left(R - \frac{j}{\omega C} \right).
\end{aligned}
$$

Take the modulus, defined for some complex number, where $ze^{j\phi}$ and $z = a + bj$, as $|ze^{j\phi}| = |z|e^{j\phi} = |z| = \sqrt{a^2 + b^2}$

$$
\begin{aligned}
\frac{V_0}{I_0} &= \sqrt{R^2 + \frac{1}{(\omega C)^2}} \\
\tan(\phi) &= \frac{1}{\omega RC}.
\end{aligned}
$$

Then we will look at the response at various ω.

$$
\begin{aligned}
V_C &= \frac{I_0}{j\omega C} \cdot e^{j(\omega t+\phi)} \\
&= \frac{V_0}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}} \cdot \frac{1}{\omega C} \cdot \frac{1}{j} e^{j(\omega t+\phi)} \text{ (Take magnitude)} \\
|V_C| &= \frac{1}{\sqrt{1 + \frac{1}{(\omega RC)^2}}}.
\end{aligned}
$$

For the Voltage across across the resistor we have

$$\begin{aligned}
V_R &= I_0 R e^{j(\omega t+\phi)} \\
&= \frac{R}{\sqrt{R^2+\frac{1}{(\omega C)^2}}} V_0 \cdot e^{i(\omega t+\phi)} \\
&= \frac{1}{\sqrt{1+\frac{1}{(\omega RC)^2}}} \cdot e^{j(\omega t+\phi)} \\
|V_R| &= \frac{1}{\sqrt{1+\frac{1}{(\omega RC)^2}}}.
\end{aligned}$$

We will now discuss the quantitative differences between these high-pass and low-pass configurations. Let τ=RC be the time constant of the RC circuit. Then we have that $\omega_T = \frac{1}{RC}$ is the transition frequency between high-pass and low-pass filters. In other words, if $\omega < \omega_T$ then we have that $|V_C| > |V_R|$. If $\omega = \omega_T$ then we have that $|V_C| = |V_R|$. If $\omega > \omega_T$ then we have that $|V_C| < |V_R|$. Then we have at a low value of ω we have that $\omega RC < 1$ and $|V_C| > |V_R|$, which implies that $\tan(\phi) = \frac{1}{\omega RC}$ and $\frac{\pi}{4} < \phi < \frac{\pi}{2}$. Let us summarize these findings.

At low frequencies, the capacitor has enough time to to enough charge and take on most of the voltage in the circuit. At low frequencies, the resister does not have much current (charge/time), so V_R is small. Therefore, low-pass RC filters have V_C is the output and decreases from an initial amount and crosses the high-pass filter at the crossover frequency.

For a high ω, $|V_R| > |V_C|$, so $\tan(\phi) < 1$ and we have $0 < \phi < \frac{\pi}{4}$. At high frequency, a small time means a lot of current is running through the circuit, so V_R is large. But at a high frequency, the capacitor does not have enough time to charge/discharge so there is a low value of V_R. Thus for a high-pass RC filter we have V_R is the output and increases from an initial amount; it crosses over the low-pass filter at the crossover frequency.

Now that we have been exposed to some of the behavior of low-pass and high-pass filters, we will discuss the terminology applied to components of signals under filtration.

7.4 Filter Terminology

We are interested in understanding the visual components of frequency responses in real filters in order to form a quantitative system to measure these elements. The following definitions are useful in describing the behavior of filtered frequencies.

Definition 7.4.1. The **passband** is a frequency band in which signals are transmitted by a filter without attenuation.

Definition 7.4.2. The **stopband** is a band of frequencies that are attenuated by a filter.

Definition 7.4.3. The **transition-band** is a range of frequencies that allows a shift between a passband and stopband of a filter.

Definition 7.4.4. The **gain** refers to the amount of maximum amplification of the signal in the passband.

Definition 7.4.5. The **stopband attenuation** is the difference in dB between the passband gain and stopband gain.

Definition 7.4.6. The **passband ripple** is the maximum fluctuation or variation in the filter's frequency magnitude response curve in the passband, measured in dB.

The amount of fluctuations relate to the kind of filter being used, which in turn depends on how many poles or zeros there are on the frequency-axis. Seeing as we are only using a *Butterworth* derived based filter (transfer function with only poles on the frequency-axis) for our case study, there is no ripple in the passband or the stopband response of this type of filter.

Definition 7.4.7. The **stopband ripple** is the maximum fluctuation in the frequency response in the stop band.

The stopband ripple is negligible so long as the stopband attenuation is met, and for our purposes will be considered negligible for the reason stated following definition 7.4.6.

Definition 7.4.8. The **roll-off rate** is the steepness of the slope in the transition band.

The roll-off rate is usually in multiples of 20 dB/decade, where 20 dB/decade=6dB/octave, where an octave is a doubling in frequency.

Definition 7.4.9. The **order** is the number of poles in the system function $H(s)$.

Poles are singularities where the function would be undefined and occur when a value would cause the equation to be some value divided by 0. The higher the order the steeper the roll-off rate and the shorter the transition band. Poles are only of concern when using another type of filter, with singularities not confined to the frequency-axis, which indicates a more complicated system function is in use. For the purposes of our case study we will omit a discussion on poles.

Definition 7.4.10. The **cutoff frequency** is the edge of the passband.

Definition 7.4.11. The **Q factor**, or the **quality factor**, is the sharpness of the peak in a band-pass filter, it is defined as the center frequency divided by the half-power bandwidth and is a measure of how close poles are on the $j\omega$-axis (imaginary numbers on the y-axis and angular velocity on the x-axis).

Poles are places where the system function has singularities, meaning where the system function is not defined. Poles are important in certain areas of digital signal processing, but for our purposes we will not be discussing how poles play a role in filter design. These definitions will be important in understanding the process for designing our digital filter.

8
Case Study

8.1 Preliminary Measures

Our goal is to create a digital filter which replicates the effects of an analog filter known as the Enhanced Pultec EQP1-A Equalizer. We will refer to this analog filter as the Manley Pultec Equalizer for brevity, named after the company Manley Laboratories Incorporated that created the filter. The process of creating a digital filter facsimile of an obsolete analog filter could be arduous. However, using technology accessible to modern acousticians, one has the tools necessary to do the job. We must detail the functions of the technology used and the reasons for selecting the options deemed most suitable for accomplishing our goal.

Acousticians utilize a variety of computer programs to analyze continuous periodic signals from the time domain to the frequency domain. One of these programs is called Audacity. Audacity is an audio engineering program that may be downloaded online for free. Audacity is a multi-use platform that allows audio signals to be produced, captured, or analyzed with a great deal of flexibility for the user. Of the many mathematical methods built into Audacity, one of the most useful for the purposes of filtering is plotting the frequency spectrum of the signal by choosing spectrum as the algorithm. This creates a spectrum of frequencies using a discrete-time Fourier transform (DTFT). The functionality of the other options of this feature in Audacity will be covered in more depth when deemed appropriate to our discussion of mathematical methods pertaining to the analysis of the output spectra.

Audacity is also capable of producing chirps, pink noise, frequency sweeps, and tones. For the purposes of this project, tones were produced and deemed sufficient for the input signal. However, it was essential that certain parameters were fulfilled in order for the most meaningful results. The parameters are as follows: Waveform, Frequency (Hz), Amplitude, Duration (time). For the reader's convenience, an explanation of these terms is provided.

The 'Waveform' of a signal refers to the curve showing the shape of the wave at a given time. The options for 'Waveform' are 'Sine, Square, Sawtooth, Square (no alias).' These options are important because each wave serves a different purpose in audio engineering. The 'Sine' option yields a sinusoidal curve that describes a smooth repetitive oscillation. It contains no overtones so it sounds very "pure" because it is only one frequency. Overtones are musical notes that are part of the harmonic series above a fundamental note, whose frequency is defined as the lowest frequency of the periodic waveform. The 'Square' option has fixed maximum and minimum values of identical duration with a vertical line between each alternating maximum and minimum value. The curve contains every other overtone. Square waves are unique in that they have identical peak and root mean square (RMS) levels so their waveforms are displayed as entirely blue when zooming out of the display window. The 'Sawtooth' option has a gradual upward incline of positive slope followed by a gradual downward incline of negative slope. It contains all overtones so it sounds "bright" and penetrating. The 'Square (no alias)' option is similar to the Square wave but it does not produce *aliasing* distortion, which causes the program to take a longer time generating the tone.

Aliasing is an effect that causes different signals to become indistinguishable from one another when sampled. *Aliasing* occurs when the sampling rate of a signal is too slow in comparison to the signal being measured. Signals which only differ in variations at frequencies higher than the sampling rate will yield the same sample. Since our sample rate is high enough we may neglect the any concerns regarding aliasing.

The ‘Frequency’ option refers to the number of cycles the waveform will oscillate per unit time, often denoted ‘f’ or ‘ν.’ We will maintain the convention used in the previous chapters and use the letter ‘f’ to denote frequency. The period, denoted ‘T,’ is the duration of one cycle, and is related inversely to frequency. The SI unit of frequency is ‘hertz (Hz);’ for example, one hertz means that the wave repeats once per second.

The ‘Amplitude’ option represents the height of the continuous-time signal. It is a scalar that multiplies the signal and “scales" all occurrences of the signal by the given factor; we disregard the trivial case when amplitude is 0.

‘Frequency’ may be chosen at the acoustician’s discretion. ‘Amplitude’ is restricted to a range from 0 to 1. In this program, ‘Amplitude’ is a fraction of the maximum amplitude. ‘Duration’ has a variety of parameters, some of which are notated by the abbreviation ’hh:mm:ss’ which corresponds to hours, minutes, and seconds. In addition to having the option of whether or not to use these standard time units for the length of the sample, there were also other options that relate time units to samples or frames, depending on the medium of the audio. The options for ‘Duration’ were ‘seconds, hh:mm:ss, dd:hh:mm:ss, hh:mm:ss+hundredths, hh:mm:ss+milliseconds,hh:mm:ss+samples, samples, hh:mm:ss+film frames (fps), film frames (24 fps), hh:mm:ss+NTSC drop frames, hh:mm:ss+NTSC non-drop frames, NTSC frames, hh:mm:ss +PAL frames (25 fps), PAL frames (25 fps), hh:mm:ss+CDDA frames (75 fps), CDDA frames (75 fps).’ The parameters chosen were as follows:

Waveform: Sine

Frequency (Hz): $25, 50, 100, 200, 400, 800, 1600, 3200, 6400, 12800$

Amplitude (0-1): 0.8

Duration: hh:mm:ss + samples

'Duration' was chosen by using the default setting, as was 'Amplitude.' The reason for the choice of the amplitude was arbitrary but not trivial, seeing as an amplitude of 0 would yield no results. The decision for 'Duration' was based on the sample rate of the project; with the whole project at a sample rate of 44100 Hz and the parameter for 'Duration' set to 30 seconds, the duration for the continuous-time signal was 44100 Hz × 30 seconds = 1323000 samples. This resolves the issue of taking an FFT over a noninteger period because the number of samples are in increments which correspond to the frequency of the input signal. A sinusoidal wave was chosen for the 'Waveform' option in order to produce a smooth signal with no overtones. In this way, a signal whose amplitude in voltage is charted with respect to the time domain may be transformed into a dB-scale with respect to the frequency domain. Before we proceed, a brief explanation of logarithms, decibels, and intensity has been provided for the reader's convenience.

Logarithms are used to compare two signals and establish a ratio between the two.

Example 8.1.1. Take two signals I_1 and I_2. In decibels we have

$$x \text{ dB} = 10 \log_{10}\left(\frac{I_1}{I_2}\right). \tag{8.1.1}$$

If $I_1 = 100$ Hz and $I_2 = 10$ Hz then in decibels we have

$$10 \log_{10}\left(\frac{100}{10}\right) = 10 \text{ dB}. \tag{8.1.2}$$

Note that every 10 dB increase multiplies the signal by a factor of 10.

Example 8.1.2. Let D dB $= 10 \cdot \log_{10}\left(\frac{I_1}{I_2}\right)$ Then we have

$$\begin{aligned} \frac{D}{10} &= \log_{10} \frac{I_1}{I_2} \\ 10^{\frac{D}{10}} &= \frac{I_1}{I_2} \\ I_1 &= I_2 \cdot 10^{\frac{D}{10}} \end{aligned} \tag{8.1.3}$$

Suppose I_1 was boosted 3 dB. Then we have $I_1 = I_2 \cdot 10^{0.3} = 2I_2$. Now suppose I_1 was boosted 8.5 dB. Then we have $I_1 = I_2 \cdot 10^{0.85} = 7.1I_2$.

Now that we have discussed the settings for the input signal, we will describe the setup of the equipment. The computer running the input signal using Audacity was connected to the Manley Pultec Equalizer with a USB cable. The analog filter was connected to was connected to a speaker system so the different frequencies could be heard qualitatively in order to ensure precision. Another computer running a more sophisticated audio engineering program called Pro-Tools was connected to the output and used to record the output signal modified by the filter with the same duration as the input signal. It is worth noting that although the functionality and mathematical capabilities of Pro-tools exceed those of Audacity, any audio engineering program of the same caliber as Audacity would have been suitable for our purposes. Pro-tools was readily available at the time, so it was the option chosen for the job. Now that we have discussed the setup of the study, we will elaborate on the way the analog device works.

8.2 Functionality of the Manley Pultec Equalizer

[9, Page 5] shows a diagram of the interface. Each of the controls to the filter are labeled alphabetically. Control A is a switch providing the option of whether to select *In*, denoting input, or *Bypass* which bypasses the equalizer section while the amplifier stays connected in the circuit. This option is not useful to us because there are no other analog elements of this circuit which the amplification option could be of use. We also wish to gain information from the equalizer option, so bypassing it would be counterintuitive.

Controls B, C, and D pertain to the functionality of a low pass equalizer. Control B is a knob which controls the continuous variable boost from 0 dB to +15 dB to the selected cut-off frequency. A continuous variable boost means that all frequencies up to the cut-off are boosted by the same amount. This knob ranges from 0 to 10. The cut-off frequency is selected using the knob labeled control C. The options for the cut-off frequencies are '20 Hz, 30 Hz, 60 Hz, 90 Hz, 120 Hz.' With a maximum boost, lower frequencies may receive an additional 3 dB. Control D is a knob which controls the low frequency cut from 0 to +16 dB, reducing frequencies below the selected cut-off frequency by the selected amount. The option to cut these low frequencies was not exercised because modifying the low frequencies other than with a boost would not provide as much information since there was not much low frequency content in the input signal below the max for the selection.

Controls E, F, and G apply to the functionality of a high-pass filter combined with an equalizer. Control E is a knob which controls the *bandwidth* of the high frequency selected with control G to boost by a certain amount determined by control F. Turning control E counterclockwise results in a narrow peak or high Q. Turning control E clockwise is a larger bandwidth, wider peak or lower Q. Control E ranges from 0 to 10. Control F provides a continuously variable boost from 0 to +20 dB to frequencies within a certain range (determined by Q) of the center frequency chosen using the select knob labeled control G. Control F ranges from 0 to 10 The quality factor of the high frequency impulse response is altered by control E, so controls E, F, and G alone function strictly as an equalizer used to boost frequencies in the neighborhood of one main boosted frequency. The options for the boosted frequencies are '1 kHz, 1.5kHz, 2 kHz, 3 kHz, 4 kHz, 5 kHz, 8 kHz, 10 kHz, 12 kHz, 14 kHz, 16 kHz.' It is important to note that the bandwidth tends to become slightly narrower with higher frequencies which are boosted, which is standard for passive equalizers. This is due to the fact that passive equalizers have equally lengthened increments for each discrete frequency sample point which will appear narrower at high frequencies when graphed using a logarithmic plot.

Control H provides a continuously variable cut from 0 to -21 dB of the frequencies higher than the selected cut-off frequency determined by control I. Control H ranges from 0 to 10. Control I has the options '4 kHz, 8 kHz, 12 kHz, 16 kHz, 20 kHz.'

Control J is an LED which is red when the power first turns on, indicating that the equalizer (EQ) is muted until the hardware has been on long enough. Around 20 seconds later when the LED turns green the EQ will accept audio input.

8.3 Settings used for the Analog Filter Measurements

The knob for boost for was kept at '8.5' for a +12.75 dB boost for all low-pass measurements and +17 dB for all high frequency boosts. Separate measurements were taken for each frequency setting while keeping the other settings of the filter untouched in order to preserve the effects of each variable completely isolated for the purposes of modeling the digital filter. The measurements for the low-pass filter were '20 Hz, 30 Hz, 60 Hz, 90 Hz, 120 Hz.' The measurements for the high frequencies with Q = 1 were '1 kHz, 1.5kHz, 2 kHz, 3 kHz, 4 kHz, 5 kHz, 8 kHz, 10 kHz, 12 kHz, 14 kHz, 16 kHz.' Certain frequencies were set to have a Q factor of '10.' These measurements were taken in addition to the measurements having Q = 1, and the measurements with Q = 10 were '1kHz' and '4kHz.' The frequencies that were cut that were set to '8.5' and lowered by -17.85 dB were '4 kHz, 8 kHz, 12 kHz, 16 kHz, 20 kHz.'

8.4 Intricacies Regarding the Filtered Frequency Selections

The quality factor must be solved for in order to gain more information about the functionality of the analog filter. The FWHM (full width at half maximum) aids in finding the quality factor. The quality factor is $Q = \frac{f_0}{\triangle f_0}$, where $\triangle f_0$ is the difference between the frequencies at half of the center frequency. A quality factor of 1 indicates no change in width. In an effort to boost the central frequency and mitigate those around the central frequency, the quality factor may be increased in order to narrow the width of the peak of the central frequency and increase the accuracy of the boost of the desired frequency. The whole purpose of these measurements is to compare the analog filter with the digital filter, in order to see what adjustments must be made in order for the quality factor on the digital filter to match with the quality factor of the analog filter. The window function used by Audacity to modify the spectrum for its discrete-time Fourier transform (DTFT) was a Hanning window.

The most important observation of the non-filtered input signal is to note the values of the peaks of the signal. This is necessary for when it will be time to match the boosted spectrum of signals with the non-filtered signals so an idea of the mathematical formula can be captured.

The Fourier transform is important in filtering because certain frequencies may be selected as input and they may be boosted accordingly, while other frequencies besides this frequency are nearly completely mitigated due to the nature of the sinc function generated when convolving a finite Fourier sine transform with the Hanning window to create an *instrument function.* The instrument function determines the amplitude and the rate of the cut-off of the graph of the function which resembles a single sinc function, as well as the quality factor. This is exciting, particularly because comparing the frequencies from the non-filtered spectrum of the aforementioned frequencies and the boosted frequencies, to see how the filter operated, will reveal how the formula will be created. In other words, the non-filtered frequencies are compared to the filtered frequencies from the Manley Equalizer manual which is used to create an ideal filter, which will be compared to the results of the data taken from the analog filter.

Let $I, A\colon \mathbb{R} \longrightarrow \mathbb{R}$ be functions. We have

$$I(k) = \int_{-a}^{a} \cos(2\pi kx) A(x) dx \tag{8.4.1}$$

where $I(k)$ is the instrument function and $A(x)$ is the apodization (window) function for all $k, a \in \mathbb{R}$. For the Hanning window we have

$$A(x) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \tag{8.4.2}$$

and

$$I(k) = \frac{a \operatorname{sinc}(2\pi ak)}{(1 - 4a^2k^2)} = a[\operatorname{sinc}(2\pi ak) + 0.5 \operatorname{sinc}(2\pi ak - \pi) + 0.5 \operatorname{sinc}(2\pi ak + \pi)] \tag{8.4.3}$$

Equation 8.4.3 is the instrument function of the Hanning window convolved with the sine Fourier transform, which is the equivalent of a sinc function with side lobes. In making the digital filter, this function is used for the purposes of boosting the high frequencies, while a normal low-pass filter formula is utilized for the low boosted frequencies.

8.5 The Digital Filter Effects Versus the Analog Filter Effects

We chose to focus on the low frequency analysis of the digital filter compared to the analog filter because there were more variables we could isolate for compared with the high frequency EQ section and the high frequency cut section of the analog filter. The analog filter data shows that there are peaks that are more emphasized below the cut-off frequency, which is what we would expect from our understanding of how the filter should function. However, the levels of the frequencies are not precisely what we would expect when compared with the ideal digital filter found by using a curve fit to the Manley Pultec Equalizer curves. In order to model what the Manley Filter should have done, we consulted a spectra sheet found the Manley Pultec manual [9, Page 8] that gave amplification factors over the full range of possible input frequencies. To fit this empirical curve, we chose a Lorentzian peak shape, and adjusted the parameters until it was well-matched to the spectra sheet graph. The result was the following function:

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by

$$f(x) = \frac{12.75}{1 + 0.1\left(\frac{x}{f}\right)^2} \tag{8.5.1}$$

where x is the frequency being boosted and f is the cut-off frequency. 12.75 in the numerator denotes the +12.75 dB boost received by each frequency bin at the lowest frequencies. In this instance, the Lorentzian is our instrument function.

Below are graphs of each low-pass cut-off frequency and a description of the occurrence of each peak. For the unprocessed input signal of each graph, the peaks occur at 25 Hz, 50 Hz, 100 Hz, 200 Hz, 400 Hz, 800 Hz, 1600 Hz, 3200 Hz, 6400 Hz, 12800 Hz. However, since the levels are different for each peak, these points will be included in the analysis.

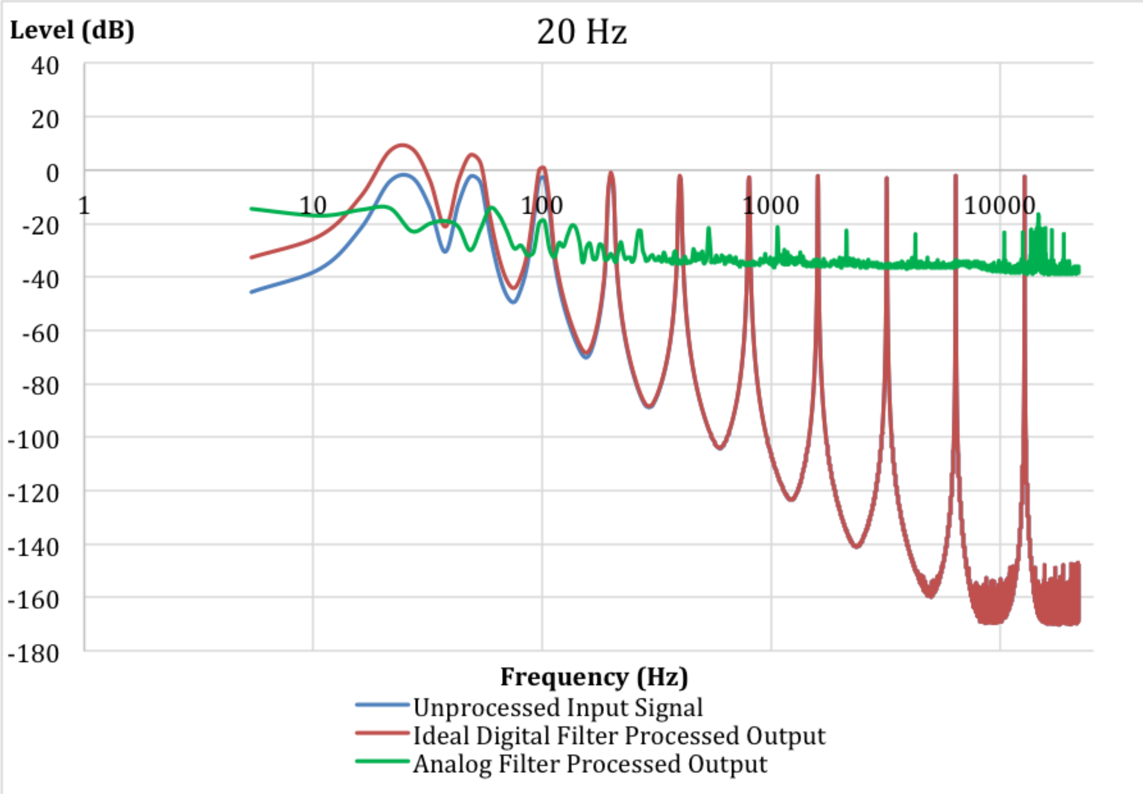

Level (dB)
20 Hz
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
1
10
100
1000
10000
Frequency (Hz)
Unprocessed Input Signal
Ideal Digital Filter Processed Output
Analog Filter Processed Output

Figure 8.5.1.

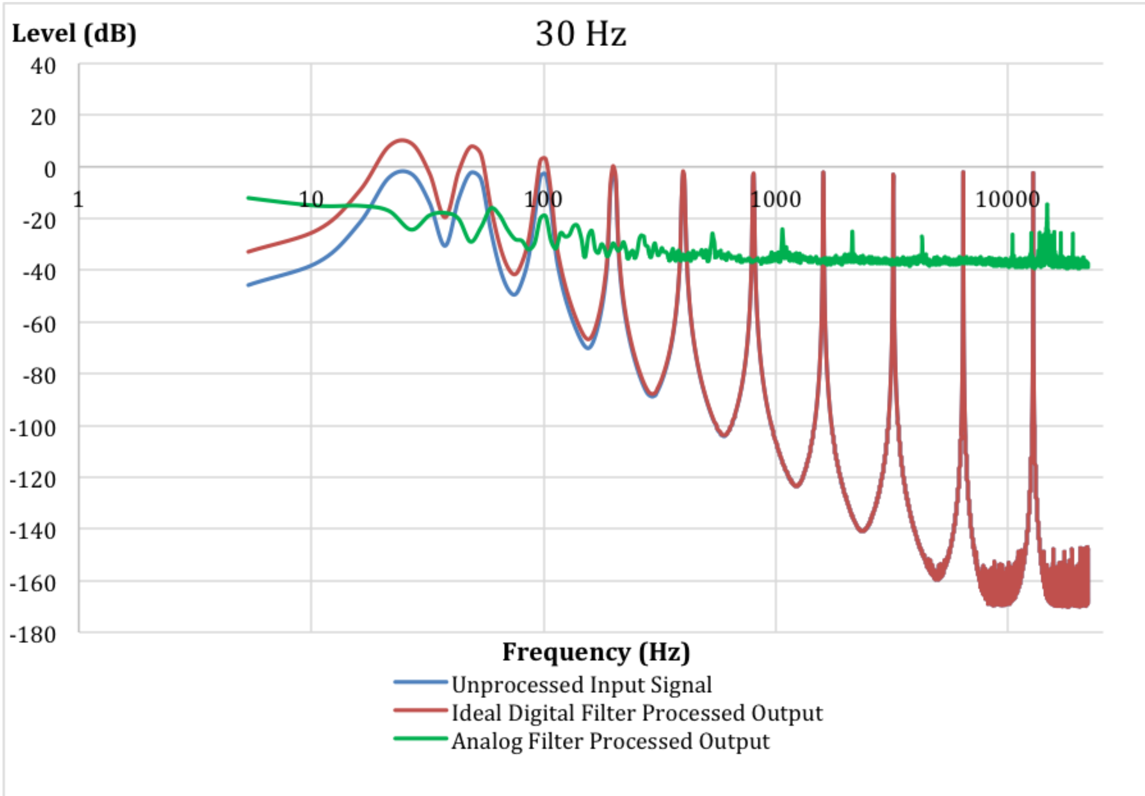
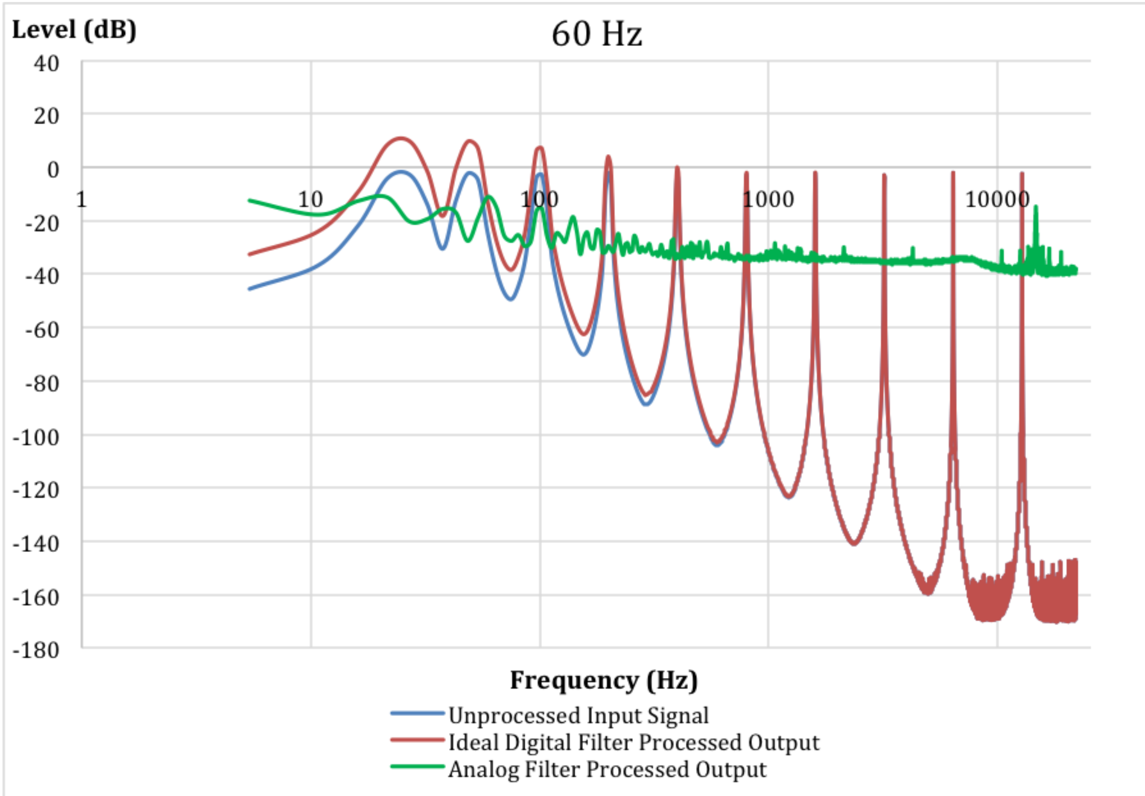

Level (dB)
30 Hz
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
1
10
100
1000
10000
Frequency (Hz)
Unprocessed Input Signal
Ideal Digital Filter Processed Output
Analog Filter Processed Output

Figure 8.5.2.



60 Hz
Level (dB)
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
1
10
100
1000
10000
Frequency (Hz)
Unprocessed Input Signal
Ideal Digital Filter Processed Output
Analog Filter Processed Output

Figure 8.5.3.

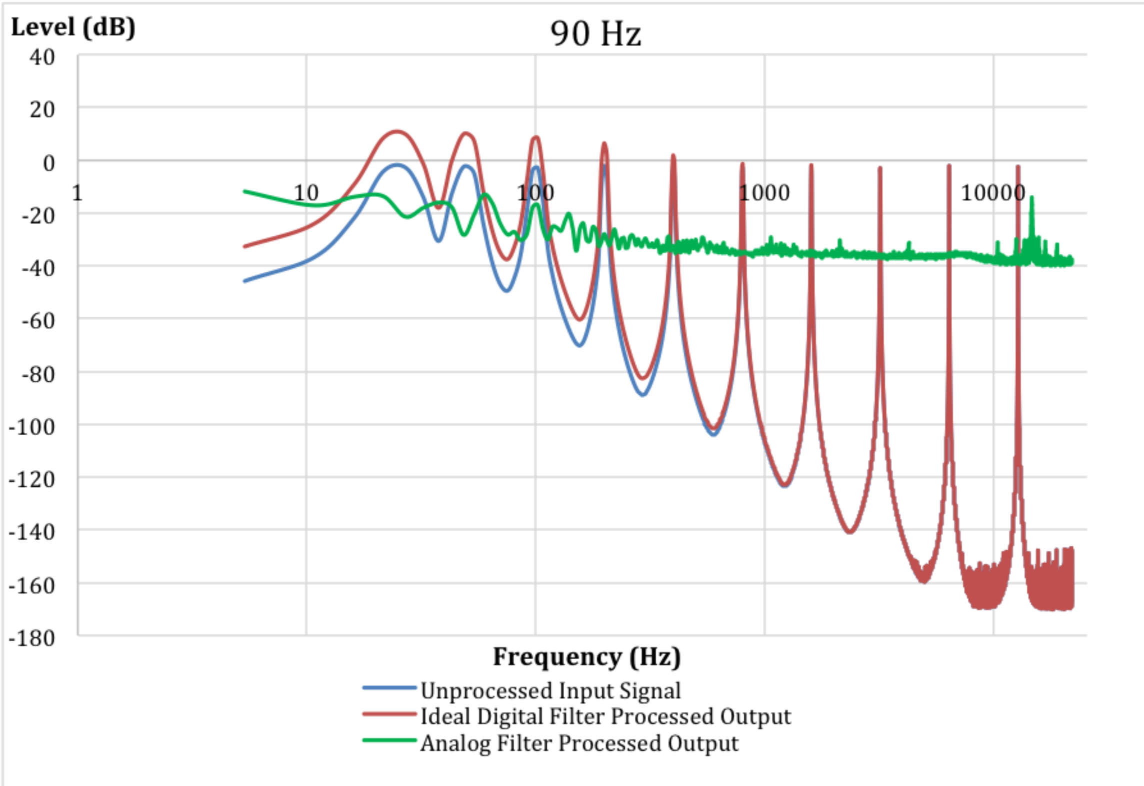

Level (dB)
90 Hz
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
1
10
100
1000
10000
Frequency (Hz)
Unprocessed Input Signal
Ideal Digital Filter Processed Output
Analog Filter Processed Output

Figure 8.5.4.

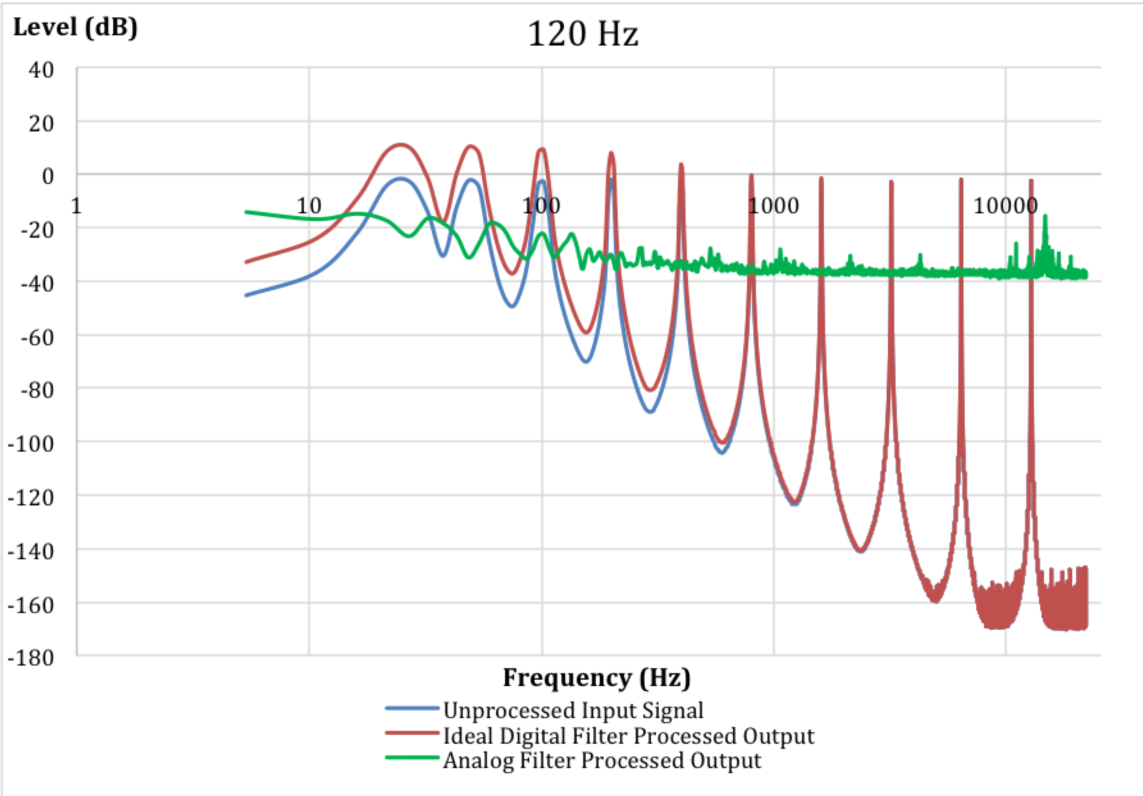

Level (dB)
120 Hz
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
1
10
100
1000
10000
Frequency (Hz)
Unprocessed Input Signal
Ideal Digital Filter Processed Output
Analog Filter Processed Output

Figure 8.5.5.

Unprocessed input signal peaks Figure 8.5.1:

25 Hz at -1.3 dB

50 Hz at -1.6 dB

100 Hz at -1.2 dB

200 Hz at -1.6 dB

400 Hz at -1.6 dB

800 Hz at -1.4 dB

1600 Hz at -1.6 dB

3200 Hz at -1.6 dB

6400 Hz at -2.0 dB

12800 Hz at -1.8 dB

Ideal digital filter processed output peaks Figure 8.5.1:

25 Hz at 9.7 dB

50 Hz at 6.2 dB

100 Hz at 2.4 dB

201 Hz at -0.45 dB

400 Hz at -1.28 dB

800 Hz at -1.32 dB

1600 Hz at -1.58 dB

3200 Hz at -1.595 dB

6400 Hz at -1.998 dB

12800 Hz at -1.7996 dB

Analog filter processed output peaks Figure 8.5.1:

18 Hz at -16 dB

35 Hz at -19.1 dB

61 Hz at -16.7 dB

100 Hz at -19 dB

135 Hz at -20.7 dB

266 Hz at -20.8 dB

533 Hz at -21 dB

1066 Hz at -21.1 dB

2133 Hz at -21.7 dB

4266 Hz at -21.6 dB

14765 Hz at -19.8 dB

Unprocessed input signal peaks Figure 8.5.2:

25 Hz at -1.3 dB

50 Hz at -1.6 dB

100 Hz at -1.2 dB

200 Hz at -1.6 dB

400 Hz at -1.6 dB

800 Hz at -1.4 dB

1600 Hz at -1.6 dB

3200 Hz at -1.6 dB

6400 Hz at -2.0 dB

12800 Hz at -1.8 dB

Ideal digital filter processed output peaks Figure 8.5.2:

25 Hz at 10.6 dB

50 Hz at 8.38 dB

100 Hz at 4.84 dB

201 Hz at 0.72 dB

400 Hz at -0.92 dB

800 Hz at -1.22 dB

1600 Hz at -1.555 dB

3200 Hz at -1.588 dB

6400 Hz at -1.997 dB

12800 Hz at -1.799 dB

Analog filter processed output peaks Figure 8.5.2:

34 Hz at -18.2 dB

61 Hz at -18.6 dB

100 Hz at -20 dB

135 Hz at -22.9 dB

266 Hz at -26.2 dB

533 Hz at -25.5 dB

1066 Hz at -23.8 dB

2133 Hz at -24.7 dB

4266 Hz at -25.3 dB

14764 Hz at -21.6 dB

Unprocessed input signal peaks Figure 8.5.3:

25 Hz at -1.3 dB

50 Hz at -1.6 dB

100 Hz at -1.2 dB

200 Hz at -1.6 dB

400 Hz at -1.6 dB

800 Hz at -1.4 dB

1600 Hz at -1.6 dB

3200 Hz at -1.6 dB

6400 Hz at -2.0 dB

12800 Hz at -1.8 dB

Ideal digital filter processed output peaks Figure 8.5.3:

25 Hz at 11.23 dB

50 Hz at 10.32 dB

100 Hz at 8.77 dB

201 Hz at 4.41 dB

400 Hz at 0.74 dB

800 Hz at -0.72 dB

1600 Hz at -1.42 dB

3200 Hz at -1.555 dB

6400 Hz at -1.988 dB

12800 Hz at -1.797 dB

Analog filter processed output peaks Figure 8.5.3:

17 Hz at -14 dB

35 Hz at -17.5 dB

60 Hz at -16 dB

100 Hz at -18.4 dB

137 Hz at -22.2 dB

263 Hz at -28.4 dB

533 Hz at -27.7 dB

1066 Hz at -26.7 dB

2132 Hz at -28.1 dB

4267 Hz at -27.7 dB

14764 Hz at -20.4 dB

Unprocessed input signal peaks Figure 8.5.4:

25 Hz at -1.3 dB

50 Hz at -1.6 dB

100 Hz at -1.2 dB

200 Hz at -1.6 dB

400 Hz at -1.6 dB

800 Hz at -1.4 dB

1600 Hz at -1.6 dB

3200 Hz at -1.6 dB

6400 Hz at -2.0 dB

12800 Hz at -1.8 dB

Ideal digital filter processed output peaks Figure 8.5.4:

25 Hz at 11.35 dB

50 Hz at 10.76 dB

100 Hz at 10.14 dB

201 Hz at 6.91 dB

400 Hz at 2.69 dB

800 Hz at 0.032 dB

1600 Hz at -1.21 dB

3200 Hz at -1.4999 dB

6400 Hz at -1.97 dB

12800 Hz at -1.794 dB

Analog filter processed output peaks Figure 8.5.4:

17 Hz at -14.5 dB

34 Hz at -16.9 dB

60 Hz at -16.4 dB

100 Hz at -18.9 dB

137 Hz at -22.8 dB

266 Hz at -28.6 dB

533 Hz at -28.7 dB

1066 Hz at -28.5 dB

2132 Hz at -29.5 dB

4266 Hz at -29.2 dB

14764 Hz at -20.1 dB

Unprocessed input signal peaks Figure 8.5.5:

25 Hz at -1.3 dB

50 Hz at -1.6 dB

100 Hz at -1.2 dB

200 Hz at -1.6 dB

400 Hz at -1.6 dB

800 Hz at -1.4 dB

1600 Hz at -1.6 dB

3200 Hz at -1.6 dB

6400 Hz at -2.0 dB

12800 Hz at -1.8 dB

Ideal digital filter processed output peaks Figure 8.5.5:

25 Hz at 11.39 dB

50 Hz at 10.93 dB

100 Hz at 10.72 dB

201 Hz at 8.356 dB

400 Hz at 4.439 dB

800 Hz at 0.94 dB

1600 Hz at -0.92 dB

3200 Hz at -1.42 dB

6400 Hz at -1.955 dB

12800 Hz at -1.788 dB

Analog filter processed output peaks Figure 8.5.5:

16 Hz at -14.8 dB

33 Hz at -16.4 dB

62 Hz at -18.3 dB

100 Hz at -21.7 dB

134 Hz at -22.3 dB

267 Hz at -27 dB

533 Hz at -27.7 dB

1065 Hz at -28 dB

2132 Hz at -30.5 dB

4306 Hz at -33.6 dB

14765 Hz at -19.3 dB

The peaks at octaves of one another are expected, but a certain peak at 14764-14765 Hz is prevalent in all of the analog processed output spectra and is inexplicable from the data provided. However, the qualitative effect of this peak on the audio content of the signal is negligible since this value exceeds the limit of middle-aged adult hearing (20 Hz to 14 kHz).

It is important to note that, in Figure 8.5.1, Figure 8.5.2, Figure 8.5.3, Figure 8.5.4, and Figure 8.5.5 the reason for lower levels in amplitude of each frequency is due to signal loss characteristic of all passive equalizers. This loss occurs because filters do not amplify the original signal, they reduce the input signal by greater or lesser amount; as a result the level of the processed output signal is reduced. This is why the filtered output signal appears to be weaker than the unfiltered input signal. Additionally, the noise floor of the analog filter is a certain amount, estimated to be around -34 dB to -36 dB, which is why the minimum levels of the processed signal do not approach the minimum levels of the unprocessed signal. Perhaps the most shocking observation is that the peaks of the filtered spectra do not align with the peaks of the unfiltered spectra in the frequency domain, indicating that the analog filter was not precise in which frequencies it boosted. This inconsistency necessitates some explanation.

There are many variables which may affect the functionality of the analog filter and explain why the peaks of the unfiltered input signal and the input signal processed by the analog filter do not align in the frequency domain. One reason may be that the THD+Noise (Total Harmonic Distortion plus Noise) which comes from all sources other than the input source may contribute various levels to the signal, even though this constitutes very little of the filtered output signal (allegedly 0.015 percent). However, over time this percentage will increase due to wear on the internal circuitry of the analog filter. To determine whether or not THD+Noise was a significant factor would require further testing.

The way which the analog filter stores each frequency bin in contrast to how Pro-Tools stored each frequency bin and Audacity stored those frequency bins is another possibility explaining the mismatch of frequency peaks. If either one of these did not align, mainly Pro-Tools and the analog filter, then there would be a mismatch in frequency bins which would be transcribed to Audacity once the recordings were exported from Pro-Tools. The general procedure for making a digital filter designed after an analog filter is summarized below.

1. Record continuous-time input signal using analog filter and receive modified output spectrum.

2. Create an ideal digital filter transfer function using input signal frequency spectrum modifications based on the observations of the ideal behavior of the analog filter provided by a manual.

3. Design digital filter using the formula so that the modified input signal frequency spectrum is the same as the modified frequency content of the analog filter.

9
Conclusion

The applications of digital signal processing in the study of acoustics is an intricate subject in which physics, mathematics, computer programming, and musicality conjoin at a juncture where each one is equally as important in accomplishing the goal of a project. The focus of this project was on developing an understanding of the methods used for analog-to-digital conversion and then the filtering the converted signal. This is one particular aspect of digital signal processing which comprises a large part of modern audio engineering.

The case study revealed how a particular analog filter differs from the ideal digital filter, as well as the general protocol one would follow to design a digital filter to replicate the effects of an analog filter. A major component omitted from this study was the successful execution of a digital filter designed after this analog filter which implemented formulae discussed in the study. In order to do this, extensive knowledge of the use of digital signal processing software and knowledge of computer programming applicable to audio engineering is required. In addition to the knowledge required to execute designing this replica, more data must be recorded with an exhaustive number of settings on the analog filter to gain a more complete picture of the filter's functionality. Another area which was not explored was the optimization of this digital filter design, which would require more data to be recorded using a greater degree of precision for measurements.

Major accomplishments of this project were gaining an understanding of the evolution of signal processing and the mechanics behind the mathematical methods which are essential in signal processing of any kind. After establishing the fundamentals of signal processing, the objective of this project was to expound on how certain practices are vital to remedying problems encountered in editing audio engineering projects. This paper provides the general information of digital signal processing and certain important aspects of this field which are necessary in guiding a more in depth examination of the methods used in audio engineering software. This paper has presented a comprehensive approach to learning the rudimentary techniques of digital signal processing methods and the functionality of audio engineering software, enabling the reader to lead more advanced investigations of the mathematical and scientific methods involved in the study of acoustics.

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