The Use of Stochastic Assumption in Accelerator-Multiplier Business Cycle Theory

I. Introduction

What will be considered in this paper is the use of probabalistic assumptions in the construction of Accelerator-Multiplier models. An examination of a possible generating relation for one of the parameters of these business cycle models is undertaken and it is shown that the parameter has attributes that makes it a random variable. The little this paper contributes is to economics, not to mathematics or statistics; in fact, the paper ends where the mathematics begins.

Linear accelerator-multiplier models have been around in business cycle analysis for some time: certainly since 1938. However, all such linear models suffer from a common defect: no matter what values are assigned to the consumption and investment parameters the time series that the model generates is unsatisfactory. Various modifications have been suggested to circumvent this difficulty: we can mention (a) the substitution of non-linear accelerator forms for the linear forms (by Goodwin and Hicks); (b) the addition of a "synchronized" autonomous investment impulse-identifying the impulse with Schumpeter's innovations (also done by Goodwin) and (c) the addition to a damped linear form of a "random" energy source (as suggested by Frisch in his 1933 Propagation and Impulse Problems paper: This was also taken up by Fisher in the A. E. R. in 1952.[1] The Fisher paper will be discussed in some detail below).

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[1] G. H. Fisher: Some Comments on Stochastic Macro-Economic Models: A.E.R. (Sept. 1952) p. 519-539.

In this paper I wish to examine up the stochastic way out of the difficulty inherent in linear models. In order to do this I wish to distinguish between an "error" approach and a "random parameter" approach to such models. The above mentioned paper by G. H. Fisher is an example of the Error approach to the use of stochastic variables in business cycle theory.

II. The stochastic error approach: The Fisher Paper.

Consider a Hicks type (induced investment is a linear function of the change in income) accelerator-multiplier model in which the accelerator coefficient $\beta < 1$. The time path of income is damped. If α the consumption coefficient is sufficiently small for the given β the time series generated will be damped oscillatory. This implies that the cycle will die out: therefore unless the model is modified it is unsatisfactory for business cycle analysis. Fisher modifies this model by imposing upon this damped cycle an outside energy source under the guise of a "random shock."

Briefly, Fisher considers a consumption function and an investment function subject to random errors: that is,

1) $C_t = \alpha Y_{t-1} + \mu_t$ and

2) $I_t = \beta (Y_{t-1} Y_{t-2}) + \nu_t$

The μ_t and ν_t are random errors. We therefore have

3) $Y_t = (\alpha + \beta) Y_{t-1} - \beta Y_{t-2} + \omega_t + \lambda$ the exogenous government investment.

where $\mu_t + \nu_t = \omega_t$. Fisher assumes that ω_t is normally distributed with a mean of zero and a variance $\sigma^2_{\omega_t}$. Fisher also assumes $\alpha = .7$, $\beta = .5$ so that the non-stochastic part of the model generates a damped-oscillatory time series. In order to examine how such a model subject to random shocks behaves, Fisher computed Y_t for 100 time periods, obtaining his μ_t by random selections from a simulated normal population of the μ_t. The error model in contrast to the mechanical model exhibited a persistent cycle: the random shocks counteracted the damping effect due to the assumed values of α and β.

However, Hicks in his volume on the Trade Cycle concluded that "the theory of damped fluctuations and erratic shocks prove unacceptable ... "for the correlation between successive cycles is quite small." This objection is true but irrelevant. The rejection is based upon the unwarranted assumption that the period generated by such a random shock model is basically the period of the cycle due to the accelerator-multiplier mechanism.

The random shocks in this case are samples drawn from a universe. In such drawings you expect to have runs of various lengths of similar valued shocks. Such runs will tend to build up the amplitude of a deviation from the equilibrium level of income. These large deviations will lead to the persistence of the oscillatory movement. If the random shocks are of a significant size relative to the equilibrium level of income, the resulting time series would tend to show a small correlation between the corresponding terms of successive cycles as determined by the accelerator- multiplier mechanism. Nevertheless the time series would exhibit the varying amplitudes and periods that are characteristics of observations.

Fisher's approach however is vulnerable to a second comment made by Hicks: "quite moderate reduction in the investment coefficient leaves us with fluctuations that are mainly random: with fluctuations, that is, that remain unexplained." Fisher assumed that the standard deviation of the random shocks σ_{u_t} is 5 billion: his equilibrium level of income is 57 billions. Assuming the successive shocks are independent, a positive shock equal to or greater than one standard deviation will on the average occur about 16 per cent of the time: in Fisher's simulated population such shocks occur 19 ½ per cent of the time. Let us assume that $Y_t = Y_{t-1} = Y_{t-2} = 57$ billion and that two successive shocks of + 5 occur. In this event, $Y_{t-1} = 62$ and $Y_{t+2} = 68$. In Fisher's population the mean value of the shocks ≥ 5 is 7.7. Using this mean value we get $Y_{t+1} = 65$ and $Y_{t+2} = 74$. By inspection of the time series exhibited in Fisher's article we get that an income ≥ 68 billion occured 6 times and ≥ 74 billion 3 times. The population would on the average yield an income ≥ 68 billion by means of two successive positive shocks of 5 billion or more 4 times out of 100. Taking into account that different lengths of runs of similar signed shocks which could yield the extreme deviations we conclude that Fisher's series is primarily the result of the random shocks. Essentially the effect of the accelerator-multiplier model is to make each period's income a weighted average of two previous incomes plus or minus a shock. This Fisher model therefore is vulnerable to the contention of Hicks that it leads to an unexplained cycle: it is therefore similar to the "tomorrow's income will be today's income" school of business cycle forecasters.

It is also obvious that if the variance of the shocks were smaller, for the same α and β that Fisher assumed, the result would be a damped cycle: Only by leaving a great deal of the cyclical phenomena unexplained can Fisher achieve his result.

The approach to the use of stochastic processes in economic analysis which Fisher used and which Hicks criticised assumes that random shocks are attached to a systematic generating function. This approach to economic analysis can be imputed to Haavelmo's paper on the "Probability Approach to Econometrics." The ideology of Haavelmo's approach is given by the following quotations:

> What we want are theories that, without involving us in direct logical contradiction, state that the observations will as a rule cluster in a limited subset of the set of all conceivable observations, while it is still consistent with the theory that an observation falls outside this subset now and then...,[1] and
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> The question is not whether probabilities exist or not but whether -- if we proceed as if they existed we are able to make statements about real phenomena that are correct for practical purposes.[2]

The approach embodied in those two quotations can be derived from two sources: (1) the residual variations in regression analysis after the systematic effect of the "variables" has been eliminated, and (2) errors of observation where the fallibility of humans and of the measuring instruments combine to yield observations which do not, in detail, conform to the "real world" values. The Haavelmo approach leads to the formulation of economic problems in the light of statistical testing techniques. This is an appropriate transformation of economic models where the problem is to apply such

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[1]Haavelmo, T. "The Probability Approach in Econometrics, Econometria Vol. 12, Supplement (July 1944) p. 40.

[2]*Ibid*.

tests to economic data. However, it is not the appropriate approach to the construction of a stochastic model.

III. The Random Parameter Approach

As an alternative to the Haavelmo errors of observation and unexplained residuals approach to the use of stochastic processes in economics, we will contrast a formulation of an accelerator-multiplier model in which the parameters have stochastic properties. Such a model assumes that the behavior of the economy can be best explained in terms of elements which are in their very nature random variables. This will be embodied in statements which assert that the values of certain attributes of the elementary economic units, firms or households, after allowing for the constraints of market conditions, technological, production, or utility relations, and specified behavior principles, may still take on any of a set of values. These attributes will be characterized by a probability distribution. Therefore, in any model in which such an attribute enters as a parameter, the variables of the model are not strictly determinate. Paraphrasing O. Lundberg we have that: "To characterize the economic process with the aid of a random process implies that certain parameters, e.g., the output of a firm, its profitability during a given period, its investment decisions are regarded as variables that with given probabilities assume given values; i.e., they are considered random variables. The probability distribution of a random variable or of a combination of such random variables at a certain moment is determined by the past of the economic process."[3]

In attempting to set up models for investment behavior, economists usually rely upon expected values to achieve a meaningful statement. Expectation relations are inherently of the nature where for the different economic units different expectations can co-exist; and the 'distribution of expectations' becomes an element in the aggregate investment relation. The investment relation: the amount of investment forthcoming during any period of time is one that is not strictly determined by the observable and measurable variables of the economic system.

If we attempt to apply the random process approach as stated above to an accelerator-multiplier model of income determination, we have, naturally, to assume that the α's and the β's are random. For the observable and measurable determinants of income in these models are the previous periods income (ignoring "measurement errors"). Present income is not strictly determined by these historic variables, if we assume that the parameters α and β of the income-generating model are random. In what follows we will specialize by taking up only the induced-investment coefficient. Obviously similar considerations enter into the determination of the consumption coefficient.

A model of the economy which yields an accelerator coefficient that is of the nature of an element drawn from a probability

[3] Paraphrase of a statement of O. Lundberg, "On Random Processes and their Applicability to Sickness and Accident Insurance," Almquist and Wicksell's Boktryckeri, A. B. Uppsala, 1940, p. 3.

distribution can be easily constructed. Let us assume that each firm is an element in a Marshallian Industry, that it is a unit in a set of firms producing a homogenous product. The firms in the industry vary in a manner which is consistent with the doctrine of the representative firm: differences in their cost structure, production function, and in the nature (perhaps spead) of their reaction to changes. The economy consists of many such industries, and it is assumed that in each industry the behavior of the firms is determined by the industry parameters and not by the situation in other industries.

A change in income implies that the set of demand curves for the products of the particular industries shifts. However, firms are the investing units. What is needed for each industry is a transformation of the shift in the industry demand curve into a change in a parameter upon which the firms in the industry base their investment decisions. The impact upon a firm of a shift in the industry demand curve depends upon the market structure of the industry. In a competitive industry a shift in demand affects firms by means of a change in the market price of the product. This change in price implies that at the old price a quantity different from the quantity actually taken would now be taken. Let us assume that the investment decision of firms is based upon the firm's estimate of the change in the quantity that the market will take at the price that ruled prior to the shift in demand. Each firm estimates the quantity of the product which it would be profitable for it to produce by allowing its plant size to vary. The investment by a particular firm which is induced by a change in income will be the change in fixed capital necessary to alter its plant size plus whatever change in working capital that is needed to produce the new optimum output. Such induced investment in a competitive industry may take place by means of a change in the number of firms in the industry rather than by means of an alteration in the size of the plants of existing firms.

We therefore have a particular firm investment relation of the form:

4) $i_{\lambda\rho}(t) = \gamma[Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)]$ where

i = investment by a particular firm

λ = industry index

ρ = firm index

γ = coefficient of induced investment for a particular firm

$Q_{\lambda\rho}(t-1)$: estimate by the ρ firm of the quantity demanded at the price of t-2 during the period t-1

$Q_{\lambda}(t-2)$: quantity actually taken at the price of t-2 during the period t-2

$Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)$: the firm's estimate of the industry demand curve shift.

The amount of investment that takes place in an industry will be given by:

5) $i_{\lambda}(t) = \sum_{\rho} i_{\lambda\rho}(t) = \sum \gamma_{\lambda\rho}[Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)]$.

Heroically assuming that all firms in the λ industry estimates of $Q_{\lambda\rho}(t-1)$ are the same[4] we have

$$6) \quad i_{\lambda}(t) = [Q_{\lambda}(t-1) - Q_{\lambda}(t-2)] \sum_{\rho} \gamma_{\lambda\rho}$$

The amount of investment induced in the economy is the sum of the investment of the different industries:

$$7) \quad I_t = \sum i_{\lambda}(t) = \sum_{\lambda} [(Q_{\lambda}(t-1) - Q_{\lambda}(t-2)) \sum_{\rho} \gamma_{\lambda\rho}].$$

However, by the aggregate accelerator relation we also have that $I_t = \beta (Y_{t-1} - Y_{t-2})$. Therefore we have that

$$8) \quad \beta_t (Y_{t-1} - Y_{t-2}) = \sum_{\lambda} [(Q_{\lambda}(t-1) - Q_{\lambda}(t-2)) \sum \gamma_{\lambda\rho}]$$

$$9) \quad \beta_t = \frac{\sum [(Q_{\lambda}(t-1) - Q_{\lambda}(t-2)) \sum_{\rho} \gamma_{\lambda\rho}]}{Y_{t-1} - Y_{t-2}}$$

If the set of shifts in industry demand curves which is implied by a change in income is determinate, and if the impact of those shifts in industry demand curves upon particular firms investment is determinate, then β_t, the coefficient of induced investment will be non-random. For the aggregate coefficient of induced investment to be a constant we have to assume that each $\gamma_{\lambda\rho}$ is independent of the size and direction of the shift in its particular industry demand curve and that the shift in each demand curve $Q_{\lambda}(t-1) - Q_{\lambda}(t-2)$ is a fixed ratio to $Y_{t-1} - Y_{t-2}$. Then we would have that

$$10) \quad \beta_t = \frac{\sum [(Q_{\lambda}(t-1) - Q_{\lambda}(t-2)] \sum \gamma_{\lambda\rho}}{Y_{t-1} - Y_{t-2}}$$

$$11) \quad \sum_{\rho} \gamma_{\lambda\rho} = \bar{\gamma}_{\lambda} \quad \text{a constant}$$

$\bar{\gamma}_{\lambda}$ is the "industry" coefficient of induced investment

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[4] That is, every firm in the λ industry has the same estimate of the elasticity of demand for the product.

12) $\dfrac{Q_{\lambda}(t-1) - Q_{\lambda}(t-2)}{Y_{t-1} - Y_{t-2}} = \xi_{\lambda} = \text{constant}$: as $p_{\lambda}(t-2)$ is used to estimate $Q_{\lambda}(t-1)$, ξ_{λ} is equivalent to the marginal propensity to consume a particular good.

13) $\beta_t = \sum_{\lambda} \xi_{\lambda} \gamma_{\lambda}$ = constant. The above are the implied assumption in any "constant" accelerator coefficient formulation.

If the set of shifts in industry demand curves which is associated with a change in income is determined by a process which can be considered as analogous to sampling, then at any time the shift in a particular industry's demand curve, which is the immediate cause of inducing investment, can be considered as a sample drawn from a universe. In addition, the amount of investment which a given shift in an industry demand curve will induce can be interpreted as depending upon the reactions of the affected firms, and the reaction of firms to particular stimuli may be characterized by a probability distribution. In both circumstances the β_t coefficient for the economy is a random variable.

If an industry consists of a large number of firms, and if the probability distribution of reactions by firms is the same, then aggregate investment will be a summation of the reaction of a large number of firms. By "laws of large numbers" the summation of random variables which is the aggregate investment relation will tend to be stable. If an industry consists of a small number of firms, the summation of random variables which is the aggregate investment relation will tend to be unstable. This can be interpreted as implying that for competitive industries the stochastic elements in the determination of β is relatively unimportant, whereas for oligopolistic industries the stochastic element will tend to be more significant.

Combining the above arguments we have that the probability distribution of β_t depends upon: a) the probability distribution of particular shifts in industry demand curves given a particular change in income $Y_{t-1} - Y_{t-2}$; b) the probability that a particular set of firms ω from the set of all firms being affected by a particular change in income $Y_{t-1} - Y_{t-2}$. If the above probability relations apply in the determination of β_t we would no longer expect a deterministic relation to exist between a change in income and a change in investment.

In order to contrast the above stochastic process with the error process which is Fisher's approach, we assume that β is a random variable whose value at any moment of time is drawn from a probability distribution which depends upon a) the structure of demand curve shifts which result from a given change in income; b) the set of firms for which the resultant demand curve shifts imply investment; c) the relation between output and capital stock for each affected firm. As β is a function of a subset of firms drawn from the set of all firms, it is a random variable.

If we assume that the structure of demand curve shifts which result from a given change in income is independent of the level of income or of the change in income (that the marginal propensity to consume particular goods is constant), and if we assume that the magnitude of the individual-firms-accelerator coefficient is independent of the magnitude of the shift in the industry demand curves, then we have that the aggregate accelerator probability distribution is independent of the level or change in income. The probability distribution of β will be independent of the time path of income. Such a model of the accelerator process results in statements that given the value of α, the realized value of β will be in the interval which results in the economy being in either damped asymptotic, damped oscillatory, explosive oscillatory, or explosive states a determinate percentage of time. Such a proposition is stochastic, as it is based upon a frequency distribution of the β's from which the observed values are drawn.

For example, in an accelerator-multiplier model of the type $Y_t = (\alpha + \beta)Y_{t-1} - \beta(Y_{t-2})$ the following table gives the range of values of β which, for given α's, place the economy in each state:

Values of α	Values of β States of the economy			
	DAMPED		EXPLOSIVE	
	monotonic	oscillatory	oscillatory	monotonic
.9	0-.47	.47-1	1-1.73	1.73-
.8	0-.30	.30-1	1.2.10	2.10-
.7	0-.20	.20-1	1-2.40	2.40-
.6	0-.14	.14-1	1-2.66	2.66-
.5	0-.08	.08-1	1-2.92	2.92-

The probability of the economy being in any state depends upon the probability of β having the value appropriate to that state. For example, with $\alpha = .9$ the probability of the economy being explosive cyclical, depends upon the probability of β having a value of between 1 and 1.73. As we are using a Hicks type model, the probability of the economy being damped or being explosive depends solely upon the value of the β coefficient.

In an attempt to illustrate how such a random β would affect the operations of the accelerator and multiplier model, two test runs were made using the values of the constant, λ, of the marginal propensity to consume α, and of Y_{t-1} and Y_{t-2} that Fisher used in his 'random variable' model. In the first run β was assumed to have a rectangular distribution, with the values of 0, .25, .50, .75, 1.0, 1.5, 2.0, 3.0, and 4 all being equally probable. The resultant series exhibited a great amplitude of fluctuation in the first half of the series. Then because of a run of values of β coefficients which lead to a highly damped movement of income, the series exhibited a very damped cycle. Of course in such a series if $Y_{t-1} = Y_{t-2} = \frac{\lambda}{1 - \alpha}$, the cycle would die out. The damping of the series was so great that in the latter part of the sample the cycle well nigh disappeared.

A second test of β as a random variable was made using a triangular frequency distribution of β. The frequency distribution from which the sample of β values were drawn was:

β	Relative Frequency
0	1
25	2
50	3
75	4
1.00	4
1.50	4
2.00	3
3.00	2
4.00	1

The time series which resulted does not exhibit the extreme fluctuations that the time series derived from a rectangular distribution of β exhibited. The fifty period time series also did not show the 'damping' of the cycle that the rectangular distribution exhibited. The reason is obvious: with the probability distribution in the second case, the chances of a 'run' of values of the β coefficient which lead to a highly explosive or a highly damped movement is much lower than in the rectangular distribution. As a result the extreme fluctuations and the extreme damping associated with the rectangular distribution do not occur.

We could continue to analyse the implications of β being a probability distribution independent of the level of income, or of the path of income by experimenting with additional frequency distributions of β, taking samples with replacements from these frequency distributions and observing, for specified values of α, the resultant time series. However, the assumptions that were made:- that the structure of demand curve shifts which result from a given change in income is independent of the level of income or of the change in income and that the magnitude of the individual firms accelerator coefficient is independent of the the magnitude of the shift in the industry demand curve (in order to derive the probability distribution of β independently of the level or the change in income) - are strong. Let us assume that the expected value (mean) of the frequency distribution of β_t depends upon the change in income and the difference between last periods income and the previous peak income.

14) $\overline{\beta_t} = Q(Y_{t-1}, Y_{t-2}, Y_{t-1} - \overset{*}{Y})$. This model, where $\overline{\beta_t}$ is the mean of the frequency distribution of β, is a non-linear stochastic model. Let us assume that even though

$\overline{\beta_t}$ changes the variance of β is a constant, and that the mean and the variance are the only relevant moments of the frequency distribution of β_t. We can now write the income generating function as:

$$15) \quad Y_t = \lambda + \alpha(Y_{t-1}) + (\overline{\beta_t} + \mu\sigma_{\beta_t})[Y_{t-1} - Y_{t-2}] = \lambda + \alpha(Y_{t-1}) + \overline{\beta_t}(Y_{t-1} - Y_{t-2}) + \mu\sigma_{\beta_t}(Y_{t-1} - Y_{t-2})$$

where the actual value of the accelerator coefficient at time t is $\beta_t = \overline{\beta_t} + \mu\sigma_{\beta_t}$. Depending upon the relation between σ_{β_t} and $\overline{\beta_t}$, and upon the frequency distribution of β_t we have that we can assign a probability to β_t falling within any range $\beta_0 < \beta_t \leq \beta_1$ e.g., $P[\beta_t > \beta_0$ and $\beta_t \leq \beta_1]$. This can be written as:

$$16) \quad P_\beta = \int_{\beta_0}^{\beta_1} \varphi(\overline{\beta}_t, \sigma_{\beta_t})\, d\beta_t .$$

As the probability distribution of β_t is a function of the time path of income, we have that the probability of β_t being such as to place the economy in each of its four states is a function of the path of income. If the variance of the probability distribution is small, we have that the probability of the economy remaining in the explosive state, where the value of $\overline{\beta_t}$ is high, is greater than the probability of the economy remaining in damped oscillation, where the value of $\overline{\beta_t}$ is small.

We have assumed that the effect of the rate of change of income is to shift the mean value of the frequency distribution of the accelerator coefficient, leaving the variance of the frequency distribution of the accelerator coefficient unchanged. We have also assumed that this variance of the frequency distribution of the accelerator coefficient is small in respect to the 'explosive' values of the accelerator coefficient and relatively large with respect to the damped values of the accelerator coefficient. We therefore have a model in which the probability that random variation will lead to a change in the direction of the movement of income is high when income is changing slowly, but the random process has a small probability of affecting the value of the accelerator coefficient sufficiently to change the state of the model when income is changing rapidly.

A succession of high values of the accelerator coefficient in relation to the expected value of the accelerator coefficient may, if the economy is in a damped state, lead to an explosive movement. A succession of small values of the accelerator coefficient in relation to the expected value of the accelerator coefficient may, by decreasing the rate of growth of income, lower the expected value of the accelerator coefficient through a number of time periods so that if the economy had been in an explosive state, it enters a damped state. Such a formulation of the accelerator generation process can be combined with a model of the accelerator generating relation which leads to either explosive or stagnant states as stable states. Such an inflation-stagnation model does not contain a satisfactory mechanism which would result in a change of the economy from a damped to an explosive state and vice versa. A combination of the random element and the systematic element makes it unnecessary to posit external 'shocks' or 'crises' of the magnitude of the stock market crash of 1929 or of World War II in order to have the economy shift from one of its stable states to another of its stable states. Although not a determinate relation, the variance of the accelerator coefficient is an endogenous economic phenomenon, for it is simply a statement to the effect that the investment reaction of a particular economic unit to a given economic change (a change in income) is, to some extent, indeterminate. As a result over-all economic behavior which is due to the coefficients of macro-economic models, such as the accelerator coefficient, is, to some extent, indeterminate.

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