Diffeomorphisms of Time

Talia Willcott
Bard College

Diffeomorphisms of Time

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Talia Willcott

Annandale-on-Hudson, New York
May, 2022

Abstract

According to Newton, time is a consistent and reliable tool used to measure various physical quantities. According to Einstein, time is subjective and dependent on a number of external factors. There are many ways to analyze and bridge the gab between the two theories, one being mathematically. In this project, I examine three specific notions of time by representing time as a line, using projecive geometry to relate the line to a circle, transforming the circle, and analyzing the physical implications of these transformations.

Contents

Acknowledgments

I would like to acknowledge and give my warmest thanks to my advisors Hal Haggard and Ethan Bloch for guiding and supporting me through this paper. Your council made this project possible. I also would like to thank Matthew Deady and Lauren Rose for igniting my interest in physics and math. I would not have considered studying either subject if it weren't for your warm welcome and contagious passion. I want to extend gratitude to the rest of my amazing professors at Bard, as well. Also thank you to my family for loving and supporting me through college and life.

1 Introduction

I became fascinated with time in the first physics course I took at Bard, *Time Examined*. On the first day the professors asked, "What is time?" This question stuck with me. Responses to this question range widely, addressing different facets of this idea; there is no 'correct' answer. There is, however, a way to understand time in a concrete and thorough way: mathematically. In this section, I will provide a brief overview of the history behind the scientific notion of time and illustrate how this rigorous concept of time evolved with science over the decades.

1.1 A Brief History of Time

The importance of viewing time mathematically comes from the history of time itself. In this section, we will run through a brief overview of how our notion of time has developed with science and how that has shaped the ways we view time today.

In the 1600's, Gottfried Leibniz, a German mathematician and philosopher, proposed the theory of vis viva, the first notion of kinetic energy. This theory seemed to contradict René Descartes' theory of conservation of quantity of motion which claimed that the total amount of motion of an object remains constant. Isaac Newton combined both definitions to create his third law of motion which states that two interacting bodies apply forces to each other that are equal and opposite. Newton reluctantly shattered this definition of a force by introducing a new kind of force that did not require physical interaction. He proposed a force that acted from a distance.

This idea sounded impossible; forces necessitated touch. But Newton pushed back on the old theory and even posited the equation $F = G\frac{m_1 m_2}{d^2}$, which states that the gravitational force between two objects is equal to the gravitational constant times the mass of the two objects, m_1 and m_2, divided by the distance squared. This equation was groundbreaking because it offered an explanation for a previously unexplainable phenomena concerning an object's tendency to fall to the ground. He named this new super-force gravity. Surprisingly, this claim was not actually proven experimentally until 100 years later by Henry Cavendish who calculated the value of G.

Forty years earlier, a similar claim was made by Michael Faraday, an English scientist, about two other contact-less forces called the electric and magnetic fields. He introduced the notion of a field and stated that between two charges there is a set of 'lines,' called a field, connecting them. Faraday's field lines described the direction of force on an object at any point around that object. James Maxwell then provided the mathematical structure to support the physical argument of the electric and magnetic fields. He created the concept of electromagnetic radiation and field theory, and the famous field equations based on Faraday's observations.

Another student of electromagnetic theory was Oliver Heaviside, an English mathematical physicist. Heaviside helped further develop and improve Maxwell's theories and introduced a crucial aspect of physics today: the vector field. A vector is defined as a quantity with direction and magnitude and is often used to represent the force acting on an object. A vector field is a group of vectors that assigns each point its own vector. The difference between a field line (introduced by Faraday) and a vector field (introduced by Heaviside) is that vector fields are tangent to field lines at a point. These two concepts are closely related but not the same.

In the late 1780's, Charles-Augustin de Coulomb then came up with an equation for the force between two electrically charged particles that was incredibly similar to Newton's equation: $F = k\frac{q_1 q_2}{d^2}$. It says that the force between two charges is equal to Coulomb's constant, k, times the charge of two particles, q_1 and q_2, over the distance between them squared. Around 1905, Albert Einstein found a connection between Newton's, Maxwell's, and Coulomb's equations that established a new aspect to the classical laws of motion. One important characteristic of

Maxwell's law is that the force between the two charges is *not* instantaneous; it takes a time $\frac{d}{c}$ for a charge, q_1, to feel the force of another charge, q_2. Einstein claimed that the force of gravity is not instantaneous either and extended the idea of field theory to that of Newton's notion of gravitational force. With this, he discovered relativity.

In the next section, we will compare Newtonian time to Einstein's notions of time through an exploration of classical Newtonian mechanics, special relativity and general relativity.

1.2 Notions of Time

Before we discuss the theories, I will explain how physicists often choose to use time. Two clocks are often used to keep track of how time changes in different circumstances. The first clock is the reference clock. The reference clock stays still and keeps track of time consistently. This clock is the one on the wall that we look at to check the time or use to count units of time. We say that this clock is in the reference frame. A reference frame is a coordinate system in which we view an object. We sometimes call this frame the lab, ground, or Earth frame. This frame is also still. The second clock is the moving clock. We will watch this clock from the lab frame and compare it to our reference clock. The moving clock is in the inertial frame because it is in motion. The inertial frame moves with the clock. The importance of these clocks and their respective frames will become apparent in the next section.

Now we will look at three theories that will allow us to describe time in three different ways. For each theory, we will look at the reference and moving clock and illustrate their relationship using a vector field.

1.2.1 Newtonian Time

The basic model of time is Newtonian time. It is linear, it is constant and it is the same everywhere no matter what. Seconds are seconds whether you are here or there, moving or still. This is the time that most people know and love. The reference clock is sufficient to show how

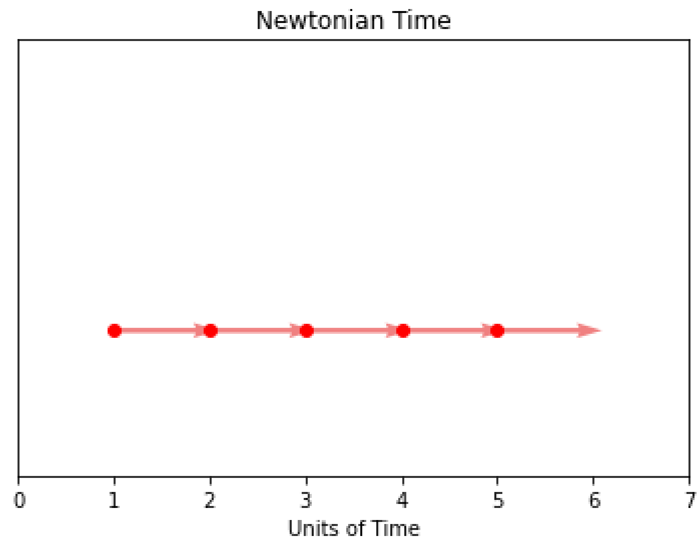

Figure 1.2.1: A model of Newtonian time as a vector field.

time moves with Newtonian mechanics, which are based on Newton's three laws and implement the idea that time, mass, and distance are absolute.

We can represent this model of time as a group of arrows, or a vector field. The red dots in Figure 1.2.1 represents seconds passing in the lab frame. The arrows represent how a moving clock ticks relative to the reference clock, the distance of the arrow representing the duration between seconds as experienced in the lab frame. In this case, the same amount of time appears to pass between both clocks so the arrows of Newtonian time are constant and all have a length of one unit.

1.2.2 Special Relativity

As well founded as Newton's mechanics were, Einstein was not convinced that they told the whole story. In 1905, Einstein published a paper claiming that an object's speed affects its mass and time, and the space around it. This proposition stemmed from an interest in the behavior of light. When scientists began measuring the speed of light, they realized it was the same everywhere despite the earth's motion. Note that according to Newtonian mechanics, the speeds of objects add up, therefore the speed of light should change depending on its reference frame. As an example of these mechanics, think about walking on a train. Say you walk at 1 mile per hour on a train moving at 40 miles per hour. Then you are moving at 41 miles per hour relative to the earth. In terms of reference frames, we can say that in the earth frame you are moving at 41 miles per hour but in the train frame you are moving at 1 mile per hour. This should have held for the speed of light according to Newtonian mechanics; however, when scientists ran this experiment on light, they found that the speed of light is constant all the time.

Einstein's curiosity of this predicament led to a number of 'thought experiments' that included scenarios like the one I just mentioned, except instead of the train traveling at 40 mph it travels close to the speed of light.[1] Through these thought experiments, Einstein concluded that since light can only move at one speed, simultaneity does not exist. This may seem like a jump, but believe it or not light and time are fundamentally connected. When an event[2] happens somewhere, say a streetlight changing from red to green down the street, it will take some time for the light to travel from the streetlight all the way to you. Then you process the event sometime after it happens. The farther you are from the event, the longer it takes for the light to reach you and the later you process the event. Therefore, simultaneity cannot exist.

Another characteristic of Einstein's discovery is that time moves slower when in motion. A common saying is "moving clocks run slow." This basically means that when an object is in motion relative to an observer in another frame, the object appears to experience a shorter duration of time over the course of the movement than a stationary object in the frame of the observer. This theory has even been proven experimentally a number of times.[3] We can imagine an experiment with our reference clock and our moving clock. Say two people are running the experiment in the lab frame. One person watches the reference clock and the other person watches the moving clock. They will come to find that the reference clock appears to tick more times over the course of the measurement period than the clock that is in motion. The faster that the moving clock travels, the longer the duration between ticks appears to become and the

[1] The speed of light is 3×10^8 meters per second or 6.7×10^8 miles per hour.

[2] Something happening at a particular time and place.

[3] Please see the Rossi-Hall experiment or the Frisch-Smith experiment for more information behind this theory.

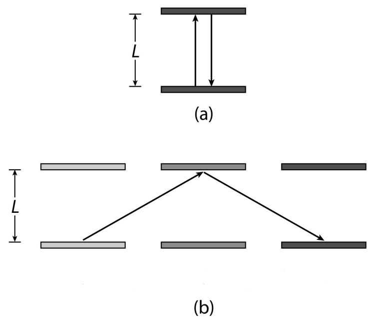

Figure 1.2.2: The mirror clock in the ground frame is show in (a). The mirror clock on a moving train as seen from the ground frame is shown in (b). (Harms 2019)

less ticks the second observer in the lab frame records. This is because it takes a longer period of time for the light to reach a moving object; when the clock moves close to the speed of light, it essentially appears that the clock has stopped ticking.

The factor of γ, pronounced gamma and called the Lorentz factor, is introduced when talking about relativity. To illustrate how γ works, let's look at a thought experiment. Say we have two mirrors, one on the ground and one above it. We send a beam off from the bottom mirror to the top mirror and back. This is illustrated from the ground frame in (a) in Figure 1.2.2[4]. Now say we set up the light mirror on a train moving at 40 mph. In the ground frame, the observer will see the light travel a greater distance, as illustrated in (b). The distance between the top mirror and the bottom mirror is given by L. As the mirrors move on the train, the light traverses a greater distance as seen from the ground frame.

Let us call event 1 the light beam shooting off from the bottom mirror and event 2 the light beam hitting the bottom mirror on descent. We can say that the relationship between the time measured between the two events in scenario (a), called t_0, and the time measured between the two events in scenario (b), called t, differ by a factor of γ as defined by

$$\Delta t = \gamma \Delta t_0 \tag{1.2.1}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, v is the speed of the object and c is the speed of light. For very small velocities, like the velocity of the train, we see that the second factor in the square root, $\frac{v^2}{c^2}$

[4]Harms, Thomas. 2019. *time dilation, space contraction.* Diagram. The Free University of Berlin. http://page.mi.fu-berlin.de/cbenzmueller/2019-Goedel/SlidesHarms.pdf

practically vanishes so the denominator becomes $\sqrt{1} = 1$ so $\gamma = 1$ and then $t = t_0$. So the time recorded on the moving clock is the same amount of time as the reference clock. As the moving clock's speed reaches the speed of light, $\frac{v^2}{c^2} \approx 1$. Then the denominator becomes smaller and smaller and γ becomes bigger and bigger. Then $t_0 < t$ as the moving clock's speed increases, implying that the units of time on the moving clock become a factor of γ longer than the units of time on the reference clock. If we measure the two events and find that it took 10 seconds on the reference clock for the beam to travel up and back down, we will measure from the ground frame that it took $\frac{10}{\gamma}$ seconds on the moving clock.

The chart below shows how γ affects the seconds on a moving clock compared to the reference clock. We will say that the time between two events in the ground frame is 10 seconds. Then the time that we measure on the moving clock, as the speed changes, will change by $\frac{\Delta t}{\gamma} = \frac{10}{\gamma}$.

Speed (meters / second)	γ	Seconds (reference)	Seconds (moving)
0 m/s	1	10	10
20 m/s	1.0000000000000022	10	10
$100,000$ m/s	1.000000056	10	10
$0.1c$ m/s	1.005	10	9.9503
$0.9c$ m/s	2.29	10	4.3368
$0.999c$ m/s	22.4	10	0.446429
c m/s	Infinite	10	0

So we can see that as the speed of the moving clock increases, the time between two events as measured on the moving clock measure *less time* than the amount of time between the two events as measured on the reference clock. We can represent this idea again by a vector. We can say that the units of time will change linearly by some amount at certain speeds. At higher speeds, the arrows will grow bigger. In other words, the arrows from Newtonian mechanics is now just magnified by some amount depending on the objects' speed. The new arrows of special relativity can be represented as shown in Figure 1.2.3. Again, the red dots represent units of time as measured by a reference clock in the ground frame and the arrows represent the change in units of time as measured by a moving clock in the inertial frame. To illustrate this idea in a clearer way, I have separated the points and their respective arrows ever so slightly in order to show the overlapping of the pink arrows with the red dots as shown in Figure 1.2.4.

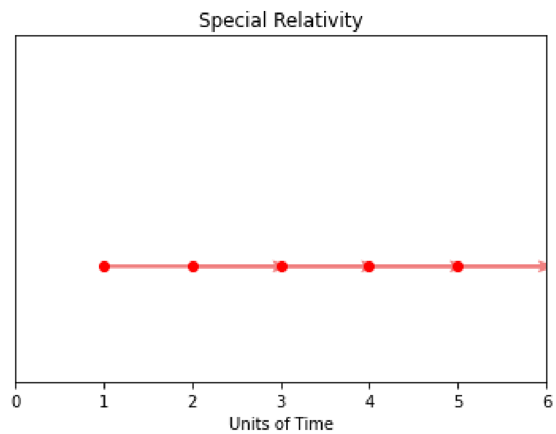

Figure 1.2.3: A model of time in the context of special relativity represented by a vector field.

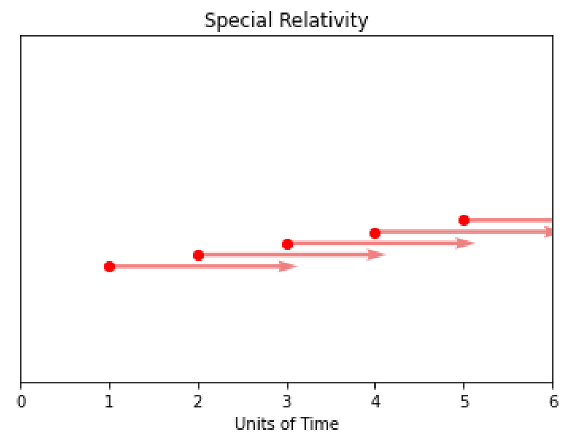

Figure 1.2.4: A model of time in the context of special relativity represented by a vector field with spaces.

Since the arrows are longer than the distance between the dots, more time will appear to pass between ticks on the moving clock compared to the reference clock. If we imagine that the two scientists record that the time between two events is 5 seconds according to the reference clock, the time between the two events as measured on the moving clock will only appear to be about 4 seconds. This is again due to the distance that the light has to travel to reach either clock.

1.2.3 General Relativity

Lastly, we will discuss general relativity. Einstein published a paper on general relativity 10 years after his paper about special relativity, which basically took his arguments about space and time and extended them to include gravity, as well. This paper concluded that massive objects, like planets, warp the fabric of spacetime and in turn create a gravitational effect. We can think about this idea like putting something on a trampoline. The amount that the trampoline bends around the object depends on the object's weight. From this, we can conclude that time acts differently depending on the objects around the area we are interested in.

As we discussed in Section 1.2.2, units of time appear grows for a clock in motion as seen from another reference frame. So in the theory of general relativity, as the gravity of an object increases, spacetime warps, and the separations in units of time increase. We can again think about a similar γ factor but in this case time is also affected by the gravitation field. What this means is that the closer a clock is to a gravitational mass, the slower the seconds will appear to tick on that clock. This theory is fascinating and can also be proven experimentally.[5] Much like time dilation in special relativity, we can characterize this gravitational time dilation by the equation

$$\Delta t = \Delta\gamma t_0 = \sqrt{1 - \frac{2GM}{rc^2}} t_0 \tag{1.2.2}$$

where G is the gravitational constant, M is the mass of the object creating the gravitational field, r is the distance from the center of the gravitational mass to the reference clock, and c is the speed of light. The Δ characterizes big changes, so the object has to be massive to see any change in the time. This factor is a little harder to explain but the idea is the same. The fraction in the square root will give something less than 1, so the square root will also always be less than 1 and $t < t_0$. We can see that our reference clock, in this case a clock not in a gravitational field, will measure some amount of time between two events. Our other clock, in this case a clock deep in a gravitational field instead of a moving clock, will measure a shorter amount of time than the reference clock between the two events.

In the context of a vector field, we can imagine general relativity like a vector field with arrows of different sizes as shown in Figure 1.2.5. The red dots show units of time on a reference clock and the pink arrows represent how the second clock will tick in warped spacetime compared to the reference clock.

Again, to make this model more clear we can add a bit of separation between the arrows as shown in Figure 1.2.6. Please note that the directionality remains the same, we are just moving the points up a tiny bit in order to see how the arrows look compared to the red dots.

The difference in the sizes of the arrows is due to the curvature of spacetime and the bending of light. We can imagine a clock moving through this warped spacetime. The duration between the units of time will change inconsistently in comparison to the reference clock. The separation between units of time varies based on the location in spacetime.

[5] The Hafele and Keating Experiment demonstrated this phenomena through an experimental lens.

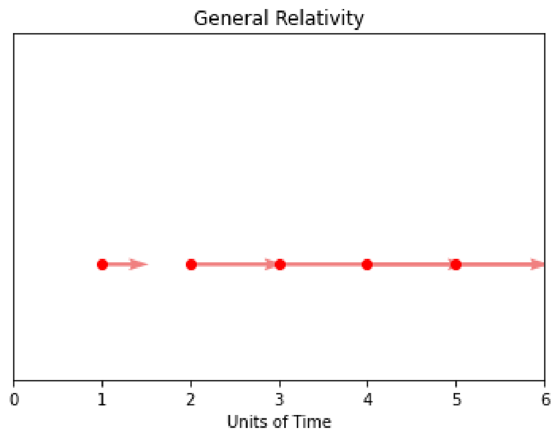

Figure 1.2.5: A model of time in the context of general relativity as a vector field.

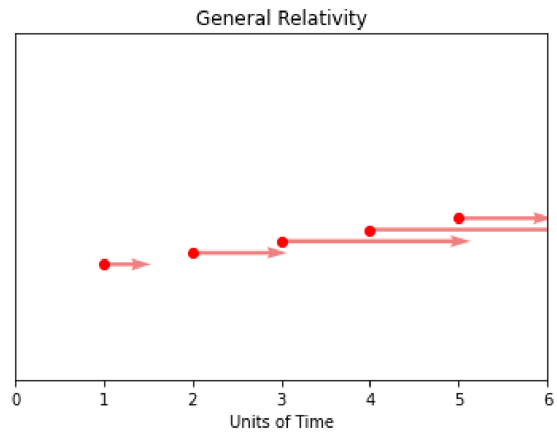

Figure 1.2.6: A model of time in the context of general relativity as a vector field with spaces.

Now that we are familiar with our three theories of time, we can discuss the role they play in the rest of the project.

1.3 Introduction to the Project

In this project, I want to explore the physical implications of these ideas by transforming a line mathematically and relating the findings back to the vector fields. We will present time as the line and explore how the transformations reflect time in different situations. We will primarily use the Mobius transformation during this research. The last chapter will focus on a counterpart to the Mobius transformation, the Schwarzian derivative. I will end the project with a few open ended questions about topics to be explored and how they tie back into time.

Let us imagine our linear line of time. Let the extension out to infinity come to a point. Now take the two points at infinity and connect them like a necklace so the line of time is now a loop, as illustrated in Figure 1.3.1. We can think of this new loop as simply a different way to view the line. We can now transform this loop mathematically on itself by moving it around and rotating it. Keeping in mind the original line that we had, we can take our new transformed circle and lay it out like a line again and compare it to our original line. We can imagine these transformations as a different way to represent the three theories above. We will then derive

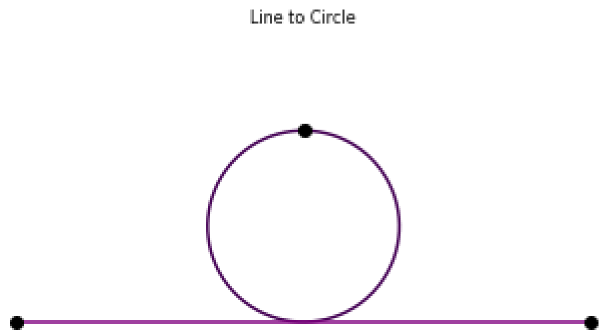

Figure 1.3.1: Changing from a line to a circle and vice versa.

properties of these transformations that can be used to capture the relationship between the transformations.

There has been research done on this topic and many of my arguments are based on papers discussing this research. But I do not intend to restate what has already been said. Instead, I want to bridge the gap between the general and the complicated by providing insight into these complex ideas in a digestible way while implementing the mathematical tools can be used to understand them.

The paper will be divided into three sections. The first section is solely background about the mathematical methods used throughout the project. We will introduce, or review, manifolds, diffeomorphisms, projective geometry, and stereographic projection. The second section will be more math-based and will walk through the actual mathematical transformations of the circle through the use of the Mobius transformation. The last section will be about the Schwarzian derivative and how it is related to the Mobius transformations and diffeomorphisms. I will then summarize the findings and provide a a basis for further exploration of this topic in the conclusion.

2
Fundamentals

Before jumping into math, it is important to understand the basic concepts that will be used throughout the project. In this section, I will give qualitative definitions and examples of manifolds, diffeomorphisms, projective geometry and stereographic projection. This chapter is geared towards readers with less mathematical background. If you feel comfortable with the concepts, please skip ahead to Chapter 2 Mobius transformation.

2.1 Manifolds

To begin, we should outline the basic definition of a manifold. A manifold is a kind of topological space, a space made up of a set of points and corresponding neighborhoods with axioms relating points and neighborhoods. In simpler terms, a neighborhood is an area around a point where you can generalize the behavior of said point. A manifold is then a type of topological space that is locally Euclidean, meaning that if we only focus on one point on the manifold, the neighborhood resembles the traditional two- or three- dimensional Euclidean space. A great way to understand this condition is to think about the earth. Since we are so much smaller than the planet, we see the earth locally. On average, the horizon is roughly 3 miles away from where you stand,[1]. So we are able to see around 27 square miles of surface area at any given point, about 0.000134% of

[1] Please note, this number changes depending on how high you are off the earth and is being taken as an average from sea level, (BBC Science Focus Magazine, April 22, 2020)

the planet. It often feels like we see much more of the earth than we actually do, which is why so many people believe that the earth is flat. When we look out at the horizon, we don't see the curve of a sphere but instead a long, flat surface as far as we can tell. That is only because we are viewing the earth on small scales, we are seeing it locally. So connecting this back to our original definition, the earth is like a manifold, our location is like a point in the manifold and the area of the earth that we see is like the neighborhood.

2.1.1 Smooth Manifolds

Now that we know what a manifold is, a kind of locally Euclidean set of points that make up a space, we can focus on a specific type of manifold called a smooth manifold. Mathematically, smooth manifolds are manifolds that are differentiable everywhere for all orders. This means that every point has infinite derivatives. In the first order, every point on the graph has a line that is tangent to it that with. You have probably seen a graph of a tangent line before but to remind readers with less experience, a tangent line at a point will look like the graph in Figure 2.1.2. The straight line is the tangent line of the point on the curve that touches the straight line. Going back to the earth example, if we zoom out and ignore all the little mountains and trenches on the earth's surface we can think of the earth as a ball. Now imagine taking a really long ruler and resting it on the earth. The ruler will touch the earth at one point then go off in a straight line. The ruler represents the line tangent to the earth at whatever point it is touching. Since the earth is like a ball in the scenario we are imagining, we can touch the ruler to the earth at every point and find a tangent line. The earth is then differentiable at every point and the first order of derivatives exist. We will find that for any sphere, like the earth, every order of derivatives exist so it is a smooth function. This definition is pretty intuitive: a manifold is smooth if the curve is continuous, if it is literally smooth, and is differentiable for all orders of derivatives.

One way to tell that a function is NOT differentiable is when there is a sharp edge or a break. For example, imagine the function $f(x) = |x|$: See Figure 2.1.1. This function is differentiable everywhere except at $x = 0$. As shown, there is a sharp point at 0 which means the derivative does not exist at that point. This is because there can be infinite tangent lines at $x = 0$ so we cannot assign a well defined tangent line. Note that there is a tangent line at every other point, but because of the lack of tangent line at $x = 0$ the line is not continuous and is therefore not a smooth manifold. Looking back at the earth example, we can imagine including all the mountains and trenches. All the sharp points on the earth's surface are *not* continuous so the manifold of the earth would *not* be smooth.

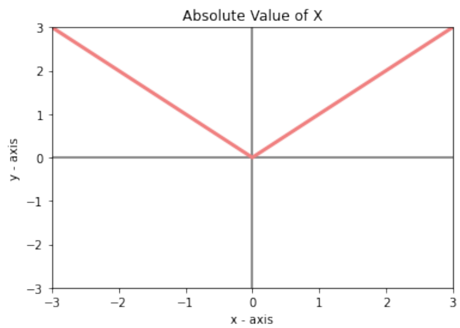

Figure 2.1.1: The function $f(x) = |x|$.

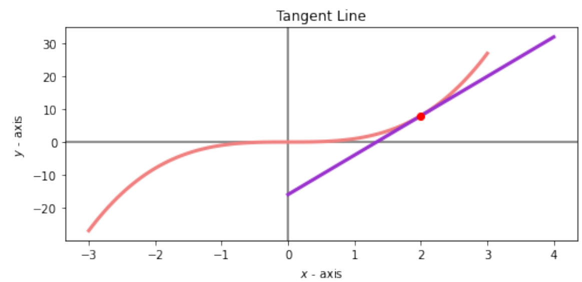

Figure 2.1.2: A graph of the tangent line of $f(x) = x^3$ at the point $x = 2$.

A function can have a first derivative, but not a second derivative. For example, we can building a function that has a derivative of $|x|$ by taking the integral. Then we will get a parametric equation whose first derivative exists at the point $x = 0$, but second does not. This function is also *not* smooth. For a smooth function, all orders of derivatives must exist.

Some good examples of smooth manifolds are a line, a circle and a sphere. A line is a smooth manifold in one dimension, a circle is a smooth manifold in two dimensions and a sphere is a smooth manifold in three dimensions. The smooth manifold that we will be focusing on moving forward will be the two dimensional circle.

2.1.2 Choice of Coordinates

Another way to study manifolds is to look at how we describe them. We are able to basically 'break' the manifold into pieces and describe those pieces with coordinate systems. Since manifolds are locally Euclidean, each piece and their corresponding coordinate system has a local description, which means it does not apply to the whole manifold but only the piece that we are concerned with. If we have a complicated manifold, we can look at individual pieces that make it up, and use whatever coordinate system would be easiest to describe each individual piece. We then are able to put all the pieces together again by 'gluing' them mathematically, which creates a function between each piece and allows us to go between coordinate systems / pieces. We can use each individual piece and coordinate system to describe a manifold as a whole.

Einstein was interested in this idea and used it as a tool in his research. One of Einstein's goals was to be able to describe spacetime without a dependency on coordinate systems. This method of breaking different manifolds into sections, studying their independent coordinate systems, and rebuilding the entire structure allowed Einstein to escape the constraints of coordinates entirely. We will discuss Einstein's use of manifolds more in Section 2.2 and will use this idea to analyze our results at the end of the paper.

2.2 Diffeomorphism

The next definition that we should outline is a diffeomorphism. Rigorously, a diffeomorphism is an isomorphism between two smooth manifolds.

There is a lot to unpack here so let's just begin with 'isomorphism.' An isomorphism is a structure-preserving mapping between two structures of the same kind that can be reversed by an inverse mapping. Let the two structures be sets of numbers, let's call them X and Y. The function f sends one group to the other, X to Y. These two groups are isomorphic if they are injective (one-to-one)[2] and surjective (onto)[3] In terms of diffeomorphisms, there is a function that maps every point of the first manifold to a point on the second and every point on the second has a corresponding point on the first. As stated above, we will be focusing on two dimensional circles, so our goal is to learn about the mappings between two circles.

One way to view diffeomorphisms is through vector fields. Electric fields are vector fields that show the force on a charge when placed in the electric field. Magnetic fields show how a particle

[2] An injective function is a function that maps each x one set of numbers to a y in another set of numbers.

[3] A surjective function is a function that maps a set of numbers to another set of numbers such that every y value is mapped to.

will move when placed in the vector field. The vector fields that we looked at in Section 1.2 are similar and show how units of time change when placed in different spacetime orientations. Those are a kind of diffeomorphism.

There are two kinds of diffeomorphisms that will be of interest in this project: active diffeomorphisms and passive diffeomorphisms. An active diffeomorphism is when you make a transformation by taking a point and moving it to another point then finding that mapping. We can think of this like picking up a glass of water and moving it from one table, point a, to another, point b, and then finding the path that the glass moved, from a to b. That path is the active diffeomorphism of the glass. This kind of transformation is illustrated in Figure 2.2.1.

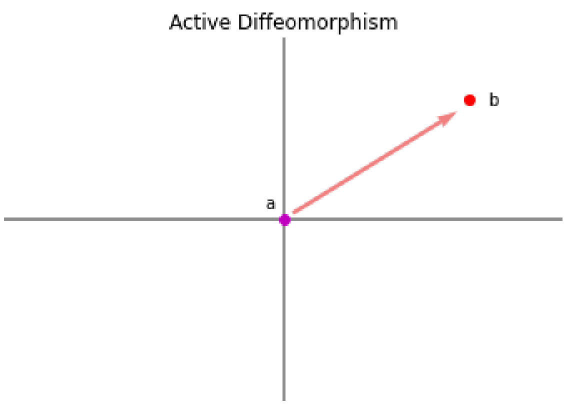

Figure 2.2.1: An active diffeomorphism.

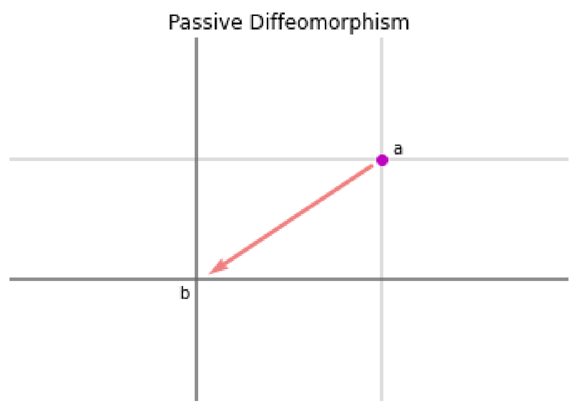

Figure 2.2.2: A passive diffeomorphism.

A passive diffeomorphism is when you move the whole coordinate system to make a transformation. Comparing it to the active diffeomorphism example, think of watching the glass of water from a satellite in space. We are standing still in space and see the earth move as normal but the glass does not move with it and instead stays in the same spot as far as we can tell from the satellite. The earth represents the coordinate system and the glass represents the point. Then the passive diffeomorphism is the movement of the coordinate system from coordinate a to coordinate b relative to the point. This kind of transformation is illustrated in Figure 2.2.2. We will focus on active diffeomorphisms throughout the project.

Einstein was interested in this characteristic of diffeomorphisms. As stated before, he wanted to find a way to describe spacetime without a dependence on the coordinate system. A theory that Einstein was particularly interested in was gauge theory,[4], which he used in his theory of relativity. He was able to make sense of spacetime by comparing gravity with gauge theory. In Einstein's theory, we can recognize gauge theory as an invariance under diffeomorphisms. In a 2011 essay, Hans Hagen Goetke explores the use of diffeomorphisms in Einstein's work. Using invariances under diffeomorphisms, Einstein was able to somewhat escape the constraints of coordinate systems. In the next few chapters, we will explore transformations and invariances and connect it back to Einstein's notion of time.

2.3 Projective Geometry

Projective geometry is a branch of mathematics that is concerned with the mapping of objects onto surfaces. We can think about this mapping like holding a flashlight up to something and analyzing the shadow it projects. This kind of mapping, from an object to its 'shadow,' can be thought of as an extension of Euclidean geometry. Originally created in the context of art during the Italian Renaissance, projective geometry became increasingly relevant in mathematics in the 18th century when mathematicians including Jean-Victor Poncelet realized a few key features of this new kind of geometry. One important feature is that geometric measurements, like angles and distances, can be ignored when analyzing projections. By doing so, projective geometry can be thought of as a kind of extension of the Euclidean plane, namely by including the point at infinity.

A projective plane contains a set of points, a set of lines and an incidence relation[5] that satisfies the following two conditions:

1. For any two points, there is one line that is incident with both of them.
2. For any two lines, there is one point where the lines intersect.

We will be using projective geometry to explore transformations of circles. Imagine holding a flash light at the top of a circle and projecting its 'shadow' to a line that runs through or below it. We can map each point where the light hits the circle to a point on the line. To find the

[4] A type of field theory in which the system does not change under local transformations.
[5] A relationship between a point lying on a line, or a line contained in a plane, etc.

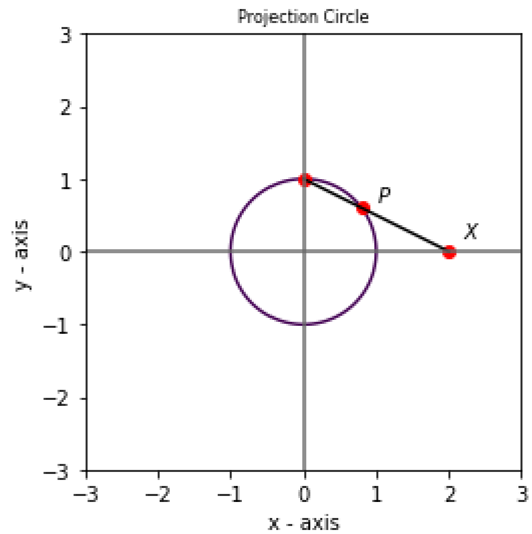

Figure 2.3.1: An example of stereographic projection of a circle.

specific points, imagine shooting a laser beam through the circle. Let the point on the circle that is hit by the beam be P and the projected point on the line, call it X. Then an example of projecting a circle down to a plane is shown in Figure 2.3.1.

This specific kind of projective geometry is called stereographic projection. As we will show in the following chapter, we can derive the one-dimensional Mobius transformation of a two dimensional circle using this process.

2.4 Stereographic Projection

Stereographic projection is a specific kind of projective geometry that maps points on a sphere to a plane, or in our case the points on a circle to a line. This type of geometry is especially important because it creates a unique relationship between the points on a circle to the points on a line. By using this, we can switch between the circle and the line using mathematical relation. We are then able to study the behavior of what is being projected in a new way and potentially discover new information about the original object.

We can imagine the laser beam shooting down from the top of the circle as shown in Figure 2.4.1. This point is called the north pole or the projection point. We can draw lines outward from the north pole that connect points on the sphere down to a plane. Imagine that the light

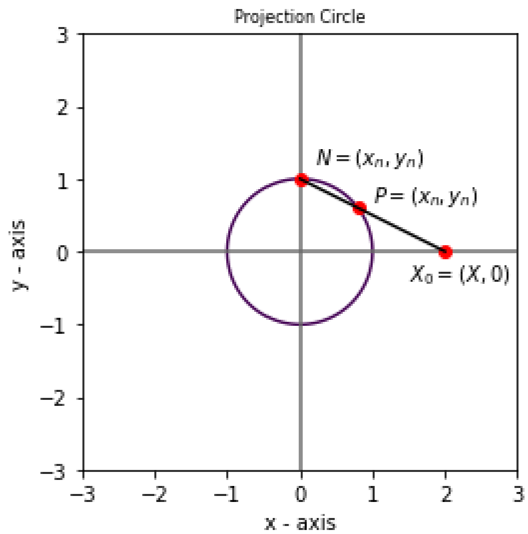

Figure 2.4.1: An example of stereographic projection of a circle.

source is a laser like before, one thin beam of light. The beam at the north pole, called N, which has coordinates (x_n, y_n), will hit exactly one point on the sphere, $P = (x_p, y_p)$ and one point on the plane, X_0. We then know that there is a relation between point P and point X_0. If we do that to every point on the sphere, we would end up with a stereographic projection of a line, like in Figure 2.4.1.

Now that we have a grasp on the techniques we will be using to look at the transformations, we can move on to the transformations themselves. In the next chapter, we will introduce and derive the Mobius transformation and relate it back to our three theories of time.

3
The Mobius Transformation

The Mobius transformation is an important mathematical tool used mostly in geometry and complex analysis. Rigorously, a Mobius transformation is a projective transformation of the complex projective line. We can think of a projective transformation in a similar way to what we discussed previously. A projection is a type of transformation from shape a to shape b. We can think of our body standing in the sunlight as shape a and our shadow that is projected down onto the ground as shape b. How the shape changes is a projective transformation.

Now we want to shift our thinking from physical objects to mathematical objects, say a straight line. That straight line is the projective line. We can think about a projective line as a normal, linear line that extend to infinity. As the line extends further and further, it 'shrinks' towards infinity. Now imagine taking the two points at infinity and connecting them to form a circle, like how we thought about clasping a necklace in Section 1.3. Now we are looking at the same line, but in a new way. The line is now a circle that contains every point in between $-\infty$ to ∞. We can find a mapping between every point on the circle and point on the line. This shift is simple yet powerful. It allows us to take an infinite line, something relatively unmanageable, and turn it into a finite circle, something that is strikingly easier to work with. Using this switch, we are able to discover new characteristics of the line that were previously hidden.

Our goal now is to find how the line and the circle are related mathematically. Just like with the shadow, we can imagine placing a light at the top of the circle. A 'shadow' of the circle will be projected down from the top and form the original line. Once we have the basic mapping between the circle and the line, we can further explore their relationship by moving the circle and seeing how its projection changes. The Mobius transformation describes what happens when we change the circle.

Mathematically, the Mobius transformation can be written as

$$M(z) = \frac{Az + B}{Cz + D} \tag{3.0.1}$$

in the complex plane, where A, B, C and D are any real numbers, and z is a changing variable, real or complex, along the circle. One requirement of the Mobius transformation is that $AC - BD \neq 0$, but we often impose the condition that

$$AC - BD = 1. \tag{3.0.2}$$

This condition is not required, but it is useful. It is like how we set the radius to 1 when studying the unit circle; it is not essential but it simplifies without losing any real generality.

We will be working exclusively with real numbers in the project so the general Mobius transformation will look like

$$M(x) = \frac{Ax + B}{Cx + D}. \tag{3.0.3}$$

This general transformation is a combination of three specific kinds of transformations. The first transformation is translation. When we translate an object, we move it up or over by some amount. The second transformation is dilation. This transformation makes the circle bigger or smaller. The last transformation is rotation. We can imagine performing this transformation by simply taking the circle and rotating it by θ. In the next section, we will be working through the derivation of these equations.

The importance of these derivations comes from how we choose to view time. During the derivations, I will be using variations of X to represent points along the projective line. Physically, the projective line corresponds to the linear line of time and the points along the projective line correspond to moments in time. At the end of the derivations, I will explain the physical implications of the transformations.

As well as the derivation of the Mobius transformation, we will be looking for specific Mobius transformations that allow all four coefficients to be unrestrained in their range. Our goal is to achieve a full Mobius transformation. It is our assumption that A, B, C and D can be any real number, so we want to find expressions that show our assumption is true.

3.1 Deriving Relations

We can derive the Mobius transformation by projecting a circle down to a plane using stereographic projection. We then transform it onto itself by changing the circle however we want as long as we preserve the circular shape. Then we perform stereographic projection once more back to the line. This will allow us to find a relation between the original X point and the $X_{transformation}$ point after the transformation.

General Relations

The most general circle is pictured in Figure 3.1.1. The figure shows a circle defined by the equation

$$(x_g - a)^2 + (y_g - b)^2 = r^2$$

where r is the radius of the circle and the circle is centered at (a, b)[1]. A projection line, called the beam, relates a point $P_g = (x_{gp}, y_{gp})$ on the circle to a point $X_g = (X, 0)$ on the projective line. We will use similar triangles to find a relation between the points. The triangle that we will use is shown in Figure 3.1.1. The points in Figure 3.1.2 show the similar triangles between north pole, $N_g = (x_{gn}, y_{gn}) = (a, r + b)$, the point where the beam intersects the circle, P_g, and the point where the beam meets the projective axis X_g.

Our goal is to find equations for x_{gp} and y_{gp} in terms of X in order to give the most general coordinates for P_g in terms of X so we can find a way to mathematically describe the relationship

[1]Note that a, b can be any real number, and r can be any real number except 0.

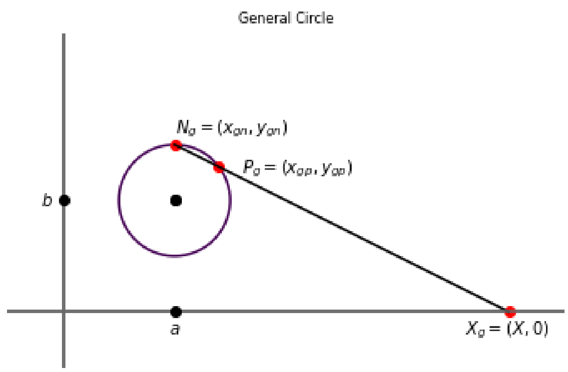

Figure 3.1.1: A circle centered at (a, b).

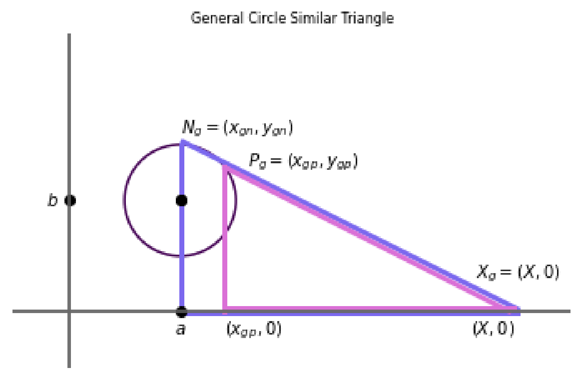

Figure 3.1.2: Similar triangles between three points and the projective axis.

between the circle and the projective line. We can then transform the circle and derive equations for $X_{transform}(X)$ that returns values along the projective axis after the transformation given initial X values.

By the similar triangles,

$$\frac{X - x_{gn}}{y_{gn}} = \frac{X - x_{gp}}{y_{gp}},$$

$$X = \frac{x_{gp}y_{gn} - x_{gn}y_{gp}}{y_{gn}y_{gp}}. \tag{3.1.1}$$

From the equation of our circle, we can find general equations for x_g and y_g.

$$x_{gp} = \sqrt{r^2 - (y_{gp} - b)^2} + a$$

$$y_{gp} = \sqrt{r^2 - (x_{gp} - a)^2} + b$$

Now we can plug these values into into equation (3.1.1) and solve for x_{gp} and y_{gp} in order to find general equations for the coordinates in terms of X. First solving for x_{gp}, we find

$$x_{gp} = \frac{a^3 - 2Xa^2 + a^2 + X^2a - ar^2 + 2Xbr + 2Xr^2}{a^2 - 2Xa + b^2 + 2br + X^2 + r^2}. \tag{3.1.2}$$

Now we can do the same thing with y_{gp}.

$$y_{gp} = \frac{(b + r)(a^2 - 2Xa + b^2 - r^2 + X^2)}{a^2 - 2Xa + b^2 + 2br + r^2 + X^2} \tag{3.1.3}$$

We now have the general equations for the coordinates of any point along the circle in terms of an X point on the projective axis.

These equations give us a general formula for finding specific translation and dilation transformations. We can now follow a similar process to find a general expression for rotation of any circle. We will use Figure 3.1.3 and Figure 3.1.4 to derive new coordinates.

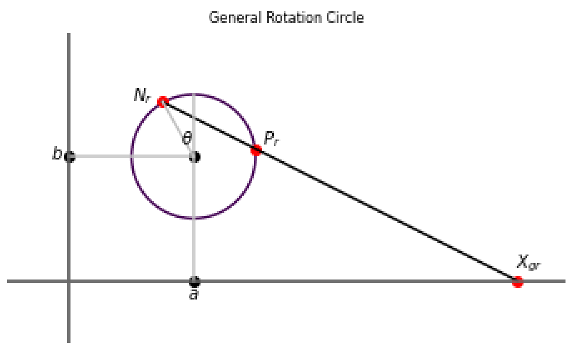

Figure 3.1.3: A circle centered at (a, b) rotated by θ.

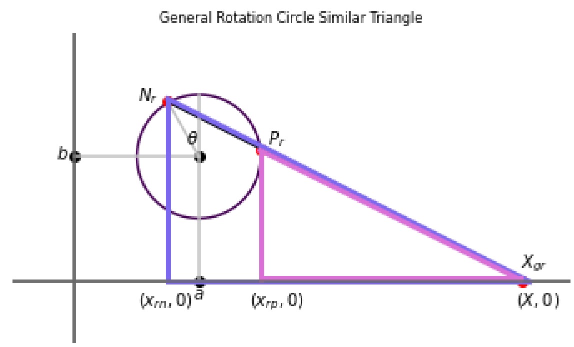

Figure 3.1.4: Similar triangles between three points and the projective axis.

The equation for our circle remains the same since the actual circle has not changed. The equation of the circle will be $(x_{rp}g - a)^2 + (y_{rp} - b)^2 = r^2$. Using the geometry of the rotated circle, we can conclude that the new north pole will have coordinates

$$N_r = (a - d\sin\theta, b + d\cos\theta).$$

Similarly, we can see that the rotated point P_r will have coordinates

$$P_r = (a + dx_{gp}\cos\theta - dy_{gp}\sin\theta, b + dx_{gp}\sin\theta + dy_{gp}\cos\theta) = (x_{rp}, y_{rp}).$$

Note that we could use a rotation matrix to find new the coordinates, but since the center of the circle is not around the origin $(0, 0)$, it is easier to use the geometry of the circle to derive the new coordinates. Now we need to plug these equations into our similar triangle relation, which will have the same structure as the general circle.

$$X_{gr} = \frac{(a + dx_{gp}\cos\theta - dy_{gp}\sin\theta)(b + d\cos\theta) - (a - d\sin\theta)(b + dx_{gp}\sin\theta + dy_{gp}\cos\theta)}{(b + d\cos\theta) - (b + dx_{gp}\sin\theta + dy_{gp}\cos\theta)}$$

$$= \frac{dX + b(\sin\theta + X\cos\theta) + a(\cos\theta - X\sin\theta)}{\cos - X\sin\theta}$$

$$X_{gr} = \frac{(d + b\cos\theta - a\sin\theta) + b\sin\theta + a\cos\theta}{-\sin\theta X + \cos\theta} \tag{3.1.4}$$

We can confirm that this works by plugging in $a = 0$, $b = 0$, $r = 1$, $\cos(0)$, and $\sin(0)$ to check that $X_{gr} = X$. This check verifies that a circle that undergoes no translation, dilation or rotation will give the X values back that we plug in.

Using this equation, we can now derive any transformation and obtain any orientation of the circle. Then we have found the most general expression of $X_{gr}(X)$. In the next few sections, I will derive all transformations and combinations of transformations and provide the circle's orientation after undergoing the transformations.

Unit Circle

To check that our equations work, we can use the simplest case of the unit circle, shown in Figure 3.1.5. The unit circle is centered at $(0, 0)$ with a radius of $r = 1$. The north pole is located at $(0, 1)$. We can plug $a = b = 0$ and $r = 1$ into equations (3.1.2) and (3.1.3) to find the coordinates of P_u in terms of X. The similar triangle corresponding to the unit circle is shown in Figure 3.1.6.

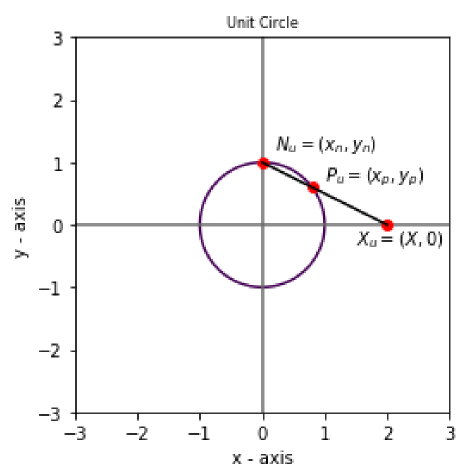

Figure 3.1.5: The unit circle centered at $(0, 0)$.

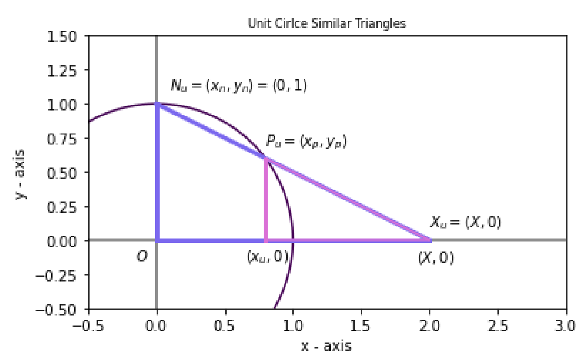

Figure 3.1.6: The similar triangle between three points and the projective axis.

We find that

$$x_p = \frac{0^3 - 2X(0)^2 + (0)^2 + X^2(0) - (0)(1)^2 + 2X(0)(1) + 2X(1)^2}{(0)^2 - 2X(0) + (0)^2 + 2(0)(1) + X^2 + (1)^2} = \frac{2X}{X^2+1}, \tag{3.1.5}$$

$$y_p = \frac{((0)+(1))((0)^2 - 2X(0) + (0)^2 - (1)^2 + X^2)}{(0)^2 - 2X(0) + (0)^2 + 2(0)(1) + (1)^2 + X^2} = \frac{X^2-1}{X^2+1} \tag{3.1.6}$$

are the general coordinates for the point P_u. We can verify that these are the correct coordinates by going through the same similar triangle argument as before. The equation of the unit circle is $x_p^2 + y_p^2 = 1$. Then

$$x_p = \sqrt{1 - y_p^2}, \tag{3.1.7}$$

$$y_p = \sqrt{1 - x_p^2}. \tag{3.1.8}$$

Then using similar triangles. we see that

$$X = \frac{x_p}{1 - y_p}.$$

Then we can plug in our expressions in equations (3.1.5) and (3.1.6) to find an equation for x_p and y_p in terms of X_u.

$$x_p = \frac{2X_p}{X_p^2+1} \tag{3.1.9}$$

$$y_p = \frac{X^2-1}{X^2+1} \tag{3.1.10}$$

The equations match so we have the correct general equations for x_p and y_p in terms of X.

Now that we have these general equations, we can perform transformations on the circle and derive equations that relate points on the projective before and after the transformations. We will be using the unit circle as a starting point when we derive the three transformations and will use the general starting point when we perform transformation combinations of the three transformations.

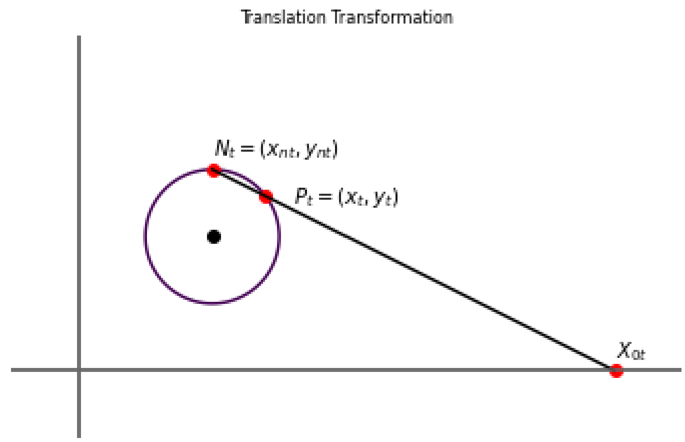

Figure 3.1.7: The unit circle after the translation transformation.

Translation

We will analyze the translation transformation shown in Figure 3.1.7. Again, this transformation simply moves the circle from its original position to a new one. The original position in this case will be the unit circle centered at the origin. The new north pole is N_t with coordinates (x_{nt}, y_{nt}), the new intersection between the beam and the circle at point P_t has coordinates (x_t, y_t), and the intersection point between the beam and the projection line is labeled X_{0t} with coordinates $(X_t, 0)$.

Please note that there are two ways to go about deriving $X_t(X)$. We can either use our general equations that we found in the last few sections and plug in certain values to find the transformation that we want or we can re-derive the expressions. I am going to re-derive the expressions.

Since this transformation is just a translation, we know that the points on the circle are going to move along the x-axis by some amount a and along the y-axis by some amount b. We can find the coordinates of the new north pole, (x_{nt}, y_{nt}) and the coordinates of the intersection point (x_t, y_t) using this relation

$$N_u = (0, 1) \to N_t = (0 + a, 1 + b) = (a, 1 + b) = (x_{nt}, y_{nt}). \tag{3.1.11}$$

$$P_u = (x_p, y_p) \to P_t = (x_p + a, y_p + b) = (\frac{2X}{X^2+1} + a, \frac{X^2-1}{X^2+1} + b) = (x_t, y_t) \tag{3.1.12}$$

We can check that these relations are correct by plugging in $a = b = 0$. We should get back our original coordinates. When we do that we see that the coordinates for the north pole are $N_t = (0, 1+0) = (0, 1)$ and the coordinates for the intersection point of the beam and the circle are$P_t = (x_p + 0, y_p + 0) = (x_p + y_p)$. So our new coordinates work!

Now we want to find an expression for X_t in terms of x and y. We can use the similar triangle relation and plug the new values in.

$$X_t = \frac{(\frac{2X}{X^2+1} + a)(1+b) + (a)(\frac{X^2-1}{X^2+1} + b)}{(1+b) - (\frac{X^2-1}{X^2+1} + b)} = (1+b)X + a \tag{3.1.13}$$

This is one kind of Mobius transformation! Certain values can be plugged in for the coefficients A, B, C, and D to find this specific Mobius transformation. We can see that when we plug $A = (1+b)$, $B = a$, $C = 0$ and $D = 1$ into $\frac{Az+B}{Cz+d}$ we get equation (3.1.13).

We can find the range of the two coefficients by looking at the possible values. We see that $a \in \mathbb{R}$[2] and $b \in (-1, \infty)$. Then we have found an unrestricted value for B with $B = a$.

Dilation

Next, we will derive the Mobius transformation for dilation shown in Figure 3.1.8. On the graph, we see that the orientation of the circle has not changed, but the coordinates have. As before, the new north pole is labeled N_{nd} and has coordinates (x_{nd}, y_{nd}), the point of intersection between the beam and the circle is at $P_d = (x_d, y_d)$, and the point where the beam intersects the projective line is $X_{0d} = (X_d, 0)$.

Just looking at the graph, we can see that, since a dilation just multiplies the original graph by a variable, the new coordinates are just the old ones multiplied by some variable d. Then we see that the new north pole has coordinates $N_{nd} = (d)(0,1) = (0,d) = (x_{nd}, y_{nd})$ and the new intersection point P_d has coordinates $P_d = (d)(x_p, y_p) = (dx_p, dy_p) = (x_d, y_d)$. We can again

[2] This notation is often used in mathematics. This symbol represents all real numbers from $-\infty$ to ∞.

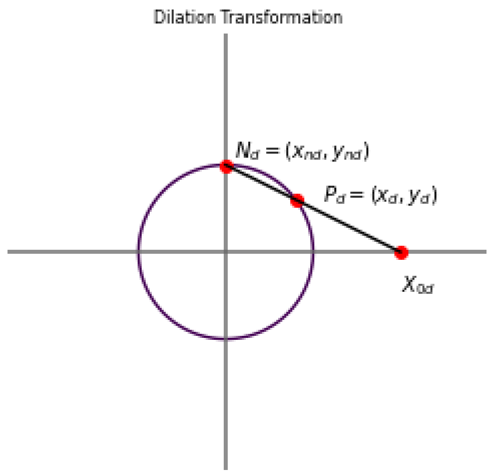

Figure 3.1.8: The unit circle after the dilation transformation.

check that these equations work by plugging in $d = 1$ and seeing if we get our original N and P coordinates back.

Next we can follow a similar geometric process to find an equation for X_d in terms of x_d and y_d then plug in our known values.

$$X_d = \frac{y_{nd}x_d}{y_{nd} - y_d}$$

Now plugging in our values,

$$X_d = \frac{(d)(dx_p)}{d - dy_y} = \frac{d^2x_p}{d(1 - y_p)} = \frac{d(\frac{2X}{X^2+1})}{1 - (\frac{X^2-1}{X^2+1})} = dX \tag{3.1.14}$$

which is, once again, a simple Mobius transformation. We can plug in $A = d$, $B = 0$, $C = 0$ and $D = 1$ to obtain this specific Mobius transformation. We see that the range of d is $d \in (0, \infty)$. Then the variable $A = d$ is still restricted.

Rotation

The last transformation we will analyze is a rotation. As stated previously, a rotation can be performed by simply taking the circle and rotating it about the origin by θ. This process is depicted in Figure 3.1.9. The purpose of θ is to illustrate the amount that we rotate our reference circle.

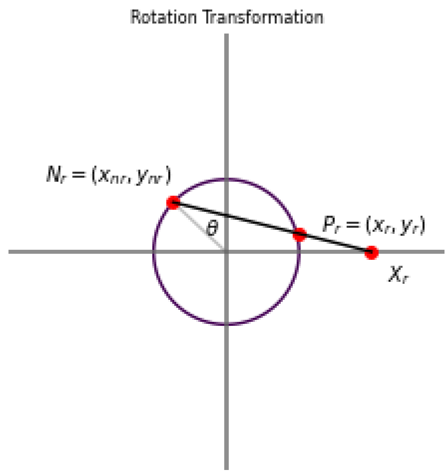

Figure 3.1.9: The unit circle after the rotation transformation.

We will walk through a similar process as before in order to find the new relationship between X_r and $P_r = (x_r, y_r)$ that allows us to find X_r given x_r and y_r using the geometry of the new circle. We can use the same triangle relation:

$$X_r = \frac{x_r y_{nr} - x_{nr} y_r}{y_{nr} - y_r}$$

We can check that this relation is correct by plugging in $x_{nr} = \cos(0) = 1$ and $y_{nr} = \sin(0) = 0$. We get back our original north pole coordinates. Then we get back the north pole of the unit circle, our original unit circle value for X in terms of x and y.

A rotation matrix can be used to find the coordinates for the new north pole, X_{nr}, and the new coordinates for P_r at (x_r, y_r). To find the new north pole, we take the old coordinates, $N_u = (0, 1)$ and multiply them by the rotation matrix

$$N_u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} x_{nr} \\ y_{nr} \end{pmatrix}. \tag{3.1.15}$$

We can now do a similar process to find the new coordinates (x_r, y_r).

$$P_r = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{2X}{X^2+1} \\ \frac{X^2-1}{X^2+1} \end{pmatrix} = \begin{pmatrix} \frac{2X\cos\theta - X^2\sin\theta + \sin\theta}{X^2+1} \\ \frac{2X\sin\theta + X^2\cos\theta - \cos\theta}{X^2+1} \end{pmatrix}$$

$$P_r = \begin{pmatrix} x_r \\ y_r \end{pmatrix} \tag{3.1.16}$$

Now using our new coordinates, we will plug (x_r, y_r) and (x_{nr}, y_{nr}) back into the equation we found for X_r.

$$X_r = \frac{x_r y_{nr} + x_{nr} y_r}{y_{nr} - y_r} = \frac{(\frac{2X\cos\theta - X^2\sin\theta + \sin\theta}{X^2+1})(\cos\theta) + (-\sin\theta)(\frac{2X\sin\theta + X^2\cos\theta - \cos\theta}{X^2+1})}{(\cos\theta) - (\frac{2X\sin\theta + X^2\cos\theta - \cos\theta}{X^2+1})}$$

$$X_r = \frac{X}{\cos\theta - X\sin\theta} \tag{3.1.17}$$

This equation is a Mobius transformation! If you don't see it, try plugging in values for A, B, C, and D and see if you can get it into the form that we found. It works!

Looking at the possible values of θ, we see that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then $C = -\sin\theta$ and $D = \cos\theta$ are still restricted. This means that, although we did find a Mobius transformation, we have not yet found a version of the Mobius transformation without restrictions. We can also combine transformations. This may allow us to find a general Mobius transformation without restrictions.

3.2 Exploring Combinations

We will begin with combining two of the transformations and then combine three. I am going to begin with dilation and translation. Recall that our dilated circle will look like Figure 3.2.1 and have the coordinates $N_t = (a, 1+b)$ and $P_t = (x_p + a, y_p + b) = (x_t, y_t)$.

We can not perform the dilation transformation on this circle. It is much easier to move the circle back to the origin, dilate, and then move it back than to dilate around the point (a, b). Our new coordinates will be $N_{D(T))} = (d(a-a)+a, d(1+b-b)+b) = (a, d+b)$ and $P_{D(T))} = (d(x_p + a - a) + a, d(y_p + b - b) + b) = (dx_p + a, dy_p + b)$. The circle now looks like Figure 3.2.2

Now we will use the similar triangles relation to derive a formula for $X_{D(T)}(X)$, substituting equations (3.1.5) and (3.1.6) in for x_p and y_p.

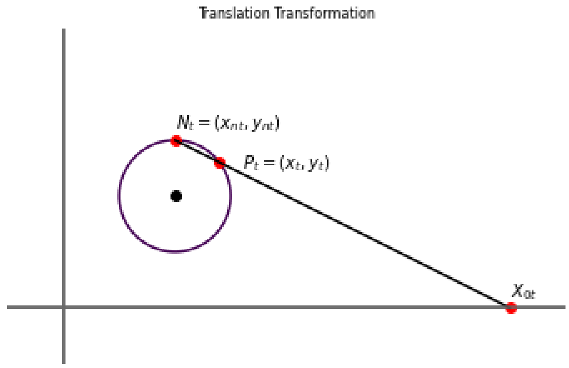

Figure 3.2.1: The unit circle after the translation transformation.

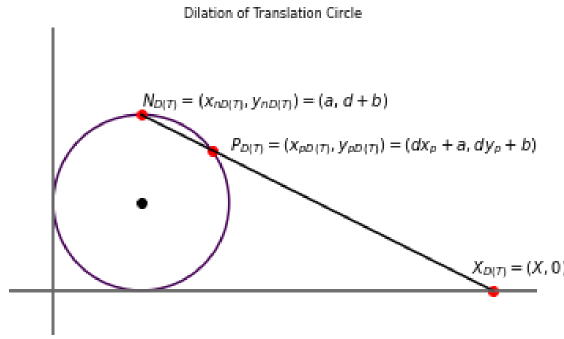

Figure 3.2.2: The unit circle after the dilation of translation transformation.

$$X_{D(T)} = \frac{x_p y_n - x_n y_p}{y_n - y_p} \tag{3.2.1}$$

$$= \frac{(dx_p + a)(d + b) - (a)(dy_p + b)}{(d + b) - (dy_p + b)} \tag{3.2.2}$$

$$= (d + b)X + a \tag{3.2.3}$$

We see that $a \in \mathbb{R}$ and $b \in (0, \infty)$. Then this gies an expression for a general A but a restricted B. Our range is also the same as the two separate transformations. As a way to know which function goes where, I will write the future equations like Dilation(Translation) or D(T). This is often said "D of T" or "Dilation of Translation," meaning we are performing the dilation transformation on the translation transformation.

We can follow this same process for the rest of the combinations. The combinations and their equations into the table below.

Combination	$\text{X}_{transform}(X_u)$
$T(D)$	$(b + d)X + a$
$R(D)$	$\frac{(1+b)X+a}{-\sin\theta X+\cos\theta}$
$D(R)$	$\frac{(1+b)X+a}{-\sin\theta X+\cos\theta}$
$T(R)$	$\frac{(1+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$R(T)$	$\frac{(1+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$

The rotation of translation transformation is an interesting combination because it should allow us to rotate the circle all the way around. If we think about it, our rotation is restricted because our north pole can approach, but can't touch, the projective axis. If we first move the circle away from or off of the axis, our circle should be able to rotate all the way around. So our θ is unrestricted for $R(T)$. Unfortunately because nothing is multiplying $\cos\theta$ and $\sin\theta$, our $C = -\sin\theta$ and $D = \cos\theta$ are still restricted.

We want to find all the coefficients A, B, C and D of the Mobius transformation into a form such that there are no restrictions on them. From Rotation and Translation,

$$T(R) = \frac{(1 + b\cos\theta - a\sin\theta)X + b\sin\theta + a\cos\theta}{-\sin\theta X + \cos\theta}$$

The distinct coefficience are $A = [1 + b\cos\theta - a\sin\theta]$, $B = [b\sin\theta + a\cos\theta]$, $C = [-\sin\theta]$ and $D + [\cos\theta]$. Since $a \in \mathbb{R}$, we know that coefficients A and B have unrestricted ranges.

Now we can try combining all three in different ways following a similar process. Please note that I am skipping the derivations but the equations follow the same process as the combinations of only two Mobius transformations.

Combination	$\mathrm{X}_{transform}(X)$
$T(D(R))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$T(R(D))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$D(T(R))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$R(D(T))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$D(R(T))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$
$R(T(D))$	$\frac{(d+b\cos\theta-a\sin\theta)X+b\sin\theta+a\cos\theta}{-\sin\theta X+\cos\theta}$

The combination of the transformations gives the same equation for every combination. Note that the process followed the same steps with a different order of transformations so the equations along the way varied based on the order that we performed the transformations.

We can see that the A, B, C, and D values are consistent for these general Mobius transformations. $A = d + b\cos\theta - a\sin\theta$, $B = b\sin\theta + a\cos\theta$, $C = -\sin\theta$, and $D = \cos\theta$. The values for A and B are unrestricted but the values of C and D are restricted by θ. In theory, we should be able to find a combination that gives unrestricted C and D values. This is one aspect of the transformations that we did not uncover in the project and that needs to be further explored.

The next thing we are going to do is find the vector fields for these equations and relate them back to our notions of time from Section 1.2.

Finding Mobius Vectors

We can now take the derivations that we found in Section 3.2 and connect them back to the vector fields of time from Section 1.2. In this section, we will be deriving the vector fields for the Mobius transformations that we found in the last section.

How to Find Vectors

Rigorously, a vector is a quantity with magnitude and direction. For example, velocity is a vector but speed is not. Velocity tells you the speed at which an object is travelling as well as the direction. How do we find that vector? Say we are at position a and we want to know the velocity at which we are traveling. We can calculate how much distance we cover over a certain period of time by finding the distance between two points and seeing how long it takes to travel from point a to point b. Another way to find the velocity vector is by taking the derivative of position. One characteristic of derivatives is that they tell us how much a function is changing at a point. This will be a key feature that we will use while deriving the vectors for the different Mobius transformations. Once we find the vectors, we can find their vector fields. These fields will tell us how the behavior of the line changes when undergoing these transformations, which corresponds to time.

We can now derive the vectors for the translation, dilation, and rotation transformations. Please note that for the sake of simplicity I will continue to use X, but as we begin to unpack the physical meaning behind the math we will come to see that X represents the time that we measure on our clocks.

3.2.1 Translation

Recall that the Mobius transformation that we found for our translation transformation is

$$X_t = a + X(1 + b).$$

The derivative of this equation, denoted by X_t', is

$$X_t' = 1 + b. \tag{3.2.4}$$

We can now find the vector field of this equation. We can do this by plugging in different values of X and solving for X'. As we can see, X' is completely independent of X and will therefore be unaffected by it. On the other hand, the equation is dependent on the value b. Recall that b tells us how far up our circle moves from the origin.

Suppose we move our circle up by $b = 2$. Then

$$X_t' = 1 + b = 1 + 2 = 3.$$

Then we see that for all values of X, $X_t' = 3$.

Let's run through a full example of the transformation and how to find its vector field. First we should transform the circle and plot the new points. Let us take 5 points on the projective axis, say -2, -1, 0, 1, and 2. We will move the unit circle centered at $(0, 0)$ over by $a = 2$ and up by $b = 2$. The circle before the transformation and our X values are shown in Figure 3.2.3. When we move the circle over by $a = 2$ and up by $b = 2$, the graph will look like Figure 3.2.4. I am going to skip the steps where I derive the equations of the lines, but I will provide them in the table below. I used the point at the north pole, in this case $(2, 3)$, and the five distinct points along the projective axis given by the X_t equation to find the equations of the lines for each point.

X	X_t	Line Equation
-2	-4	$y_{-2} = \frac{1}{2}x + 1$
-1	-1	$y_{-1} = x + 1$
0	2	2
1	5	$y_1 = -x + 1$
2	8	$\mathrm{y}_2 = -\frac{1}{2}x + 1$

Notice that all the points projected down to the projective line are separated by a distance of 3. This is directly related to how much we move the circle off the projection line. Note that when we move the circle left or right by any factor of a, the distance between the points remains the same.

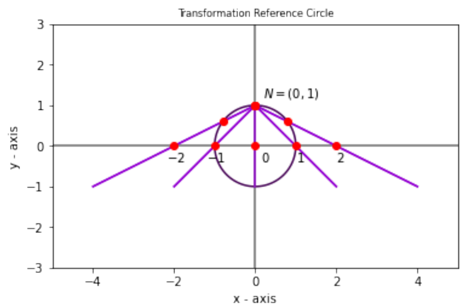

Figure 3.2.3: The unit circle with beams from the north pole to the projective line.

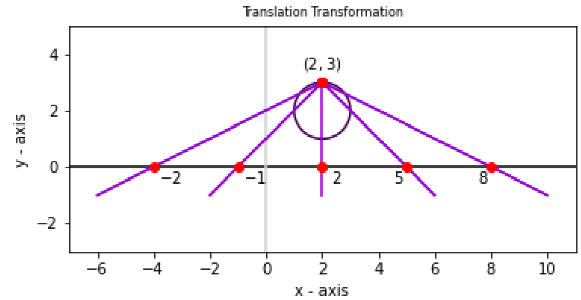

Figure 3.2.4: The unit circle after undergoing a translation transformation. The circle is moved over by $a = 2$ and up by $b = 2$.

Then our vector field will look like Figure 3.2.5. Like before, we can add a little distance between the points in order to see how the arrows are related to the red dots. This is shown in Figure 3.2.6. We can now relate our findings back to time.

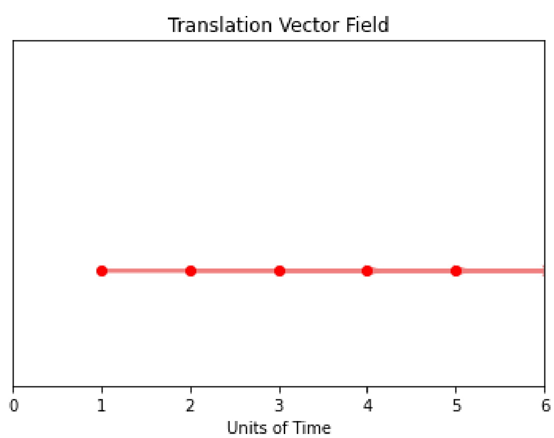

Figure 3.2.5: A vector field of the translation transformation. The arrows have a length of $(1 + b) = 3$.

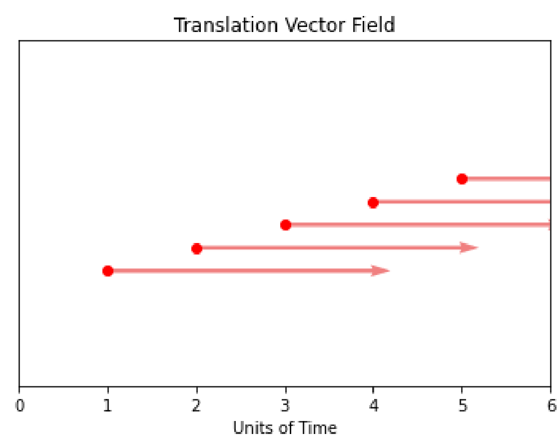

Figure 3.2.6: A vector field of the translation transformation with spaces.

Does this vector field look familiar? This vector field closely resembles the vector field of special relativity. Then this vector field shows that the distance between units of time can be mathematically represented by the distance from the north pole to the projective axis. The separation between units of time will always be $1 + b$, as given by equation (3.2.4). So this translation transformation represents a clock under special relativity. This means the moving clock represented in the vector fields could be traveling at high speeds or could be at have a different gravitational potential[3].

3.2.2 Dilation

We can follow a similar process with dilation. Recall that the Mobius transformation for dilation is

$$X_d = dX.$$

Then the derivative will be

$$D'_d = d. \tag{3.2.5}$$

So our vector for dilation is $X'_d = d$. Again, we see a lack of X dependence. Then the separation between points is solely dependent on d, the amount that we dilate the circle.

We can see how this circle looks after the dilation transformation by going through a similar example. Let's dilate the circle by $d = 2$ and use the points $X = [-2, -1, 0, 1, 2]$ as before. See Figure 3.2.3 for what our circle will look like before our transformation. We can plug the starting X values into our equation for X_D in order to find the points along the projective axis after the transformation and then use those points and the point at the north pole to find the equations of the beams. I will leave it to the reader to go through these steps and will instead give you the new points and graph. Then the circle after the dilation transformation will look like Figure 3.2.7.

X	X_d	Line Equation
-2	-4	$y_{-2} = \frac{1}{2}x + 2$
-1	-2	$y_{-1} = x + 2$
0	0	0
1	2	$y_1 = -x + 2$
2	4	$\mathrm{y}_2 = -\frac{1}{2}x + 2$

[3]In special relativity, time dilation can occur due to either a difference in speeds of two clocks or a difference in gravitational potential. A clock at high speeds will seem to run slow, as well as a clock high off the Earth's surface.

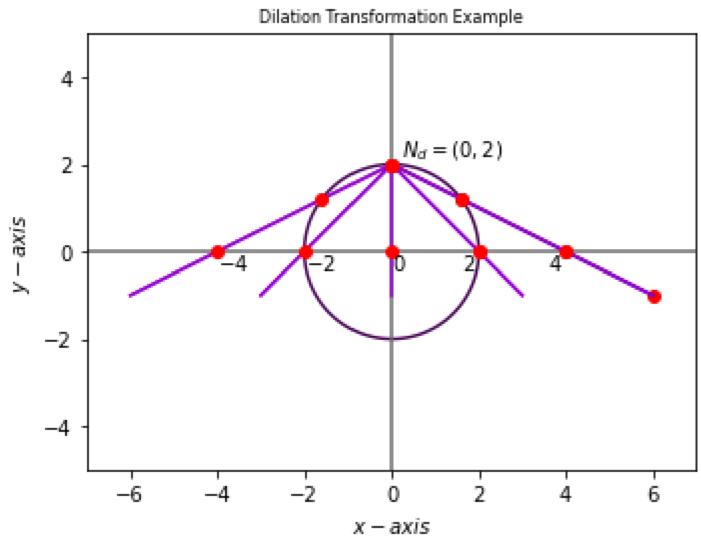

Figure 3.2.7: The unit circle after undergoing the dilation transformation. The circle is dilated by $d = 2$.

We can clearly see that the separation between our points is 2, which is the value we chose for d. So the distance between the points is directly related to how much we dilate the circle.

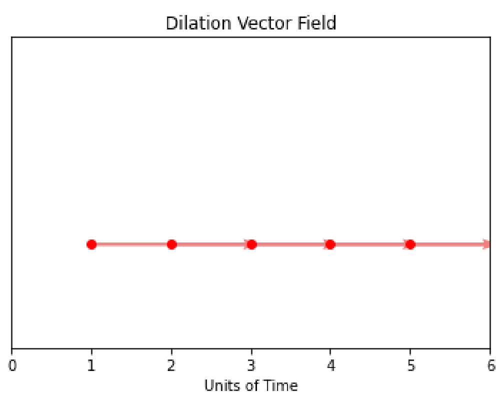

Figure 3.2.8: A vector field of a dilation transformation. The arrows will have a length of $d = 2$.

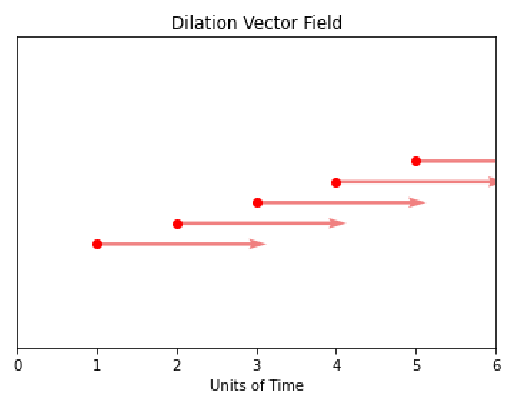

Figure 3.2.9: The vector field of the dilation transformation with spaces.

Our vector field for this transformation will then look like Figure 3.2.8. Physically, this has a very similar meaning as the translation vector field. We can again think of the red dots in the vector field representing units of time on our reference clock and the arrows as the change in units of time on a moving clock in comparison to a reference clock. Then we see that the way that units of time change on the moving clock are dependent on d, the dilation of our circle. So we can think of this as representing special relativity. Again, this could imply that the moving clock is moving at very high speeds or could be far off the earth's surface.

3.2.3 Rotation

We can now look at the rotation vector. Recall that the Mobius transformation for rotation is

$$X_r = \frac{X}{\cos\theta - X\sin\theta}.$$

Then the derivative is

$$X_r' = \frac{\cos\theta}{(X\cos\theta - \sin\theta)^2} \tag{3.2.6}$$

This derivative is much more complicated than the other two, but can be understood by plugging in terms for X, $\sin\theta$, and $\cos\theta$.

For example, let's plug in the same values, $X = [-2, -1, 0, 1, 2]$, and use $\theta = \frac{\pi}{6} = 30°$. Then the values of X_D are shown in the table below. Please note that I have rounded the values to the fifth decimal place, so our graphs will not be exact but will be close enough to see what is happening.

X	X_d	Line Equation
-2	-1.071	$y_{-2} = 1.51457x + 1.62331$
-1	-0.732	$y_{-1} = 3.73205x + 2.73205$
0	0	$y_0 = -\sqrt{3}x$
1	2.732	$y_1 = -0.26795x + 0.73205$
2	-14.928	$y_2 = 0.06002x + 0.89604$

We can now graph the equation lines that we have found on our circle and see how it will look. So our new graph is shown in Figure 3.2.10. We can see that the distance between the points varies unlike before. This is because of the X dependence in the denominator of our X_r' equation. Then our vector field will look like Figure 3.2.11.

Please note that we are using estimations to produce what our vector field will look like. Rigorously, the process of finding the actual vectors is more complicated than before. To be accurate, we have to take infinitesimally close points. The closer the points are, the more

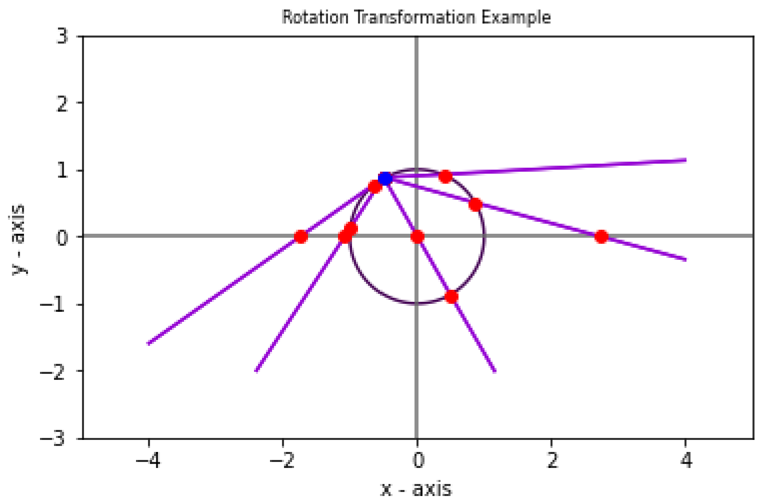

Figure 3.2.10: The unit circle after the rotation transformation. The circle is rotated by $\theta = \frac{\pi}{6}$.

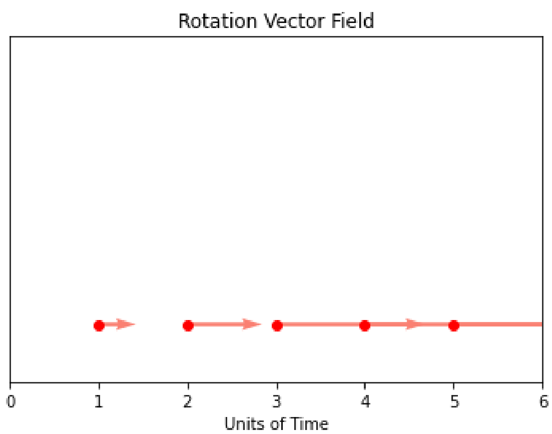

Figure 3.2.11: A vector field of the dilation transformation.

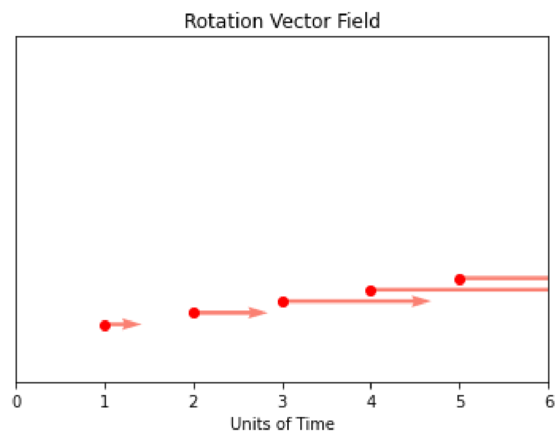

Figure 3.2.12: A vector field of the dilation transformation with spaces.

accurate the vector will be but the smaller it will get, as well. Unfortunately, making a precise vector field for this equation would prove challenging. We will briefly go over how the process of finding the vectors works in a few sections, but since the accuracy is dependent on how close the points are, and therefore how small the vector is, I will not include a vector field for the precise values.

Then we see that this vector field fits best in the model from general relativity. The changes in time as measured on the moving clock by the reference clock are not equidistant. Then this transformation could represent some form of gravitational time dilation due to the warping of spacetime. The amount that we rotate the circle corresponds to different factors in physical situations, like gravity, mass or speed. The amount that we rotate the circle corresponds to an orientation in spacetime.

We can now find the vector fields for the combined transformations.

3.2.4 Combination Vector Fields

We can now find the vector fields for our combination equations. Since this process is quite tedious, I am going to provide a chart with the equation lines and vector fields for the equations with the same starting conditions as before. We will set θ to $\frac{\pi}{6}$ and use points $X = [-2, -1, 0, 1, 2]$.

Combinations of Two

Transformation	Transformation Equation	Vector Equation	Figure Number
Rotation of Translation	$\frac{X_u(1+b\cos\theta-a\sin\theta)+b\sin\theta+a\cos\theta}{X_u-\sin\theta+\cos\theta}$	$\frac{b+\cos\theta}{(x\sin\theta-\cos\theta}$	3.2.13, 3.2.14
Translation of Rotation	$\frac{X_u(1+b\cos\theta-a\sin\theta)+b\sin\theta+a\cos\theta}{-X_u\sin\theta+\cos\theta}$	$\frac{b+\cos\theta}{(X_u\sin\theta-\cos\theta}$	3.2.15, 3.2.16
Rotation of Dilation	$\frac{X_u(1+b)+a}{-X_u\sin\theta+\cos\theta}$	$\frac{d\cos\theta}{(X_u\sin\theta-\cos\theta)^2}$	3.2.17, 3.2.18
Dilation of Rotation	$\frac{X_u(1+b)+a}{-X_u\sin\theta+\cos\theta}$	$\frac{d\cos\theta}{(X_u\sin\theta-\cos\theta)^2}$	3.2.19, 3.2.20
Translation of Dilation	$\frac{a+X_u+bX_u}{\cos\theta-\sin\theta}$	$b+d$	3.2.21, 3.2.22
Dilation of Translation	$\frac{a+X_u+bX_u}{\cos\theta-\sin\theta}$	$b+d$	3.2.23, 3.2.24

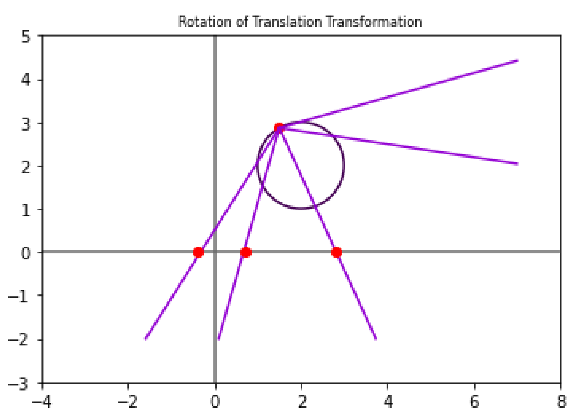

Figure 3.2.13: The unit circle after the rotation of translation transformation.

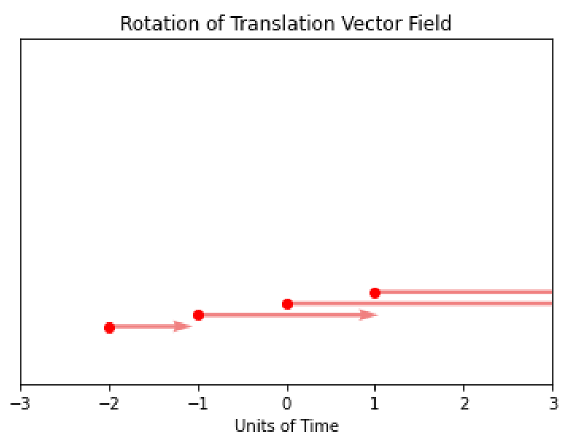

Figure 3.2.14: The vector field of the rotation of translation transformation with spaces.

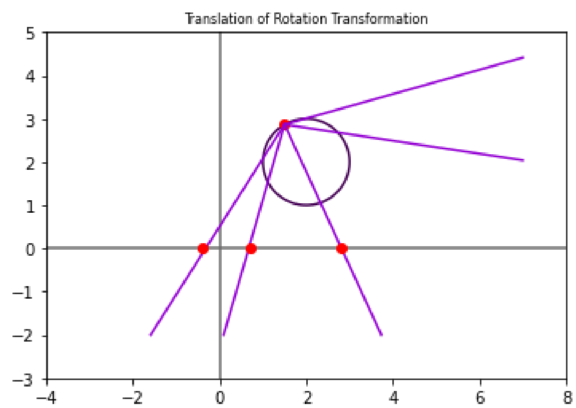

Figure 3.2.15: The unit circle after the translation of rotation transformation.

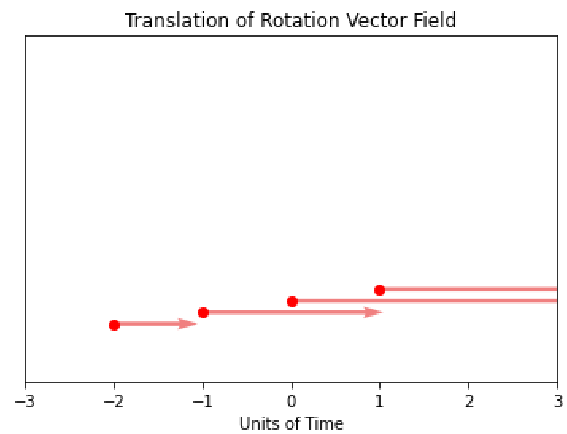

Figure 3.2.16: The vector field of the translation of rotation transformation with spaces.

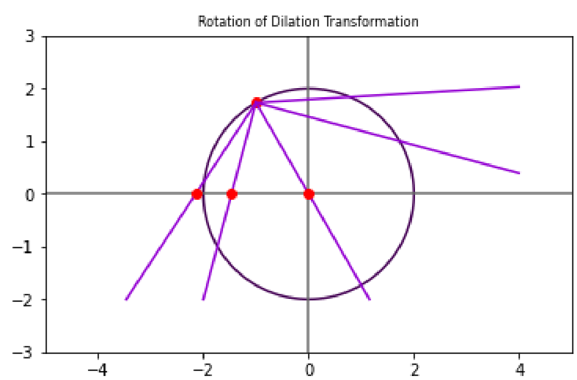

Figure 3.2.17: The unit circle after the rotation of dilation transformation.

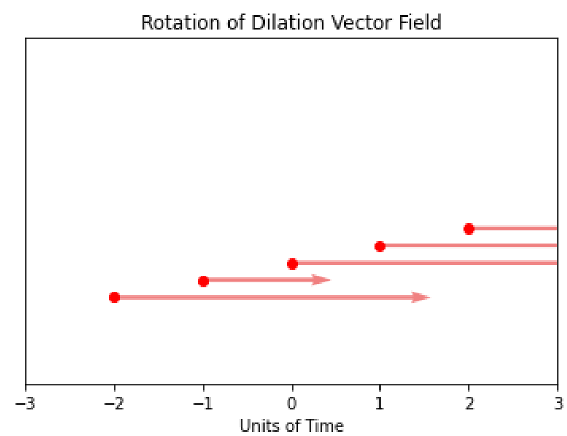

Figure 3.2.18: The vector field of the rotation of dilation transformation with spaces.

Combinations of Three

Transformation	Transformation Equation	Vector Equation	Figure Number
$R(T(D))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.25, 3.2.26
$T(R(D))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.27, 3.2.28
$R(D(T))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.29, 3.2.30
$D(R(T))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.31, 3.2.32
$T(D(R))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.33, 3.2.34
$D(T(R))$	$\frac{(d+b\cos\theta-a\sin\theta)X_u+b\sin\theta+a\cos\theta}{-\sin\theta X_u+\cos\theta}$	$\frac{b^2\sin\theta+b\cos\theta\sin\theta+d\cos\theta}{(\sin\theta X_u-\cos\theta}$	3.2.35, 3.2.36

As we can see, most of the fields are comparable to the vector fields of general relativity. The vector fields of functions *without* X dependence in their derivative are similar to the special

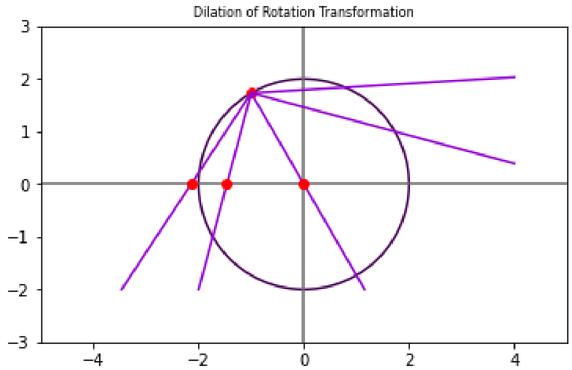

Figure 3.2.19: The unit circle after the dilation of rotation transformation.

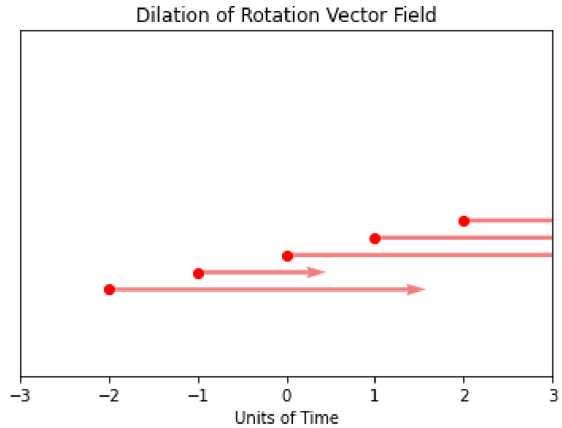

Figure 3.2.20: The vector field of after the dilation of rotation transformation with spaces.

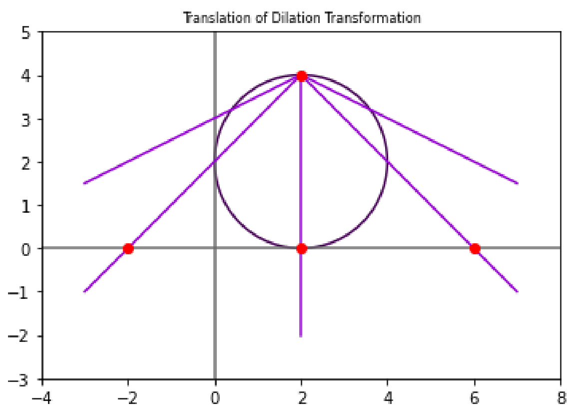

Figure 3.2.21: The unit circle after the translation of dilation transformation.

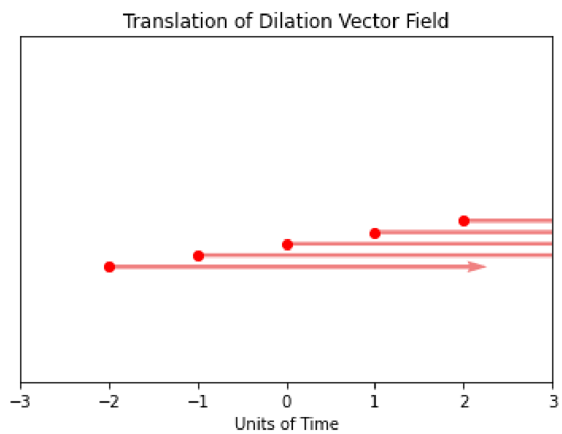

Figure 3.2.22: The vector field of the translation of dilation transformation with spaces.

relativity vector fields and the vector fields for functions *with* an X dependence correspond to the general relativity vector fields. This dependence is caused by the rotation transformation. The rotation transformation provides a mathematical interpretation of the warping of space-time, while the translation and dilation transformations of the circle represent time dilation in relativity.

We are then able to describe these two different kinds of time dilation through a mathematical depiction, either as a vector field or as a circle. This gives us two new ways to describe time. The importance of this ties back to diffeomorphisms and coordinate systems. As we discussed previously, one way to represent diffeomorphisms is through vector fields, like the electric or

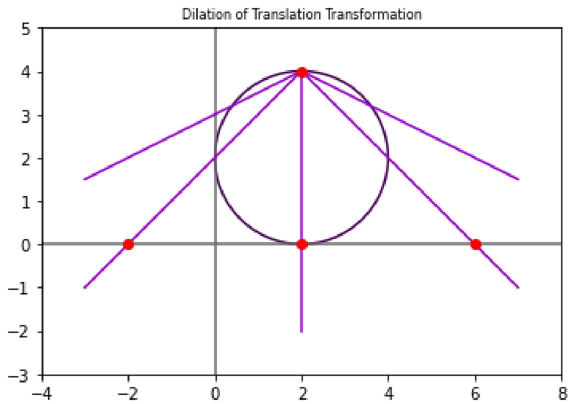

Figure 3.2.23: A vector field after the dilation of translation transformation.

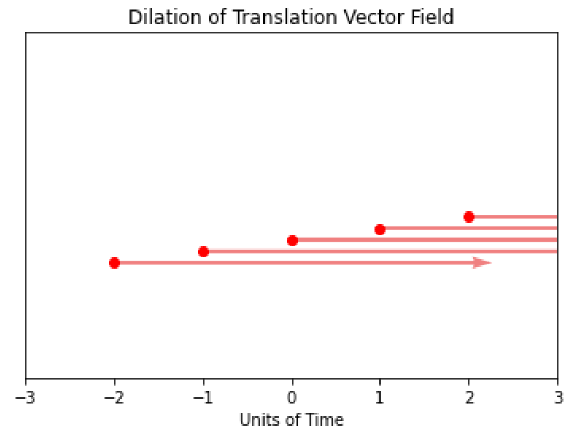

Figure 3.2.24: The vector field of the dilation of translation transformation with spaces.

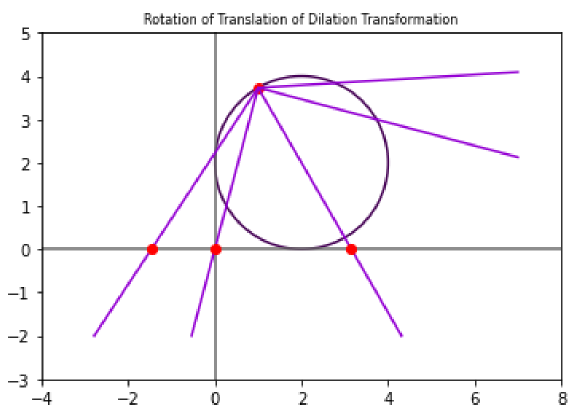

Figure 3.2.25: The unit circle after the rotation of translation of dilation transformation.

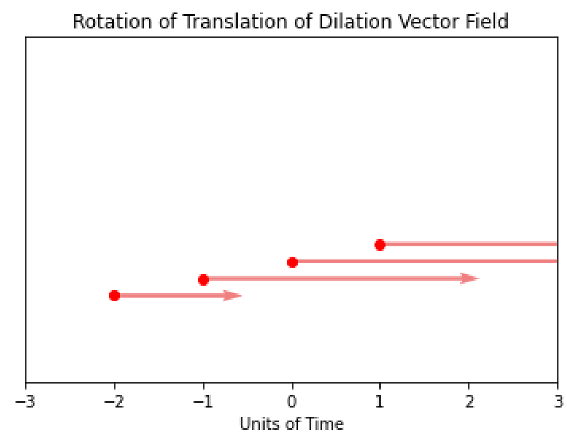

Figure 3.2.26: The vector field of the rotation of translation of dilation transformation with spaces.

magnetic fields. We can interpret active diffeomorphisms as vector fields that describe how a point is going to change in relation to a coordinate system and passive diffeomorphisms as vector fields that describe how a coordinate system will change in relation to a point. We have found a way to use vector fields to describe the dilation of time, which then allows us to describe time as diffeomorphisms. The algebra of transformations is an entire branch of mathematics that will not be deeply explored in this paper but that plays an important role in this project. In the case of diffeomorphisms, this branch is called Virasoro algebra.

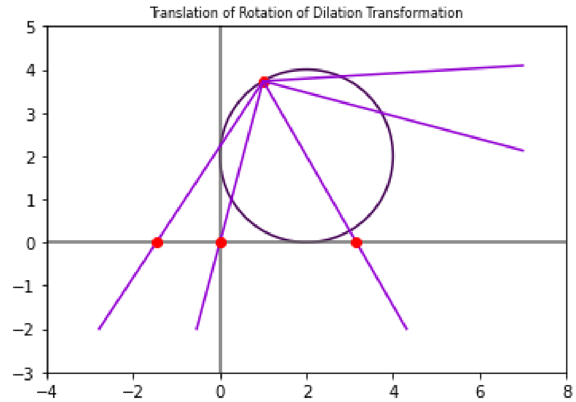

Figure 3.2.27: The unit circle after the translation of rotation of dilation transformation.

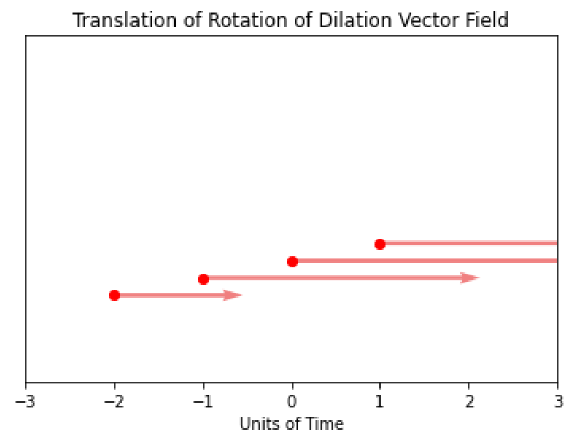

Figure 3.2.28: The vector field of the translation of rotation of dilation transformation with spaces.

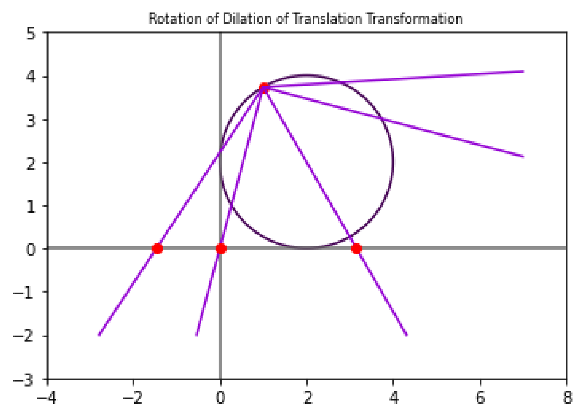

Figure 3.2.29: The unit circle after the rotation of dilation of translation transformation.

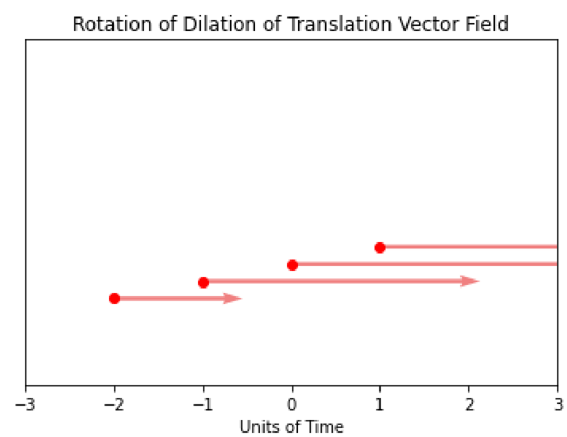

Figure 3.2.30: The vector field of the rotation of dilation of translation transformation with spaces.

Before we move on, I mentioned after the rotation vector field that to find an accurate vector field, we need to do more than simply use the derivative like in translation and dilation. In the next section, I will go over how to accurately find the vectors for derivatives with X dependence.

3.2.5 How to Find Accurate Vectors for X Dependent Derivatives

If you try to calculate the vector fields I have provided using the derivative equations like we did in the example, you will find that the spacing is all off and the vector fields are actually incorrect. This is because to calculate an accurate vector field for functions with an X dependence in their

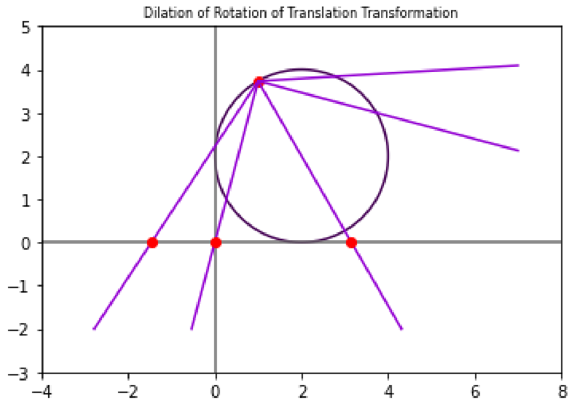

Figure 3.2.31: The unit circle after the dilation of rotation of translation transformation.

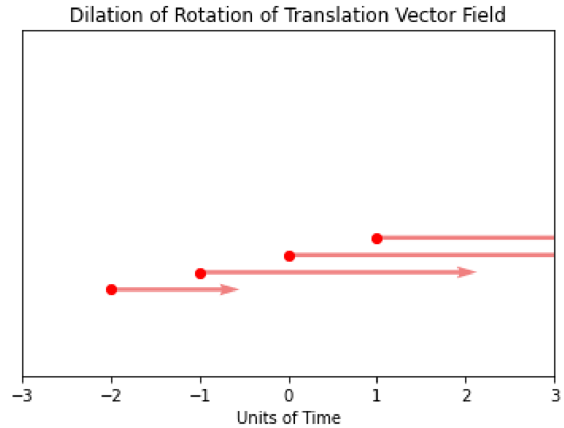

Figure 3.2.32: The vector field of after the dilation of rotation of translation transformation with spaces.

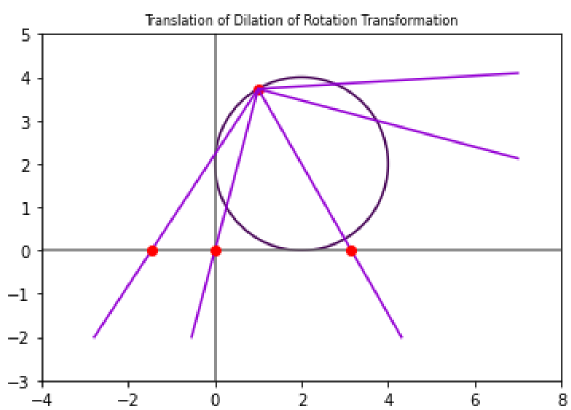

Figure 3.2.33: The unit circle after the translation of dilation of rotation transformation.

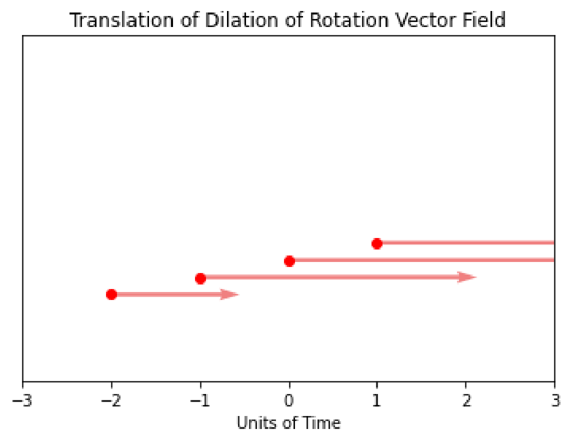

Figure 3.2.34: The vector field of the translation of dilation of rotation transformation with spaces.

derivative, we need to use infinitesimally close points. In this section, I will demonstrate how to find an accurate vector field.

We have been thinking about vectors as a kind of derivative. The basic definition for a derivative is

$$f'(x) = \frac{\lim}{h \to 0} \frac{f(x+h) - f(x)}{h},$$

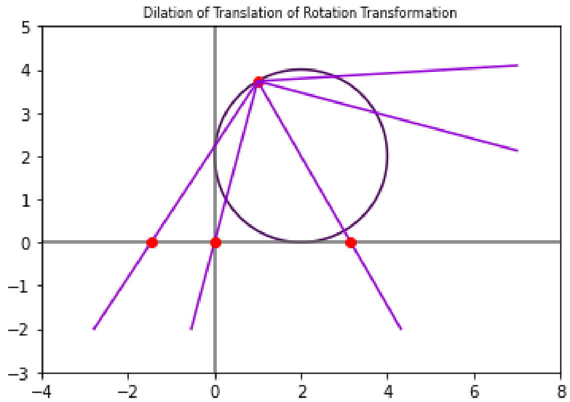

Figure 3.2.35: A vector field after the dilation of translation of rotation transformation.

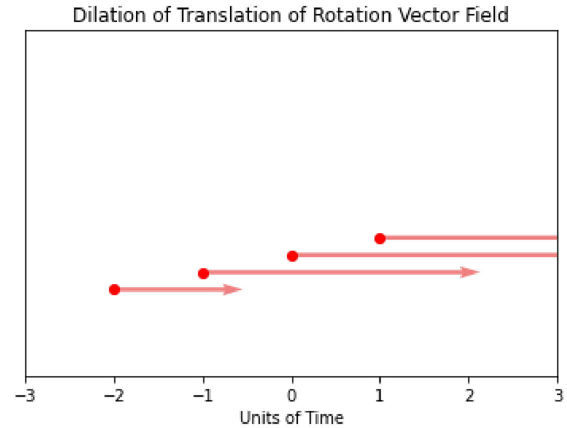

Figure 3.2.36: The vector field of the dilation of translation of rotation transformation with spaces.

which says that the derivative of a function $f(x)$ with respect to a variable x is defined by the difference between $f(x+h)$ and $f(x)$ divided by h as h goes to 0. This definition tells us that we are interested in infinitesimal changes of the function. Although we can't algebraically compute actual infinitesimal changes, we can choose points that are close together. In this case, we can think of our original Mobius transformation equation for rotation X_r as our function. To find the slope, we want to focus on infinitesimal changes of the function. So to find the vectors at the points -2, -1, 0, 1, and 2, we need to find the infinitesimal change of those points. We can expand equation (3.2.5) into

$$X_r' = \frac{dX_r}{dX} = \frac{\cos\theta}{(X\sin\theta - \cos\theta)^2}.$$

So to find our vectors, we need to move dX over to the other side.

$$dX_r = \frac{\cos\theta}{(X\sin\theta - \cos\theta)^2} dX \tag{3.2.7}$$

Let us look at $X = 2$. We can use $X = 2$ and $X = 2.0001$ as our two X values. Then

$$X_r(2) = -14.9282,$$

$$X_r(2.001) = -14.8801.$$

So the difference $X_r(2) - X_r(2.001)$ is 0.048. Now we can use equation 3.3.4 to find the value of the vector at that point. So

$$
\begin{aligned}
dX_r &= \frac{\cos(\frac{\pi}{6})}{((2)\sin(\frac{\pi}{6}) - (\cos\frac{\pi}{6}))^2} dX \\
&= 48.248 * (2.001 - 2) \\
&= 48.248 * 0.001 = 0.0482248
\end{aligned}
$$

As we bring the two points, 2 and 2.001 closer together, dX_r will be closer and closer to $X_r(X + h) - X_r(X)$. This shows us that the vectors depend on infinitesimally close points. Because this is not a feasible calculation that we can and interpret through a vector field, I simply took the difference between the points along the projective axis and estimated what a vector field might look like for those equations.

Vectors encode an abundant amount of information about what is happening physically. In this case, we are able to take transformations of a circle, a purely mathematical concept, and extract physical implications behind what the math is showing us. Through vector fields, we can see how different transformations of the circle illustrate different physical situations in regards to the dilation of time.

Because we have derived all the transformations and their combinations, we can achieve any orientation of the circle. Then we can find some orientation of the circle that describes a particular situation of time, whether that be high speeds or gravitational potential. Therefore, we can find a way to mathematically describe dilated time in terms of a transformed circle or a vector field. This then gives us a new way to describe time.

4
The Schwarzian Derivative

I want to introduce a new mathematical object that is deeply connected to the Mobius transformation. This particular object of interest is called the Schwarzian Derivative. Due to its natural connection with the Mobius transformation, we most often see the Schwarzian derivative in geometry and complex analysis. In this chapter, I want to begin unpacking the relationship between these two mathematical objects and provide a basis for further exploration of the topic.

As indicated by the name, this operator is a special kind of derivative. The formula for this derivative is

$$S(f(x)) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{4.0.1}$$

In this equation, $f(x)$ represents a function with x dependence where x can be any number, real or imaginary. The importance of the function $f(x)$ is that it can be any diffeomorphism. Then the Schwarzian derivative can be used to characterize of *all* diffeomorphisms. It does, on the other hand, have a unique relationship with one specific family of diffeomorphisms: the Mobius transformations.

4.1 Schwarzian Derivative and Mobius Transformation

The Schwarzian derivative's connection to the Mobius transformation is simple but important. The Schwarzian of some function equals zero if and only if that function is a Mobius transformation. That means that any function whose Schwarzian is zero is a Mobius transformation and all Mobius transformation have a zero Schwarzian. We will quickly run through a proof of this.

4.1.1 Proof of the Schwarzian Derivative and Mobius Transformation Relationship

To fully prove this theorem, we need to perform a proof both ways. I will walk through the proof that for any Mobius transformation, the Schwarzian will be 0 and leave the second proof to the reader.

$$S(f(x)) = S(M(x)) = \frac{M'''(x)}{M'(x)} - \frac{3}{2}(\frac{M''(x)}{M'(x)})^2 = 0$$

Now we can find the derivatives of the Mobius transformation.

$$M'(x) = \frac{AC - BD}{(Cx + D)^2}$$

Recall from Chapter 3 that one condition that we often impose on the Mobius transformation is that $AC - BD = 1$. Then

$$M'(x) = \frac{1}{(Cx + D)^2}. \tag{4.1.1}$$

Now we can take the next two derivatives.

$$M''(x) = \frac{-2C}{(Cx + D)^2} \tag{4.1.2}$$

$$M'''(x) = \frac{6C^2}{(Cx + D)^4} \tag{4.1.3}$$

Now we can plug those into our Schwarzian Derivative.

$$\frac{M'''(x)}{M'(x)} - \frac{3}{2}(\frac{M''(x)}{M'(x)})^2 = \frac{\frac{6C^2}{(Cx+D)^4}}{\frac{1}{(Cx+D)^2}} - \frac{3}{2}(\frac{\frac{-2C}{(Cx+D)^2}}{\frac{1}{(Cx+D)^2}})^2 = 6C^2 - \frac{3}{2}(-2C)^2 = 6C^2 - 3(2C^2) = 0$$

Therefore, for any Mobius transformation its Schwarzian will be 0. As stated above, I will leave the second half of the proof to the reader.

The relationship between the Mobius transformation and the Schwarzian derivative is important because it separates the family of Mobius transformations from other diffeomorphisms. The Schwarzian is special because it can describe all diffeomorphisms and the Mobius is special because it has this unique relationship with the Schwarzian.

Another important characteristic of the Schwarzian derivative is the cross-ratio. In the next section, we will explore the purpose of the cross-ratio, its connection to the Schwarzian derivative and the Mobius transformation, and the way it connects back to our vector fields.

4.2 The Cross-Ratio

In Chapter 2, I introduced invariances under projective transformations. This concept stems from the relationship between the Schwarzian derivative and the Mobius transformation and has to do with the cross-ratio.

The cross-ratio is a fundamental characteristic of projective transformations and turns out to carry important information about diffeomorphisms, too. Let us take 4 points that are projected down from the circle to the projective line, t_1, t_2, t_3 and t_4. Then the cross-ratio of those 4 points is given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)}. \qquad (4.2.1)$$

The amount that a diffeomorphism changes the cross-ratio of a transformation is given by the equation

$$[f(t_1), f(t_2), f(t_3), f(t_4)] = [t_1, t_2, t_3, t_4] - 2\epsilon^2 S(f)(t) + O(\epsilon^3). \qquad (4.2.2)$$

If f is a Mobius transformation, we know that the Schwarzian will be 0 so the second term in the equation will vanish. The value ϵ is very small and represents the distance between infinitesimally close t values. Our values of t are $[t_1, t_2 = \phi_\epsilon t_1, t_3 = \phi_{2\epsilon} t_1, t_4 = \phi_{3\epsilon} t_1]$. The third term, $O(\epsilon^3)$, represents the factor of ϵ^3. Since ϵ is so small, the term is often negligible. Then the cross-ratio is preserved for Mobius transformations.

Note that for any other kind of diffeomorphism, the cross-ratio will *not* be preserved and will not be invariant under projective transformations. From "What is... a Cross Ratio?[1], two quadruples[2] are equivalent under a linear transformation[3] if and only if they have the same cross-ratio.

Geometrically, the cross-ratio will look like Figure 4.2.1[4]. for a general projective transformation. We can see that the 4 points A, B, C, and D are colinear and the 4 points A', B', C', and D' are also colinear. We can use the cross ratio between them to understand relationship between the two quadruples. For the Mobius transformation, the cross ratio will look more like Figure 4.2.2.

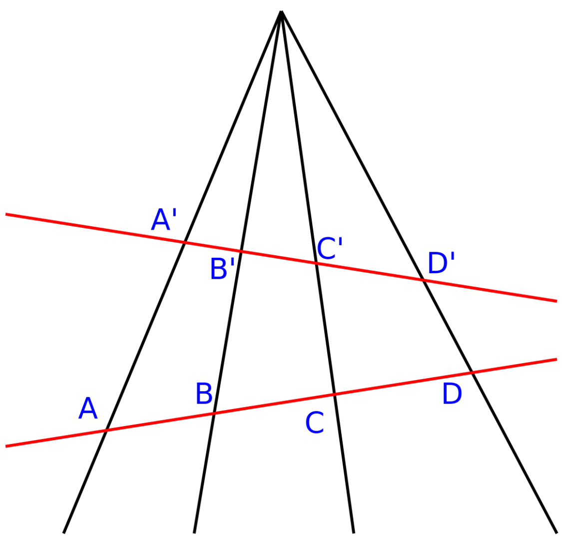

Figure 4.2.1: The cross-ratio of a projective transformation from a circle to a line (Vedala 2014)

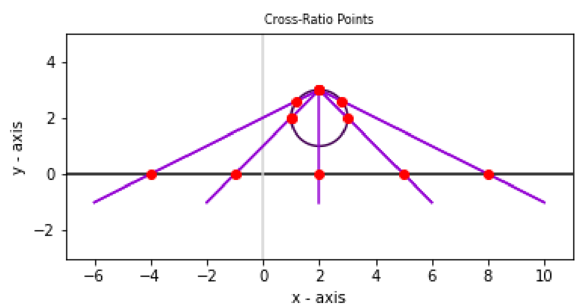

Figure 4.2.2: The cross-ratio of a projective transformation from a circle to a line.

We can now prove that the cross-ratio is invariant for Mobius transformations using the equations we found in the previous chapter and equation (4.2.2).

[1] François Labourie, "What Is... a Cross Ratio?," accessed May 3, 2022, https://www.ams.org/notices/200810/tx081001234p.pdf.

[2] A set of 4 points that all fall on the same line

[3] A general transformation that includes the Mobius transformation

[4] Vedala, Krishna. 2014. *Projection geometry.* Graph. Wikipedia. https://en.wikipedia.org/wiki/Cross-ratio/media/File:Projection$_{g}eometry.svg$

Translation

Recall from Chapter 3, we derived an equation that gave the points on the projective line after a translation transformation. We found that

$$X_t = X(1 + b) + a.$$

To find the cross-ratio for translation, we can take 4 points, say $[1, 2, 3, 4]$, and find the new X_t points after the translation transformation. As we did before, let $a = b = 2$. Then our new points are $[4, 7, 10, 13]$. Recall that since the translation diffeomorphism is a Mobius transformation, the Schwarzian will go away. Now that we have our $[t_1, t_2, t_3, t_4]$ and $[f(t_1), f(t_2), f(t_3), f(t_4)]$ points, we can now find the cross-ratio:

$$\begin{aligned}
[f(t_1), f(t_2), f(t_3), f(t_4)] &= [t_1, t_2, t_3, t_4] \\
\frac{(f(t_1) - f(t_3))(f(t_2) - f(t_4))}{(f(t_1) - f(t_2))(f(t_3) - f(t_4))} &= \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)} \\
\frac{(4 - 10)(7 - 13)}{(4 - 7)(10 - 13)} &= \frac{(1 - 3)(2 - 4)}{(1 - 2)(3 - 4)} \\
4 &= 4
\end{aligned}$$

Then the cross ratio is preserved! By looking at Figure 3.2.4, we can see that since the points are equidistant it makes sense that the cross ratio is invariant under this transformation.

Dilation

We can follow a similar process for our dilation transformation. The equation that we derived that gave point on the projective line after the dilation transformation is

$$X_d = dX$$

Let us use the same 4 points, $X = [1, 2, 3, 4]$, and find their X_d points, $X_d = [2, 4, 6, 8]$, for a dilation by a factor of $d = 2$. Then the cross ratio is

$$\begin{aligned}
[f(t_1), f(t_2), f(t_3), f(t_4)] &= [t_1, t_2, t_3, t_4] \\
\frac{(f(t_1) - f(t_3))(f(t_2) - f(t_4))}{(f(t_1) - f(t_2))(f(t_3) - f(t_4))} &= \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)} \\
\frac{(2-6)(4-8)}{(2-4)(6-8)} &= \frac{(1-3)(2-4)}{(1-2)(3-4)} \\
4 &= 4.
\end{aligned}$$

Then our cross ratio is invariant under the dilation transformation. This can be seen by looking at Figure 3.2.7. The points are equidistant along the projective axis so accoding to Figure 4.2.1, we can expect the cross ratio to be invariant under this transformation.

Rotation

Finally, we can follow the same process for the rotation transformation. Recall that the equation we found for finding the points on the projective axis after the rotation transformation is

$$X_r = \frac{X}{\cos(\theta) - X\sin(\theta)}.$$

For the points $X = [1, 2, 3, 4]$ and a rotation by $\theta = \frac{\pi}{6}$, our X_r points will be $X_r = [1.00926, 2.03732, 3.08469, 4.15194]$. I have rounded the numbers to the 5th decimal place so the cross ratio will be off by a very small amount. Our cross ratio will be

$$\begin{aligned}
[f(t_1), f(t_2), f(t_3), f(t_4)] &= [t_1, t_2, t_3, t_4] \\
\frac{(f(t_1) - f(t_3))(f(t_2) - f(t_4))}{(f(t_1) - f(t_2))(f(t_3) - f(t_4))} &= \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)} \\
\frac{(1.00926 - 3.08469)(2.03732 - 4.15194)}{(1.00926 - 2.03732)(3.08469 - 4.15194)} &= \frac{(1-3)(2-4)}{(1-2)(3-4)} \\
3.99999 &= 4.
\end{aligned}$$

If we had the correct decimals, the cross ratio would be exactly 4.

But wait, how is the cross-ratio remaining fixed when the points on the projective line after the transformation are unevenly spaced? The cross-ratio is not actually illustrated in our figures contrary to how it may look. The graphical representation of a cross-ratio is much harder to include on projections and is not actually portrayed in our figures. Nevertheless, it is invariant under all of our transformations. We will not go through this in the paper, but all of our Mobius combinations have invariant cross-ratios. You can find these by going through the same process as we did for translation, dilation and rotation. Note that because most equations include the rotation transformation, the calculations will be decimals so the cross ratio will only be accurate if every decimal place is included.

The cross ratio is important because it contains information about the shape of the object that is being transformed. A shape that undergoes projection will display a warped image of the original shape onto some other surface. The surface in this case represents the line and the object represents the circle. We use points along that line to calculate the cross ratio of points projected down from the circle. How the cross ratio changes for points before and after undergoing a transformation is given by the Scwarzian derivative which for Mobius transformations is 0 so the cross ratio is invariant. This unlocks a new path to be explored that would be interesting to look at. Unfortunately, we will not be exploring this topic in too much detail but there is much more to lean about the cross ratio.

In the next section, I want to introduce one more mathematical tool that is closely related to the Schwarzian derivative and therefore the Mobius transformation.

4.3 The Sturm-Liouville Equation

One last mathematical tool that we will briefly look at is the Sturm-Liouville Equation. This equation is fundamentally connected to the Schwarzian derivative and the Mobius Transformation. The Sturm Liouville equation is the simplest second order differential equation given by

$$\phi''(x) + u(x)\phi(x) = 0. \tag{4.3.1}$$

In the equation, $u(x)$ is a smooth function[5] which can be real or complex. The space of solutions for this function are two dimensional and spanned by any 2 linearly independent solutions ϕ_1 and ϕ_2[6]. The Sturm-Liouville equation captures a whole family of second order linear differential equations that can be written in that form. This equation is applicable in many areas of mathematics and physics because of this property.

We can rewrite the Sturm Liouville equation as

$$\frac{d}{dx}[p(x)\frac{dy}{dx} + q(x)y] = -\lambda w(x)y$$

where p, q and w are coefficient functions such that $p > 0$ and $w > 0$, and y is a function of x. All homogeneous second order linear ordinary differential equations can be written in this form.

To explore one particular property of the Sturm Liouville equation, let's define the function $f(x)$ as $f(x) = \frac{\phi_1}{\phi_2}$. According to "What is... The Schwarzian Derivative?," we should find that the potential gives $u = \frac{1}{2}S(f)$. This quotient $\frac{\phi_1}{\phi_2}$ is an affine coordinate[7] of the projective line. The Schwarzian derivative then takes a parameterized curve and reconstructs the Sturm Liouville equation from this curve. We also find that one solution to the Sturm Liouville equation is given by

$$(S(g))(x) = 2u(x) \tag{4.3.2}$$

where $g(x) = \frac{\phi(x)_1}{\phi(x)_2}$. $S(g))(x)$ is the Schwarzian derivative for some function $g(x)$. We can reverse this argument and reconstruct the Sturm Liouville equation from the Schwarzian derivative, as well. This equation allows us to use the Schwarzian derivative to turn a special class of non-linear differential equations that cannot be solved into the form of the Sturm Liouville equation which can be solved. This is powerful because it allows us to solve equations that were previously unsolvable. Because of the Schwarzian derivative's connection to the Mobius transformation, we can say that the Sturm Liouville equation is also related to the Mobius transformation in this way, as well.

[5] A smooth function is a function for which derivatives of all orders exist. For more information, see Section 2.1.

[6] Quoted from "What is... the Schwarzian Derivative?."

[7] Euclidean space independent of distance and angles but maintaining parallelism and ratio lengths.

The relationship between the Schwarzian derivative and the Sturm-Liouville equation and the Mobius transformation is much deeper than we are able to cover in this project. This topic is rich and has room for more investigation. The main take away that we should remember is the the Schwarzian labels different kinds of diffeomorphisms by showing how the cross-ratio changes. For the Mobius transformation, the Schwarzian is 0 but for all other diffeomorphisms, the Schwarzian is not 0 so the corss ratio is *not* invariant under projective transformations. Then we can connect the Sturm Liouville to the changes in the cross ratio through the relation in equation (4.3.2). Then we have found a way to characterize diffeomorphisms by means of the Sturm Liouville equation. This relation holds more information about diffeomorphisms than we are able to explore in this project, but the project presents a basis for further exploration of the subject.

The Sturm Liouville equation can be written in terms of the Schwarzian derivative, an equation that has a special relation ship to the Mobius transformation. We found a way to characterize time through projective transformations using the Mobius transformation. Now we could extend this idea further to the cross ratio, which is another way to characterize projective transformations. Since the Mobius transformation causes the Schwarzian to vanish, the cross-ratio will be invariant under projective transformations. This could imply that the changes in spacetime, as we explored through Mobius transformations, are ultimately invariant under the cross-ratio.

5 Conclusion

By viewing time mathematically, we unlock a new way to describe time. Newtonian time, special relativity and general relativity utilize time in unique ways that can be represented through mathematical transformations which allow us to study the theories in a clean and concise way. By viewing time as a line, we are able to switch between the infinite line and the finite circle. We can then use the Mobius transformation to move the circle. We use projective geometry to find a relationship between the line and the circle. By differentiating, we are able to find expressions for the vector field around that circle. We use vector fields to switch between the mathematical and physical. As shown in Chapter 2, the vector fields encode information about physical situations. We can use them to describe time under specific conditions. By analyzing the vector field, we extend the characteristics observed in the mathematical representations of time to physical situations.

This process allows us to relabel time. Einstein's interest in diffeomorphisms stemmed from a desire to break free of coordinate systems. The concept of passive diffeomorphisms aided in developing his theory of relativity and continues to be relevant in the ways that we describe time today. Using these mathematical tools, specifically the Mobius transformation and the Schwarzian derivative, we are able to view time in a way that is free of coordinate systems. For the most part, we viewed the transformations as active. Since the transformations are a kind of diffeomorphism, we can reverse the way we transformed our points and view them as passive diffeomorphisms. We can image doing everything we did, but instead changing the coordinate system instead of the point. This allows us to follow in Einstein's footsteps and take a step towards breaking the chains of coordinate systems.

Although we did derive relations for the transformations of the Mobius transformation and looked at other mathematical objects connected to the transformations, most of the project is conceptual connects ideas without showing rigorous mathematical proof. There is still more to learn about the Mobius transformation and diffeomorphisms of time. We have not yet found a variation of the Mobius transformation that is unrestrained. There is more to learn about the importance of cross ratios. The Sturm Liouville equitation relates what we did in this project to the rest of physics by incorporating second order linear differential equations. These ideas will have to wait until another project.

From our exploration of the Mobius transformation, we were able to find orientations of the circle that describe Newtonian time, special relativity and general relativity. We found that Newtonian time can be described using the unit circle. The unit circle lies at the center of our coordinate system and has a radius of 1. Then when we stereographically project down from the north pole, we find that the units of time on the transformed circle math the units of time on the reference circle. We found that we can describe time dilation in special relativity by translating or rotating the circle. These transformations move the north pole where we project down from further away from the circle which causes a kind of stretching. Then the physical implication of this turns out to resemble that of time dilation. We found that rotating the circle causes the units of time after the transformation to change inconsistently. Physically, this resembles time dilation in general relativity. We fond that we can describe time through these transformations or through vector fields. As discussed in the introduction, one way to view diffeomorphisms is through vector fields. Then these vector fields can be read as diffeomorphisms of the points on this circle which represent units of time. Then we have derived a conceptual way to view diffeomorphisms of time.

Bibliography

[1] Megan Bartels, *Einstein's Theory of General Relativity*, https://www.space.com/17661-theory-general-relativity.html.

[2] John Cook, *Schwarzian Derivative*, https://www.johndcook.com/blog/2020/09/22/schwarzian-derivative/.

[3] Stephanie Glen, *Differential Operator, Pseudodifferential, Total Forms*, https://www.calculushowto.com/differential-operator/#Schwarzian.

[4] Hanns Hagen Goetzke, *Einstein's Hole Argument and its Legacy*, https://physicsworld.com/a/a-very-brief-history-of-relativity/.

[5] Jobn Gribbon, *A Very Brief History of Relativity*, https://doi.org/10.1007/bf02786934.

[6] David Hestenes, *Gauge Theory Gravity with Geometric Calculus.*, https://doi.org/10.1007/s10701-005-5828-y.

[7] Mihaela Iftime, *The Hole Argument for Covariant Theories*, https://doi.org/10.1007/s10714-006-0303-4.

[8] François Labourie, *What Is... a Cross Ratio?*, https://www.ams.org/notices/200810/tx081001234p.pdf.

[9] Brad Osgood, *The Schwarzian Derivative, Conformal Connections, and Möbius Structures*, https://doi.org/10.1007/bf02786934.

[10] Valentin Ovsienko, *What is the Schwarzian Derivative*, http://ovsienko.perso.math.cnrs.fr/Publis/what-is.pdf.

[11] Carlo Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge, UK, 2004.

[12] ———, *The Order of Time*, Riverhead Books, New York City, NY, 2018.

Bibliography

[1] Vicky Stein, *Einstein's Theory of Special Relativity*, https://www.space.com/36273-theory-special-relativity.html.

[2] Nikki Withers, *How Far Away Is the Horizon?*, https://www.sciencefocus.com/planet-earth/how-far-away-is-the-horizon/.