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Examining Stellate Unions

Julia Erin Crager
Bard College

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by
Julia Crager

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Abstract

Stellate neighborhoods are created by gluing half disks together along their straight edges. A 1-stellate neighborhood is a half disk, a 2-stellate neighborhood is a disk, a 3-stellate neighborhood is 3 half disks glued together to make a star-like shape, and so on. For a topological space X, and for each $n \in \mathbb{N}$, the n-stellate subspace of X is the set of all points in X that have a neighborhood homeomorphic to an n-stellate neighborhood. I will be examining topological spaces called stellate unions, where each point in the space is contained in an n-stellate subspace for some $n \in \mathbb{N}$. All surfaces and surfaces with boundary are stellate unions, yet there are many stellate unions that are not surfaces or surfaces with boundary. I will explore some stellate unions called extended graph twists and examine their orientability.

Contents

Acknowledgments

To Ethan Bloch, thank you for helping me every step of the way with this project, I could not have done it without you.

To my parents, thank you for always being a phone call away. I would not be where I am without your love and support.

1
What are Stellate Unions?

1.1 Preliminary Definitions

Before we begin we will go over some basic concepts of topology. These definitions come from [2] and [5].

Definition 1.1.1. Let $I \subseteq \mathbb{R}$ be the unit interval. △

Definition 1.1.2. Let X be a topological space. Let $x \in X$ be a point. A **neighborhood** of x is an open subset N of X such that $x \in N$. △

A commonly used neighborhood in topology is an open disk in $\mathbb{R}^2$. To understand what an open disk is we must first define some notation.

Let S^1 and D^2, the unit circle and open unit disk in $\mathbb{R}^2$, respectively, be defined by

$$S^1 = \{p \in \mathbb{R}^2 \mid |p| = 1\} \quad \text{and} \quad D^2 = \{p \in \mathbb{R}^2 \mid |p| < 1\}.$$

Definition 1.1.3. Let X be a topological space. Let $N \subseteq X$. The set N is an **open disk** if N is homeomorphic to D^2. △

Now with our definition of an open disk, we can begin to define half disks. This definition is inspired by [1].

Definition 1.1.4.

1. Let $H = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.

2. Let $P_1 = D^2 \cap H$; the set P_1 is referred to as a **half disk**.

3. Let $L = \{0\} \times (-1, 1) \subseteq P_1$.

4. Let $n \in \mathbb{N}$. Let P_n be the space obtained by gluing n copies of P_1 along the line L. We will call these **stellate neighborhoods**.

$\triangle$

See Figure 1.1.1 for P_1, and Figure 1.1.2 and 1.1.3 for the stellate neighborhood P_n.

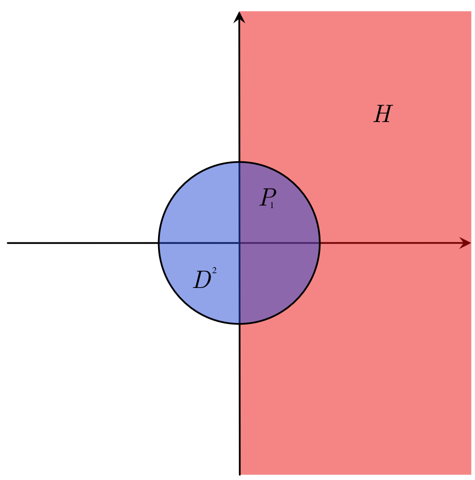


Figure 1.1.1: P_1.

We will note that the line segment L does not include its endpoints. We will also note that P_1 is not an open set in $\mathbb{R}^2$. However P_2 is homeomorphic to D^2, which is open in $\mathbb{R}^2$.

We will use these concepts to take a look at some common topological spaces. First, we will examine a torus.

Example 1.1.5. Let X be a torus. We know that for every point x in the torus, there exists a neighborhood N of x where N is homeomorphic to D^2. Thus, for every point x in the torus there exists a neighborhood N of x where N is homeomorphic to P_2. See Figure 1.1.4. $\diamondsuit$

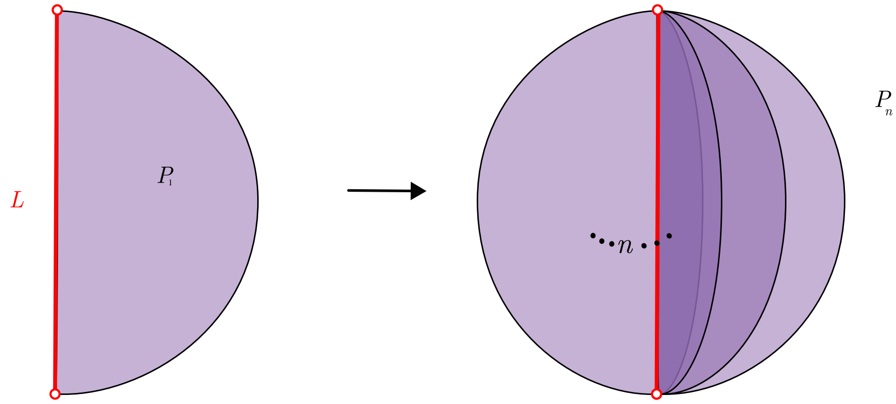


Figure 1.1.2: Gluing n half disks along the line h to make P_n.

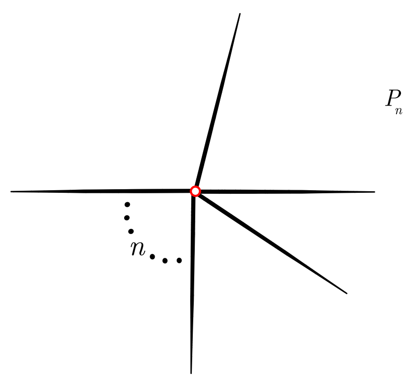


Figure 1.1.3: Top view of P_n.

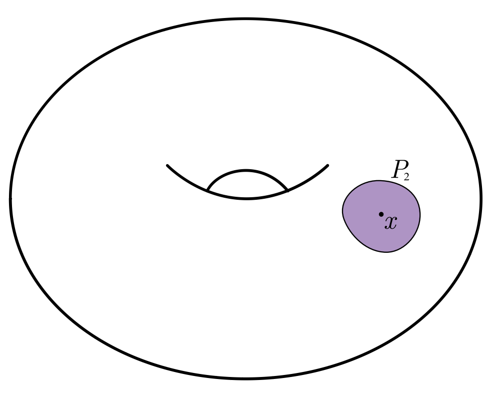


Figure 1.1.4: Torus containing point x with the neighborhood P_2.

Example 1.1.6. Let X be a Möbius Strip. Let $M \subseteq X$ be an open Möbius Strip. Note that for all points $x \in M$ there exists a neighborhood N of x where N is homeomorphic to P_2. Let $B \subseteq X$ be the boundary of the Möbius Strip. If we look at a point $y \in B$, there does not exist any neighborhoods of y that is homeomorphic to P_2. However, we find that for each point $y \in B$ there exists a neighborhood N' of y such that N' is homeomorphic to P_1. Notice that $X = M \cup B$ and $M \cap B = \emptyset$. See Figure 1.1.5. $\Diamond$

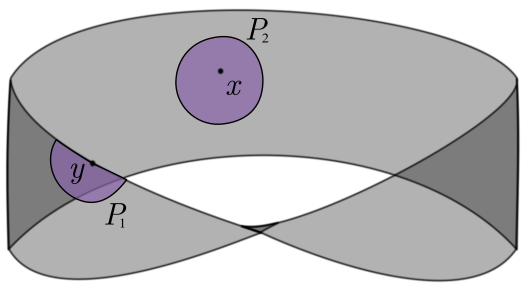


Figure 1.1.5: Möbius Strip containing points x and y with the neighborhoods P_2 and P_1 respectively.

Definition 1.1.7. Let X be a topological space. Let $n \in \mathbb{N}$. Let

$$G_n = \{x \in X \mid \text{ there exists a neighborhood } N \subseteq X \text{ of } x \text{ such that } N \approx P_n\}.$$

The set G_n is referred to as the **n-stellate subspace** of X. If we need to specify the topological space, then we write $G_n(X)$. $\triangle$

We will apply Definition 1.1.7 to Example 1.1.6. Since for all points $x \in M$ there exists a neighborhood N of x where N is homeomorphic to P_2, and for all points $y \in B$ there exists a neighborhood N of y where N is homeomorphic to P_1, then $G_2 = M$ and $G_1 = B$. See Figure 1.1.6. Hence $X = G_1 \cup G_2$.

We will now define topological concepts in terms of n-stellate subspaces.

Definition 1.1.8. Let X be topological space. If $X = G_2$, then X is a **2-manifold**, also called a **surface**. $\triangle$

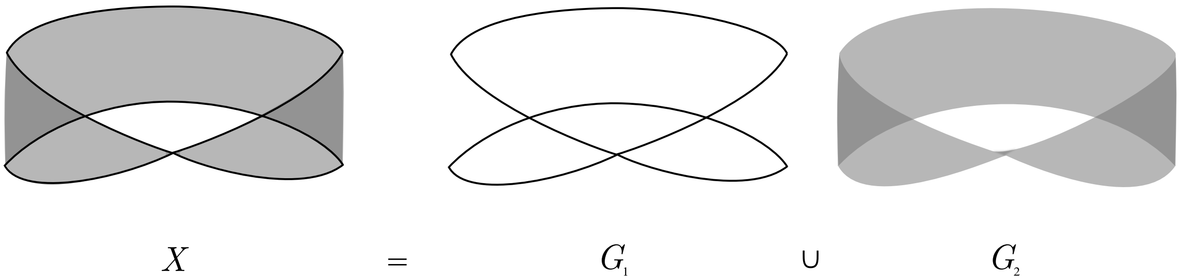


Figure 1.1.6: Möbius Strip separated into G_1 and G_2.

Definition 1.1.9. Let X be a topological space. If $X = G_1 \cup G_2$ then X is a **surface with boundary**; the boundary of X, denoted ∂X, is the set G_1. △

We state the following lemma without proof.

Lemma 1.1.10. *Let X be a topological space. Let $n, k \in \mathbb{N}$. Suppose that $n \neq k$. Then $G_n \cap G_k = \emptyset$.*

1.2 Stellate Unions

With our definitions stated, we can now define the spaces we will be looking at.

Definition 1.2.1. Let X be a topological space. The space X is a **stellate union** if

$$X = \bigcup_{i=1}^{\infty} G_i.$$

△

While some examples of stellate unions are a Möbius Strip (as seen in Figure 1.1.6) and a torus (as seen in Figure 1.1.4), these topolgical spaces can get much more complicated. To better understand these objects, we will look at some more examples.

Example 1.2.2. Let Y be the topological object pictured in Figure 1.2.1. Let point x be as in the Figure. Observe that no neighborhood of x is homeomorphic to a stellate neighborhood. Therefore $x \notin G_n$ for any $n \in \mathbb{N}$. Thus X is not a stellate union. ◊

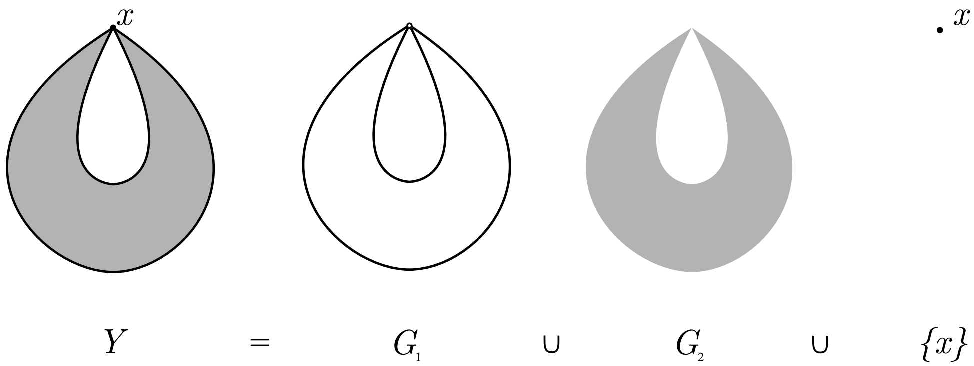


Figure 1.2.1: Topological space Y that is not a stellate union.

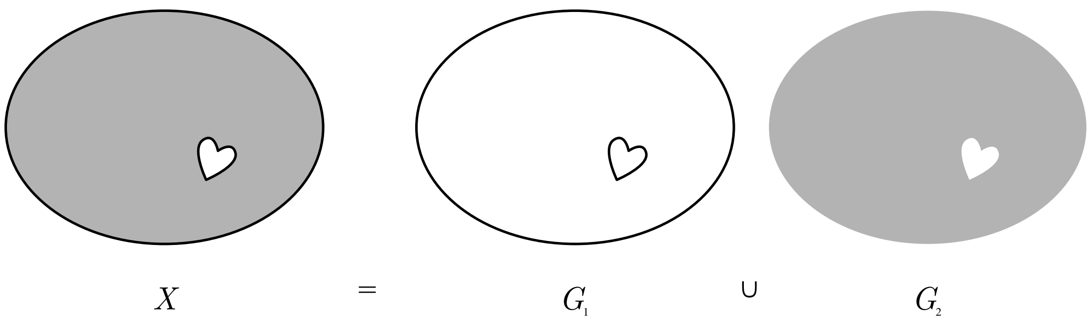


Figure 1.2.2: Surface with boundary X separated into G_1 and G_2.

Example 1.2.3. Let X be a surface with boundary as pictured in Figure ref. By Definition 1.1.9, we know that $X = G_1 \cup G_2$. Hence X is a stellate union.

$\Diamond$

We can use this logic and Definitions 1.1.8 and 1.1.9 when examining our next lemma. We will state the following lemma without proof.

Lemma 1.2.4.

1. *Every surface X is a stellate union, where $G_2 = X$ and $G_n = \emptyset$ when $n \neq 2$.*

2. *For all surfaces with boundary, we have $G_n = \emptyset$ when $n > 2$.*

2
Non Trivial Examples of Stellate Unions

2.1 Stellate Washers

We will now see stellate unions that are not surfaces or surfaces with boundary.

Example 2.1.1. Imagine three strips of paper all glued together along one edge. We will label the 3 strips a_1, a_2 and a_3, respectively. Now imagine gluing the top of each strip to the bottom of the same strip so that the end of a_1 would attach to the beginning of a_1, the end of a_2 would attach to the beginning of a_2, and so on. See Figure 2.1.1.We will call this space $R_{(3,0)}$.

Notice that if we break this object up into n-stellate subspaces, then G_1 is the union of three disjoint circles, G_2 is the union of three disjoint washers, and G_3 is one circle. Also note that this object equals $G_1 \cup G_2 \cup G_3$. Thus, it is a stellate union. See Figure 2.1.2. $\diamond$

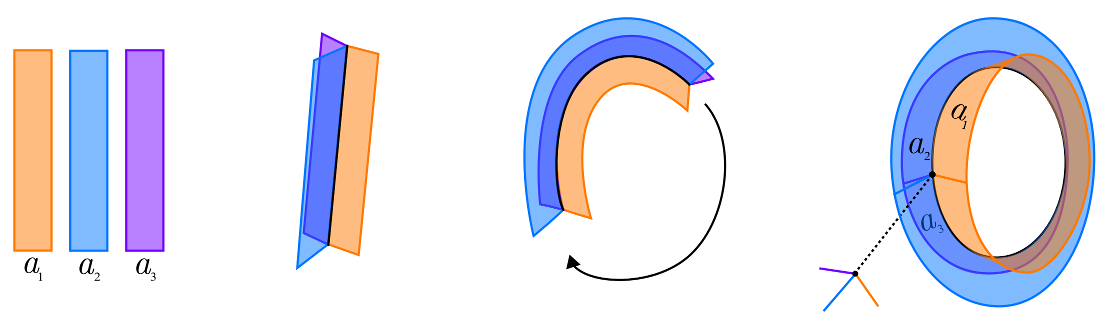


Figure 2.1.1: Constructing a 3-flap stellate washer.

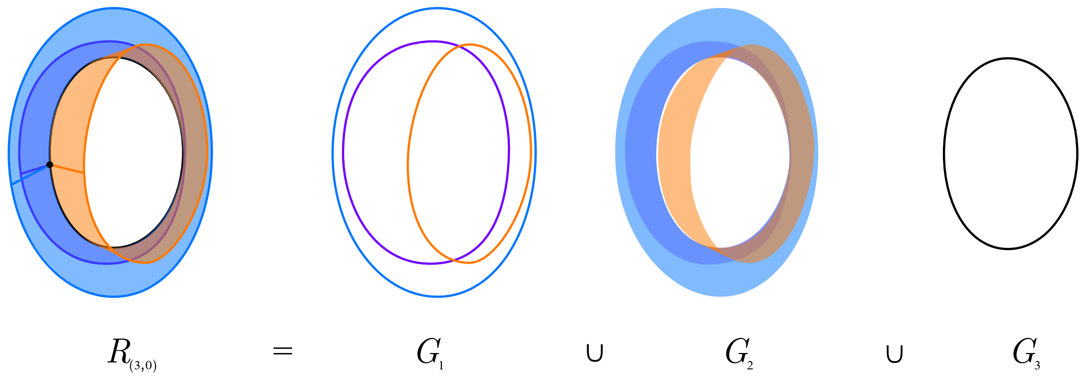


Figure 2.1.2: 3-flap stellate washer separated into G_1, G_2 and G_3.

Example 2.1.2. Imagine the three strips of paper glued along one edge as before. However, before gluing the ends together we will first twist the structure so that the end of a_2 will be glued to the beginning of a_1, the end of a_3 will be glued to the beginning of a_2, and the end of a_1 will be glued to the beginning of a_3. See Figure 2.1.3. We will call this a $R_{(3,1)}$.

If we split the the object up into its n-stellate subspaces, we see that G_1 is one big circle, G_2 is one big twisted washer, and G_3 is one circle. See Figure 2.1.4 $\Diamond$

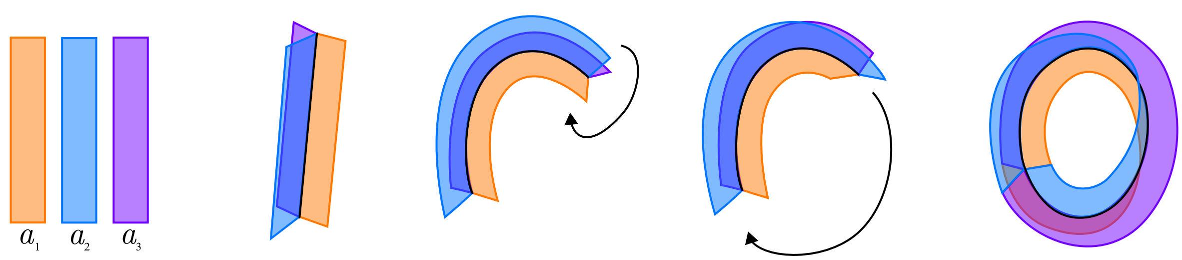


Figure 2.1.3: Constructing a 3-flap stellate washer with a twist.

After these examples to help us with our intuition, we will state a formal definition of stellate washers.

Definition 2.1.3. Let $I \in \mathbb{R}$ be the unit interval. Let $n \in \mathbb{N}$. Glue n unit intervals at $\{0\} \in I$ to create a star-like shape denoted T_n as seen in Figure 2.1.5. For each $i \in \mathbb{N}$ such that $1 \leq i \leq n$, we will call the corresponding unit interval a_i. $\triangle$

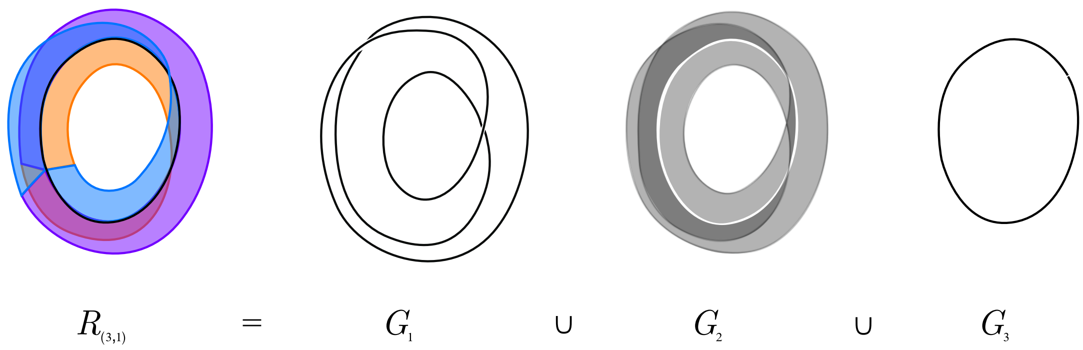


Figure 2.1.4: 3-flap stellate washer with a twist separated into G_1, G_2 and G_3.

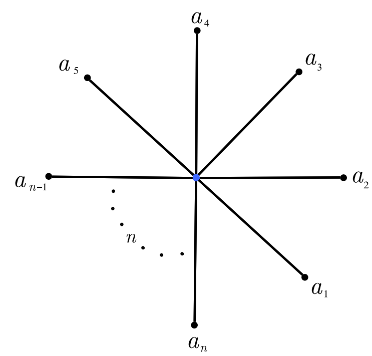


Figure 2.1.5: Constructing T_n.

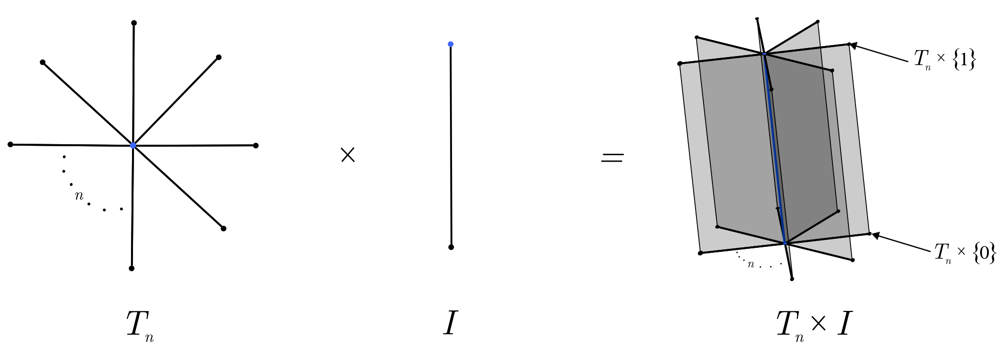


Figure 2.1.6: $T_n \times I$.

Definition 2.1.4. Let $n \in \mathbb{N}$ and let $r \in \{0, 1, \ldots, n-1\}$. The **$(n, r)$-stellate washer**, denoted $R_{(n,r)}$, is a quotient space of $T_n \times I$ defined by gluing $a_i \times \{0\}$ to $a_{i+r} \times \{1\}$ for all $i \in \{1, 2, \ldots, n\}$, where addition is mod n. △

The space $T_n \times I$ is consisted of n strips all glued together along $(0, x) \in a_i \times I$ for all $1 \leq i \leq n$ as seen in Figure 2.1.6. The variable r determines *how* twisted the stellate washer is through

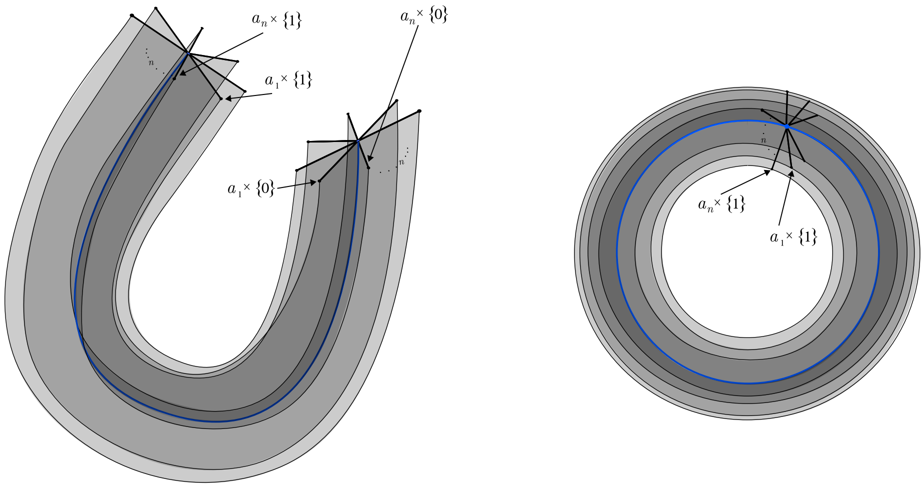


Figure 2.1.7: Constructing $R_{(n,0)}$.

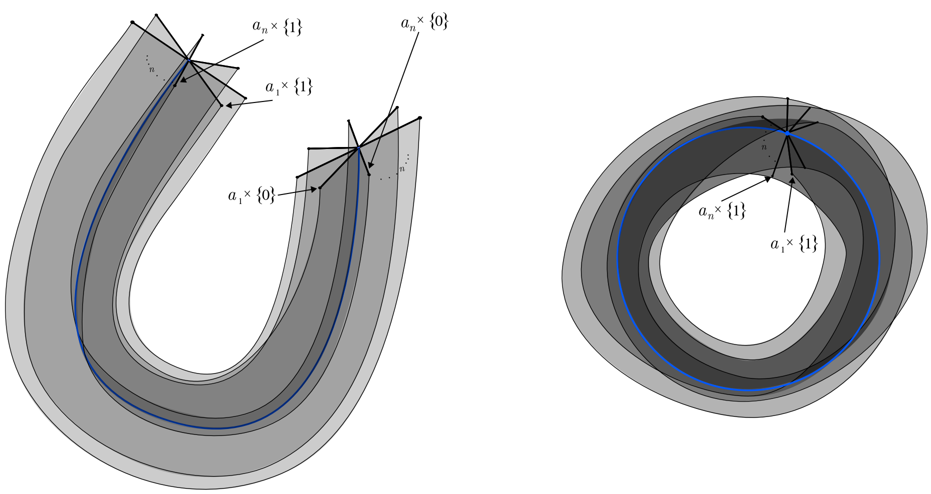


Figure 2.1.8: Constructing $R_{(n,r)}$.

the gluing of $a_i \times \{0\}$ to $a_{i+r} \times \{1\}$ as seen in Figure 2.1.8. Note that if $r = 0$ then a stellate washer is created *without* a twist as seen in Figure 2.1.7.

It's also important to note that the flaps of the stellate washer are glued under mod n. For this reason we decide $r \neq n$ because $i + r = i + n \equiv i \pmod n$ and thus $a_i \times \{0\}$ would be glued to $a_i \times \{1\}$ for all $i \in \{1, 2, \ldots, n\}$. Then, all (n, n)-stellate washers would be structurally the same as $(n, 0)$-stellate washers and even contain the same stellate subspaces with the distinction that (n, n)-stellate washers are twisted while $(n, 0)$-stellate washers are not.

We can also see that depending on r, because T_n is symmetrical, Definition 2.1.4 produces some stellate washers that are homeomorphic. For instance, in Example 2.1.2, we examine $R_{(3,1)}$ which has one circle in G_1, one washer in G_2, and one circle in G_3. If we examine $R_{(3,2)}$, we will find the stellate subspaces to be homeomorphic. This is because if we look at twisting T_3 2 notches, it is the same as if we twisted T_3 1 notch backwards.

Lemma 2.1.5. *Let $n \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-1\}$.. The stellate subspaces of $R_{(n,r)}$ are homeomorphic to the stellate subspaces of $R_{(n,n-r)}$.*

Essentially, Lemma 2.1.5 states that the gluing of stellate washers is symmetrical. Now we will examine the stellate subspaces of stellate washers.

Theorem 2.1.6. *Let $n \in \mathbb{N}$ such that $n \geq 2$ and $r \in \{0, 1, \ldots, n-1\}$. Let $m = \gcd(n, r)$. Then $R_{(n,r)}$ has one circle in G_n, m circles in G_1 and m washers in G_2.*

Proof. First, we will prove that G_n contains one circle. Let v be the center vertex of T_n. We know $\{v\} \times \{0\}$ will always be glued $\{v\} \times \{1\}$ in the construction of (n, r)-stellate washer. Thus, the endpoints of the line $v \times I$ are glued together to create a circle. Note that for every point in the circle there exists a neighborhood N of x where N is homeomorphic to P_n. Thus, by Definition 1.1.7, there is one circle in G_n. Because there are no other points $x \in R_{(n,r)}$ where there exists a neighborhood N of x where N is homeomorphic to G_n, then we know there is only one circle in G_n. With this in mind, we will remove this circle.

Now we will prove that G_1 contains m circles and that G_2 contains m washers. By Lemma 1.1.10, we know that the circle created by the gluing of $\{v\} \times I$ is not in G_1 or G_2. Note that we can denote each flap of $R_{(n,r)}$ as an element of the finite cyclic group $\mathbb{Z}_n$ with the generator $a = 1$. Let $\langle [r] \rangle$ be the cyclic subgroup of $\mathbb{Z}_n$. Because $m = \gcd(n, r)$ and $[r] \in \mathbb{Z}_n$, we know that $\langle [r] \rangle$ has $\frac{n}{m}$ elements, as stated in [4]. In terms of our $R_{(n,r)}$, this means that there are $\frac{n}{m}$ flaps per washer. Thus, in order to glue all the flaps together, there must be m washers total.

Each of these washers are half open and half closed (due to removing the circle in G_n. The remaining boundary of each washer creates one circle. For every point in these m circles there exists a neighborhood N of x where N is homeomorphic to P_1. Thus, by Definition 1.1.7, there are m circles in G_1. By Lemma 1.1.10, we know these m circles are not in G_2. With this in mind, we will remove them, Thus, we now have m open washers. For every point in these m washers there exists a neighborhood N of x where N is homeomorphic to P_2. Thus, by Definition 1.1.7, there are m circles in G_2.

Because there are no other points $x \in R_{(n,r)}$ where there exists a neighborhood N of x where N is homeomorphic to either G_1 or G_2, we know there is nothing else in G_1 and G_2. □

2.2 Layered Strip Spaces

Let's take a look at another type of stellate union called a layered strip space. To understand these spaces, we will first look at some examples.

Example 2.2.1. Let e_1 be an edge with vertices v_1 and v_2. We will call this graph H_1. See Figure 2.2.1.

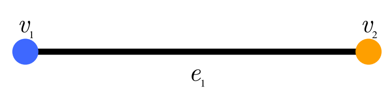


Figure 2.2.1: Constructing H_1.

Now, consider $H_1 \times I$. Notice that the space created is one strip. In order to glue the two ends to one another, we must use two functions: one that glues each vertex in $\{v_1, v_2\} \times \{0\}$ to a vertex in $\{v_1, v_2\} \times \{1\}$, and another function that glues $e_1 \times \{0\}$ to $e_1 \times \{1\}$.

Since there are two vertices, we can either glue $\{v_1\} \times \{0\}$ to $\{v_1\} \times \{1\}$ and $\{v_2\} \times \{0\}$ to $\{v_2\} \times \{1\}$ *or* we can glue $\{v_1\} \times \{0\}$ to $\{v_2\} \times \{1\}$ and $\{v_2\} \times \{0\}$ to $\{v_1\} \times \{1\}$. Since there is only one edge in H_1, we are only able to glue the edge to itself. Thus there are only two ways to glue $H_1 \times \{0\}$ to $H_1 \times \{1\}$. See Figures 2.2.2 and 2.2.3.

Note that if we glue the vertices with their identity function, we create a washer with boundary but if we glue the vertices by switching them, we create a Möbius Strip with boundary. ◊

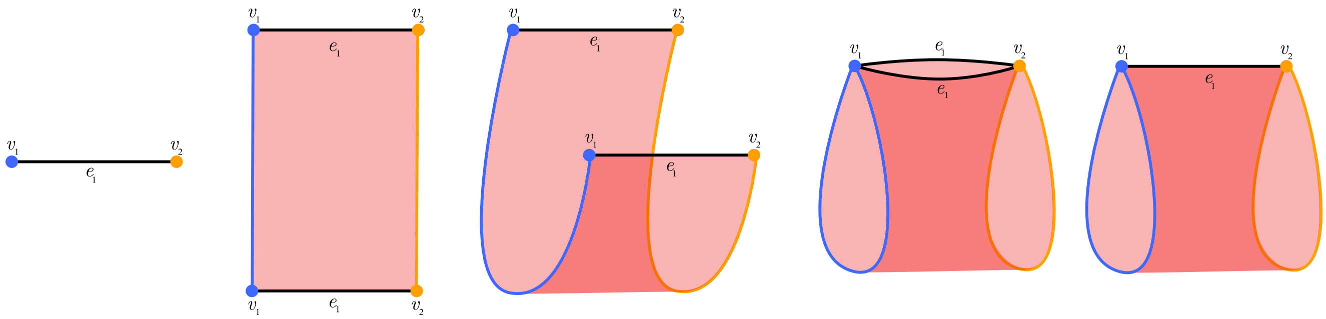


Figure 2.2.2: Making a 1-layered strip space with the vertices glued with the identity, which is a washer.

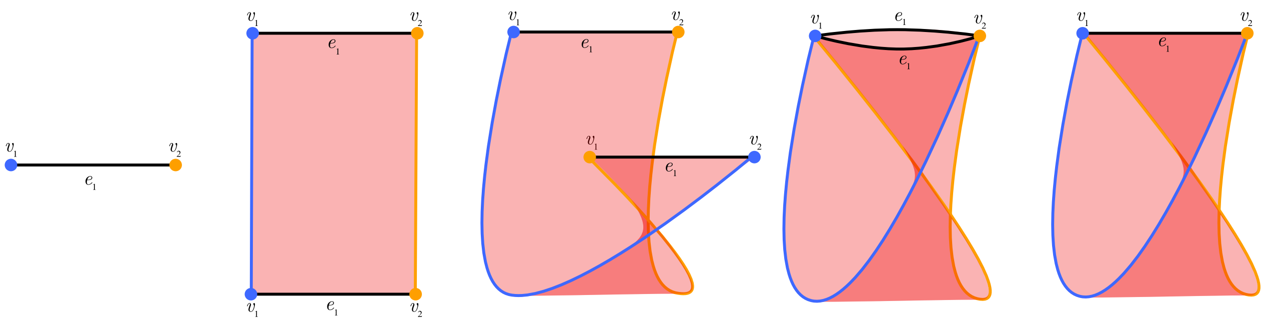


Figure 2.2.3: Making a 1-layered strip space with the vertices glued switched which is a Möbius strip.

Now, let's look at an example of an layered strip space that has more than one edge.

Example 2.2.2. Let e_1 and e_2 be two edges with the same two vertices v_1 and v_2. Note that this graph not a simple graph. We will call this graph H_2. See Figure 2.2.4. Now, cross H_2 with I. Notice that the space created is two strips glued together along their sides.

As in Example 2.2.1, there are two functions to glue $\{v_1, v_2\} \times \{0\}$ to $\{v_1, v_2\} \times \{1\}$. We can either use the identity or we can glue the vertices switched. See Figures 2.2.5 and 2.2.6. To glue the edges together, we can either glue $\{e_1\} \times \{0\}$ to $\{e_1\} \times \{1\}$ and $\{e_2\} \times \{0\}$ to $\{e_2\} \times \{1\}$ *or* we can glue $\{e_1\} \times \{0\}$ to $\{e_2\} \times \{1\}$ and $\{e_2\} \times \{0\}$ to $\{e_1\} \times \{1\}$. Thus, with two ways to glue the vertices and two ways to glue the edges, there are four ways to glue $H_2 \times \{0\}$ to $H_2 \times \{1\}$.

Note that if you glue the vertices and the edges with the identity, you get a torus, and if you glue the vertices with the identity but the edges switched, then you get a Klein bottle. If you glue the vertices switched and the edges with the identity, then you get a Klein bottle, and if you glue the vertices switched but the edges switched, then you get a torus. See Figures 2.2.7, 2.2.8, 2.2.9,and 2.2.10. ◊

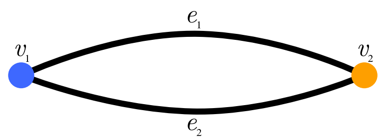


Figure 2.2.4: H_2.

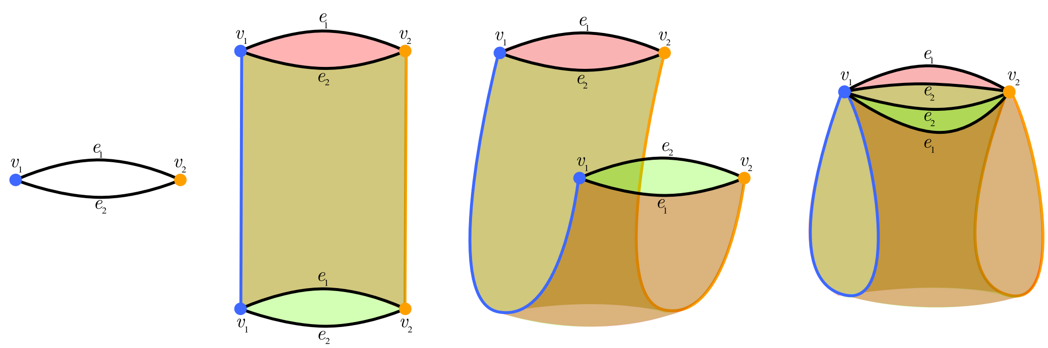


Figure 2.2.5: Gluing the vertices together with the identity.

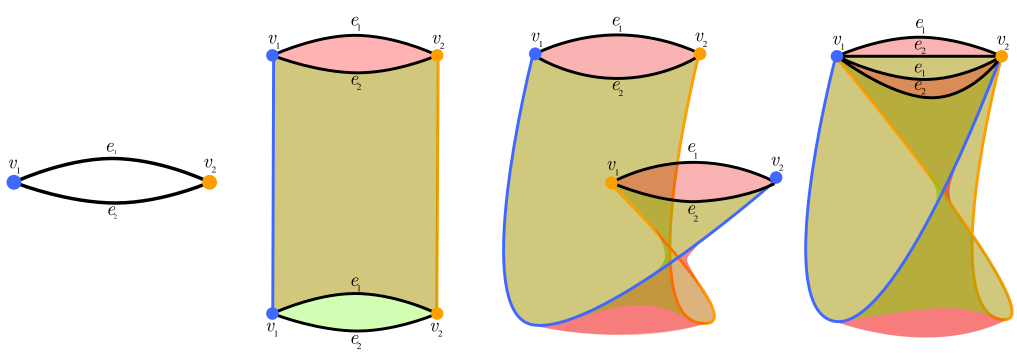


Figure 2.2.6: Gluing the vertices together switched.

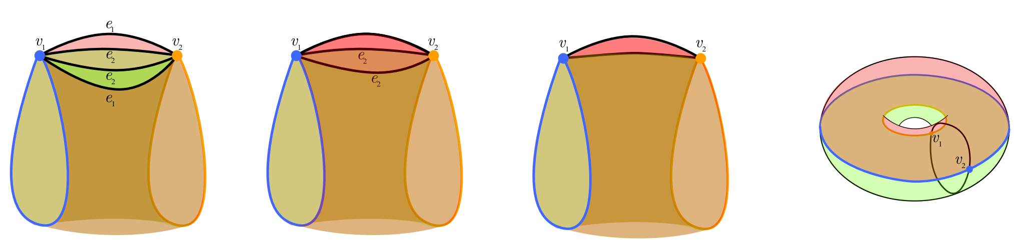


Figure 2.2.7: Gluing the vertices and the edges with the identity to make a torus.

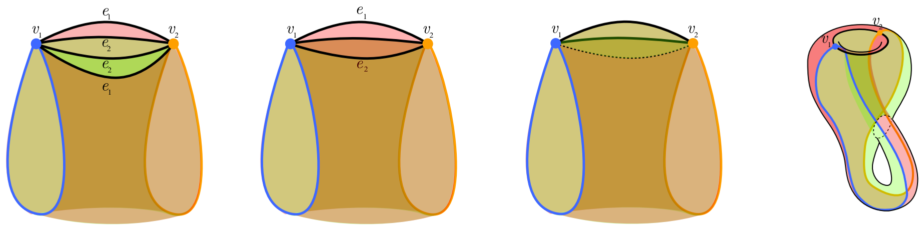


Figure 2.2.8: Gluing the vertices with the identity and the edges switched to make a Klein bottle.

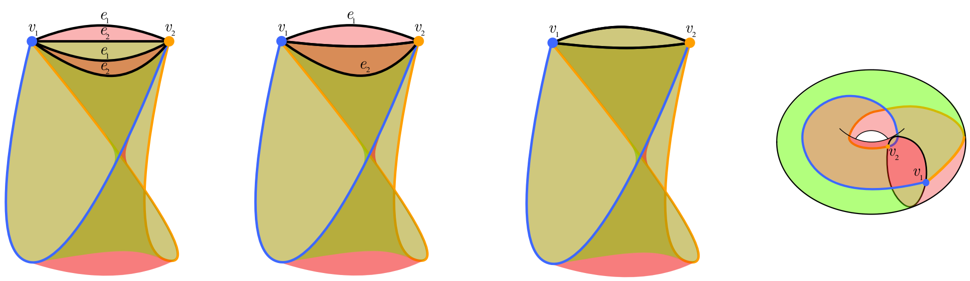


Figure 2.2.9: Gluing the vertices switched and the edges with the identity to make a torus.

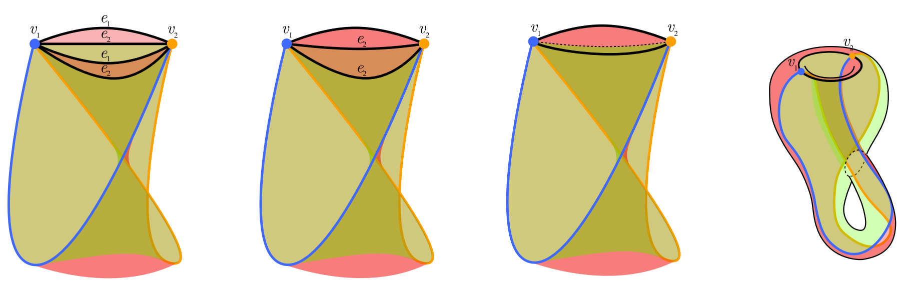


Figure 2.2.10: Gluing the vertices and the edges switched a Klein bottle.

Now that we have an understanding of how a layered strip space is constructed, we will state a formal definition of these spaces. To describe layered strip spaces, we will be using permutations described with cyclic notation. Unconventionally, we will be including cycles of one element in our notation for clarity later on. We will also be using standard notation for cyclic groups.

Definition 2.2.3. Let $n \in \mathbb{N}$. Glue n intervals together at $\{0\} \in I$ and $\{1\} \in I$ to create two vertices connected by multiple edges, denoted H_n, as seen in Figure 2.2.11. For each $i \in \{1, 2, \ldots, n\}$ we will call the corresponding edge e_i. $\triangle$

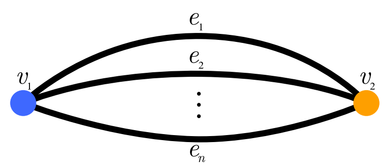


Figure 2.2.11: H_n.

Definition 2.2.4. Let $V = \{v_1, v_2\}$ be the set of vertices of H_n. Let $\tau : \{1, 2\} \to \{1, 2\}$ be a permutation. Let $V_\tau : V \times \{0\} \to V \times \{1\}$ be the function defined by $V_\tau((v_i, 0) = (v_{\tau(i)}, 1)$ for all $i \in \{1, 2\}$. △

There are only 2 permutations in S_2, which are i_2 as the identity permutation and $\tau = (1, 2)$. We will denote the identity function on the vertices as V_i. Informally, the two permutations cause V_i to glue each vertex to themselves, and V_τ to glue each vertex switched.

Definition 2.2.5. Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of H_n. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Let $E_\sigma : E \times \{0\} \to E \times \{1\}$ be the function defined by $E_\sigma((e_i, 0)) = (e_{\sigma(i)}, 1)$ for all $i \in \{1, \ldots, n\}$. △

Note that there are $n!$ permutations in the symmetric group S_n on $\{1, \ldots, n\}$, thus there are $n!$ possible functions in Definition 2.2.5. We will define i_n as the identity permutation and denote the identity function on the edges as E_i. Informally, each function glues each edge in $H_n \times \{1\}$ to another edge in $H_n \times \{1\}$.

Definition 2.2.6. Let $n \in \mathbb{N}$. Let $\tau \in S_2$ and let $\sigma \in S_n$. A n-**layered strip space**, denoted $L(n, \tau, \sigma)$, is the quotient space of $H_n \times I$ defined by gluing the vertices and edges of $H_n \times \{0\}$ to the vertices and edges of $H_n \times \{1\}$ under the functions V_τ and E_σ. △

The space $H_n \times I$ is consisted of n strips all glued together along $\{v_1\} \times I$ and $\{v_2\} \times I$ as seen in Figure 2.2.12. See Figures 2.2.13 and 2.2.14 to see how the layered strip spaces are glued with V_i and V_τ.

Example 2.2.7. Using the formal definition of layered strip spaces, we will now examine all 3-layered strip spaces.

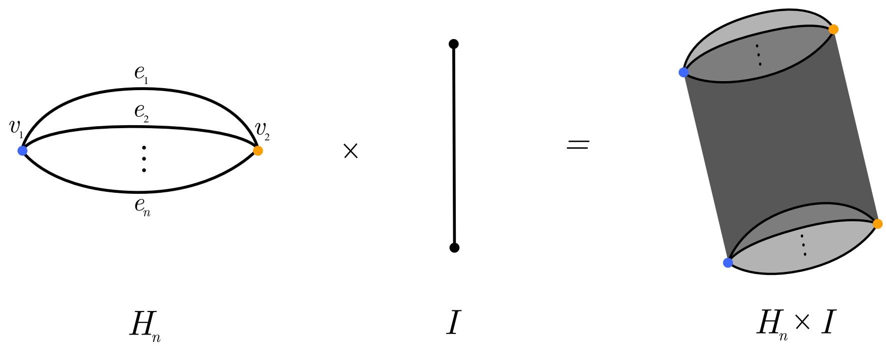


Figure 2.2.12: $H_n \times I$.

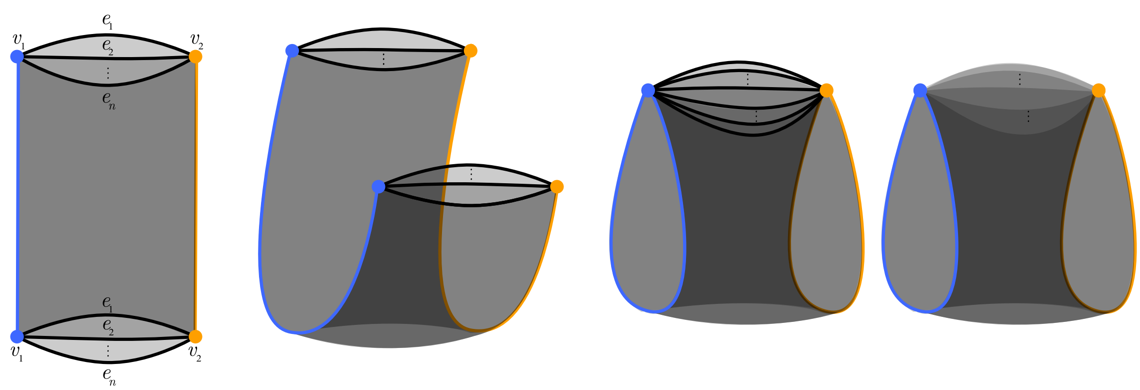


Figure 2.2.13: $L(n, \tau, \sigma)$ when τ is the identity.

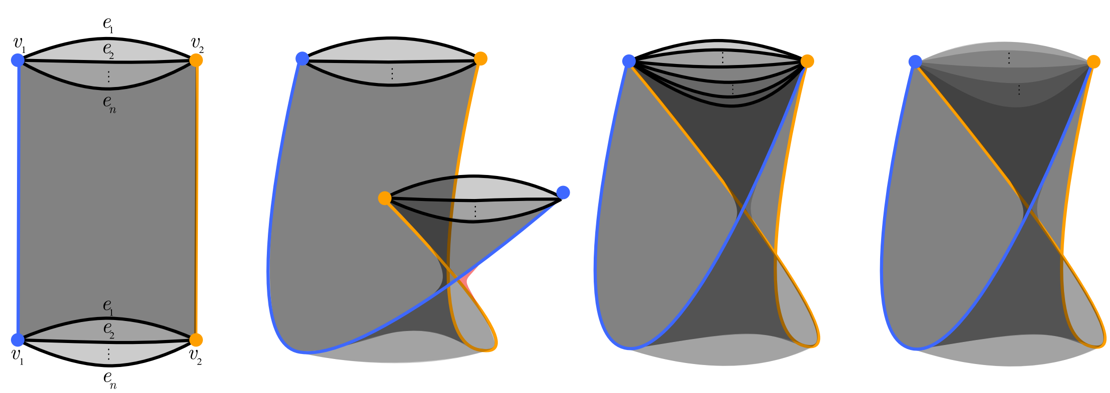


Figure 2.2.14: $L(n, \tau, \sigma)$ when τ is not the identity.

Imagine $H_n \times I$. As always, we can use either V_i or V_τ to glue the vertices. Because there are 3 edges in H_3, there we can use one of the 6 permutations in S_3 to glue the edges. Thus there are 12 possible ways to glue $H_3 \times \{0\}$ to $H_3 \times \{1\}$.

Rather than representing these layered strip spaces as their true shape embedded into $\mathbb{R}^3$, we will look at representations of layered strip spaces as 3 strips (which might be twisted) where all edges with the same color are glued together. See Table 2.2.1. $\Diamond$

We saw that all 1-layered strip spaces are either a washer or a Möbius strip. We saw that all 2-layered strip spaces are either a torus or a Klein bottle. We will now examine the stellate subspaces of non-trivial layered strip spaces.

Theorem 2.2.8. *Let $n \in \mathbb{N}$. Suppose $n \geq 3$. Let $i_2 \in S_2$ be the identity permutation and let $\sigma \in \ S_n$. Let $C(\sigma)$ be the number of cycles in σ. The following hold for $L(n, i_2, \sigma)$.*

1. *There are two circles in G_n.*

2. *There are $C(\sigma)$ washers in G_2.*

3. *$G_i = \emptyset$ if $i \neq 2$ and $i \neq n$.*

Proof. 1. The vertices of $H_n \times \{0\}$ and $H_n \times \{1\}$ are glued using the identity function. This means for all $i \in \{1, 2\}$ the two endpoints of the line $\{v_i\} \times I$ are glued together to create a circle. Note that for every point x on the circle there exists a neighborhood N of x where N is homeomorphic to P_n. Thus, by Definition 1.1.7, there are two circles in is G_n.

   There are no other points $x \in L(n, i_2, \sigma)$ where there exists a neighborhood N of x where N is homeomorphic to G_n. Hence, there are only two circles in G_n.

2. Since the two circles described in Part 1 of the proof are in G_n, we know by Lemma 1.1.10 that they are not in G_2. With this in mind we remove these two circles. Without these circles in $L(n, i_2, \sigma)$, we are left with n strips glued together by E_σ. Let $r = C(\sigma)$. Let $c_1, \ldots, c_r$ be the cycles in σ.

   Let $j \in \{1, \ldots, r\}$. Let $k \in \mathbb{N}$ be the length of c_j. Note that $k \leq n$. Let $s_1, \ldots, s_k$ be the edges in H_n in the cycle c_j. Thus, the edge s_p will be glued to s_{p+1} for all $p \in \{1, \ldots k\}$, where addition is mod k.

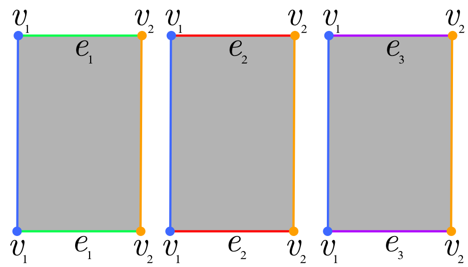
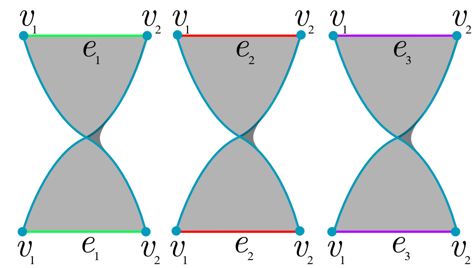
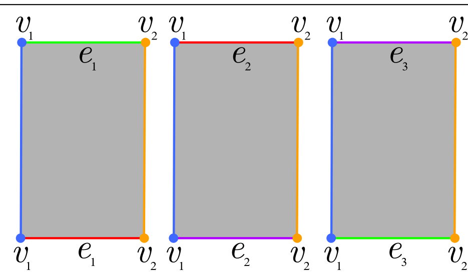
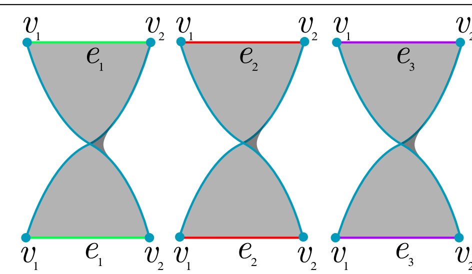
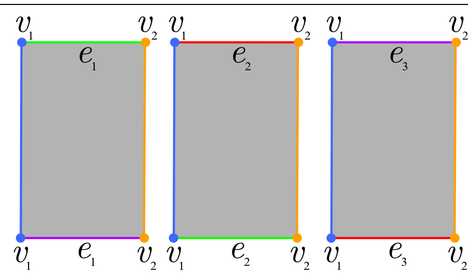
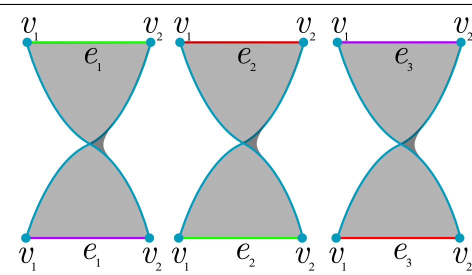
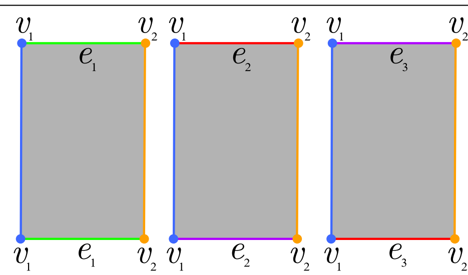
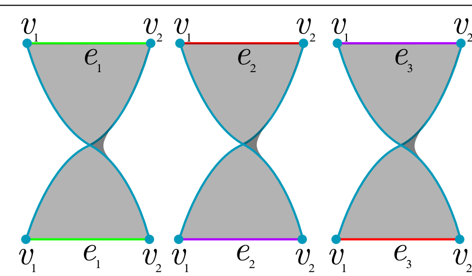
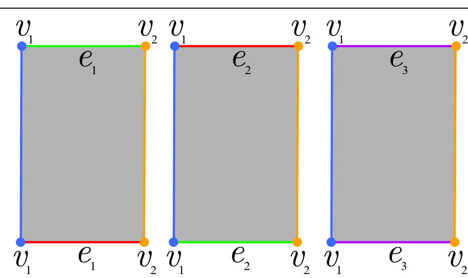
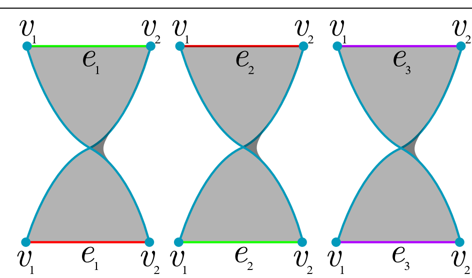
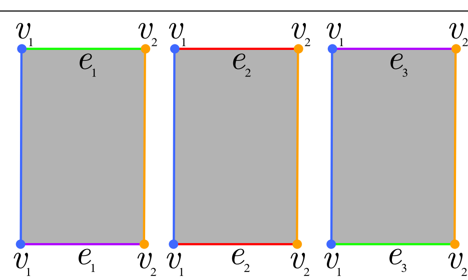
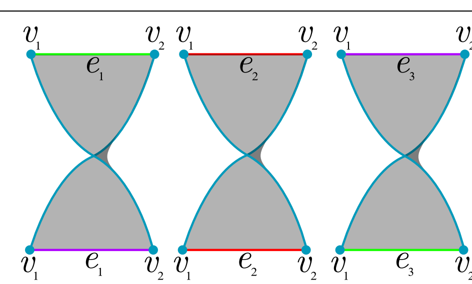
Edge Function	$L(3, i_2, \sigma)$	$L(3, \tau, \sigma)$
$\sigma = (1)(2)(3) = i_n$		
$\sigma = (123)$		
$\sigma = (321)$		
$\sigma = (1)(23)$		
$\sigma = (12)(3)$		
$\sigma = (13)(2)$		

Table 2.2.1: All $L(3, \tau, \sigma)$.

Because the vertices are glued using V_i, we know that each strip will not be twisted when glued to the next strip. Thus when all of the edges are glued together except for one, one long strip is created. Then, when the last edges are glued together the long strip becomes a washer. Thus we know that each cycle in σ creates a washer. Hence there are $C(\sigma)$ washers. (Note that because we removed the two circles in G_n, the washers are without boundary).

Because for each point x in an open washer there exists a neighborhood N of x such that N is homeomorphic to P_2, we know each washer is in G_2 by Definition 1.1.7. Hence there are $C(\sigma)$ washers in G_n.

Since there are no other points $x \in L(n, I, \sigma)$ where there exists a neighbourhood N of x where N is homeomorphic to P_n, there is nothing else in G_n. Hence, there are only $C(\sigma)$ washers in G_n.

3. Because there is no other point $x \in L(n, I, \sigma)$ where there exists a neighbourhood N of x where N is homeomorphic to P_i if $i \neq 2$ and $i \neq n$, we know that $G_i = \emptyset$.

□

Theorem 2.2.9. *Let $n \in \mathbb{N}$. Suppose $n \geq 3$. Let $\tau \in S_2$ not be the identity permutation and let $\sigma \in S_n$. Let $E(\sigma)$ be the number of even cycles in σ, let $O(\sigma)$ be the number of odd cycles in σ and let $C(\sigma)$ be the total number of cycles in σ. The following hold for $L(n, \tau, \sigma)$.*

1. *There is one circle in G_n.*
2. *There are $E(\sigma)$ washers and $O(\sigma)$ Möbius strips in G_2.*
3. *$G_i = \emptyset$ if $i \neq 2$ and $i \neq n$.*

Proof. 1. The vertices of H_n are glued not using the identity function. Thus the vertices must be glued together switched. This means that the endpoint $(v_1, 0)$ on the line $\{v_1\} \times I$ is glued to the endpoint $(v_2, 1)$ on the line $\{v_2\} \times I$. Note that the gluing of two lines together makes one long line. This long line has the endpoints $(v_2, 0)$ and $(v_1, 1)$ which are also glued together under V_τ to make one big circle. Note that for every point x in this circle there exists a neighborhood N of x where N is homeomorphic to P_n. Thus, by Definition 1.1.7, there is one circle in G_n.

There are no other points $x \in L(n, \tau, \sigma)$ where there exists a neighborhood N of x where N is homeomorphic to G_n. Thus, only one circle in G_n.

2. Since the circle described in Part 1 of the proof is in G_n, we know by Lemma 1.1.10 that it are not in G_2. With this in mind we will remove the circle. Without the circle in $L(n, \tau, \sigma)$, we are left with n strips glued together by E_σ. Let $r = C(\sigma)$. Let $c_1, \ldots, c_r$ be the cycles in σ.

   Let $j \in \{1, \ldots, r\}$. Let $k \in \mathbb{N}$ be the length of c_j. Note that $k \leq n$.

   Case 1: Suppose c_j is an even cycle. Note that then k is even. Let $s_1, \ldots, s_k$ be the edges in H_n in the cycle c_j. Thus, the edge s_p will be glued to s_{p+1} for all $p \in \{1, \ldots k,\}$ where addition is mod k.

   Because the vertices are glued using V_τ, we know that each strip will be twisted once before being glued to the next strip. Thus, when all of the edges are glued together except for one, one long strip is created that is twisted k times. Since k is even, we know that when the last edges are glued together the long strip becomes a twisted washer.

   Case 2: Suppose c_j is an odd cycle. Note that then k is odd. Let $s_1, \ldots, s_k$ be the edges in H_n in the cycle c_j. Thus, the edge s_p will be glued to s_{p+1} for all $p \in \{1, \ldots k,\}$ where addition is mod k.

   Again, because the vertices are glued using V_τ, we know that each strip will be twisted once before being glued to the next strip. Thus, when all of the edges are glued together except for one, one long strip that is twisted k times is created. Since k is odd, we know that when the last edges are glued together the long strip becomes a Möbius strip.

Thus we know that each even cycle in σ creates a washer and each odd cycle in σ creates a Möbius strip. Hence there are $E(\sigma)$ washers and $O(\sigma)$ Möbius strips. (Note that because we are ignoring the two circles in G_n, the washers and Möbius strips are without boundary).

Because for each point x in an open washer or open Möbius strip there exists a neighborhood N of x such that N is homeomorphic to P_2, we know each washer and Möbius strip is in G_2 by Definition 1.1.7. Hence there are $E(\sigma)$ washers and $O(\sigma)$ Möbius strips in G_n.

Because there are no other points $x \in L(n, \tau, \sigma)$ where there exists a neighborhood N of x where N is homeomorphic to P_n, there is nothing else in G_n. Hence, there are only $E(\sigma)$ washers and $O(\sigma)$ Möbius strips in G_n.

3. Because there is no other point $x \in L(n, \tau, \sigma)$ where there exists a neighborhood N of x where N is homeomorphic to P_i if $i \neq 2$ and $i \neq n$, we know that $G_i = \emptyset$.

□

Corollary 2.2.10. *Let $n \geq 3$. Let $\tau \in S_2$ be the permutation other than the identity permutation, and let $\sigma \in S_n$. For $L(n, \tau, \sigma)$, if n is odd then there exists a Möbius strip in G_2.*

Proof. Suppose n is odd. Then there must be at least one odd cycle in σ. Thus, by Theorem 2.2.9, we know that there exists a Möbius strip in G_2. □

2.3 Extended Graph Washers

An extended graph washer is created when a connected graph G is crossed with I and $G \times \{0\}$ is glued to $G \times \{1\}$ using a function to glue the vertices together and a function to glue the edges together. Both stellate washers and layered strip spaces are examples of extended graph washers. We will informally define extended graph washers and talk about some possibilities for further study.

First we will state some preliminary definitions on graph automorphisms. This definition comes from [6].

Definition 2.3.1. Let G be a simple graph. An **automorphism**, denoted ϕ of G, is a bijective mapping of the vertex set of G with itself with the property that $\phi(v)$ and $\phi(w)$ are adjacent whenever v and w are, for all vertices v and w of G. △

Note that this definition only applies to simple graphs. For our purposes, we state an alternate definition.

Definition 2.3.2. Let G be a graph. An **automorphism** ϕ of a graph G consists of two functions as follows.

1. Let V be a bijective mapping of the vertex set of G with itself with the property that $\phi(v)$ and $\phi(w)$ are adjacent whenever v and w are, for all vertices v and w of G.

2. Let E be a bijective mapping of the edge set of G with itself with the property that if the edge $\phi(e)$ is adjacent to vertex $\phi(v)$ then e is adjacent to vertex v, for all edges e of G and vertices v of G.

△

Definition 2.3.3. The **automorphism group** of a graph G, denoted $\Gamma(G)$, is the group of automorphisms of G. △

In order to create an extended graph washer, we must glue together $G \times \{0\}$ to $G \times \{1\}$ using an automorphism of G. The size of $\Gamma(G)$ determines how many different extended graph washers can be made out of G. Note that extended graph washers made from a simple graph would only need to use the standard definition of automorphisms.

Lemma 2.3.4. *If graph G contains one or more vertices connected by multiple edges, then* $|\Gamma(G)| > 1$.

Proof. Note that G is not a simple graph. As was the case for layered strip spaces, we can glue the vertices of G using the identity and still glue the edges switched. Thus we know that there will always more than one automorphism in which to glue G. □

Note that Lemma 2.3.4 does not mean that all non-simple graphs G have more than $|\Gamma(G)| > 1$. For example, an otherwise simple graph with a loop would have only one automorphism.

Some extended graph washers that deserve further exploration are those made from a graph G where $|\Gamma(G)| = 1$. Graphs with only one automorphism are called identity graphs or asymmetric graphs. See [3] for more on these kinds of graphs. On the other end of the spectrum, we should also examine extended graph washers made from graphs with many automorphisms which, as we saw for stellate washers and layered strip spaces, can create many variations of extended graph washers, and focus on how each gluing affects its stellate subspaces.

Conjecture 2.3.5. *The stellate subspaces of all extended graph washers (made from connected graphs) are made up of circles, and washers or Möbius strips.*

While there is no proof of Conjecture 2.3.5 yet, based off of the two types of extended graph washers we explored earlier in the chapter, both of which support this proposition, we can start to see why this would be the case. Imagine that X is a stellate washer or layered strip space constructed out of graph G and that v is a vertex in G connecting n edges. Because v is essentially a point in G, then $\{v\} \times I$ is a line in $G \times I$. If this line is glued to itself, it creates a circle. If the endpoints of this line is glued to the endpoints of another line, it will also create a circle. Thus no matter how $G \times \{0\}$ is glued to $G \times \{1\}$, G_n will have a circle in it. We can use similar logic to see how the edges create either a washer or a Möbius strip. Let e be an edge of G. Because e is essentially a line in G, then $\{e\} \times I$ is a rectangle in $G \times I$. If two opposite edges of this rectangle are glued to themselves, then it creates either a washer or a Möbius strip (depedning on how many times it is twisted before gluing). If a rectangle is glued to another rectangle along one edge, and then the two opposite edges of the now one, long rectangle are glued together, it also creates either a washer or a Möbius strip. Thus no matter how $G \times \{0\}$ is glued to $G \times \{1\}$, G_2 will have either a washer or a Möbius strip in it. Since graphs are always made up of vertices and edges, there is no reason to believe these would not be the same outcomes for all extended graph washers.

3
Orientability and Connected Sums

3.1 Orientability

We will take a look at one way of defining 2-dimensional non-orientable objects.

Definition 3.1.1. Let X be a 2-dimensional topological space. The space X is **non-orientable** if there exists a Möbius Strip $M \subseteq X$; otherwise, the space X is **orientable**. $\triangle$

With this definition of non-orientable, we can take a look at how it applies within the context of our stellate unions.

Definition 3.1.2. Let X be a stellate union. Suppose X is non-orientable.

1. The space X is **strongly non-orientable** if there exists a Möbius Strip M in G_2.
2. The space X is **weakly non-orientable** if there exists a Möbius Strip M in X that is not contained in G_2.

$\triangle$

Note that, because of Lemma 1.2.4, any non-orientable surface is strongly non-orientable. Similarly, any non-orientable surface with boundary is strongly non-orientable. This is because the Möbius strip in the surface must be in G_2.

We will now look at some non-trivial examples of strongly and weakly non-orientable spaces.

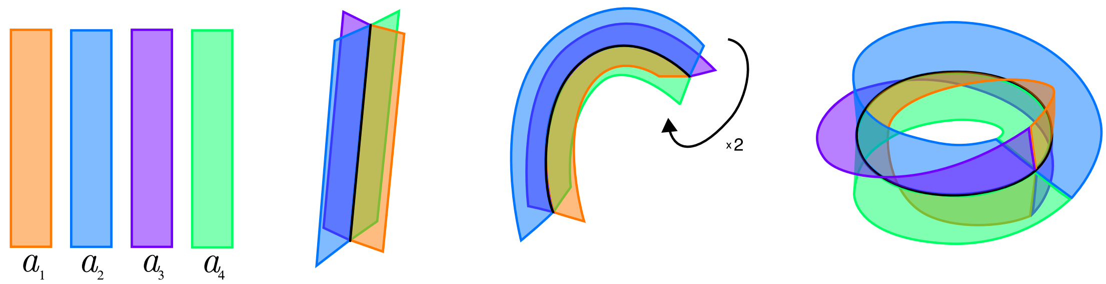


Figure 3.1.1: Creating $R_{(4,2)}$.

Example 3.1.3. We will examine $R_{(4,2)}$ as shown in Figure 3.1.1. Note that G_1 contains two circles, G_2 contains 2 twisted washers and G_4 contains one circle. While the washers in G_2 are twisted they are still orientable, meaning there is no Möbius Strip found within them. Thus, we know $R_{(4,2)}$ cannot be strongly orientable.

Now let's take a look at $a_2, a_4 \in R_{(4,2)}$. If we connect these flaps using the circle in G_4, we create a Möbius strip M_1. See Figure 3.1.2. Similarly, if we connect a_1, a_3 and G_4, we get another Möbius strip M_2. For now, we will focus on M_1. Note that because $M_1 = a_2 \cup a_4 \cup G_4$ of $R_{(4,2)}$ and $a_2, a_4, G_4 \in R_{(4,2)}$ that $M_1 \in R_{(4,2)}$. Thus, $R_{(4,2)}$ is weakly non-orientable. $\Diamond$

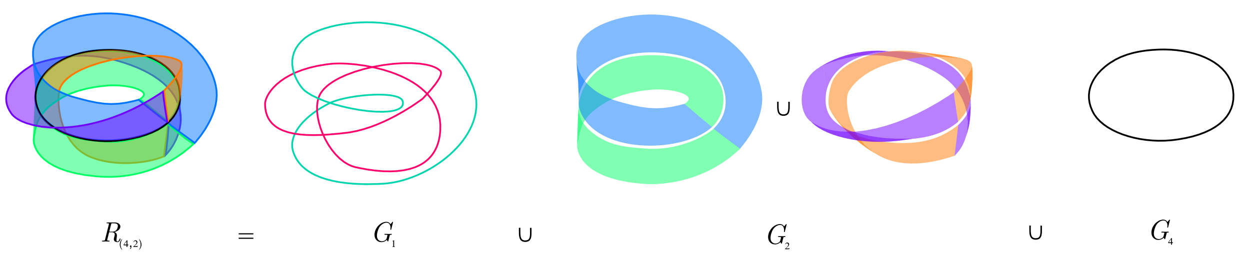


Figure 3.1.2: $R_{(4,2)}$ separated into G_1, G_2 and G_4.

Now we will look at another, less obvious example of a weakly non-orientable surface.

Example 3.1.4. We will examine $R_{(3,1)}$. We know that $R_{(3,1)}$ is not strongly non-orientable from Example 2.1.2 because G_2 does not contain a Möbius strip.

To show that $R_{(3,1)}$ is weakly non-orientable, we will place a letter "F" on the space and move it until it returns to the same starting place. We will track the movement of the F with various gray Fs in previous positions. First, place F on a_1 facing outward. Move the F downward until it moves onto a_2. Then slide the F across G_3 back to a_1. Now the F is facing inwards. Thus, we know there is a Möbius strip contained in $R_{(3,1)}$. Hence, $R_{(3,1)}$ is weakly non-orientable. See Figure 3.1.5. $\Diamond$

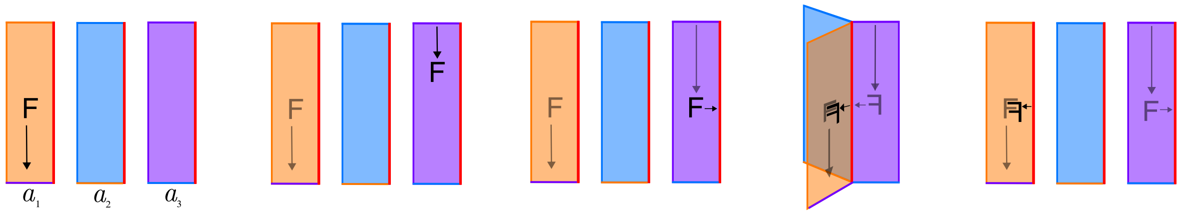


Figure 3.1.3: Finding a Möbius strip in $R_{(3,1)}$.

Now we will look at an example of a strongly non-orientable stellate union.

Example 3.1.5. Let let $\tau \in S_2$ be the permutation other than the identity permutation, and let $\sigma = (123)$. We will be examining $L(3, \tau, \sigma)$ as seen in Figure 3.1.4.

As seen in the Example 2.2.7, we know that there is one circle in G_n and one Möbius strip in G_2. Thus $L(3, \tau, \sigma)$ is strongly non-orientable. $\Diamond$

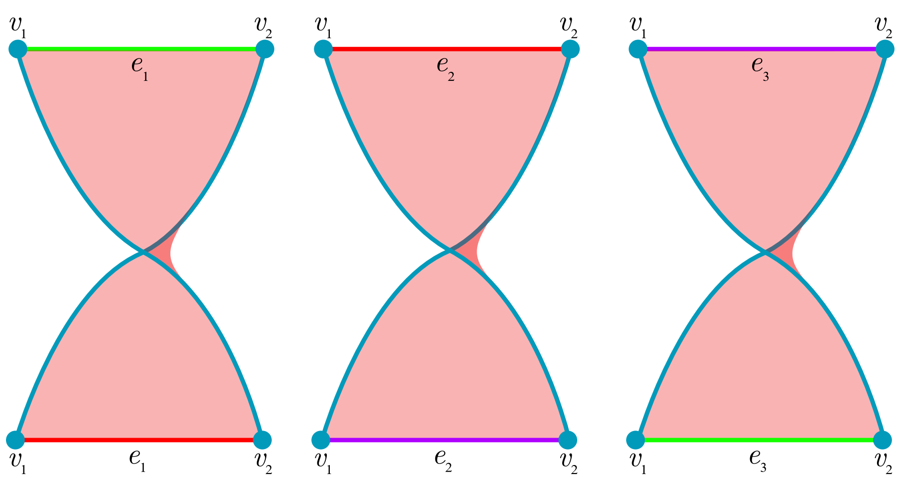


Figure 3.1.4: $L(3, \tau, \sigma)$.

Using the same technique used in Example 3.1.4, we can see that all stellate washers with a twist are weakly non-orientable. Note that this strategy does not work if the stellate washer does not have a twist.

Theorem 3.1.6. *Let* $n \in \mathbb{N}$ *and let* $p \in \{0, 1, \ldots, n-1\}$*. For* $R_{(n,p)}$*, if* $p \neq 0$ *then* $R_{(n,p)}$ *is weakly non-orientable. If* $p = 0$ *then* $R_{(n,p)}$ *is orientable.*

Using the same technique used in Example 3.1.4, we can also see that layered strip spaces where either the function on the vertices or the function on the edges is not the identity are also weakly non orientable.

Example 3.1.7. Let $n = 3$, let $i_2 \in S_2$ be the identity permutation, and let $\sigma \in S_3$ be $\sigma = (321)$.

We know from Theorem 2.2.8 that there is not a Möbius strip in G_2. Thus we know $L(3, i_2, \sigma)$ is not strongly non-orientable.

To show that $L(3, i_2, \sigma)$ is weakly non-orientable, we will place a letter "F" on the space and move it until it returns to the same starting place. First, place F on $\{e_1\} \times I$ facing towards v_2. Move the F downward until it moves onto $\{e_3\} \times I$. Then slide the F across $\{v_2\} \times I$ back to $\{e_1\} \times I$. Now the F is facing v_1. Thus, we know there is a Möbius strip contained in $L(3, i_2, \sigma)$. Hence, $L(3, i_2, \sigma)$ is weakly non-orientable. $\diamondsuit$

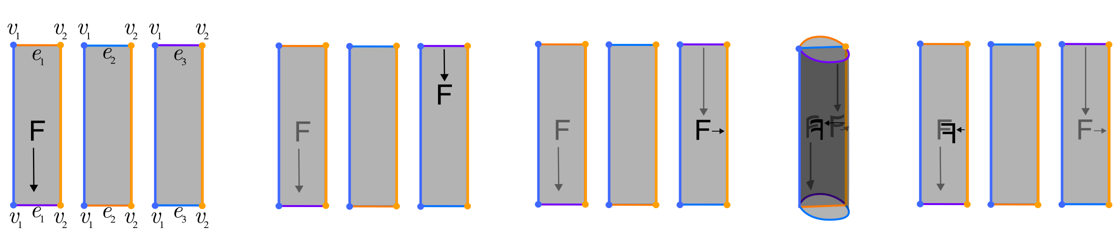


Figure 3.1.5: Finding a Möbius strip in $R_{(3,1)}$.

Theorem 3.1.8. *Let* $n \in \mathbb{N}$*. Let* $\tau \in S_2$ *and* $\sigma \in S_n$ *Let* $O(\sigma)$ *the the number of odd cycles in* σ*. The following hold for* $L(n, \tau, \sigma)$

1. *If both* τ *and* σ *are the identity permutations, then* $L(n, \tau, \sigma)$ *is orientable.*

2. *If τ is not the identity permutation and $O(\sigma) \geq 1$ then $L(n, \tau, \sigma)$ is strongly non-orientable.*

3. *All remaining layered strip spaces are weakly non-orientable.*

Proof. 1. Suppose both $\tau \in S_2$ and $\sigma \in S_n$ are their identity permutations. Then $L(n, \tau, \sigma)$ is essentially a torus with n layers. Because τ is the identity, we know that each layer is not twisted before being glued. Thus, by Theorem 2.2.8, we know there is not a Möbius strip in G_2. Because σ is the identity, we cannot use the F strategy to go to another strip and move back to the original strip to find a Möbius strip, because each strip is connected through only the circle in G_n. Hence, no Möbius strip can be found in $L(n, \tau, \sigma)$, so it must be orientable.

2. Suppose that τ is not the identity permutation and $O(\sigma) \geq 1$. Because τ is not the identity permutation, we know that each strip is twisted once before gluing. Thus we know it is possible for there to exist a Möbius strip. Because $O(\sigma) \geq 1$, by Theorem 2.2.9, we know there is at least one Möbius strip in G_2. Hence, by Definition 3.1.2, we know $L(n, \tau, \sigma)$ is strongly non-orientable.

3. Suppose $L(n, \tau, \sigma)$ is not one of the two previous cases. Then either τ is the identity permutation and σ is not the identity permutation, or τ is not the identity permutation and $O(\sigma) = 0$.

   Case 1: Suppose τ is the identity permutation and σ is not the identity permutation. By Theorem 2.2.8, we know that there cannot exist a Möbius strip in G_2. Thus, we know $L(n, \tau, \sigma)$ cannot be strongly non-orientable. Because σ is not the identity permutation, we know we can use the F strategy to find a Möbius strip in $L(n, \tau, \sigma)$. Thus, $L(n, \tau, \sigma)$ is weakly non-orientable.

   Case 2: Suppose τ is not the identity permutation and $O(\sigma) = 0$. By Theorem 2.2.9, we know that there cannot exist a Möbius strip in G_2. Thus, we know $L(n, \tau, \sigma)$ cannot be strongly non-orientable. Because $O(\sigma) = 0$, we know that σ cannot be the identity permutation, because then $O(\sigma) = n$. Thus, we can use the F strategy to find a Möbius strip in $L(n, \tau, \sigma)$. Hence, $L(n, \tau, \sigma)$ is weakly non-orientable.

□

Corollary 3.1.9. *Let $n \in \mathbb{N}$. Let $\tau \in S_2$ and $\sigma \in S_2$. Let $L(n, \tau, \sigma)$ be a layered strip space. Suppose n is odd and suppose τ is not the identity permutation. Then $L(n, \tau, \sigma)$ is strongly non-orientable.*

Corollary 3.1.9 is an extension of Corollary 2.2.10.

3.2 Connected Sums

Another potential area of exploration for stellate unions is how stellate unions are effected under connected sums. Connected sums are created when two surfaces are glued together through deleting a circle from each surface and gluing together these holes with a tube. The beauty of this method is that the two holes can be created at any location on each surface and the connected sum will stay the same. With stellate unions however, it is not as simple.

Example 3.2.1. Let $\tau \in S_2$ not be the identity permutation, and let $\sigma \in S_3$ such that $\sigma = (1, 2)(3)$. We will be creating connected sums out of $L(2, \tau, \sigma)$ and a torus.

Imagine we cut out a circle from $\{e_3\} \times I$ and glue it to a torus with a circle cut out. Then in G_2 there will be a washer, and a Möbius strip connected to a torus.

Now imagine that we cut out a circle from $\{e_1\} \times I$ and glue it to a torus with a circle cut out. Then in G_2 there will a washer connected to a torus, and a Möbius strip.

Thus, we can see that the two different placements of where we cut out the circle has an impact on the stellate makeup of the connected sum. ◊

Here we can tell how applying connected sums to stellate unions prove to be much more complicated. Some further questions could be: Are there positions of the holes that cause the number of components to change in stellate subspaces? Is there an extended graph washer where the positions of the holes does not change the connected sum?

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