From Constant to Stochastic Volatility: Black-Scholes Versus Heston Option Pricing Models

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Abstract

The Nobel Prize-winning the Black-Scholes Model for stock option pricing has a simple formula to calculate the option price, but its simplicity comes with crude assumptions. The two major assumptions of the model are that the volatility is constant and that the stock return is normally distributed. Since 1973, and especially in the 1987 Financial Crisis, these assumptions have been proven to limit the accuracy and applicability of the model, although it is still widely used. This is because, in reality, observing a stock return distribution graph would show that there is an asymmetry or a leptokurtic shown in the stock return.

Therefore, we propose that by introducing the Heston Model, we can tackle these two problematic assumptions in the Black-Scholes Model. The Heston Model considers the leverage effect and the clustering effect, which allows the volatility itself to be random and also allows it to take the non-normally distributed stock return into account.

In our project, we aim to show whether the Heston model can actually improve the option pricing estimates by using the $S\&P$ 500 Index European Call Option to compare it to the Black-Scholes Model. We find that even though the results show that the Heston Model performs worse than the Black-Scholes Model when the option expiration date is soon to expire, the Heston Model significantly outperforms the Black-Scholes Model in almost all combinations of moneyness and maturity scenarios. There remains further work to improve the Heston Model.

Contents

Dedication

I want to dedicate this project to my parents in Taiwan, Anna Cheng and Tai-Ju Wu. I am really grateful for their supports, love, and encouragements. I am really appreciated that I have very wonderful parents who give so many guidelines and always supports me to do a lot of different things. I really appreciate that I decided to come to Bard to study both math and music in 2015. It is a very tough program, but I really learn so much at Bard. I grow so much and learn how to find my own path here. I am glad that I make it to the place where I am today. I hope I can dream higher and be more fearless in trying and learning different new things in my life.

Acknowledgments

I thank for my advisor, Stefan for working with me on this topic. I thank my parents for their endless supports and love. I thank Jane for always helping me when I need the supports, thank for Paul meeting with me to improve my project, and thank professor Li on the other side of the world to answer my questions and provide the guidelines. I cannot finish this project without my beloved two friends Alisa and Corrina. We have too many memories throughout the 4 years at Bard. I appreciate all these supports I got from Bard. I thank for professor Gautam for suggesting this topic for me. This really opens a new door for me and I would like to explore more applied math in finance and consider working in quantitative finance. At last, I thank Michael, for always being by my side, supporting me and encouraging me when I am down.

1 Introduction

The purpose of my project is to compare the accuracy of two financial models–the Black-Scholes-Merton (BSM) model and the Heston Model–that are used to estimate the price of an option. I will give an example of what a call option is first and will move into the backgrounds of these two models.

Imagine one day that you cannot wait to get the new iPhone for your little brother for Christmas, but the store is running low on stock and you are afraid the price will go up. And you only have budget of \$200 and the price of the iPhone is \$220. Someone unknown comes up to you and says "if you give me \$4, I can guarantee that you can buy the new iPhone on December 21st for \$195." If on December 21st, the price is cheaper than \$195, you will just buy it directly from the market since the price is cheaper than \$195 offered by the stranger. In this case, you will just lose \$4 for accepting this stranger's offer.

Paying the \$4 is called buying a call option. Call options are a formal contract between a buyer and a seller. The holder of a call option (in this case, you) has the right, but not the obligation, to buy a commodity at a specified price on a specified date. Options are important not only because they can serve as an insurance to protect your investment when the market is not doing well but also you can own a commodity without really paying the actual commodity price until the option expires. This saves you money and you can use that money to make other investments. Thus options are very a flexible and powerful tool for investors to make money from the market.

In my example, the price of the commodity is \$220, the maturity is December 21st, and the price which you (the buyer) and the stranger (the seller) agree on while you accept the offer (called a strike price) is \$195. If all these quantities are fixed, the question becomes what is a fair price for the option? If we can have a way to calculate a fair option price, then we can see whether the option price being offered (in this case, \$4) to you is worth it or not before you agree to buy the option.

The Black-Scholes-Merton (BSM) model for option pricing was developed by Fischer Black, Myron Scholes and Robert Merton in 1970s. This impacted the financial world because it became possible to price options using a relatively simple explicit closed-form formula provided by the BSM for the first time. But this simple model comes with many assumptions that are not consistent with what is observed in the financial market, which we will explain later on. This model is still widely used not just because it becomes the standard and traditional way to calculate the , and was awarded the 1997 Nobel Prize in Economics. But also it is easy to implement. If everybody is using the BSM to price options, it make sense to take a look at what assumptions this model are based on and how the model is implemented.

There are only five parameters needed for the BSM to estimate the, which are stock price, strike price, interest rate, maturity date, and implied volatility (the standard deviation or the amplitude) of the stock. The five parameters are represented mathematically as follows:

$$BSM(\text{stock price, strike price, interest-rate, maturity}, \sigma) = \text{theoretical option price}$$

The output of the BSM, in its initial formulations, gives a theoretical option price that is not rooted in realistic situations. The first four parameters, stock price, strike price, interest rate, maturity, are observable market data. These variables can be easily determined by looking at market data. The only parameter left, σ, represents the implied volatility of the stock return, which is not easily observable in the market. Thus the way to solve the implied volatility is to set the market price as the fair option price and to solve the BSM backward to find the appropriate

σ that makes the output of the BSM equal to the market . That is why it is called “implied” volatility because it is the value that is implied by the market option price.

But the simplicity of the formula comes with crude assumptions. I will mainly deal with two major assumptions in the BSM in this project–the constant stock volatility (a measure of how far the stock return is from the mean; the standard deviation) and the lognormal stock returns (the normal distribution,or bell-curved distribution, of the log stock returns). As described in [1], since the 1987 stock market crash, these assumptions have been proven wrong. A volatility smile or skew observed in the market shows that the BSM assumption on constant volatility is wrong.

As pointed out in [3], the volatility smile is the consequence of the lognormal stock return assumptions in BSM proposed by John C. Hull and Nattenburn. Also the stock return in the real market is actually not normally distributed. There are asymmetry and fat tails (the longer tails extended on the both sides of the distribution) shown in the stock return distribution as mentioned in [3]. It is obvious that both the constant volatility and the lognormal stock return in the BSM cannot represent the real financial market. The question becomes how to price options in a way that takes the real financial market into account.

One of the approaches to tackle this problem is to allow the volatility to be random, or stochastic. There have been many models proposed for stochastic volatility. But Heston remains one of the most popular ones due to its closed-form solutions for the European options. Scott (1987), Hull and White (1987), Wiggins (1987) generalized the model to allow volatility as a stochastic process. However, as described in [4], their models require quite an amount of numerical computations and do not really take into account the fact that volatility is correlated with stock returns.

In 1993, the Heston Model was developed by Steven Heston. It provides a stochastic model that extends the BSM. It addresses both the constant volatility and the lognormal stock returns assumptions by relaxing these assumptions. It allows the volatility itself to be a random variable and also allows the log of stock returns to be nonstandard normal distribution. By allowing non-constant volatility and non standard normal distribution of the stock returns, the Heston Model can take the asymmetry, the fat tails observed in the stock return distribution, and the volatility smile into account.

As described in [28], the Heston Model assumes both the stock price and the variance of the stock price follow the Brownian motions (a random, or stochastic process). And these two Brownian motions are correlated. In reality, the correlation is usually negative in the equity world. That means increases/decreases in the stock price tend to be coupled with a decrease/increase in volatility, which is called the leverage effect in finance. Besides taking into account the correlation between stock price and volatility, the main attraction of the Heston Model is its closed-form solution. This makes the calibration of the model feasible as described in [6]. The Heston parameters can be obtained by calibrating to market data. We used the *lsqnonlin* nonlinear least-squares optimization method in MATLAB to calibrate the Heston Model in the project.

Besides the *lsqnonlin* method, there are many different optimizations for calibrating the Heston Model, including but not limited to the Adaptive Simulated Annealing(ASA), Generalized Reduced Gradient, and Genetic Algorithm(GA). All calibration algorithms search for a region of the parameters while trying to minimize the error metrics. We can see [24] and [28] for more detailed information about using different optimisations to calibrate the Heston Model. But both [24] and [28] prove the *lsqnonlin* nonlinear least-square method in MATLAB is faster in obtaining the five unknown parameters in the Heston Model, and that the five unknown parameters obtained by using the *lsqnonlin* method generate more accurate option prices among the other optimizations.

In this project, we need to deal with a total of 3,774 $S\&P$ 500 Index European call option data for the whole month. Thus we choose *lsqnonlin* due to its speed and accuracy. But the method is sensitive to the choice of the initial point. The way to address the sensitivity is to define the range of acceptable solutions, which at least guarantees that the solutions are not just mathematically feasible but also make sense economically.

My main focus will be the comparison of the estimates given by the Heston Model, the Black-Schoels model, and the actual market option price from iVolatility.com. I use the mean absolute percentage error (MAPE) as my accuracy comparison criteria. The smaller the number is, the more accurate it is. My results show that the Heston Model is more accurate than the BSM in estimating the $S\&P$ 500 Index European call option price.

It is worth mentioning that most of the papers testing the accuracy of the BSM and the Heston Model do not use enough data sets to calibrate the Heston Model. They only choose certain segments of data sets without fully taking all possible scenarios into account. Thus their calibration processes are less time-consuming and easier because they do not use enough market data to calibrate the Heston Model. Thus the five unknown parameters (initial volatility, long-term volatility, mean-reversion speed, the correlation between stock price and volatility) cannot fully explain the market behaviors, even though the results might still show that the Heston is more accurate in predicting option prices compared to the BSM.

Thus we want to consider as many scenarios as possible in trading European call options. So I use a total of 3,774 $S\&P$ 500 Index European call options which covers a total of 96 different maturities and of 98 different strike prices traded in the market from February 4th to March 4th in 2019 to calibrate the Heston Model in order to accommodate more trading call option situations by using more different maturities and strike prices. This is not an easy task because I use a larger amount of data compared to many papers which only use less than 300 data to run the results. I divide a total of 3,774 data into 15 files; each file represents one scenario depending on the combination of moneyness (the degree of whether you make money or lose money) and maturities. By doing this, I can show a clearer picture of how both models perform under these 15 different scenarios. It is worth mentioning that using the whole month data to calibrate the Heston Model requires a great deal of computer memory than a standard Macbook to find the five unknown parameters. I needed to run the algorithm on an iMac in the lab since a standard Macbook crashed due to lack of memory. Thus the whole process for analyzing the whole day data traded in the market is quite time-consuming.

Many researchers only use a few maturities and strike prices to calibrate the Heston Model, and use the five unknown parameters obtained from the calibration in the Heston Model to estimate the option prices. But, without using enough data sets to calibrate the Heston Model, the results can only reflect the behavior of the market under those segments of data, which fails to give more realistic and more accurate estimations that take enough trading situations into account. Thus in my project, we try to take more different strike prices and maturities into account and hope to obtain the five unknown parameters that reflects more different trading situations observed in the market into account by using the whole month $S\&P$ 500 Index European call option data. After obtaining these five parameters, we use them afterwards in the Heston Model and compare the Heston's estimates, the BSM's estimates and the real market data for a total of 3,371 call options traded on March 19th in 2019.

The purpose of my project is to compare the accuracy of the BSM and the Heston Model. Thus our focus will be on the implementation of these two models. For the details of the derivation of these two models, see [2] and [4]. For the data, we will mainly deal with $S\&P$ 500 Index European call options. The basic concepts for the options will be presented in Chapter 2.

1.1 Method

To test the accuracy of the BSM and the Heston Model for Index, I first obtained the data from iVolatility.com on March 19th , which includes stock price, maturity, strike price, mid price, bid price, ask price and implied volatility. For the interest rate, I looked up 1-year yields from the treasury.gov. I filtered out the data with too short or too long maturities, negative volatility, and the very deep-in-the-money (the position where you make a lot of money from the options) and very deep-out-of-the money (the position where you lose a lot of money), since these cases are quite rare and short-lived in the market. I was left with 3, 371 call options with a total of 23 maturities and a total of 103 different strike prices. I categorized them into 15 files in Excel depending on their maturity and the moneyness.

It is a lot easier to implement the BSM since it has simple formula and it does not require extra calibration before implementing the model as the Heston Model does. Thus for the BSM, we use the MATLAB code in the Appendix *bsm_call* to estimate the $S\&P$ 500 Index European call options under the BSM. But for the Heston Model, it requires a calibration process to find the five unknown parameters which I will introduce later. Once obtaining the five unknown parameters, we have all the input parameters needed in the Heston Model and thus can implement the Heston Model using the function *call_heston_cf* in MATLAB (see Appendix A).

The approach for the calibration process is to calibrate the Heston Model to the real market data to obtain the five unknown parameters, which are the initial volatility (V_0), the long-term volatility ($\bar{V}$), the volatility of the variance process (η), the mean-reverting speed for the volatility process (a), and the correlation between the stock price and the volatility (ρ). For the calibration data, I obtained one month of $S\&P$ 500 Index European call options from iVolatility.com from February 4th to March 4th in 2019. I filtered out the negative volatility, too short, too long maturity, very deep-in-the-money and very deep-out-of-the-money. I was left with 3,774 call options with a total of 96 different maturities and a total of 98 different strike prices. I used the *lsqnonlin* nonlinear least-square method in MATLAB to search for the parameters by minimizing the squared error between the model price and the market price. Once I had the five parameters, I plugged those values into *call_heston_cf* in MATLAB to estimate the values of the option prices. Finally, we can compare the estimates of the Heston Model and the estimates by the BSM to the market data we obtained on March 19th by using the forecast error called the mean absolute percentage error (MAPE).

1.2 Structure

This project is divided into the following chapters. In Chapter 2, we introduce the basic finance concepts and the Black-Scholes Model, but the attention will be given to the volatility and the limitations of the BSM, which will lead to the assumptions of the Heston Stochastic Volatility Model. In Chapter 3 , we introduce the Heston Stochastic Volatility Model and discuss the influence of the five missing parameters in the Heston Model in great detail. The ways to obtain these missing five parameters will require the calibration process. Thus in Chapter 4, we introduce the different optimizers to calibrate the Heston and uses the standard approach–*lsqnonlin* nonlinear least-squares method in MATLAB–to calibrate the Heston Model in order to obtain the five missing parameters. In Chapter 5, we present the results in the comparison of the accuracy between the BSM and the Heston Model using mean absolute percentage error (MAPE). Chapter 6, we present the conclusion and discuss what further work remains to be done in order to improve my results. Appendix A contains all the MATLAB codes used for the calculations throughout the project.

1.3 Terminology

The following terms and notations will be used a lot in the project, but will be explained fully in the following chapter. This section is mainly cited from [33].

- Call, C: An option that gives the holder the right to buy a share of stock on a given date at a predetermined price.
- Put, P: An option that gives the holder the right to sell a share of stock on a given date at a predetermined price.
- Strike Price, K: The price at which the holder can buy or sell the underlying stock.
- Expiration data or Maturity date, t: The date at which the underlying stock is selling at date t.
- Option price: The price at which the option is sold or bought.
- Risk-free interest rate, r: The rate of of an investment with no risk of financial loss, over a given time period.

2
The Black-Scholes Option Pricing Model

In this chapter, we introduce the finance terms and the basic financial concepts that are fundamental to the Black-Scholes Merton (BSM) model and the Heston Model. In this project, we only focus on pricing European call options in order to align with the BSM assumption in options being European style, or European options. In brief, European options only can be exercised on the expiration date, whereas American options can be exercised before, or on, the expiration date. The main attention will be drawn to the Black-Scholes Model session and its limitations session.

2.1 Derivative Contracts

Derivatives are contracts based on the the underlying asset price. They can be applied to almost any types of asset such as oil, gasoline, gold, commodity and stocks. In this project, we will focus on derivatives based on stocks. That is stock options. Thus we will introduce the definitions of stock and stock return first, and the main attention will be given to European call options. This section is mainly cited from [7], [13], [8], [27], and [34].

Definition 2.1.1. As described in [7], A **stock** also known as “shares” or “equity” is a type of security that signifies proportionate ownership in the issuing corporation. This entitles the stockholder to that proportion of the corporation's assets and earnings. △

Definition 2.1.2. Let P_0 be a stock price at initial buying time, let P_1 be stock price at sell time, and let D be the dividends. The **stock returns** means the net lost or gain made on an investment, which is given by the formula

$$\text{Total Stock Return} = \frac{(P_1 - P_0 + D)}{P_0}.$$

△

Definition 2.1.3. As described in [13], **options** are financial derivative sold by an option writer to an option buyer. They are typically purchased through online or retail brokers. The contract offers the buyer the right, but not the obligation to buy or sell the underlying asset at an agreed upon price under a certain period of time or on a specific date. △

Definition 2.1.4. A **European call option** is a contract that gives its holder the right, but not the obligation, to buy one unit of a stock S for a specified price also known as strike price K on a time maturity date T but not before or after time maturity date T.

A **European put option** is a contract that gives its holder the right, but not obligation, to sell one unit of a stock S for a specified strike price K on a time maturity date T but not before or after time maturity date T.

The payoffs for the European call option and put option are given by

$$\text{Call} = \begin{cases} (S_T - K) - \text{call option price} & \text{for } S_T > K \\ \text{call option price} & \text{for } S_T \leq K \end{cases}$$

and for the European put option in the analogous way

$$\text{Put} = \begin{cases} \text{put option price} + (K - S_T) & \text{for } S_T < K \\ \text{put option price} & \text{for } S_T \geq K. \end{cases}$$

See Figure 2.1.1 and Figure 2.1.2. △

Example 2.1.5. Consider an investor who buys a European call option contract on Apple stock with a strike price \$100 to purchase 100 shares of the stock. Suppose that the current stock price is \$100, the maturity date of an option is in 3 months, and the price of this option to purchase one share is \$2.00. The initial investment of buying this call option is $\$100 \times \$2 = \$200$. Since it is European call option, the investor can exercise the option only on the maturity date. We divide our scenario into two cases.

- Scenario one: When the option expires, Apple is trading at \$105.

  The call option gives the buyer the right to purchase those shares at \$100 per share. In this scenario, the buyer could use the option to purchase those shares at \$100, then immediately sell those same shares in the open market for \$105. The buyer can immediately gets additional \$5 per share and make the total extra $\$5 \times \$100 = \$500$. Since the investor purchased this option for \$200, the total payoff to the buyer is $(\$105 - \$100) \times 100$ shares $-$ $\$200 = \$300$.

- Scenario two: When the option expires, Apple is trading at or below \$100.

  The investor will clearly choose not to exercise and let the contract expire worthless. So the investor will only lose the \$200 initial investment.

$\Diamond$

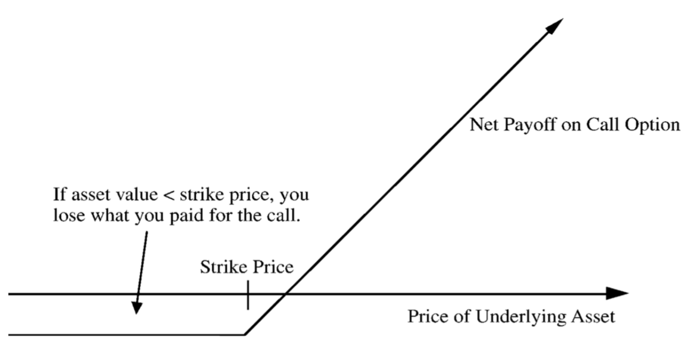

Figure 2.1.1. Payoff on Call Option

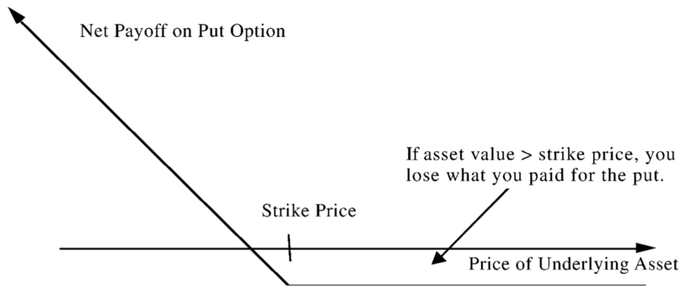

Figure 2.1.2. Payoff on Put Option

Definition 2.1.6. Moneyness is the relationship between the strike price of an options contract and the price of the underlying security at the time of maturity. The followings main three terms are in terms of call options. **In-the-money** means when underlying stock price of an option is higher than the strike price of the option. **At-the-money** means when strike price is equal to the underlying stock price of an option. **Out-of-the-Money** means the strike price is higher than the underlying stock price of an option. For put options, they will have the opposite relationship.

The **moneyness** for call options is defined as the percentage difference between the stock price (S) and the strike price (K):

$$\text{moneyness} = \frac{S}{K} - 1.$$

△

Moneyness serves as an important criteria when you are an option trader. For example, if you are buying option contracts on an underlying asset that you are expecting to move dramatically in price in a short time, then buying out-of-the-money contracts would maximize your potential profits. If you are expecting smaller movement, then in-the-money contracts would probably represent a better, less risky investment. [8] provides further information.

2.2 Black-Scholes-Merton model

The Black-Scholes-Merton (BSM) model was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes and is still widely used today. BSM is the first pricing model used to determine the theoretical value for European call options. It is based on five input parameters: stock price

S, strike price K, maturity t, risk-free interest rate r and implied volatility of stock returns, or standard deviation of stock return σ.

BSM has become the common language among option traders. It is regarded as the benchmark to determine option prices due to the fact that it is the most standard and traditional option pricing model, it has the simplest formula and it is easy to implement. Option traders use the BSM to buy call options priced higher than the BSM and sell options that are priced higher than the BSM.

2.2.1 Black-Scholes Formula

Let C be the European call option price, S be the current stock price, K be the strike price, r be the risk-free interest rate, σ be the standard deviation of the logarithm of stock return, or implied volatility, t is the maturity, N be the standard normal cumulative distribution function with mean 0 and standard deviation 1 and $N(d)$ be the value of the cumulative standardized normal distribution evaluated at d_1 and d_2.

The **The Black-Schoels Model Formula** for European call options is defined as:

$$C(S, K, \sigma, t, r) = SN(d_1) - Ke^{-rt}N(d_2) \tag{2.2.1}$$

where

$$N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \qquad \text{for all } -\infty < z < \infty,$$

$$d_1 = \frac{log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{log(\frac{S}{K}) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

and $N(d_1)$ and $N(d_2)$ are probability factors.

2.2.2 Explanations for the Black-Scholes Formula

This section is consulted from [25]. e^{-rt} is the present value factor, and reflects the fact that the strike price K on the call option does not have to be paid until expiration. The term Ke^{-rt} is the strike price K discounted back to present value. $N(d_2)$ is the probability that the stock price will be at or above the strike price when the option expires. In other words, $N(d_2)$ is the probability that the option will be exercised. Therefore, the term $Ke^{-rt}N(d_2)$ means the strike price discounted back to present value times the probability that the option is at or above the strike price when the option expires.

$N(d_1)$ is roughly, looking for the area under the bell curve up to d_1. See Figure 2.2.1. $N(d1)$ can also be interpreted as the probability that the future price will be above the strike price on the expiration date. See Nielsen [30] for the more details about $N(d_1)$ and $N(d_2)$.

In general, we can view $SN(d_1)$ as how much you get for exercising this option, or return. We can view $Ke^{-rt}N(d_2)$ as how much you pay for the option, or the cost of the option. Thus $SN(d_1) - Ke^{-rt}N(d_2)$ is the total profit you get minus the total cost of the option. Roughly, it is an investor's return minus the cost of the option.

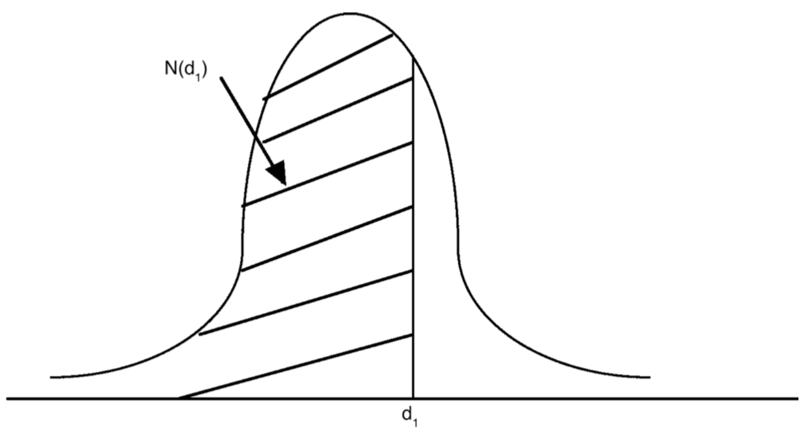

Figure 2.2.1. Cumulative Normal Distribution

2.2.3 The BSM Assumptions

When the Black-Scholes Formula was published, it was under the following assumptions. This section is mainly cited from [19].

1. The stock price S follows a stochastic process, geometric Brownian motion, $dS = \mu S dt + \sigma S dz$, with constant drift μ, standard deviation of the stock return σ and dz is a standard Wiener Process.

2. The stock return involving in the computation of the Black-Scholes formula is log-normally distributed with constant mean and variance rates.

3. No traction costs and no taxes;

4. No dividends are paid during the life of the options;

5. There are no risk-less arbitrage opportunities;

6. Based on European options;

7. Continues trading;

8. The risk-free rate of interest r is constant for different maturities.

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2.2.4 BSM Parameters

The value of a European call option in the BSM is determined by five parameters relating to the commodity and financial markets. The details for each parameter is presented. This section is cited from [34].

1. Current Stock Price S

   Options are assets that derive value from an underlying asset (stock). Therefore, changes in the value of the underlying asset affect the value of the options on that asset. Since call options provide the right to buy the underlying asset (stock) at a fixed price (strike price), an increase in the value of the asset will increase the value pf the call options. Puts, on the other hand, become less valuable as the value of the asset increases.

2. Implied Volatility σ

   Implied Volatility represents the standard deviation. The buyer of an option acquires the right to buy or sell the underlying asset (stock) at a fixed price. The higher the standard deviation in the value of the stock, the greater the value of the option. This is true for both calls and puts. While it may seem counterintuitive that an increase in standard deviation should increase value, options are different from other derivatives since buyers of options can never lose more than the price they pay for them; in reality, they have the potential to ear significant returns from large price movements.

3. Strike Price K

   Strike price is the price that the buyers and the sellers agree to buy or sell. In the case of call options, where the holder acquires the right to buy at a fixed price, the value of the call will decline as the strike price increase. We can see from Equation 2.2.1. If the $SN(d1)$ stays the same, the increase in the $K^{-rt}N(d2)$ will decrease the output of the BSM. In other words, the call option value will become less.

4. Time to Expiration, or Maturity t

   Both calls and puts are more valuable, when the maturity is greater. This is because the longer time to expiration provide more time for the value of the underlying asset (stock) to move, increasing the value of both types of options. Besides, in the case of call, where the buyer has to pay a fixed price at expiration, the present value of this fixed value decrease as the life of the option increases, increasing the value of the call.

5. Risk-free Interest Rate r

   Since the buyer of an option pays the price of the option up front, an opportunity cost is involved. This cost will depend on the level of interest rate and the time to expiration of the option. The risk-free interest rate also enters into the valuation of options when the present value of the strike price is calculated, since the strike price does not have to be paid (received) until expiration on calls or puts. Increase in the interest rate will increase the value of calls.

2.3 Implied Volatility–The Only Missing Parameter in the BSM

The Black-Scholes Model requires only five inputs into the model. These five inputs are stock price (S), strike price maturity (K), maturity (t), risk-free interest rate (r), and standard deviation of the stock return (σ), or implied volatility. The formula for the BSM can be expressed as

$$BSM(S, K, t, r, \sigma) = \text{Theoretical Call Option Price.}$$

The first four parameters S, K, t and r are easily observable in the market. They can be determined by looking at the market data. The only missing parameter in the BSM is the standard deviation of the stock return, σ, which is not observable in the market. To calculate σ, we first make the BSM equal to the market price and then solve the equation backwards to find the appropriate σ that makes the equation equal to the option market price.

In other words, **implied volatility** is the value of the volatility parameter σ that must go into the BSM formula (see Equation 2.2.1) to match the market price:

$$BSM(S, K, t, r, \sigma) = C_{market} \tag{2.3.1}$$

where C_{market} is the easily observable market call option price, S is the stock price, K is the strike price, t is the maturity, r is the risk-free interest rate and σ is the implied volatility, or standard deviation of stock return.

As we have seen that **implied volatility** is the value we get from equating the option market value to its BSM value. It reflects the volatility suggested by the market and also tells us how volatile the stock would be in the future.

2.3.1 The Newton-Raphson method

This section is mainly cited from [36]. One of the most efficient algorithms to estimate the implied volatility from the market observed price and the theoretical Black-Scholes formula is the Newton-Rahpson method. Newton-Raphson method is used to find the zeros of a real valued function. This means if there is a function $f(x)$, then the root of the function would be such that $f(x) = 0$. The Newton Method is well suited on computers because it is iterative in nature. This feature of Newton-Raphson method has attracted many scientists and many scientific application programs use Newton-Raphson method as one of the root finding tools. The **Newton-Raphson method** in one variable is implemented as follows:

Given a function $f(x)$ defined over the real x, and its derivative $f'(x)$, we begin with a first guess x_0 for a root of the function f. Provided $f'(x) \neq 0$, a better approximation x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Geometrically, $(x_1, 0)$ is the intersection with the x-axis of a line tangent to f at $(x_0, f(x_0))$. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.3.2}$$

until a sufficiently desired accurate value is obtained. See Figure 2.3.1 for the geometric interpretation of Newton-Raphson Method.

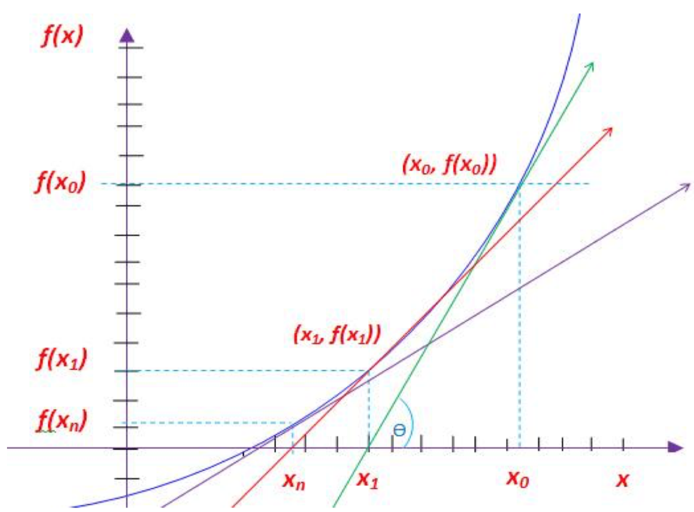

Figure 2.3.1. Geometric Interpretation of Newton-Raphson Method

2.3.2 How to apply the Newton Method to calculating the implied volatility?

In order to apply the Newton method to calculating the implied volatility, we need to find a function $f(\sigma)$ so we can use the Newton method to find the σ such that $f(\sigma) = 0$. We can view the BSM as a function called $BSM(\sigma)$ which only depends on σ, provided that the other four parameters S, K, t, r are known and they are constant. To calculate the implied volatility, we let the BSM equal to the market observed price C_{market} which is a constant. The process is presented as

$$BSM(\sigma) = C_{market}. \tag{2.3.3}$$

Then we let $f(\sigma)$ to be difference between the market observed option price C_{market} and the BSM theoretical option price such that

$$f(\sigma) = BSM(\sigma) - C_{market}. \tag{2.3.4}$$

Our objective is to estimate $f(\sigma)$ by using the Newton method until the σ value is close enough so that the method converges such that $f(\sigma) = 0$. However, when we applied the Newton method to calculate the implied volatility in MATLAB, we encountered the problem that our initial guess for σ is not "good" enough so the Newton method fails to find the root.

2.3.3 Weaknesses of Newton Method

This section is mainly cited from [37]. As we see in Equation 2.3.2, the Newton method keeps iterating until a sufficient desired accurate value is found. This kind of iterative process would make the Newton method quite analytically trackable. However, when we applied the Newton Method to calculate the implied volatility in MATLAB, we encountered the problem that our initial guess for σ is not "good" enough so the Newton method fails to find the root. So we ended up trying many different initial guess value for σ to make the Newton method work. As pointed out in [21], Newton method is indeed quite sensitive to the initial value (σ) you start with. Thus Jäckel in [21] comes up with a new way to solve the implied volatility in the Black-Scholes Model, which we will discuss in the next session.

The two main weaknesses of the Newton method are presented in this subsection, but both of them are related to the location of the initial value in the function. This section will show the fact that the Newton method is quite sensitive to the initial value. Figure 2.3.2 and Figure 2.3.3 illustrate the initial sensitivity problem.

1. When Newton method encounters a local minimum or maximum σ, the iteration is not able to continue. See Figure 2.3.2.

2. When Newton method encounters a non-convergent cycle, it produces a situation where you do not know what to predict and it is hard to recovery once within this non-convergent region. See Figure 2.3.3.

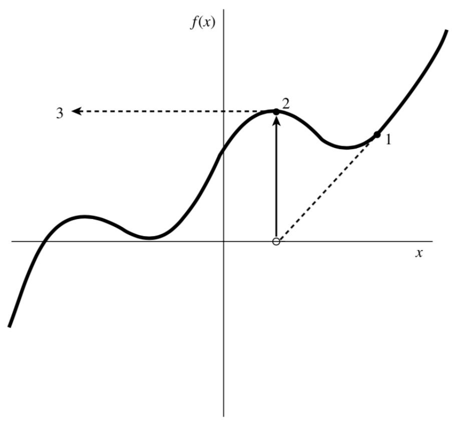

Figure 2.3.2. encounters a local extremum and shoots off to outer space

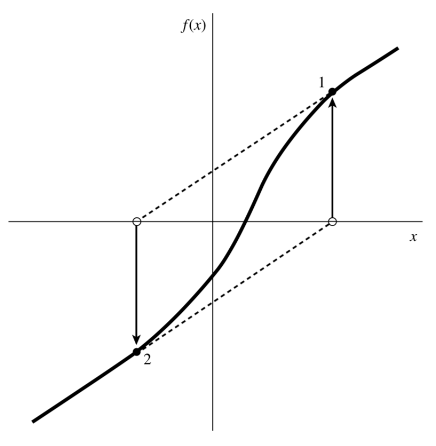

Figure 2.3.3. encounters a non-convergent cycle

2.3.4 Suggestions for improving the Newton Method

As we have shown that the Newton method has problems when the initial point is located in a non-convergent area, or in a local maximum and local minimum value for σ. Thus the Newton method fails to give a good approximation when we encounter these situations. Thus we suggest to use Peter Jäckel's method which tackles the sensitivity of the initial point problem in the Newton method. This method not only tackles the sensitivity problem in the Newton method but also requires only two iterations to get the implied volatility in the BSM with maximum attainable precision on standard hardware for all possible inputs. Peter Jäckel's Method for solving the implied volatility in the BSM has high accuracy and only requires two iterations. The main advantage of the Peter Jäckel's method is that he tackles the sensitivity in the initial point shown in the Newton method by reconstructing the BSM formula. By doing this, we can decompose the initial guess function into four branches. This avoids the problems of having initial value in non-convergent area, or in the local maximum and in the local minimum area. He uses the rational approximations for each branch. Finally, he divides the objective function that is used to find the root σ into three branches and then uses the third order iterative root-finding algorithm called Householder method to find the σ. The overall procedure is shown below:

$$\sigma(\beta) = \sigma^{HH3}\Bigg(\sigma^{HH3}\Big(\sigma_0(\beta)\Big)\Bigg)$$

where the σ_0 is the four-branch initial guess function, $HH3$ stands for the third order Householder iteration method and β is the output of the reconstructed normalized Black-Scholes formula. However, the efficiency and the attainable accuracy comes with the complexity of the method. Since the focus of the project is on the comparison of the BSM and the Heston Model, we will not go into the details for his method. For more information about his method, see [21].

2.4 The Limitations of the BSM

The BSM model has been a benchmark for option pricing and has been recognized by both the finance industry and academia. Its importance in option pricing cannot be mentioned enough. However, since the 1987 financial crisis, its drawback in inability to accurately capture the market behaviors has been widely recognized. There are a few drawbacks in the BSM mainly due to the fact that many of the BSM idealized assumptions do not hold in the real world. First, the volatility smile observed in the market contradicts with the constant volatility assumption in the BSM. Second, the asymmetry, fat tails, and high peaks reflect the fact that the lognormal stock return assumption in BSM is not sufficient enough to accommodate what is observed in the market. Third, the leverage effect and the clustering effect observed in the market are both not taken into account in the BSM. The greater details of these drawbacks will be presented in the section.

2.4.1 Volatility Smile

This section is mainly cited from [19] and [8]. Recall that definition of implied volatility from section 2.3. Implied volatility is the value of volatility parameter σ that must go into the BSM to match the market price. Based on the BSM assumptions, the implied volatility is constant regardless of which strike price we use. But, in reality, the implied volatility is different across different strike prices. This phenomena is known as the volatility skew. The U-shape curve resembling a smile in Figure 2.4.1 is called volatility smile, which is a particular kind of volatility skew. As we see in Figure 2.4.1, the implied volatility for out-of-the-money options and in-the-money options are higher than those of at-the-money options.Thus it is obvious that the BSM constant volatility assumption does not hold in the real market.

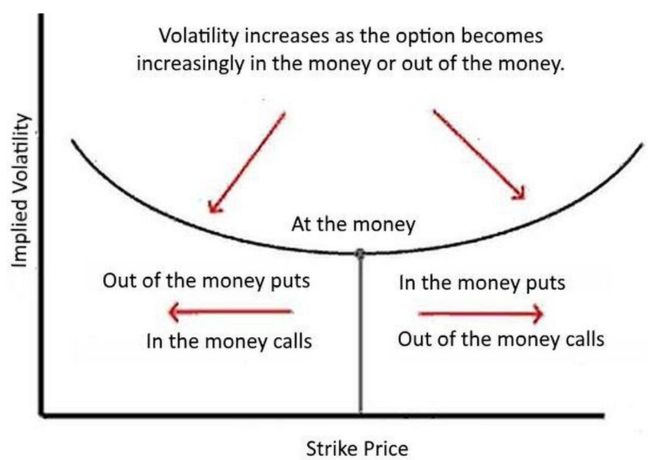

Figure 2.4.1.

Besides the volatility smile, the $S\&P$ 500 volatility index in Figure 2.4.2 also shows that volatility is not constant across different timing. Figure 2.4.2 shows that the implied volatility from 1990 to today stayed mostly below 30% with some occasional spikes. The biggest spike occurred during financial crisis in 2008. Even during some periods when the implied volatility was very low and stable, it was never constant.

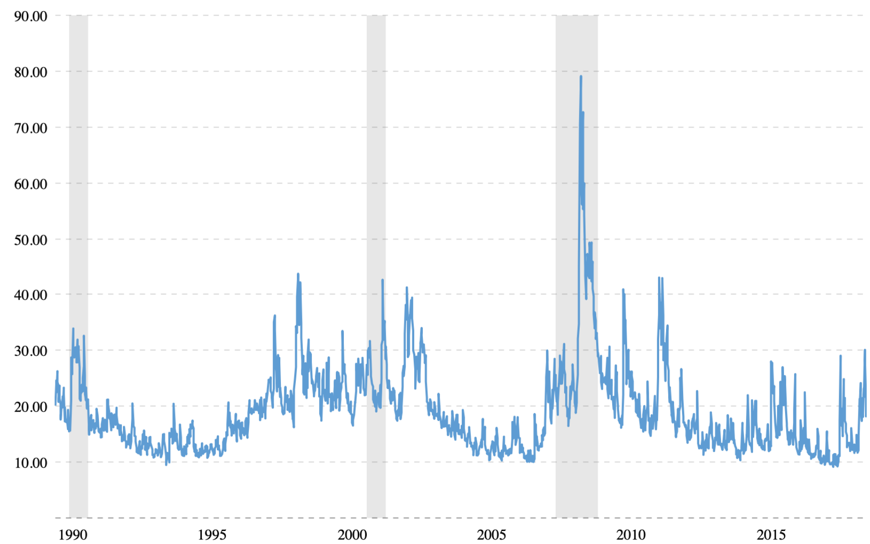

Figure 2.4.2. the daily level of the CBOE VIX Volatility Index implied by S&P 500 back to 1990

2.4.2 Shortcomings of the stock log-normal distribution

The BSM assumes the lof of stock return is normally distributed, but in reality, the stock return distribution is not normally distributed. There is asymmetry and extreme outlier along x axis compared to the normal distribution, which reflects the stock return is not normally distributed. The degree of asymmetry and outlier can be measured by skewness and kurtosis separately in statistics.

We will first give the definition of normal distribution and log-normal variable in order to give the mathematical sense of what it means for the log of stock return to be normally distributed. We can view the stock return as x in the following definitions. After we introduce the log of stock return is normally distributed, we move into the definitions of skewness and kurtosis. We will discuss how skewness and kurtosis indicate the risk of stock investment. The definitions and the figures for this section are cited from [12],[13], [11],[15], and [10] .

Definition 2.4.1. A random variable x is **normal or normally distributed** with mean μ and variance σ^2, if it is continuous with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \text{for all } -\infty < x < \infty.$$

The distribution of such an x is called the **normal distribution** with mean μ and variance σ^2. △

Definition 2.4.2. A variable x is a **log-normal random variable** with mean μ and variance σ^2 if $\log X$ is a normal random variable with mean μ and variance σ^2,which is defined by the formula

$$Y = e^x \qquad \text{with } x \text{ being a normal random variable.}$$

△

The purpose of introducing both Definition 2.4.2 and Definition 2.4.2 is to show that the fact that log of stock return (x) is normally distributed means that the stock return is log-normally distributed. But, in this project, we use the term– log-normal stock return to describe log of stock return is normally distributed.

Definition 2.4.3. . **Skewness** is a measure of how asymmetric the data are. A distribution with a skewness of zero is perfectly symmetric. In comparison, a distribution with a negative skewness will have a longer tail on the left side than on the right side; the opposite is true of positive skewness, as it is shown in the Figure 2.4.3 cited from [11].

1. **Right Skewed** or **Positive Skewed** means the tail on the right side of the distribution is longer than the left side. A relatively high positive skewness indicates stock returns deep in the right tail of the distribution as described in [15]. This means that there is more chance for gain than loss. Thus having right skewed means the investment is less risky.

2. **Left Skewed** or **Negative Skewed** means the tail on the left side of the distribution is longer than the tail of the right side. A relatively high negative skewness indicates the big

downside moves, or loss of stock return are more likely than big upside moves, or gain. Thus having left skewed means the investments are more risky.

△

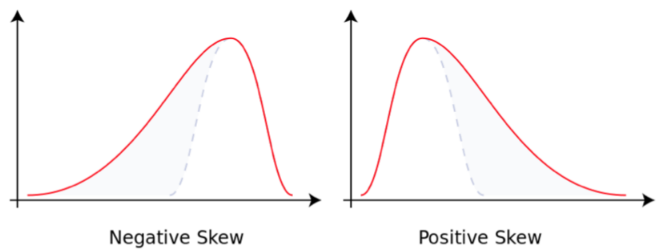

Figure 2.4.3. Skewness

Definition 2.4.4. Kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. That is, data sets with high **kurtosis** tend to have heavy tails, or outliers. Data sets with low **kurtosis** tend to have light tails, or lack of outliers. The figure 2.4.4 is cited from [13]. Kurtosis helps the investor gauge an asset's level of risk.

There are three types of kurtosis depends on the kurtosis values, which are **mesokurtic**, **leptokurtic** and **platykurtic**. Basically mesokurtic is the normal distribution(bell-curve). We will focus on leptokurtic and platykurtic, especially leptokurtic.

1. when kurtosis value is positive, it is called **leptokurtic**. It is a statistical distribution where there are extreme outliers or points along x-axis, resulting in a higher kurtosis than found in a normal distribution.

   For investors, having leptokurtic distribution for stock return means the investors will experience occasional large fluctuations more often than predicted by the normal distributions. This is due to the extreme values, or outliers which gives more chance of losing or gaining suggested by the normal distribution.

2. when kurtosis value is negative, it is called **platykurtic**. It is a particular statistical distribution with thinner tails than a normal distribution. Because this distribution has thin tails, it has fewer outliers than mesokurtic and leptokurtic distributions.

For investors, having platykurtic distribution for the stock return means there are less extreme values than the normal distribution. This means the investment is less risky and stable and has less major fluctuations.

△

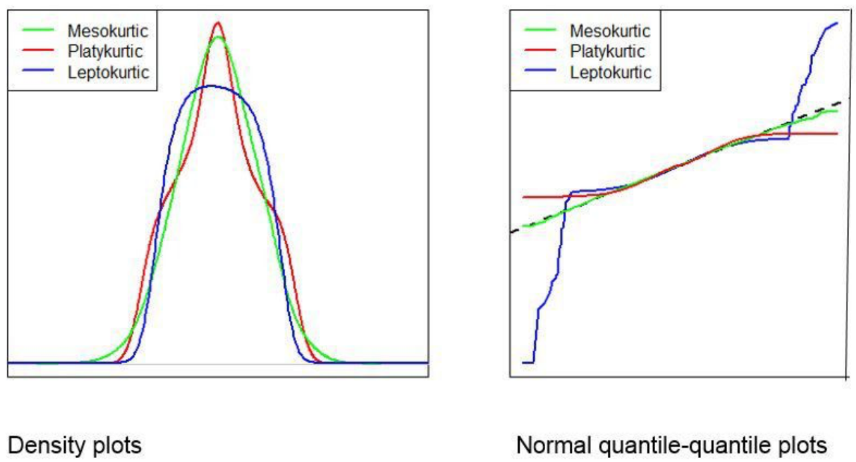

Figure 2.4.4. Kurtosis

Generally, the stock investors care more about the left skewed and the leptokurtic because having leptokurtic distribution indicates that there are higher chance of extreme values for stock return and higher chance of loss and gain, and having left skewed distribution indicates that big downsides moves are more likely than big upsides moves. We will use two figures to show that the BSM assumptions in log-normal stock return is not consistent with what is observed in the market.

Figure 2.4.5 cited from [18] is the frequency distribution of SPX log returns over 77-year period from 1928 to 2005. The x-axis is the continuously compounded stock return and the y-axis is the density. Notice that the x-axis has been extended to the left to accommodate the return. We can see that this distribution is not normally distributed; instead it is highly peaked and leptokurtic (fat-tail, outlier).

Figure 2.4.6 cited from [18] is the Q-Q plot, which shows how extreme the tails of the empirical distributions of returns are relative to the normal distribution. The plot would be a straight line

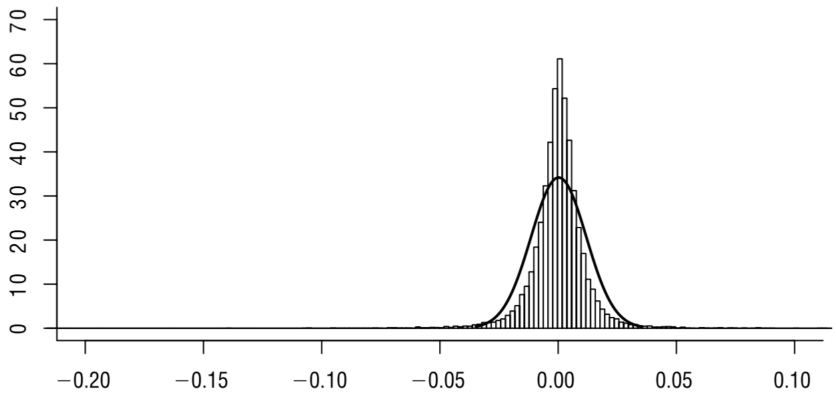

Figure 2.4.5. Frequency distribution of 77 yeas of SPX daily log-returns compared with the normal distribution.

of the empirical distribution were normal. We can see the long tail toward both sides and this is called leptokurtic as described in 2.4.4. Both Figure 2.4.5 and Figure 2.4.6 show that the BSM log-normal stock return assumption does not hold in the reality. There are usually high peak, fat tails observed in the market. As described in [18], the high peak and fat tails are the characteristics of mixture distribution with different variance. Thus it is necessary to move from the constant volatility assumption in the BSM to stochastic volatility.

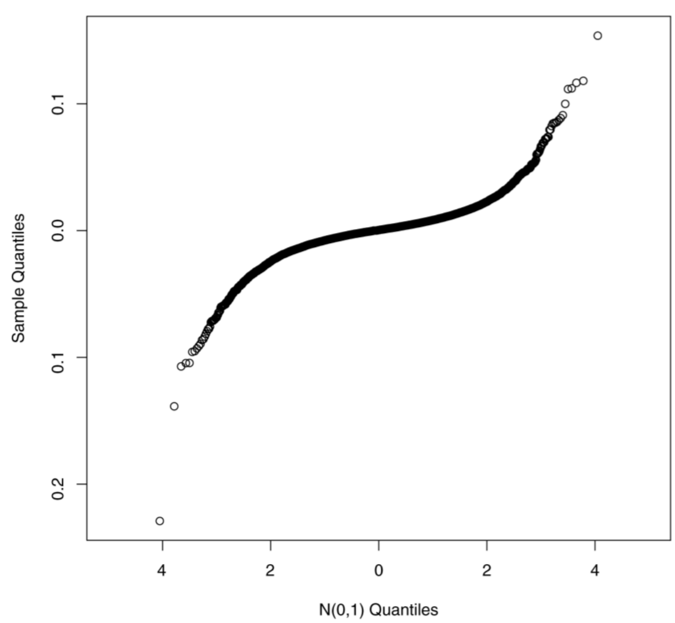

Figure 2.4.6. Q-Q plot of SPX daily log returns compared with the normal distribution. Note the extreme tails.

2.4.3 Leverage Effect and Clustering Effect

The **Leverage Effect** is the negative correlation between the stock price and the volatility. This means when there are high drops in the stock price, the volatility increases; when there in an increase in the stock price, the volatility decreases. This makes sense intuitively because the investors usually react more, when there are negative stock returns; whereas the investors become more confident, when there are positive stock returns.

The **Clustering Effect** means the larger market moves are followed by large moves. While the smaller market moves are followed by small moves. This is the feature that cannot be captured by a model assuming a constant volatility. As we see in Figure 2.4.7, it is the log return of the SPX over a 15-year period. The x-axis is the time in years and the y-axis is log return of the SPX Index. We can see the volatility is actually auto-correlated, which means the volatility process might have some extreme cases but over the long-run it self-adjust to the mean value. See Figure 2.4.7. This tendency of toward the mean is a mean-reverting volatility process.

However, despite the fact that the limitations of the BSM, which shows the idealized assumptions are not consistent with what is observed in the market, the BSM is still widely used. The main reason is its simple formula, which allows people to quickly estimate the option prices. It has become the benchmark among the option pricing models. But, it is still worth seeing how we can improve the BSM by changing its problematic assumptions, especially constant volatility to make the BSM more consistent to the reality. Thus in the next section, we will move from the constant volatility to stochastic volatility.

2.5 Moving to Stochastic Volatility

We have seen the constant volatility and the normally distributed stock return assumptions are not consistent with what is observed in the real market. The existence of volatility smile, the asymmetry and the fat tails show that the BSM leaves much room for improvement.

Many researches have done to improve its drawbacks and various models are proposed. One of the approach is to allow the volatility itself to be a random process. the Heston Model remains

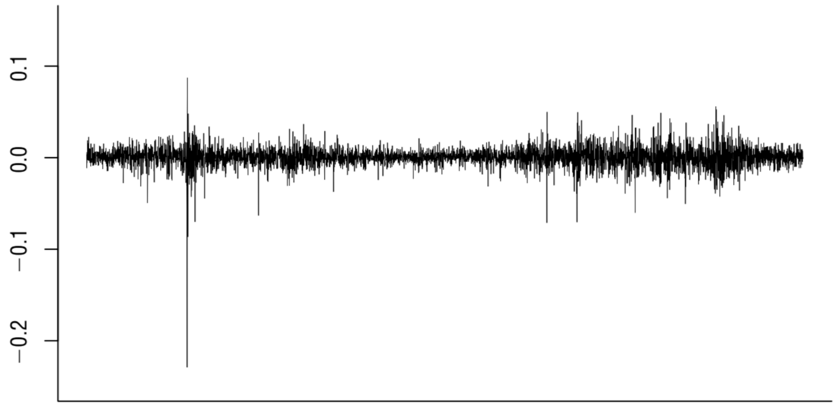

Figure 2.4.7. SPX daily log-returns from December 31, 1984 to December 31, 2004. Note the -22.9% return om October 19, 1987.

the most popular one since its many advantages, which we will discuss in the details in the next chapter. Before moving into the Heston Model, it is important to understand the basic concept of the stochastic process. Remember that in the BSM, the stock price is assumed to be a stochastic process, but not the volatility. We will present the details of stochastic process in the following session.

2.5.1 Stochastic Process

This section is mainly cite from [20]. We will move from Wiener process to the generalized Wiener process. The understanding of this section will lay out the foundation for the Heston Model because the Heston Model involves two correlated Wiener process–one for the stock price and the other for the volatility. The appearance of the Winner process is where the randomness comes in the Heston Model.

Any variable whose value changes over time in an uncertain way is said to follow a **stochastic process**. **Stochastic process** can be classified as discrete time or continuous time. A **discrete-time stochastic process** is one where the value of the variable can change only at certain fixed points in time, whereas a **continuous-time stochastic process** is one where changes can take place at any time.

A **Markov process** is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant.

The **Brownian Motion** sometimes also known as **Wiener process** is a particular type of Markov stochastic process. We consider a variable follow the **Brownian Motion** if it has the following two properties:

1. The change Δz during a small period of time Δ is

$$\Delta z = \epsilon\sqrt{\Delta t}$$

   where ϵ has a standardize normal distribution with mean 0 and standard deviation 1 $N(0,1)$.

2. The value of Δz for any two different short intervals of time, Δt, are independent. The value Δz has a normal distribution with mean of 0, standard deviation of $\sqrt{\Delta t}$, and variance of Δt.

A **generalize Wiener process** for a variable x is defined in terms of dz as

$$dx = adt + bdz \tag{2.5.1}$$

where a and b are constants and dz is Wiener process.

The adt term implies that x has an expected drift rate of a per unit of time. The bdz term can be regarded as adding noise or variability to the path followed by x. The amount of this noise or variability is b times dz (Wiener process).

A Wiener process has a standard deviation of 1. It follows that b times a Wiener process has a standard deviation of b. In a small time interval Δt, the change Δx in the value of x is given by

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t} \tag{2.5.2}$$

where ϵ has a standard normal distribution. It is shown that the change of x in any time interval T is normally distributed with mean aT, standard deviation $b\sqrt{T}$, and variance b^2T. Thus the

generalized Wiener process given by Equation 2.5.1 has an expected drift rate(average drift rate per unit of time) of a and a variance rate(variance per unit of time) of b^2. It is illustrated in Figure 2.5.1.

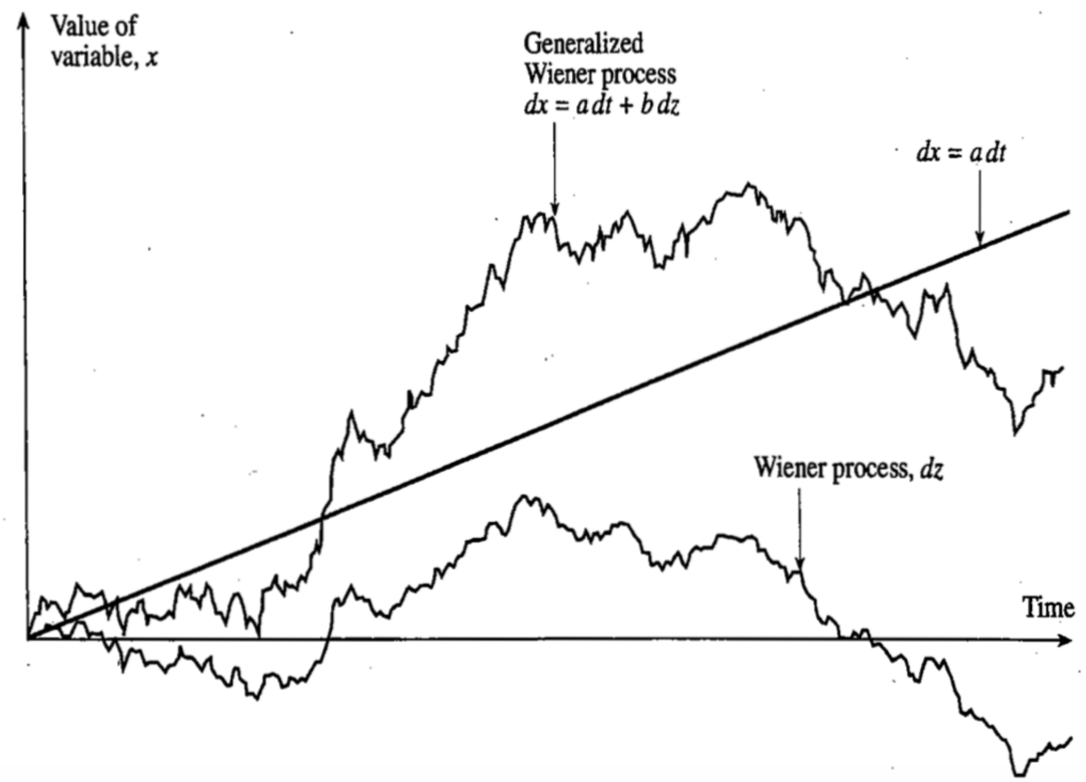

Figure 2.5.1. Generalized Wiener process with a=0.3 and b=1.5

3
The Heston Stochastic Volatility Model

Since the 1987 financial crisis, a number of models have been proposed to improve the BSM in order to reflect the market behaviors better than the BSM. The Heston Stochastic Volatility Model, for European option pricing, was developed by Steven Heston in 1993 to overcome the shortcomings of the BSM, especially the constant volatility assumption and the log-normal stock return assumption. It is one of the most popular stochastic volatility pricing models not only because it address the two major assumptions in the BSM by allowing the volatility itself to be a random variable, but also it takes the volatility smile, the leverage effect and the important mean-reverting property of volatility into account.

The Heston Model is based on nine input parameters, which are stock price S, strike price K, the risk-free interest rate r, maturity t, initial volatility V_0, long-term volatility $\bar{V}$, the mean-reverting speed for volatility a, and the correlation between stock price and volatility ρ. The first four input parameters are easily observable market data, which can be determined by looking at the market. The last five input parameters can be obtained using the calibration process which will be presented in the next chapter. The following sections are mainly cited from [24], [31], [17] and [29] .

3.1 the Heston Model

Let S_t be the price of the underlying asset at time t, η be the volatility of the volatility process, r be the risk-free interest rate, μ be the drift coefficients of the stock price, V_t be the variance at time t, $\bar{V}$ be the long-term mean of variance, a be the rate of mean-reversion, dW_t^1 and dW_t^2 be two correlated Brownian motions and ρ be the correlation coefficient.

Heston assumes the underlying asset S at time t with risk-free interest rate r follows the risk-neutral dynamics such that

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \tag{3.1.1}$$

$$dV_t = a(\bar{V}_t - V_t)dt + \eta\sqrt{V_t} dW_t^2 \tag{3.1.2}$$

$$dW_t^1 dW_t^2 = \rho dt \tag{3.1.3}$$

where $a, \eta, V_t > 0$.

Equation 3.1.1 shows that the stock price follows the stochastic process. Equation 3.1.2 shows that the volatility follows the stochastic process but also the volatility is toward the mean. Equation 3.1.3 shows that the two stochastic processes for both volatility and stock price are correlated. The details for each equation is presented in the following section.

3.1.1 Geometric Brownian Motion

Equation 3.1.1 is called geometric Brownian motion, which is the same assumption proposed in the BSM. This geometric Brownian motion can derived from viewing the generalized Wiener process in Equation 2.5.1 in terms of variable stock price S and multiplying S to the right side of the equation. By multiplying the stock price S on the right side means the stock price is proportional to its mean and to its standard deviation in the stochastic process. The μ is the expected rate of stock return and the $\sqrt{V_t}$ is the volatility of the stock price. $\sqrt{V_t} dW_t^1$ is the stochastic component of the return, or randomness component.

3.1.2 Mean-Reverting Process

Besides keeping the same assumptions about stock price follows the stochastic process, Heston adds $dV_t = a(\bar{V_t} - V_t)dt + \eta\sqrt{V_t}dW_t^2$, which is a mean-reverting process also known as Cox-Ingersoll-Ross (CIR) process. This means the volatility tend to bounce toward its long-term mean.

This assumption is consistent with the behavior observed in financial market. If volatility were not mean-reverting, markets would be characterized by a considerable amount of assets with volatility exploding or going near zero. However, in practice, these cases are quite rare and generally short-lived as mentioned in [24].

The deterministic term $a(\bar{V_t} - V_t)$ ensures the mean reversion of the volatility toward the long run value $\bar{V_t}$, with the speed of adjustment governed by the strictly positive parameter a.

The standard deviation factor $\eta\sqrt{V_t}$ avoids the possibility of negative volatility for all positive values of η and θ, When the volatility V_t is close to zero, the standard deviation $\eta\sqrt{V_t}$ becomes very small, which dampens the effect of the random shock on the volatility. In other words, when the volatility gets close to 0, its evolution becomes dominated by the deterministic term, which pushes the volatility towards the long-term mean.

3.1.3 Correlated shocks between stock price and volatility

Heston introduces a correlated relationship between stock returns and volatility. This assumption allows modelling the statistical dependence between the stock return and and its volatility, which is a prominent feature of financial markets. In stock markets, volatility tends to increase when the stock price return decrease; whereas volatility tends to decrease when the stock price return increases. This is quite intuitive because investors usually react more, which drives the volatility to be higher, when there are high drops in the stock returns. Whereas when the stock return increases, investors become more confident which leads the volatility to be less .

The Heston Model provides a modeling framework that can take many of the specific characteristics that are typically observed in the behavior of financial market into account. But, its advantages come at the expense of higher complicity. Compared to the BSM, the implementation of the Heston Model requires more sophisticated mathematics and it takes more time to calculate the option price since it involves a challenging calibration to the market price in order to find the five missing parameters. We will explain the calibration later on.

3.2 Closed-Form Solution of the Heston Model

In this section, we follow the reasoning in Heston paper [4], but we cite the new formulation of Heston solutions from [24] and cite the pros and cons of the Heston Model from [?MRZEK]. Equation 3.2.4, Equation 3.2.5, Equation 3.2.6, Equation 3.2.7, Equation 3.2.8 and Equation 3.2.9 are proposed by [24] who modifies the Heston characteristic function proposed by [18].

The present value of a European call option can be estimated using a probabilistic approach

$$C_0 = S_0\Pi_1 - e^{-rT}K\Pi_2 \tag{3.2.1}$$

,where the first term $S_0\Pi_1$ represents the present value of the underlying given an optimal exercise and the second term $e^{-rT}K\Pi_2$ is the present value of the strike price of the strike price payment. Moreover, Π_1 is the delta of the European call option and Π_2 is the conditional risk neutral probability that the asset price will be greater than K at the maturity. Both Π_1 and Π_2 represent the conditional probability of call option expiring in-the-money.

Provided that characteristic function $\psi^{Heston}_{\ln(S_t)}(w)$ are known, the terms Π_1 and Π_2 are defined via the Fourier transformation,

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{iw\ln K}\psi_{\ln S_T}(w-i)}{iw\psi_{\ln S_T}(-i)}\right]dw \tag{3.2.2}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{iw\ln K}\psi_{\ln S_T}(w-i)}{iw}\right]dw \tag{3.2.3}$$

Where Equation 3.2.2 and Equation 3.2.3 are smooth functions that decay rapidly and present no difficulties as mentioned in [4].

Heston considers the characteristic function of these probabilities in the form

$$\psi_{\ln(S_t)}^{Heston}(w) = e^{\left[C(t,w)\overline{V}+D(t,w)v_0+iw\ln S_0 e^{rt}\right]}, \tag{3.2.4}$$

and shows that

$$C(t,w) = a\left[r_- \cdot t - \frac{2}{\eta^2}\ln\frac{1-ge^{-ht}}{1-g}\right] \tag{3.2.5}$$

$$D(t,w) = r_-\frac{1-e^{-ht}}{1-ge^{-ht}}, \tag{3.2.6}$$

where

$$r_\pm = \frac{\beta \pm h}{\eta^2}; h = \sqrt{\beta^2 - 4\alpha\gamma} \tag{3.2.7}$$

$$g = \frac{r_-}{r_+} \tag{3.2.8}$$

$$\alpha = -\frac{w^2}{2} - \frac{iw}{2}; \beta = \alpha - \rho\eta iw; \gamma = \frac{\eta^2}{2}. \tag{3.2.9}$$

3.3 Implementation

1. Compute the Heston Characteristic Function $\psi_{\ln(S_t)}^{Heston}(w)$ in Equation 3.2.4
2. Substitute Heston Characteristic Function into Equation 3.2.2 and Equation 3.2.3 to compute Π_1 and Π_2.
3. After calculating probabilities Π_1 and Π_2, plug them back to Equation 3.2.1.
4. Equation 3.2.1 returns a European call option price under the Heston Model.

3.4 Influence of Parameters

The section including the figures are mainly cited from [31], [35] and [17]. There have been many empirical studies that show the stock return is not normally distributed. There are fat tails (leptokurtic) and asymmetry shown in the stock returns. It is also suggested that the correlation between stock return is negative, which is called the leverage effect. There is also the volatility clustering effect observed in the market. Thus it is important to understand the meaning of the Heston parameters which takes those above phenomena into account. The following five parameters are generated through the calibration process, which we will introduce in the next chapter.

- Initial variance parameter V_0

As mentioned in [17], changing the initial variance allows adjustment in the height of the smile curve rather than the shape. Increasing the initial volatility, $\sqrt{V_0}$ moves the implied volatility smile upwards. See Figure 3.4.1.

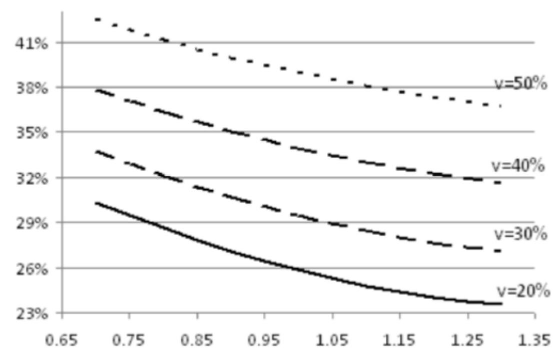

Figure 3.4.1. The effect of changing the initial variance $v = \sqrt{V_0}$

- Long term variance parameter $\bar{V}$

Changing the long-term variance $\bar{V}$ has the similar effect as changing the initial variance V_0. See Figure 3.4.2. The θ in the Figure 3.4.2 is the $\bar{V}$ here.

- Mean-reversion speed parameter a.

As mentioned in [17], the mean reversion speed can be viewed as the degree of volatility clustering. As mentioned before, volatility clustering can be observed in the market; it means that large moves are followed by large moves, while small moves are more likely followed by small

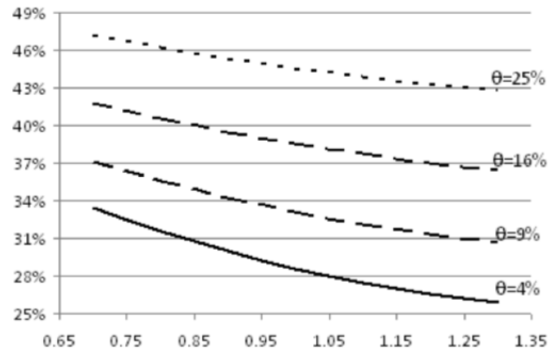

Figure 3.4.2. The effect of changing the long-term variance $\bar{V}$ which is denoted as θ

moves. The mean reversion controls the curvature of the curve. If there is a increase in the mean reversion parameter, it flattens the volatility smile. If there is a decrease in the mean reversion parameter, the effect is the opposite. See Figure 3.4.3. The k in Figure 3.4.3 is the a here.

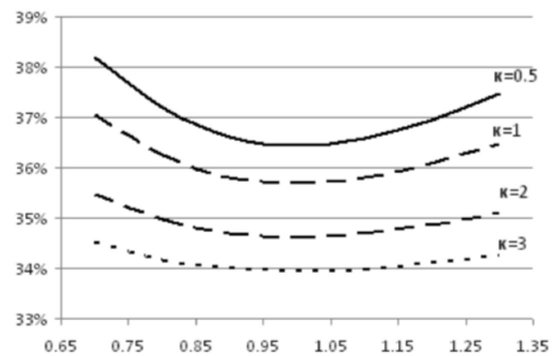

Figure 3.4.3. The effect of changing the mean reversion speed a which is denoted as k in the figure

- The volatility of volatility parameter η

The parameter η controls the kurtosis (peak) of the underlying asset return distribution. When η is zero, the volatility becomes deterministic. In other words, the volatility is constant, so the distribution of stock price follows the normal distribution. Otherwise, as η increases causes the kurtosis to increase,which creates the fat tails on the both sides. Note the higher η means the volatility is more volatile, which states that the market has a greater chance of extreme movements. See Figure 3.4.4. The σ in the Figure 3.4.4 means the volatility of variance η here.

- The correlation coefficient ρ

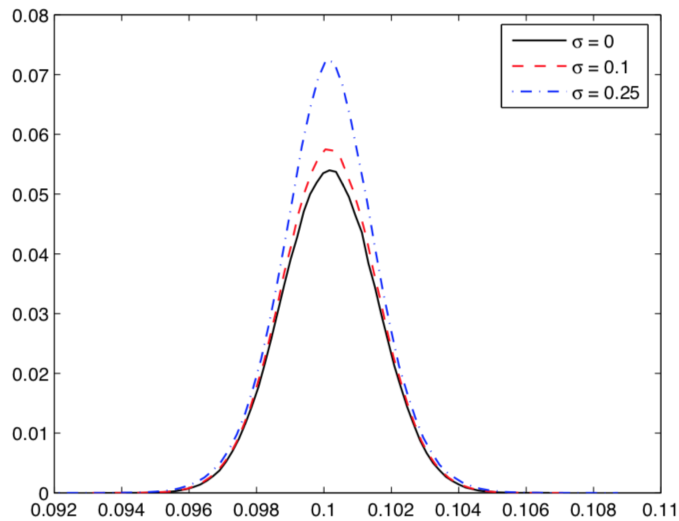

Figure 3.4.4. The effect of η which is denoted as σ on the kurtosis of the density function

The parameter ρ is the correlation between the log-returns and volatility of the stock. It affects the skewness or asymmetry of the underlying asset return distribution. See Figure 3.4.5.

If $\rho > 0$, the volatility will increase as the stock price increase. This will spread the right tail and squeeze the left tail of the distribution creating a long right-tailed distribution, which is called right skewed. See Figure 3.4.5, when ρ is 0.9, there is a right skewed.

If $\rho < 0$, the volatility will decrease while the stock price decrease.This will spread the left tail and squeeze the right tail of the distribution creating a long left-tailed distribution, which is called left skewed. See Figure 3.4.5, when ρ is -0.9, there is a left skewed.

If $\rho = 0$, there is no effect to the skewness of distribution. Thus, the distribution is normally distributed. See Figure 3.4.5, when ρ is 0, it is a normal distribution.

3.5 Advantages and Disadvantages of the Heston Model

Many researchers and scholars have studied the importance of the Heston, but this is still a model. In other words, this is a model which is used to estimate the market option price, but it is not a perfect model. This section summarize both the advantages and disadvantages of the Heston Model. We cite the information from [17] and [29].

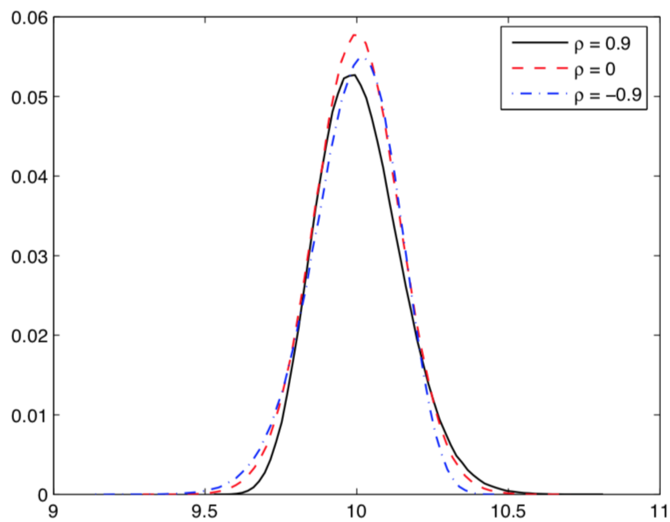

Figure 3.4.5. The effect of ρ on the skewness of the density function

The advantages of the Heston Model:

1. The closed-form solution allows the calibration.

2. Heston models takes into account the leverage effect (negative correlation of stock returns and implied volatility), and it permits the correlation between the stock price and the volatility to be changed.

3. The volatility is mean-reverting.

4. The form of the Heston Model used to model price dynamics allows for non-lognormal probability distribution.

The disadvantages of the Heston Model:

1. Since volatility is not easily observable in the market, the parameters values in the Heston Model are not easily estimated. The values depend on the algorithms being used in the calibration process.

2. The price produced by the Heston Model are quite parameter sensitive, thus the fitness of the model depends on the calibration. In other words, to have more realistic model

that take into account the overall market situation comes with a more complex model calibration.

3. The Heston Model fails to produce decent results for short maturity. To perform well, the further extensions of the model are necessary, such as adding jumps.

4
Calibration of the Heston Model

In order to estimate option prices under the Heston Model, we need to find the five unknown input parameters, which are initial volatility, long-term volatility, volatility of the stochastic volatility process, volatility mean-reverting speed and correlation between stock price and volatility. These five parameters are unknown because they cannot be easily observed in the market. The way to find these five parameters is to calibrate the Heston Model to the option market prices. In this way, we can obtain the five parameters that reflect the behaviors of the options that are traded in the real market. However, in practice, it is not possible to match exactly the observed market prices. Thus the problem of calibrating the Heston Model is formulated as an optimization problem. Our objective is to minimize the pricing error between the model prices and the market prices for a set of data. The standard approach to calibrate the Heston Model is to use the non-linear least-square method, which minimizes the squared difference between the model prices and the marker prices. The details of the non-linear least squares method will be presented in this section.

4.1 Data

In order to calibrate the Heston Model, we need the real market data. We obtained the S&P 500 Index European call options from February 4th to March 4th in 2019 from iVolatility.com.

There are a total of 228,876 call options. We consulted the methods provided in [27] and [26] to filter our data and eliminate our data to a total of 3,371 call options based on the criteria shown below. The goal for our approach to filter the data is basically to eliminate the rare and extreme option trading scenarios.

We denoted only the total number of the maturities and the strike prices in Table 4.1.1, since there are a total of 96 different maturities and a total of 98 different strike price. As the risk-free interest rate we use the 1-year Treasury Bill Quotes from *treasury.gov*. Table 4.1.1 sums up the data we use to calibrate the Heston Model from February 4th to March 4th in 2019.

1. Remove any negative volatility, since the standard deviation of the stock return cannot be negative.

2. Exclude expiration day less than 6 days to expiration day more than 120 days, since most of the options are traded between 6 to 120 days.

3. Exclude very deep-out-of-the-money whose moneyness $> 9\%$ options and very deep-in-the-money options whose moneyness $< 9\%$, since these options are not actively traded.

4.2 Motive for Using Non-Linear Least-Squares Optimizer

This section is mainly cited from [32]. There are five parameters in the Heston Model that are not easily observable from the market data which includes initial variance, long-term variance, volatility of the variance process, and the correlation between the stock price and the volatility. To determine these five parameters, we use market option prices to calibrate the Heston Model. In this way, we can determine the five parameters that match the market option prices and can be used afterwards in the Heston Model to estimate the option price that are more consistent with the real market data.

As pointed out by [24], the calibration process becomes as crucial as the model itself. There are many different ways to calibrate the model, but almost all of them are to minimize the error between the model estimated price and the real market option price. We use the *lsnonlin*–least

Day	Maturities	Strikes	Stock Price	Risk-free Interest-Rate
February 4th	22	50	2724.87	2.57%
February 5th	22	48	2737.7	2.56%
February 6th	22	38	2731.61	2.56%
February 7th	22	54	2706.05	2.55%
February 8th	22	48	2707.88	2.55v
February11th	21	38	2709.8	2.55%
February 12th	23	56	2744.73	2.55%
February 13th	21	48	2753.03	2.55%
February 14th	20	46	2745.73	2.53%
February 15th	20	45	2775.6	2.55%
February 19th	19	48	2779.76	2.54%
February 20th	18	45	2784.7	2.54%
February 21st	18	54	2774.88	2.55%
February 22nd	20	46	2792.67	2.55%
February 25th	19	47	2796.11	2.56%
February 26th	19	53	2793.9	2.55%
February 27th	19	56	2792.38	2.54%
February 28th	20	53	2784.49	2.54%
March 1st	21	80	2803.69	2.55%
March 4th	20	80	2792.81	2.54%

Table 4.1.1. Real Market Data from February 4th to March 4th

square non-linear local optimization algorithm– to calibrate the Heston Model. This algorithm searches for the five parameters by implementing a trust region reflective algorithm minimizing the squared distance between the market price and the model price. We can use the global optimizer called Adaptive Simulated Annealing (ASA) algorithm which finds the global minimum like using a bouncing ball that can bounce over mountains from valley to valley. It follows relative similar process as Simulated Annealing (SA) algorithm. As cited from [32], we start at a high "temperature", where the temperature is an ASA parameter that mimics the effect of a fast moving particle in a hot object like a hot molten metal, thereby permitting the ball to make very high bounces and being able to bounce over any mountains to access any valley, given enough bounces. As the temperature is made relatively colder, the ball can not bounce so high, and it also can settle to become trapped in relatively smaller ranges of valleys. There is also the acceptance distribution decides probabilistically whether to stay in a new lower valley or to bounce out of it. All the generating and acceptance distribution depend on temperature. For further information of ASA, see [32].

However, [24] mentions that the global optimzer in general is less trackable and less stable compared to the local optimization algorithm. Besides, both [24] and [28] show that the ASA is less efficient, less trackable mathematically than *lsnonlin* and ASA gives higher error values for the Heston Model. Thus it makes sense to use *lsnonlin* since it is more accurate than ASA and less time-consuming. In the project, we need to deal with quite an amount of data so the time-efficiency is the factor we prioritize. But also *lsnonlin* is more accurate than ASA. Thus, we decide to use *lsnonlin* because of its higher accuracy and time-efficiency.

4.2.1 MATLAB's lsqnonlin

As mentioned in [31], MATLAB's non-linear least-squares optimizer, *lsqnonlin*, is the function lsqnonlin(fun, x0,lb,ub). It minimises the vector-valued function, fun, using the vector of initial parameter values, x0, where the lower and upper s of the parameters are specified in vectors lb and ub, respectively. The result produced by lsqnonlin is dependent on the choice of x0, the initial guess. This is, therefore, not a global optimizer, but, rather a local one. Thus we are aware that local optimizers are sensitive to the initial values and cannot guarantee the solutions we found are the best available one as mentioned in [24]. However, we can set up the bounds for acceptable solutions before applying the local optimizer. In this way, we can guarantee the solutions are in the rage of our bounds. If we encounter a non-acceptable solution, we can run the algorithm with a different starting value and keep searching for solutions that obey our criteria as mentioned in [24].

The function *lsqnonlin* runs a trust-region reflective minimization algorithm. the trust region is adjusted from iteration to iteration. If the approximation fits the model well, the trust region enlarges; otherwise, shrinks a merit function that is chosen for updating the next trust region and for choosing the new iteration point. For more information, [5] provides a complete overview of the algorithm.

We follow the method provided in [24]. The main goal is to find the five unknown parameters by minimizing the squared distance between the options prices provided by the market and those generated by the Heston Model. In particular, the Heston Model has five unknown parameters $\Omega = \{V_0, \bar{V}, a, \eta, \rho\}$, where V_0 stands for the initial volatility, $\bar{V}$ stands for the long-term volatility, a stands for the mean-reversion speed, η stands for the volatility of the stochastic volatility process and ρ stands for the correlation between the stock and the volatility. By calibrating these parameters, we can obtain the five parameters that are consistent with the market behaviors, since these five parameters are the generated values of using *lsqnonlin* in the calibration process to best match the market price. This optimizer runs iteratively and continuously finds the best match by tying to minimize the difference between the market price and the model price.

In order to implement this method, we need to have a cost function (see Equation 4.2.1), which is the squared sum of the distance between the market price and the model price. We obtained the market price for different strike price and maturities, and used them as the sample that the Heston is calibrated to. Then we use the *lsqnonlin* to minimize the cost function for all strike price and maturities. The process is:

$$costf = \sum_{i=1}^{N} [C_i^{\Omega(}(K_i, T_i) - C_i^{market}(K_i, T_i)]^2 \tag{4.2.1}$$

where $\Omega = \{V_0, \bar{V}, a\eta, \rho\}$ be the set of for the five unknown parameters in the Heston Model, $C_i^{\Omega(}(K_i, T_i)$ is the option prices using the parameters set Ω for different strike prices K and maturities T and $C_i^{market}(K_i, T_i)$ is the market option prices for different strike prices K and maturities T.

The outputs of this calibration process will be the calibrated five inputs that maximize the performance of the model with respect to what is observed in the market. For the detailed of the code, see Appendix A.2 and Appendix A.3.

As we mentioned before, *lsqnonlin* is sensitive to the initial value. To tackle this problem, we define the restrictions for possible solutions before using *lsqnonlin* to make sure that the outputs are not just mathematically feasible but make sense economically . The restrictions for possible solutions are cited from [24] and are shown below:

- Long-term variance $\hat{v}$ and initial variance V_0 satisfy $,0 \leq V_0, \hat{V} \leq 1$. Given its mean-reversion, the volatility of most stocks rarely reaches the level beyond 100 %.

- Correlation ρ satisfy $\rho \in [-1, 1]$, since the statistic correltiona takes value from -1 to 1. We mentioned earlier that the correlation between stock price and volatility tends to be negative. However, positive correlation might also be possible in particular cases. Thus, the full rage of acceptable solution will be used.

- Volatility of stochastic volatility process η satisfy $0 \leq \eta \leq 5$ since volatility tends to be very volatile so we need a broader bond.

- Mean-reversion speed a as we mentioned earlier satisfy $a > 0$

- We need to ensure that the volatility of stochastic volatility process does not reach zero or negative values. According to the Feller condition, $2V - \eta^2 > 0$ guarantees the variance in CIR process is always strictly positive.

4.2.2 Calibrated Parameters

After calibrating the Heston Model using a total of 3,371 market data from February 4th to March 4th in 2019, the five parameters value are shown in Table 4.2.1. We found the initial volatility is 0.0123, that the long-term volatility is 0.0158, that the volatility of the stochastic volatility process is 0.4307, that the correlation coefficient is -0.7276 and that the mean-Reversion Speed is 5.8838. We will plug these five calibrated values back in the function *call_heston_cf* which is attached in Appendix A.1. After finding these five parameters, we have all the parameters need in the Heston Model and can implement the Heston Model to estimate the option price. We will use the results shown in Table 4.2.1 in the next chapter to implement the Heston Model.

V_0	$\bar{V}$	η	ρ	α
0.0123	0.0158	0.4307	-0.7276	5.8838

Table 4.2.1. Generated Parameters

5
Comparison

We have introduced both the BSM and the Heston Model in the previous two chapters. In this chapter, we will compare the estimates under both of the BSM and the Heston Model using mean absolute percentage error (MAPE) as our measure of prediction accuracy. We will show the accuracy of the both models under 15 scenarios, which is the total combinations of different values of moneyness and maturities. The details for each scenario will be discussed in this chapter.

5.1 Data

We used $S\&P$ 500 Index European call options traded in the market on March 9th in 2019 from iVolatiltiy.com. We consulted the methods of filtering the $S\&P$ 500 Index European call options provided in [26] and [27]. We are left with a total of 3,371 $S\&P$ 500 European call options. The procedure of filtering the data is shown below:

- First, remove any negative volatility.

- Second, exclude the options with time to maturity less than 7 days and more than 180 days, since the prices for the options whose maturity less than 7 days are very volatile and also the options whose maturity more than 180 days have very high option prices, or premium.

- Third, exclude very deep-in-the money and very deep-out-of-the-money if their moneyness is $> 9\%$ and their moneyness$< -9\%$, since these options are not actively traded.

Table 5.2.1 presents the real market data we use to compare the BSM and the Heston Model from iVolatility.com. Note we only denote the total number of maturity and strike price, since there are too many of them. There are a total of 23 different maturities and a total of 103 different strike prices.

Date	Maturities	Strikes	Stock Price	Risk-free Interest-Rate
March 19th	23	103	2831.946	2.50%

Table 5.1.1. Real Market Data

5.2 Results

After getting the data, we can implement both of the BSM and the Heston Model to estimate a total of 3,371 $S\&P$ 500 Index European call options using *bsm_price* and $call\_heston_c f$ functions in MATLAB. Then we use mean absolute percentage error as the forecasting error criteria. For the Heston Model, we plugged the results of the calibrated parameters from section 4.2.2 into the Heston Model. The MATLAB code for the BSM, the Heston Model and the MAPE for the both models can be found in Appendix A. Table 5.2.1 shows the accuracy of the BSM and the Heston Model under 15 scenarios. The range for each moneyness and maturity used is shown below:

- AT-the-Money(ATM): when moneyness between -0.02 and 0.02.

- In-the-Money(ITM): when moneyness between 0.02 and 0.05.

- Out-of-the-Money(OTM): when moneyness between -0.05 and -0.02.

- Deep-out-of-the-Money(DOTM): when moneyness less than -0.05.

- Deep-in-the-Money(DITM): when moneyness greater than 0.05.

- Short-term: when the maturity is less than 45 days.

- Middle-term: when the maturity is between 45 and 90 days.

- Long-term : when the maturity is greater than 90 days.

- Mean Absolute Percentage Error (MAPE)= $\frac{1}{n}\sum_{n=1}^{n}\frac{|\text{model price-market price}|}{\text{market price}}$, where n is the total number of data.

Maturity		Deep-out-of-the-Money	Out-of-the-Money	At-the-Money	In-the-Money	Deep-in-the-Money
Short-Term (<45 days)	Black-Scholes	0.9999%	0.5582%	0.6049%	8.0017%	48.0001%
	Heston	4.05929%	1.2787%	0.5848%	7.8392%	47.6719%
Middle-Term (45-90 days)	Black-Scholes	0.6353%	0.5307%	0.3599%	2.1863%	7.3310%
	Heston	0.6545%	0.4706%	0.3184%	2.0621%	7.1367%
Long-Term (>90 days)	Black-Scholes	0.6334%	0.4536%	0.2665%	1.1063%	2.9548%
	Heston	0.5334%	0.3944%	0.1875%	0.9492%	2.7524%

Table 5.2.1. Mean Absolute Percentage Error(MAPE)

5.2.1 Result Analysis

Table 5.2.2 shows the average mean absolute percentage error for each moneyness. The accuracy analysis under deep-in-the-money, in-the-money, at-the-money, out-of-the-money, and deep-in-the-money for both the BSM and the Heston Model is presented based on the results in Figure 5.2.1:

- Deep-in-The-Money

  Both models perform very poorly, especially for the short-term, but as the maturity goes up, both models improve significantly. However, in general, the Heston still performs a lot better than the BSM for all deep-in-the-money maturity groups.

- In-the-Money, At-the-Money

  The Heston performs better than the BSM for all maturity groups.

- Out-of-the-Money

  The Heston performs poorly for the short-term, but as the maturity goes up, Heston performs better than Black-Schoels in both middle-term and long-term.

- Deep-out-of-the-Money

  The Heston performs significantly worse than the BSM for short-term, but as the maturity goes up, the Heston improves. When it is long-term maturity, Heston performs better than the BSM

See Table 5.2.1. Both models have quite large errors in the short-term, deep-in-the-money scenario. There are many factors that lead to this result. First, deep-in-the-money is when the strike price is far over than the final stock price. This case is itself an extreme case since the moneyness is quite large at this scenario. The moneyness for most cases is between -0.02 to 0.05, but in this case is above 0.05. The data we use on March 19th in 2019 for the short-term, deep-in-the-moeny scenario has moneyness value much higher than 0.05. We can analyze more days to see how both models perform in short-term, deep-in-the-money scenario. But in general, analyzing a total of 3,774 already gives a very good indication of both models' accuracy because we have calibrated the Heston Model to the market price using the whole month data from February 4th to March 4th in 2019.

As we can see in Table 5.2.2, the Heston Model performs quite poorly both in the deep-out-of-the-money and out-of-the-money than the BSM does, but the Heston Model performs better than the BSM in at-the-money, in-the-money, and deep-in-the-money. Based on the results from Table 5.2.1, we would suggest using the Heston Model when the options are at-the-money for all maturity groups and in-the-money for all maturity groups and deep-in-the-money for middle-term and long-term maturity. The Black-Scholes can be used for short-term options that are deep-out-of-the-money and and out-of-the-money. We can see Heston, in general, can predict quite accurate option prices when the maturity is not short-term. As we can seen from Table 5.2.3, the Heston performs poorly in the short-term, but as the maturity gets longer, Heston performs a lot better than Black-Scholes for the middle-term, and long-term maturity groups.

	Deep-out-of-the-money	out-of-the-money	At-the-Money	In-The-Money	Deep-In-The-Money
Black-Scholes	0.7562%	0.5142%	0.4104 %	3.7648 %	19.4283%
Heston	1.7491%	0.7146 %	0.3636 %	3.6169 %	19.187 %

Table 5.2.2. Average MAPE values of all 3 maturities for each respective moneyness

	Short-Term	Middle-Term	Long-Term
Black-Scholes	11.633%	2.2086%	1.083%
Heston	12.2868 %	2.1284%	0.963%

Table 5.2.3. Average MAPE values of all moneyness for each respective maturity

6
Conclusion

We have pointed out in Chapter 2.4, that the two major assumptions–the constant volatility and the log-normal stock return– have been proven wrong during the stock market crash in 1987. These two assumptions contradict with what is observed in the market. We have shown that there is a volatility smile observed in the market, which shows that volatility for both out-of-the-money and in-the-money options are typically higher than those of at-the-money options. This reflects that the volatility is not always constant. Thus the constant volatility assumption is not consistent with the market reality. In addition, we also looked at the $S\&P$ 500 Index stock return which has asymmetry and fat tails. Both of the asymmetry and fat tails reflect the fact that the log of stock return is not a bell-curve, or normal distribution. Thus the log-normal distribution contradicts with the market reality.

In 1973, the Heston Model was developed by Steven Heston to tackle these two problematic assumptions. He extended the BSM and addressed these two problematic assumptions by allowing volatility itself to be a random variable and allows the non-lognormal stock return. He also took the observed leverage effect and the important mean-reverting property of volatility in the market into account. See 2.4.3 for the details of these two effects. However, the BSM does not take both the leverage effect and the mean-reverting property of volatility observed in the market into account. In other words, the Heston Model is more consistent with what is observed in the market than the BSM. After analyzing a total of 3,371 $S\&P$ 500 Index European call options traded on March 19th in 2019, our results indeed show that using the Heston Model for option pricing is more accurate than the BSM. To be specific, the Heston is more accurate than the BSM in 12 out of 15 combinations of moneyness and maturity.

We generated three tables for the accuracy comparison between the Heston Model and the BSM. Table 5.2.1 shows the mean absolute percentage Error (MAPE) for a total of 15 scenarios. Table 5.2.2 shows the average MAPE values of all maturity for each respective moneyness. Table 5.2.3 shows that the average MAPE values of all moneyness for each respective maturity.

As we see in Table 5.2.3, as the maturity gets longer, the Heston reaches 0.963% error percentage for the long-term options, which means the Heston Model has quite low pricing error when the maturity gets longer. Based on the results, we suggest using the Heston Model for options that are at-the-money, in-the-money and deep-in-the-money for all maturity groups, deep-out-of-the-money for long-term maturity, and out-of-the-money for middle-term and long-term maturity. We can use the BSM for options that are deep-out-of-the-money for both short-term and middle-term, and out-of-the-money for short-term. In general, the BSM is better at predicting short-term maturity options, and the Heston is better for predicting options that have middle-term and long-term maturity.

Based on the results, there are also two possible areas to be improved. First, the Heston Model has higher pricing error than the BSM for the short-term maturity options, even though as the maturity goes up, the Heston performs significantly better than the BSM. [29] points out that the Heston's lower accuracy in pricing short-term options is due to the fact that the Heston Model fails to capture the volatility smile for the short-term options. This is one of the disadvantages in the Heston Model. The way to improve its accuracy for the short-term options, I suggest, is to add jump process into the Heston stochastic process. By adding jumps, we can take the discontinuous jumps in the stock price behavior into account. Even though adding jumps to stochastic volatility would entail high complexity, it provides a potentially more realistic framework for the option pricing as described in [24].

Second, the Heston Model is quite parameter sensitive. Thus the calibration of the parameters is very crucial. Using a different optimizer leads to different values for the five parameters. I applied the standard way to calibrate the model–non-linear least square optimizer due to its proven accuracy and the time-efficiency. We can try using different approaches to calibrate the Heston Model such as fast Fourier method. But to improve out results even more accurate, we can compare both models' implied volatility to see whether the options are being priced higher or lower than the actual option market prices. Then we can generate the volatility surface in MATLAB to anaylze how both the BSM's estimates and the Heston's estimates compare to each other in different moneyness, maturities and implied volatility more clearly.

Appendix A
MATLAB CODE

A.1 the Heston Model Characteristic Function

This section is the Heston Characteristic Function.This code is mainly cited from [24].

```
 function y = call_heston_cf(s0, v0, vbar, a, vvol, r, rho, t, k)

% Heston call value using characteristic functions.

% y = call_heston_cf(s0, v0, vbar, a, vvol, r, rho, t, k)

% Inputs:

% s0: stock price

% v0: initial volatility (v0^2 initial variance)

% vbar: long-term variance mean

% a: variance mean-reversion speed

% vvol: volatility of the variance process

% r: risk-free rate

% rho: correlation between the Weiner processes of the stock price and

its variance

% t: time to maturity

% k: option strike
```

```
% chfun_heston: Heston characteristic function

% 1st step: calculate pi1 and pi2 % Inner integral 1

int11 = @(w, s0, v0, vbar, a, vvol, r, rho, t, k) real(exp(-1i.*w*log(k)).

    *chfun_heston(s0, v0, vbar, a, vvol, r, rho, t, w-1i).

    /(1i*w.*chfun_heston(s0, v0, vbar, a, vvol, r, rho, t, -1i)));

% inner integral1

real_integral1 = integral(@(w)int11(w,s0, v0, vbar, a, vvol, r, rho, t, k),0,100);

pi1 = real_integral1/pi+0.5; % final pi1

% Inner integral 2:

int22 = @(w, s0, v0, vbar, a, vvol, r, rho, t, k) real(exp(-1i.*w*log(k)).

  *chfun_heston(s0, v0, vbar, a, vvol, r, rho, t, w)./(1i*w));


real_integral2 = integral(@(w)int22(w,s0, v0, vbar, a, vvol, r, rho, t, k),0,100);

pi2 = real_integral2/pi+0.5; % final pi2

% 2rd step: calculate call value

y = s0*pi1-exp(-r*t)*k*pi2;

end
```

A.2 Calibration for the Heston Model

The MATLAB least-square non-linear optimization is presented here. This code is mainly cited from [24].

```
% Heston calibration, local optimization (Matlab's lsqnonlin)

%input on data.txt

% Data = [So, t, k, r, mid price, bid, ask] clear all

clear all

global data7; global finalcost;
```

```
load dat7.txt
format long
% Initial parameters and parameter s
%s [v0, Vbar, vvol, rho, 2*a*vbar - vvol^2]
% Last  include non-negativity constraint and s for mean-reversion
x0 = [.5,.5,1,-0.5,1];
lb = [0, 0, 0, -1, 0];
ub = [1, 1, 5, 1, 20];
% Optimization: calls function costf.m:
tic;
display('before least squares')
x = lsqnonlin(@costf,x0,lb,ub);
toc;
% Solution:
Heston_sol = [x(1), x(2), x(3), x(4), (x(5)+x(3)^2)/(2*x(2))]
x
min = finalcost
```

A.3 Cost Function For the Calibration

This code is mainly cited from [24].

```
function [cost] = costf(x)
global data9;
global finalcost;
% Compute individual differences
% Sum of squares performed by Matlab's lsqnonlin
for i=1:length(data9)
    cost(i)= data9(i,5) -
```

```
      call_heston_cf(data9(i,1),x(1), x(2), (x(5)+x(3)^2)/(2*x(2)),
      x(3), data9(i,4), x(4), data9(i, 2), data9(i,3));
% Show final cost
finalcost =sum(cost)^2;
end
```

A.4 Mean Absolute Percentage Error for the Heston Model

The calculation for the Heston accuracy is presented here. We use mean absolute percentage error as our accuracy criteria.

```
 clear all
 diff=[];
 %load data2.txt
 %mrpe(%)
 format long
 load mrpe_heston_319_ditm_45d.txt
 load market_heston_319_ditm_45d.txt
 y=[]
 mrpe_upper=[]
for i=1:length(mrpe_heston_319_ditm_45d)
    y(i)=call_heston_cf(mrpe_heston_319_ditm_45d(i,1),mrpe_heston_319_ditm_45d(i,2),
        mrpe_heston_319_ditm_45d(i,3),mrpe_heston_319_ditm_45d(i,4),
        mrpe_heston_319_ditm_45d(i,5),mrpe_heston_319_ditm_45d(i,6),
        mrpe_heston_319_ditm_45d(i,7), mrpe_heston_319_ditm_45d(i,8),
        mrpe_heston_319_ditm_45d(i,9));
    diff(i)=abs(market_heston_319_ditm_45d(i)-y(i));
    mrpe_upper(i)=diff(i)/market_heston_319_ditm_45d(i);
end
```

```
%k(%)
k=sum(mrpe_upper)/length(mrpe_heston_319_ditm_45d)
```

A.5 Blakc-Scholes Model

The Formula for the Black-Schoels Model is presented here. This code is mainly cited from [26].

```
function c=bsm_price(St,K,r,t,sigma)
d1=(log(St./K)+(r+0.5.*sigma.^2).*t)./(sigma.*sqrt(t));
d2=d1-sigma.*sqrt(t);
c=normcdf(d1)*St-normcdf(d2)*exp(-r*t)*K;
```

A.6 Mean Absolute Percentage Error for the BSM

The calculation for the Model accuracy is presented here. We use mean absolute percentage error as our accuracy criteria.

```
load dataotm45d.txt
load marketprice_dataotm45d.txt
length(dataotm45d)
length(marketprice_dataotm45d)
mrpe_upper=[];
format rat
for i=1:length(dataotm45d)
    y(i)=bsm_price(dataotm45d(i,1),dataotm45d(i,2),dataotm45d(i,3),dataotm45d(i,4),
        dataotm45d(i,5));
    mrpe_upper(i)=(abs(y(i)-marketprice_dataotm45d(i))/marketprice_dataotm45d(i))
end
mrpe_upper'
```

```
mrpe=sum(mrpe_upper)/length(dataotm45d)
```

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