Pumping a Transition

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Abstract

This thesis is an exploration into the physical mechanism responsible for the acceleration experienced when pumping a ramp on a bicycle, skateboard, or related human-powered vehicle. Pumping is the technique by which one propels oneself on a ramp, which is very similar to the process of pumping a playground swing. It is my hypothesis that the acceleration one experiences when pumping a ramp is primarily due to the conservation of angular momentum, affected by altering ones moment of inertia relative to the focal point of the curve through which one is traveling. This report documents my attempt to accurately describe the process mathematically, experimentally verify my predictions, analyze, and synthesize the results.

Acknowledgements

I would like to thank the Bard College Physics department, Paul Cadden-Zimansky for his encouragement and assistance, and Casey Conlin for his help with the experimental video capture.

Declaration

I confirm that the work contained in this BSc project report has been composed solely by myself and has not been accepted in any previous application for a degree. All sources of information have been specifically acknowledged and all verbatim extracts are distinguished by quotation marks.

Signed: Christopher Daniel Hallman

Date: Wednesday 1st May, 2024

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Chapter 1

Introduction

In this report I will explore the physical mechanism responsible for the acceleration experienced when pumping a transition, i.e. a ramp, while riding a human-powered wheeled vehicle – in my case a bicycle, but the mechanism will be essentially the same were one to consider a skateboard, scooter, inline skates, etc. The physics is similar to that of pumping a playground swing, although perhaps somewhat simpler. I will demonstrate through a comparison of theoretical modeling and experimental results, that the acceleration experienced by rider is the result of the conservation of angular momentum achieved by moving one's center of mass relative to the focus of the ramp (transition/curve) being traversed.

1.1 Background

As a young person I became fascinated by and enamored with the relatively new sports of skateboarding and Bicycle Motocross (BMX) bicycling. By the time I reached my teens, BMX bicycling had become my primary passion and I spent nearly all of my free time practicing and exploring the world onboard my bicycle. I was most impressed by, and most wanted to emulate, the various ways that people were able to jump, or fly, using their bikes, especially using the quarter-pipes or half-pipes that were becoming increasingly popular at the time. When my friend built a ramp in his backyard, learning to ride it, learning to fly, became an obsession, and we spent countless hours burning calories and perfecting the craft. At the time, my friend's father referred to us as "the hyperventilating squirrels."

Despite persistent parental admonition backed by confident assurance that I would soon enough grow out of my childish obsession, the thrill had become an addiction, and I became ever increasingly entrenched in the competitive and cultural world of BMX bicycling. After publishing a short-lived BMX lifestyle magazine, I found employment as a graphic designer and photographer at a sports training camp. It was there that I first recall pondering the physics involved in what we were doing. In addition to riding hours each day, I spent a lot of time observing and helping others to learn various aspects of the art form.

Pumping is a fundamental skill for anyone wanting to participate in one of these sports. I can recall groups of beginner skateboarders lined up across the flat bottom of a half-pipe, all rolling back and forth, like a line of pendulums, trying to find the rhythm of the ramp. Pumping is the primary means by which riders accelerate, but speaking as a rider it becomes subjectively much more than that; it's the way that you learn to relate to a transition. The curve can catch you like a cradle, or hurl you high into the air, and each and every one has its own temperament, its own natural frequency.

1.2 About this Thesis

This is the thesis of *Christopher Hallman*, submitted as part of the requirements for the degree of BSc Physics at Bard College, Red Hook, NY.

In this work I will develop a theoretical framework for explaining the acceleration experienced by a rider (bicyclist, skateboarder, etc.) when pumping a ramp as the consequence of the conservation of angular momentum.

1.3 Chapter List

Chapter 2 Background Research.

Chapter 3 Design: explanation of the theoretical and experimental approach.

Chapter 4 Experimental Implementation.

Chapter 5 Experimental & Theoretical Results.

Chapter 6 Conclusion.

Chapter 2

Background Research

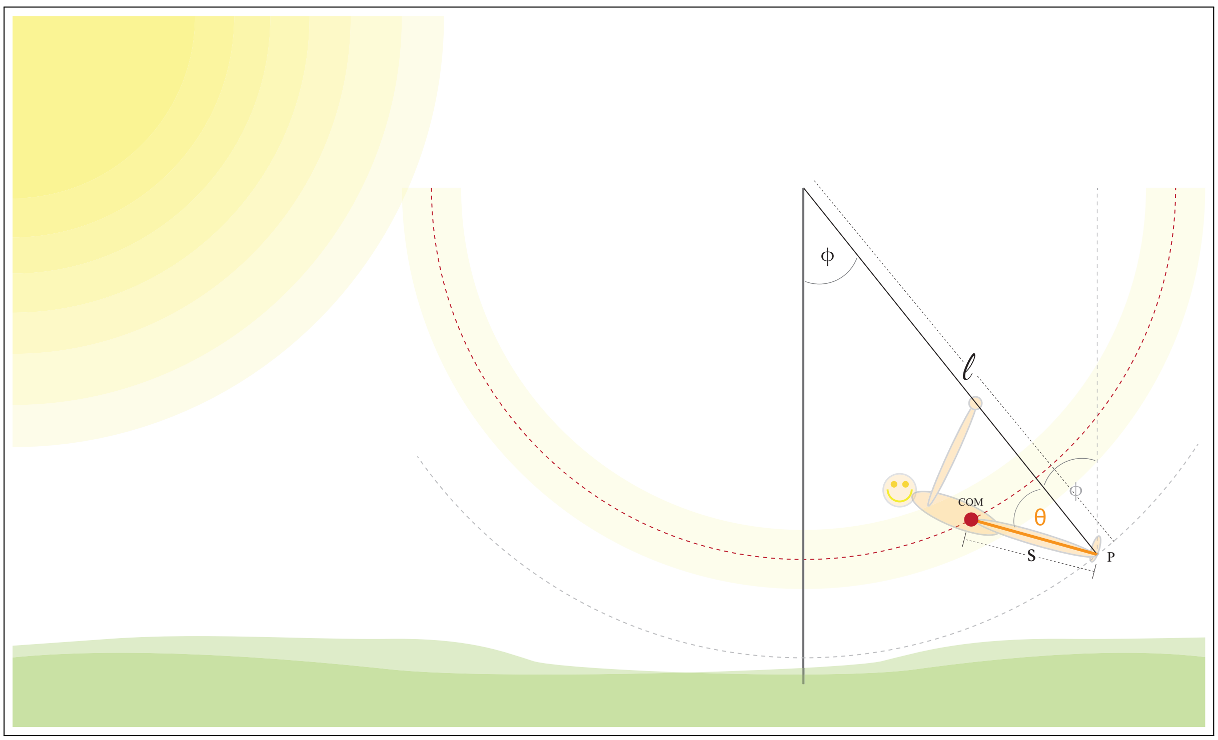

Figure 2.1: Pumping on a ramp is related to pumping on a swing.

Suspecting that pumping on a bike had a lot in common with pumping on a swing, I began by searching to see if there had been any physical studies into the mechanism of pumping a playground swing. I found, somewhat surprisingly, that there has been quite a lot of research into this deceptively simple activity. According to William Case, the author of a 1996 paper[Case 1996] on pumping a swing, "The pumping of a swing is an almost ideal example of a physical system. It is simple and transparent enough to be analyzed with confidence and complicated enough to produce almost magical results." Let me just say that one physicist's idea of simple can be profoundly confusing to a physics student; and I would argue that the continued research, with new papers published as recently as 2023, suggest that this simple activity hides a great deal of rich complexity.

I began by exploring the above mentioned paper, hoping to find an appropriate theoretical approach to the problem. In that paper, titled "The pumping of a swing from the standing position," the authors model the pumping motion of a standing swinger both as what they refer to as a harmonically driven oscillator and also as a parametrically driven oscillator. I believe that the purpose of having the swinger standing is to isolate the two motions and simplify the analysis. The technique that they refer to as harmonically driven involves a 'rigid' swinger forcing the angle between themselves and the supporting chains to vary in phase with the swing, basically leaning back and forth. By contrast, in the parametric model, the swinger pumps by standing and squatting in phase with the swing, thereby lowering and raising their center of mass. Although the parametric model seems more appropriate to the case of pumping a ramp, I wanted to explore both models to better understand the two mechanisms.

The authors employ a Lagrangian approach to derive the equations of motion for both pumping scenarios, although they did not show the derivation in detail. Therefore, to better understand their approach and hopefully learn how to adapt it for my own pumping problem, I derived the equations of motion below. In Lagrangian mechanics, one defines variables that describe the location of an object within a given space; then using that mathematical description, a mathematical algorithm can be used to arrive at the equations of motion.

Lagrangian Approach

Figure 01 illustrates the relevant parameters and relationships necessary to describe the location and movement of the center of mass for the harmonic oscillator approach.

$$x = (R - psin(\phi))cos(\phi) \longrightarrow \dot{x} = (-sin(\phi)(R - psin(\phi)) - pcos^2(\phi))\dot{\phi}$$

$$y = (R - psin(\phi))sin(\phi) \longrightarrow \dot{y} = (cos(\phi)(R - psin(\phi)) - pcos(\phi)sin(\phi))\dot{\phi}$$

In the above equations, x and y represent the position coordinates for the center of mass, whereas $\dot{x}$ and $\dot{y}$ represent the velocity of those points as they move. With this description, the Lagrangian can be constructed, and then using the Euler Lagrange equation, the equation of motion for the system can be calculated. This process is essentially a path optimization problem which results in an equation that describes the motion that an object will take. Below is the Lagrangian for the system, and the resultant equation of motion.

$$\mathscr{L} = \frac{1}{2}m[\,l^2\dot{\phi}^2 - 2sl\dot{phi}(\dot{\phi}+\dot{\theta})cos\theta + (s^2+R^2)(\dot{\phi}+\dot{\theta})^2] + mg[\,lcos\phi - scos(\phi+\theta)]$$

Equation of Motion

$$(l^2 - 2slcos\theta + s^2 + R^2)\ddot{\phi} - gl\sin\phi + gs\sin(\phi+\theta) - ls\sin\theta\dot{\theta}^2 + (ls\cos\theta - s^2 - R^2)\ddot{\theta} - 2ls\sin\theta\dot{\theta}\dot{\phi}$$

The authors of the paper then make some small-angle approximations to simplify the equation. ϕ dependence is Taylor expanded keeping only zeroth and first order terms. θ dependence is Taylor expanded keeping up to the second order. The periodic driving force supplied by the swinger periodically rocking back and forth is modeled with the following relation: $\theta = \theta_o\cos(\omega t)$. With these simplifications and the θ substitution we arrive at the following equation of motion:

$$\ddot{\phi} + \omega_0^2 = F\cos(\omega t) + A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}$$

To better understand how this equation describes the motion of the swinger, I used Python to numerically solve the equation and plot the solution (See appendix A). After this consideration and exploration, I came to believe that the simpler parametric approach would better approximate the motion undergone by a cyclist or skater when pumping a ramp. First of all, I believe that the rocking motion described in the paper is not applicable to the case of pumping a ramp, because a biker or skater can not lean forward or backward without losing balance and falling over. Put simply, a biker or

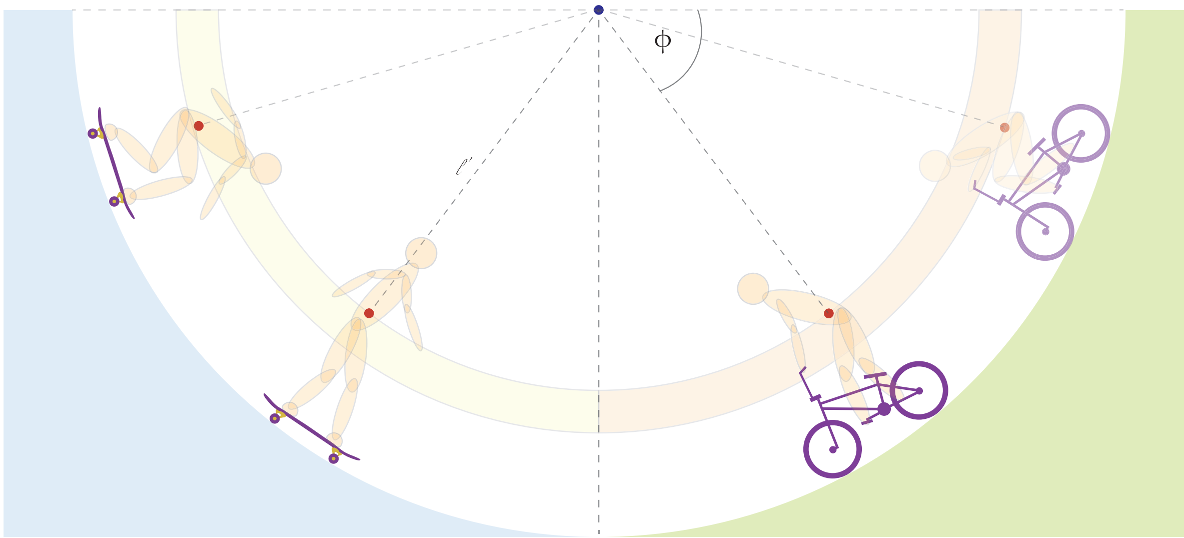

Figure 2.2: Comparison skateboard vs. bike. Notice the axis with respect to which balance must be maintained. The necessity of maintaining balance prevents any significant rotation around the rider's center of mass.

skater does not have a supporting chain to hold onto; and if they were to move their center of mass either fore or aft relative to a line drawn from the focus of the curve to their point of balance on the ramp, the wheels would roll out from underneath them, and they would fall over; an unfortunate outcome referred to as "looping out." While the complexity and horizontal extent of the biker/bike system makes this more difficult to see, the more vertically streamlined skater/skateboard system makes this more obvious. Simply put, a biker or skater must remain vertically balanced, and therefore the only relevant motion is that of their center of mass along the line from the focus of the curve to their point of balance on the ramp.

With this understanding I looked into what the authors of the paper called the parametric approach, which involves the swinger raising and lowering their center of mass along the line of the swing. There are two relevant coordinates to describe the motion of the system, the rotation from origin, which sits at the focal point of the swing and is described by the variable ϕ, and the distance l' which I have labeled in the above diagram. l' is the relevant parameter that changes in order to drive the motion of the system. With these variables defined, the position of the center of mass of the swinger can be located by defining a rotation in degrees and a distance from the origin. Both the tangential velocity as well as the radial velocity of the changing moment of inertia is considered separately, an approach which yields the following Lagrangian and equation of motion. I will use this approach later to derive a parametric model for my system.

$$\mathscr{L}' = \frac{1}{2}m(l'^2\dot{\phi}^2 + \dot{l'}^2) + mgl'\cos\phi$$

$$\text{Equation of Motion} \qquad \ddot{\phi} = -2\frac{\dot{l'}}{l'}\dot{\phi} - \frac{g}{l'}\sin\phi$$

Considering and analyzing these approaches forced me to consider pumping more critically, gave me insights into my problem, and also highlighted the significance of simplifying assumptions. The authors of this paper are arguing that swinging on a swing is better modeled as a harmonic oscillator, which again they define as the rocking back and forth motion, and which they contrast with the parametric motion of varying one's moment of inertia. However they also note that "If the amplitude of the swing is sufficiently small, the contributions from the parametric terms are negligible, the driving terms dominate, and the swing is properly considered a driven oscillator. For motion at a sufficiently great amplitude the situation is reversed, the parametric terms dominate, and the swing is properly considered a parametric oscillator" (Case 217). As evidence for their hypothesis, the go on to note that when you observe a swinger on a swing, that they are "relatively quiet" as they pass through the lowest point in the swing, either leaned all the way forward or all the way back, and that this contrasts with the parametric regime in which the swinger is most active at the lowest point, raising their center of mass.

As part of my research I went to swing on a playground swing in order to develop a better intuitive understanding of the process, trying both standing techniques as well as the traditional seated technique. Notably, when sitting and swinging, I observed that I am exerting the most effort as I pass through the lowest point, even though I am all the way leaned back and my body does not appear to be moving very much. Also when performing the standing and leaning method described by the authors, at the lowest point one needs to hold themselves up with their hands, which kinks the chains. I realized that what I am doing at the lowest point is pulling my lower body up, essentially doing a leg lift. Again, the chains at this point are noticeably kinked. In all the research that I had encountered, the researchers always simplified the problem by modeling the chains as rigid. To my understanding, this means that in reality where chains or ropes are not rigid, there is a significant parametric component even in the rocking back and forth case as the swinger lifts at least some portion of the lower body upwards.

My experiences and consideration of the previous analysis lead me to believe that, by attempting to isolate and examine the two motions separately, the authors may have overestimated the contribution of the rocking harmonic motion. Part of their qualitative argument for the case that the harmonic motion is dominant, at least at lower amplitudes, is that when you observe people swinging, everyone is rocking back and forth, and no one is stood upright on the swing, squatting and standing. In confronting my own biases, I realized that I have a preexisting belief that it is the parametric motion and not any potential rocking that is dominant in the case of pumping a ramp, a point for which I argued earlier. I still believe that this is the case, convinced by my own skateboarder argument, but I do not deny that, in the case of a bicycle with a much larger footprint, there may be some additional effects due to moving ones center of mass fore or aft on the bicycle. However, I believe that any rocking related contribution would be small, and I will focus my attention on the parametric motion.

That being said, the rocking motion is interesting. When swinging on a swing, it provides a mechanism by which to begin the swinging motion and which the authors note is most effective at smaller amplitudes. When riding a ramp, the rider does not have such a mechanism available to them, and so they are always coming at the ramp with some initial velocity. That is to say, we are generally interacting with the ramp in the large amplitude regime where the authors note that the parametric effect is dominant. This again leads me to believe that it is the parametric approach that is more relevant in the case of riding ramps.

2.1 Conclusions

With a better understanding of how playground swings work, and a clearer focus on what parameters would be important in the case of pumping a ramp, my advisor Paul Cadden-Zimansky suggested that I start with a simple model that considered the motion of a rider on a ramp as a combination of the conservation of gravitational potential energy together with the conservation of angular momentum. In this model I will consider a rider starting at the top of the ramp and dropping in. As the rider rolls down the ramp, their gravitational potential energy is transferred to kinetic energy. Then the rider pumps, pushes themselves towards the focal point of the arc, thus decreasing their moment of inertia, and by the conservation of angular momentum, increasing their velocity. I will then attempt to derive a parametric oscillator model of pumping in order to compare the two descriptions.

Chapter 3

Theoretical Approaches

$$x = (R - P\sin\phi)\cos\phi\ , \quad y = R - (R - P\sin\phi)\sin\phi$$

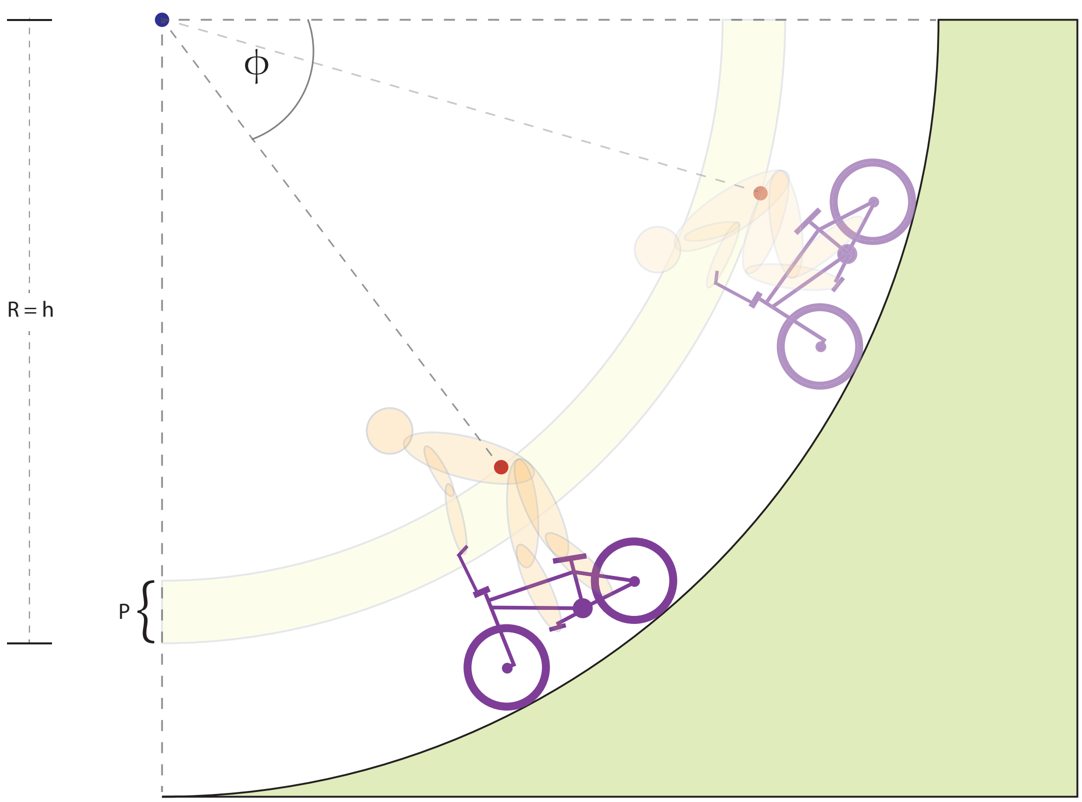

Figure 3.1: A schematic diagram for the approach to analyzing the pump. The origin is at the very bottom of the ramp, bottom left corner of this diagram. P is the displacement of the rider's center of mass, the small red dot. The position of the center of mass is given by the x and y coordinate, with P growing as a function of the angle ϕ.

First allow me to layout some terms and describe the basic theoretical approach. The essential question is how to mathematically describe and track the rider's center of mass, and hence their moment of inertia with respect to the focal point of the ramp, as they ride and pump the ramp. In this analysis I will be considering a bicyclist "pumping" a ramp with circular transitions, the transition being the curve of the ramp. Two common ramp configurations are a quarter-pipe which is in form roughly one quarter of a circle, and half-pipes which are roughly one half of a circle, which is to say that they have the shape of a "U". In the case of a half-pipe the two circular quarter-pipes are generally separated by some distance of level surface at the bottom. This level surface is referred to as the "flat bottom." I will be primarily concerned with a single quarter-pipe as it is the simplest to model and experimentally test. I will compare the velocity expected at the bottom of the ramp for a mass rolling down the ramp under only the influence of gravity, versus what might be expected after pumping.

The pump, or pumping, is the method employed to accelerate when riding a ramp, and is very similar to pumping a playground swing as discussed earlier. I will be tracking the center of mass of the rider/bicycle system as the parameter that characterizes and affects the motion of the system. I am considering a bicyclist because that is my personal experience, but I believe that this analysis will apply equally to other popular wheeled vehicles being ridden on ramps, such as skateboards, inline skates, roller skates, scooters, or even skis and snowboards in those winterized version of the activity. The commonality being the circular shaped ramps that they all utilize.

Figure 3.2: Estonian sport of Kiiking in which the parametric pump is employed to swing completely around.

As discussed earlier, I believe that it is sufficient to consider the radial displacement of the center of mass

for several reasons. Firstly, I am mainly
interested in understanding the dominant physical mechanism that drives the acceleration experienced by a rider, and due to my research, I believe that the conservation of angular momentum affected by changing ones moment of inertia is by far the dominant driver at play in this system. During my research I discovered an Estonian sport called Kiiking in which the swinger, whose feet are strapped in, is able to swing around a full 360 degrees. The pump that they employ to achieve this is the simple parametric stand-squat method, essentially what I believe is the dominant contributor to pumping a ramp. That they have chosen this method rather than some form of rocking back and forth, to my understanding, is evidence for the dominance of this technique.

I begin with a simple model that takes into account the conservation of gravitational potential energy and subsequently of angular momentum to calculate the expected velocity at the bottom of a ramp after pumping during descent. The model parameters include the radius of the ramp, the height of the ramp, the location during the descent at which the pump is performed, and the displacement of the pump (i.e. the distance that the center of mass moves perpendicular to the surface). Because I am tracking the center of mass, which sits approximately 1 meter off the ground in my case, and thus follows a circular trajectory roughly 1 meter smaller in radius than the ramp, I set the radius in my calculation accordingly. That is to say, I measure the height/radius of the ramp, and then adjust that by the position of my center of mass relative to the ramp. Also, for the sake of simplicity, I am considering ramps that are exactly one quarter of a circle, in which case the radius will be equal to the height.

Center of mass calculation To find the center of mass of the rider/bicycle system, I consider separately the center of mass of both the rider and the bicycle, and then use the center of mass formula to find their common center. I approximate the rider's center of mass based on a study published on hypertextbook.com[Elert 2024] which finds that the average center of mass for a human being is roughly located at the navel. When performing experiments I wrapped a florescent strap around my waist at the navel so that I could track it during video analysis. To locate the center of mass of the bicycle I balanced the bicycle along the top to bottom dimension and separately along the fore to aft dimension, then take the crossing point as the approximate center of mass. When analyzing the video, I measure these locations separately relative to the surface of the ramp in order to calculate the center of mass of the system.

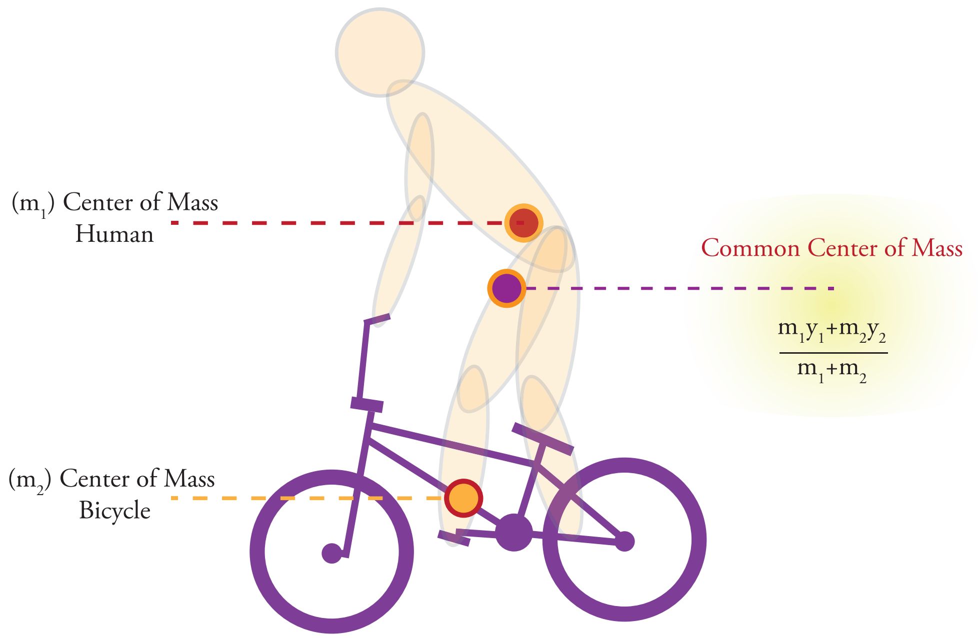

Figure 3.3: Center of mass calculation.

The initial analysis is based on the conservation of energy together with the conservation of angular momentum. If my assumption is correct, and the angular momentum is the primary mechanism at play, this approach provides a mathematically simple way to test that assumption. The analysis is based on the following physical laws and relationships:

Conservation of Energy

$$E_1 = E_2 \Longrightarrow K_i + U_i = K_f + U_f \Longrightarrow$$

$$\tfrac{1}{2}mv_i^2 + mgh_i^2 = \tfrac{1}{2}mv_f^2 + mgh_f^2$$

From which v_f can be solved for: $v_f = \sqrt{v_i^2 + 2g(h_1 - h_2)}$

Conservation of Angular Momentum L

$$L_i = L_f \implies I_i\omega_1 = I_f\omega_2 \implies$$

Moment of inertia $= I = mR^2$, and $\omega = \frac{v}{R}$

Therefore $L = mRv \implies mR_iv_i = mR_fv_f$

Solving for $v_f \implies v_f = \frac{R_i}{R_f}v_i$

With this paradigm it is possible to calculate the velocity at different points along the descent trajectory, as well as the expected final velocity at the bottom of the ramp. To begin, the velocity at any point due simply to the transformation of gravitational potential energy to kinetic energy can be calculated using the relationship $V_g = \sqrt{2gR}$. This initial velocity can then be used to calculate the change in velocity due to the conservation of angular momentum as the rider pumps $\implies v_f = \frac{R_i}{R_f}v_i$, forcing their center of mass radially inward, thus changing their moment of inertia. This second velocity can then be input into a second conservation of energy calculation to find the velocity at some point further down the ramp $v_f = \sqrt{v_i^2 + 2g(h_1 - h_2)}$. This calculation can be repeated any number of times to evaluate different patterns of pumping.

Using this approach I first calculated the expected velocity at the bottom of the ramp considering only the transformation of gravitational potential energy into kinetic energy. That is to say, simply rolling down the ramp without pumping. This could be considered the baseline or control against which theoretical predictions and experimental results can be compared. Next I considered rolling down to the bottom of the ramp, and then instantaneously pumping, the full displacement of the pump, at the very bottom. I then repeated this calculation and considered pumping all at once halfway down the transition. For this calculation I first calculated the initial velocity based on the conservation of energy due to the drop from the top of the ramp to the halfway point, then the change in velocity due to pumping at the halfway point, and then from that second velocity, I calculated the final velocity at the bottom of the ramp, again using the conservation of energy. At this point I decided to write a Python[B.1] program to perform the calculations so that I could quickly run different simulations.

Having performed an experimental trial, I had some data to enter into the equations and check against. The single pump at the very bottom of the ramp gave me a value that matched my experimental result surprisingly close, but as I added more iterations to better simulate a more realistic continuous pump, the theoretical value diverged from the experimental result, limiting towards the value I get if I calculate a single pump at the very bottom, but having first dropped only the distance $(R - P)$, rather than the full height R. While this makes sense, since, if I am pushing my center of mass to a higher level, then it can not drop the full height, the fact that it does not match experimental observations led me to suspect that this model is not telling the whole story. For this reason I decided to pursue the parametric oscillator model.

Parametrically Driven Harmonic Oscillator

A number of papers that I had found analyzing the motion of swings, including "The Pumping of a Swing from the Standing Position"[Case 1996] and "Optimal Strategies for Kiiking: Active Pumping to Invert a Swing"[Petur Bryde, Ian C. Davenport, L. Mahadevan 2010], had modeled the swing-set problem as a parametrically driven harmonic oscillator. While my first approach produced predictions that were close to my experimental results, I wanted to apply the parametric approach to pumping a ramp in order to see if it could better match my experimental results. In this model, the parameter that varies and drives the oscillation is the radius from the focal point of the ramp and the position of the center of mass. The Lagrangian for such a system has the following form:

$$\mathscr{L} = \tfrac{1}{2}m(R^2\dot{\phi}^2 + \dot{R}^2) - gR\cos\phi$$

In this approach, there are two velocities to consider in the kinetic energy, $R^2\dot{\phi}^2$ the tangential velocity, what a rider would consider their forward velocity, and $\dot{R}^2$ the radial velocity of the changing center of mass. ϕ is set to 0 at the bottom of the ramp, and R is the parametric variable which I define as $R = R_0 + P\sin\phi$. In this definition, R_0 is the radius from the focal point to the center of mass at it's shortest distance after pumping. P is the length associated with the pump, the distance that the center of mass moves while performing the pump. At the top of the ramp, at $\phi = 90°$, $\sin 90 = 1$

and the radius is at its longest extent $R_0 + P$. Then as the center of mass moves down the ramp towards $\phi = 0$, the value of sin decreases towards 0, thus shortening the length of the radius to R_0, which represents the motion of the pump. Given this definition, and substituting it into the original Lagrangian, the derivation of the equation of motion is as follows:

$$R = R_0 - P\cos\phi \qquad \dot{R} = P\sin\phi\dot{\phi}$$

$$\mathscr{L} = \tfrac{1}{2}m(R_0^2 + P^2 - 2R_0P\cos\phi)\dot{\phi}^2 - mg(R_0 - R\cos\phi - P\cos^2(\phi))$$

$$\tfrac{\partial\mathscr{L}}{\partial\phi} = \tfrac{1}{2}m(2R_0P\sin\phi)\dot{\phi}^2 - mgR_0\sin\phi - 2mgP\cos\phi\sin\phi$$

$$\tfrac{\partial\mathscr{L}}{\partial\dot{\phi}} = m(R_0^2 + P^2 - 2R_0P\cos\phi)\dot{\phi}$$

$$\tfrac{d}{dt}\tfrac{\partial\mathscr{L}}{\partial\dot{\phi}} = m2R_0P\sin\phi\dot{\phi}^2 + (R_0^2 + P^2 - 2R_0P\cos\phi)\ddot{\phi}$$

Equation of Motion

$$\ddot{\phi} = \frac{RP\sin\phi\dot{\phi}^2 - gR_0\sin\phi - gP\cos\phi\sin\phi}{R_0^2 + P^2 - 2R_0P\cos\phi}$$

I solved this equation numerically in python using solve.ivp from the scipy.integrate package, and plotted the results using matplotlib. I restricted the input from $\frac{\pi}{2} \longrightarrow 0$, to model a person dropping in from the top of the ramp and pumping, where the final output is the position and velocity at the very bottom of the ramp.

The previous model produces a realistic looking curve, but an unrealistically low final velocity. In that derivation I treated the tangential and radial velocities separately, as authors of previous papers seem to have done for their purposes. However I have defined the radial velocity to be a function of ϕ. I believe this requires the velocity vectors to be added first and then squared. So I attempted another derivation as follows:

$$\mathscr{L} = \tfrac{1}{2}m(R\dot{\phi} + \dot{R})^2 - gR\cos\phi$$

$$R = R_0 - P\cos\phi \qquad \dot{R} = P\sin\phi\dot{\phi}$$

$$\mathscr{L} = \tfrac{1}{2}m(R_0^2 + P^2 + 2R_0P\cos\phi + 2R_0Psin\phi - 2P^2\cos\phi\sin\phi)\dot{\phi}^2 - mg(R_0 - R\cos\phi - P\cos^2(\phi))$$

$$\tfrac{\partial\mathscr{L}}{\partial\phi} = \tfrac{1}{2}m(2R_0P\sin\phi + 2R_0P\cos\phi - 2P^2\cos 2\phi)\dot{\phi}^2 - mgR_0\sin\phi - mgP\cos\phi\sin\phi$$

$$\tfrac{\partial\mathscr{L}}{\partial\dot{\phi}} = m(R_0^2 + P^2 - 2R_0P\cos\phi + 2R_0Psin\phi - 2P^2\cos\phi\sin\phi)\dot{\phi}$$

$$\tfrac{d}{dt}\tfrac{\partial\mathscr{L}}{\partial\dot{\phi}} = m[(2R_0P\sin\phi + 2R_0P\cos\phi - 2P^2\cos 2\phi)\dot{\phi}^2 + (R_0^2 + P^2 + 2R_0P\cos\phi + 2R_0Psin\phi - 2P^2\cos\phi\sin\phi)\ddot{\phi}]$$

Equation of Motion

$$\ddot{\phi} = \frac{(-R_0P\cos\phi - R_0P\sin\phi + P^2\cos 2\phi)\dot{\phi}^2 - gR_0\sin\phi - gP\cos(2\phi)}{R_0^2 + P^2 - 2R_0P\cos\phi + 2R_0P\sin\phi - 2P^2\cos\phi\sin\phi}$$

I will discuss the results of these theoretical models in comparison with my experimental results in chapter 5.

Chapter 4

Experimental Implementation

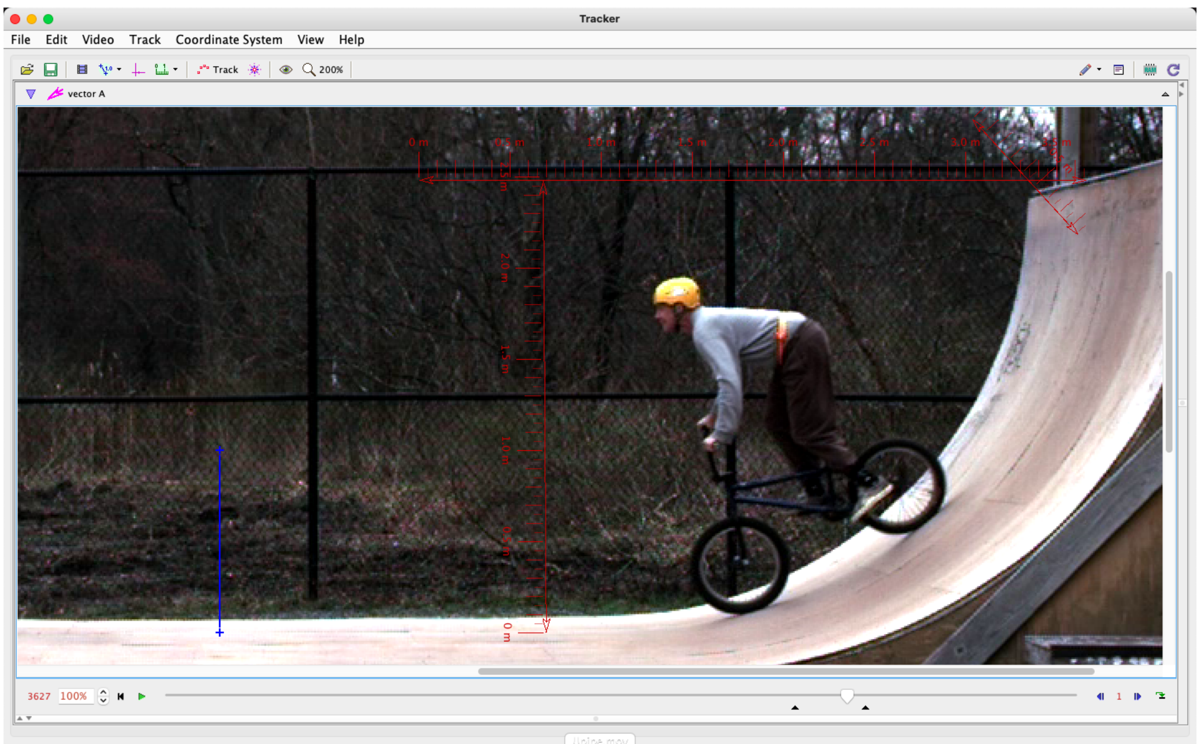

Figure 4.1: Screen shot of experimental video in *Tracker*$^{\copyright}$ video analysis software. The blue one-meter reference stick is visible on the left. With that reference, measuring tools (in red) are available to measuare other distances.

To test the theoretical predictions I filmed myself riding several different ramp configurations trying to find the best way to experimentally capture the acceleration that one experiences when pumping a ramp. After a number of trials I decided to film on a ramp that was exactly one quarter of a circle, and concentrate on one single pass, dropping in from the top of the ramp and pumping on the way down. This simplified the analysis significantly. Starting from a known height, I would be able to easily calculate the theoretical velocity that would be expected at the bottom of the ramp if one were to simply roll down the ramp under only the influence of gravity. This would give me a baseline against which to compare the recorded velocity obtained after pumping.

Care was taken to film with the camera level, positioned at the vertical midpoint, and perpendicular to the ramp at the point at the bottom of the curve where the the curve ends and the flat bottom starts. A focal length between 50mm and 85mm was used to minimize distortion.

That the form of the ramp is an exact quarter of a circle simplifies the calculation in several ways. Firstly, the radius of the ramp is equal to the height of the ramp. I first tried filming on smaller ramps and quickly realized that it was very difficult to determine the radius of the ramp. While the height was easy enough to measure, determining the radius of a partial curve would have required a much more detailed analysis. Additionally, during theoretical calculations, I would have needed to adjust the starting angle to reflect the height; in short, it would have been a much more laborious calculation. With a quarter-pipe that was exactly a quarter of a circle, a simple sine or cosine function ranging from $\phi = 0 \longrightarrow \phi = \frac{\pi}{2}$, coupled with the radius, could be employed to track the position of the rider. For my first calculations I set $\phi = 0$ to be at the top of the ramp, using $R - \sin\phi$ to track the height; while on later calculations I set$\phi = 0$ to be the bottom of the ramp and tracked the height with a cosine.

Distances were calculated in post processing by measuring the height of one meter on my bike (blue line in photo at right), and then defining that dimension to be one meter in the $Tracker^{©}$ software. Once the reference measurement is set, $Tracker^{©}$ offers measurement tools, such as tape measures, which can be used to measure other distances. In this photo I have a tape measure set to measure the height of the center of mass of my bike, and another measuring the height of the center of mass of my body.

The ramp in this case is $2.482m$ tall, which is consistent with an eight foot ramp, and the distance of the center of mass from the surface of the ramp varies from

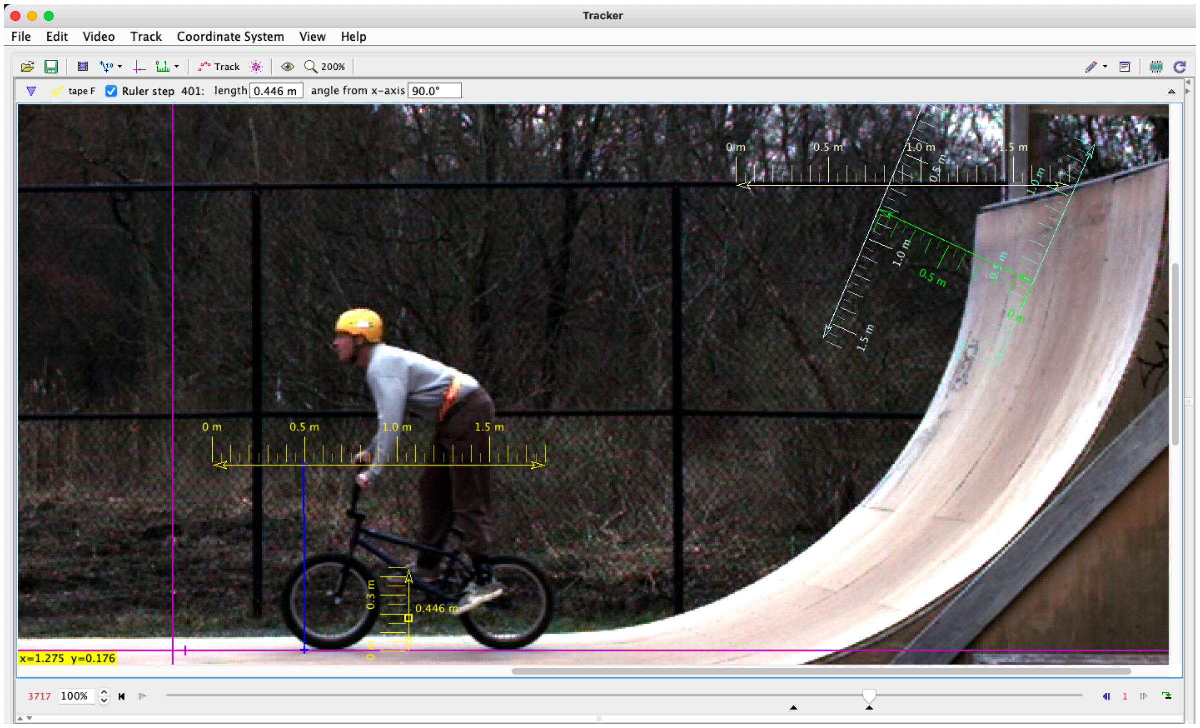

Figure 4.2: Screenshot of *Tracker*© video analysis software with coordinate axis (magenta), distance calibration stick (blue), and multiple rulers visible.

approximately 0.90m to 1.34m throughout the pump. The center of mass was calculated using a standard center of mass calculation based on my weight (78.5kg) and that of the bicycle (11.8kg) separately. My center of mass was approximated at my navel as discussed earlier, and I wore a florescent belt wrapped about my midsection at the navel in order to better track that position in the software. As mentioned, the center of mass of the bicycle was approximated by balancing the bicycle as it lay on its side, both front-to-back and side-to-side; the point of intersection being taken to be the center of mass. The height of the center of mass of the bicycle based on this calculation is 0.45m.

After comparing theoretical models to my experimental results and seeing how sensitive the calculations are to subtle changes in the moment of inertia, I felt that I needed to better refine my center of mass approximations. *Tracker*© software offered the possibility to track a center of mass based on a collection of point masses, by calculating their combined center of mass. I approximated to the best of my abilities the center of each major body part, such as head, thorax, abdomen, upper leg, lower leg, etc. I then found a kinesiology website[exrx.net 2024] with data on the average relative

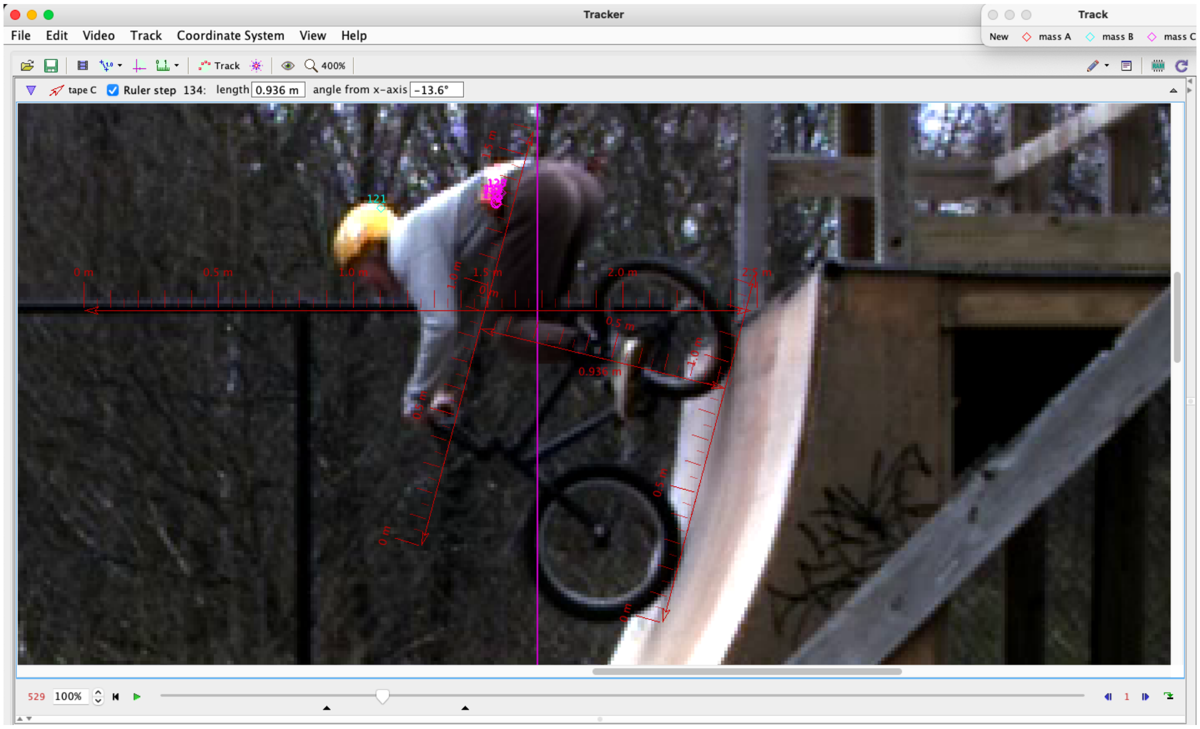

Figure 4.3: In this screenshot, you can see the process I had developed for approximating the center of mass.

mass of body parts as a percentage of total body mass. While the software said it would track the combined center of mass, I was not able to find a way to list its coordinates. So I performed a manual center of mass calculation using the coordinates of each point mass along with the relevant body weight data. When I compared the resultant x and y coordinates for my refined center of mass calculation to my previous intuitive guess in *Tracker*©, I was very surprised to find that they differed by less than 1mm! With greater confidence in the position of my center of mass, I placed a ruler at that position in *Tracker*©, and measured it's distance from the surface of the ramp. The process can be seen in the screenshot below.

I believe that uncertainty in the position of my center of mass will have the greatest effect upon the predictions of the mathematical models, although as discussed I believe that the effect should be relatively small, since it is the overall displacement of the center of mass that has the greatest effect on the change in velocity experienced. To say it another way, it is the ratio of initial radius divided by the final radius that multiplies the initial velocity and causes the acceleration. For example, if I use values from experimental trial 1 and consider a 4 inch (0.08m) discrepancy in the position of the center of mass:

$$\frac{1.569}{1.249} \quad \text{VS.} \quad \frac{1.569-0.08}{1.249-0.08}$$

$$1.2562 \quad \text{vs.} \quad 1.2737$$

The effect of the uncertainty, assuming a 4 inch discrepancy, shows up in the second decimal place, a relatively small difference for my purposes, but not insignificant. Clearly there is value in better approximating the exact center of mass. A more clearly defined singular point, or perhaps a belt with a series of well defined points, would have made this process easier.

Notably, I do not believe that this will affect the experimental measurement of the final velocity, since the bicycle and I are moving as a system, to which end I chose a point on the bicycle that was bright, small, and easy to track, and then tracked that point when it is moving purely horizontally.

One of the most difficult variables to calculate was the initial velocity, and it therefore represents one of the larger experimental uncertainties. I realized this would be an issue, and tried to plan my experiments accordingly. It is nearly impossible to start from a stand-still at the top of the ramp with my center of mass exactly equal to the top of the ramp. This was my first consideration, and I tried several approaches to mitigate this problem. In the final videos I chose to start from rest balanced on the edge of the ramp. This approach put my center of mass closest to the top of the ramp, but I was parallel rather than perpendicular to the surface of the ramp, which made determining the position of my center of mass relative to the surface of the ramp more difficult. When I drop in I rotate my body and bicycle to be perpendicular to the ramp surface. By the time my center of mass passes the top edge of the ramp, I am nearly perpendicular to the ramp, but not entirely. I did my best to locate my body's center of mass in the video analysis as discussed above, and took my bicycle's center of mass to be the same as it is on flat ground. The bicycle's center of mass might be slightly different than on flat ground, but it should be close at the moment that our center of mass passes the top of the ramp and the pump begins.

I also analyzed one run in which, rather than start from a standstill, I rode the ramp and attempted to turn around at the top edge of the ramp, given that at the peak of my maneuver, I would be motionless, essentially starting from a stand still. In each case I was trying also to maximize my body position so that I would be able to pump as much as possible, get the largest displacement of my center of mass. In retrospect, and having done more video analysis, I could have gone a little higher, which would have allowed me to enter the ramp more perpendicular, and have a better body position to pump. In post processing video analysis it's relatively easy to locate the top edge of the ramp and my position relative to it. If I had a more well-defined spot, or series of spots, marking my center of mass, I could use that to determine the initial velocity and perhaps better maximize the motion of the pump.

I filmed a number of trials runs and will analyze eight in order to obtain an average. Results will be discussed in chapter 5.

Chapter 5

Experimental & Theoretical Results

Theoretical modeling was first developed mathematically and then coded in Python in order to solve the equations and produce theoretical predictions that could then be compared against experimental results. Table 5.1 below contains the data from eight separate simulations based on initial conditions from eight experimental trials. The video was filmed at 400 frames per second. Results for each trial run vary depending on the initial velocity and pump recorded. The table lists Recorded experimental velocities as well as theoretically predicted velocities. Column V_i is the experimentally recorded initial velocity. V_{grav} is the theoretical velocity that would be expected if one were to simply roll down the ramp under the influence of gravity alone. The 45°, 50°, 75°, 90° values represent ϕ values for hypothetically pumping instantaneously at those positions in the transition; for example, 45 is halfway down, 75 is $\frac{3}{4}$ of the way down, etc.

The $\approx 50°$ column represents the height from the top of the ramp to the height of P, the pump displacement. That is, the radius that is used in the computations is the radius from the focal point to the center of mass at the top of the ramp. At the bottom of the transition after pumping, the center of mass is at a height $R-(R-P)=P$. So this column represents pumping at a height P up the transition.

I have arranged the results in order of increasing initial velocity. The last row of the table provides Averages values. I will primarily discuss these values.

The first point of interest is that the prediction for hypothetically pumping all-at-once at the very bottom of the ramp, 90 degrees, is the closest to the experimentally

Trial	V_i	V_{grav}	45°	≈ 50°	75°	90°	Cnt.	V_{exp}
6	2.23	6.02	7.18	7.63	7.54	7.83	7.06	8.64
4	2.48	6.12	7.26	7.71	7.82	7.89	7.15	8.82
2	2.68	6.20	7.43	7.87	7.99	8.06	7.31	8.11
7	2.98	6.34	7.62	8.04	8.16	8.24	7.51	7.80
8	3.23	6.44	7.79	8.18	8.32	8.39	7.68	8.65
1	3.33	6.51	7.86	8.27	8.39	8.46	8.55	8.52
5	3.44	7.08	8.63	9.05	9.12	9.18	8.53	8.56
3	4.41	7.11	8.70	8.67	9.18	9.25	8.60	8.46
Avg	3.10	6.48	7.81	8.18	8.32	8.41	7.80	8.49

Table 5.1: Recorded and Theoretical Velocities. V_i is the initial velocity, $V_g rav$ is the velocity expected due to the influence of gravity alone, the degree measurements represent pumping all-at-once that that point in the descent, $Cnt.$ represents the continuous approximation, and V_{exp} is the experimentally determined velocity.

recorded value. In fact, given the uncertainties such as locating the exact center of mass, and considering that I am ignoring friction and other forms of dissipation, this result is close enough to be confused for equivalence. This model assumes that the center of mass falls the full distance R, and then converts that kinetic energy, through the conservation of angular momentum, into the final kinetic energy. The issue is that in reality, I am pumping during the entire descent, and thus my center of mass never falls the full distance R, but rather $R - P$. Considering how the pump is performed in reality, one would expect that the continuous model should best approximate the experimental result; however, the continuous case, in fact, compares poorly with the observed velocity. This leads me to believe that this simplistic model is not capturing some aspect of the physics.

Also curious is the affect of the initial velocity on the theoretical model. When the initial velocity is smaller, all simulations underestimate the final velocity, but when the initial velocity is higher, all models instead overestimate. This to me indicates that the model is placing too much emphasis on the initial conditions, and missing some aspect essential to the process. Again, that the model produces results as close as they are leads me to believe that they are addressing the primary physical mechanism, but there seems to be more subtle aspects to how the changing moment of inertia affects that system that is being missed. I suspect from experience that it has something to do with being in resonance with the transition. When riding a ramp, the faster one goes, or the tighter the curve of the ramp, the faster one needs to pump. Also, being out of sync with the ramp can have disastrous consequences.

Having learned that resonance within a system can have profound effects on energy transfer, I had hoped that modeling pumping as a parametrically driven harmonic oscillator might help to illuminate what is happening during the pump, and account for the discrepancies in the more simplistic model. Unfortunately I was unable get my parametric models to work. I made many attempts, below is a graph of my final model.

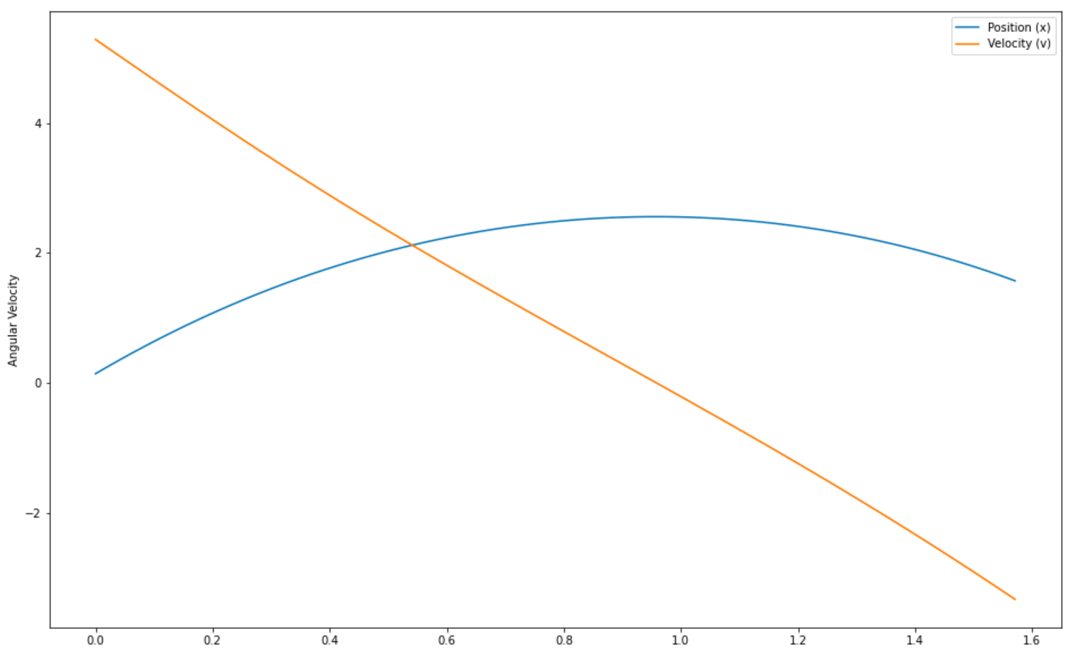

Figure 5.1: A plot of velocity and position predictions from my last parametric model.

I tried inputting different lengths for the pump, which is the say, changing the distance that the center of mass moves, and found that the model is very sensitive to changes in this parameter, which makes me think that it's capturing some aspect of the process; however, my mathematical comprehension has proven insufficient to solve the problem. I went through the Lagrangian process of deriving the equation of motion no fewer than 10 times, and each time I got different results. With each attempt I would input the resultant equation of motion into python and run the simulation. Then I would consider the results, the process, the modeling, and make another attempt. Some of the models produced certain aspects of the motion correctly; however, non made complete physical sense. This final attempt almost has the right magnitude, although the motion is in the wrong direction. At this point it feels like guesswork, and I have no sense of what part of the mathematical process I need to understand better in order to solve the issue.

I had considered that perhaps simplifying the equations might help, but the simplifications that I had seen other authors utilize, such as the small angle approximation, or removing terms of higher order, did not seem relevant in this case, since the regime I was interested in probing occurred at large angles, and the equation of motion is complex to a degree that I do not have a sense of how the different terms are contributing to the motion. I tried naively removing some of the higher order terms, but this had no positive effect on the results.

Chapter 6

Conclusion

Investigating the pump has been an interesting and educational journey. I had long suspected that pumping on a ramp was similar to pumping a playground swing, and while I may not have produced a perfectly accurate mathematical model, my research has convinced me that swing sets and ramps are indeed distantly related cousins. I have fond memories of teaching my daughter to swing; it's almost a right of passage, the transition from "push me, push me" to "I can do it myself." Pumping is a uniquely visceral and enjoyable application of a deep physical law of conservation, and we humans have found many ways to enjoy its accelerating effect, from playground swings, to spinning ice skaters, and more recently as a way to propel ourselves on ramps.

6.1 Conclusions

I believe that I've developed an effective experimental model for analyzing the pump on a ramp, I've provided evidence which links the process of pumping a ramp with that of pumping a playground swing, and I've honed in on the role that conservation of angular momentum plays in the process. Focusing the analysis on a single descent down a ramp that is exactly one quarter of a circle enables a very direct way to probe the effect of the pump, and the fact that the very simple mathematical models, considering only the conservation of angular momentum, get so close to predicting the expected velocity, I find to be compelling evidence for the central role that it plays in the acceleration experienced by a rider on a ramp.

6.2 Future Work

If someone wanted to continue this research, they could further develop the parametrically driven harmonic oscillator model. From a subjective standpoint, pumping on a bicycle feels like a very efficient way to transform physical exertion into acceleration. I find it preferable to pedaling, and have at times dreamt of city sidewalks shaped like endless rolling waves. I'm convinced that it is the conservation of angular momentum that causes the acceleration, and I find the example of kiiking to be particularly compelling entertaining evidence for that conviction. It would be interesting to see the math that describes this process, explained in a way that adds depth of understanding to the physical intuition that one has of pumping, be it on a swing, bike, skateboard, etc.

On the experimental side, if someone were interested in pursuing this research, I would suggest using a skateboarder or inline skater to film the experiments, as I believe that would simplify the analysis even further. Also, marking the center of mass more distinctly would make the task of tracking easier. Small highly contrasting spots would be best. The most difficult parameter for me to measure accurately was the initial velocity, so special attention should be given to that consideration. When I think back to watching others ride ramps, I imagine that inline skaters might be best able to start from a point where their center of mass is obviously even with the top of the ramp.

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Appendix A

Project Support

A.1 Experimental Data

Velocity data recorded from Tracker video analysis software.

Table A.1: Final Velocity data from *Tracker*© video analysis software.

Vid01	Vid02	Vid03	Vid04	Vid05	Vid06	Vid07	Vid08
8.86	9.53	8.82	9.91	9.01	9.47	8.26	9.68
9.09	8.03	8.82	9.91	8.11	8.91	9.17	9.38
8.86	9.20	8.82	9.60	8.56	8.64	9.17	9.68
8.86	9.53	9.09	9.91	9.46	7.24	8.72	8.80
9.32	8.36	6.89	9.60	9.01	6.96	7.80	8.80
9.32	8.36	6.61	8.36	8.11	8.36	8.26	9.68
9.09	8.36	8.26	8.36	8.11	8.91	10.5	8.21
9.55	8.86	8.82	9.29	8.11	8.91	10.1	7.92
8.86	9.03	8.82	8.98	8.56	8.91	7.80	9.09
8.41	8.19	8.54	9.29	8.56	8.64	8.26	9.09
7.73	8.03	7.99	9.60	9.01	8.36	9.17	8.50
9.09		6.89	8.36	8.56	9.47	8.72	7.92
9.77		7.71	8.67	8.11	8.91	7.34	9.09
9.09		8.54	9.60	7.66	6.41	7.80	9.38
		6.89	9.60	7.21	6.96	8.72	7.92
		6.89	9.60	9.01	8.64	8.72	7.04

Continued on next page

Table A.1 – continued from previous page

Vid01	Vid02	Vid03	Vid04	Vid05	Vid06	Vid07	Vid08
		7.99	9.29	8.56	9.47	8.26	7.62
		8.82	8.67	7.66	8.91	8.26	7.92
		9.09	8.05	8.11	7.52	7.34	8.21
		8.82	8.67	8.11	6.96	7.80	8.21
		9.09	8.36	7.66	7.24	8.72	8.50
		8.26	7.43	8.11	8.91	7.34	8.50
		6.89	8.98	8.11	8.91	6.42	7.33
		6.61	10.2	8.11	9.47	7.34	8.21
		7.99	9.29	8.56	8.36	7.80	7.62
		8.82	7.74	7.66	6.13	7.80	7.04
		7.71	7.74	7.21	7.80	7.80	8.21
		7.71		7.66	8.91	7.34	7.62
		7.99		7.66	8.64		
		8.26		7.21	7.52		
		8.26		8.11	7.24		
		6.34		8.11	7.80		
		6.89					
		8.54					
		9.37					
		9.37					
		9.09					
		8.26					
		6.34					
		6.89					
		8.54					
		9.37					
		7.16					
		6.89					
		8.82					

Table A.2: Initial Velocity data from $Tracker^{©}$ video analysis software. For Video01, initial velocity was calculated using $\frac{\Delta y}{\Delta y}$ with the following position and time data from Tracker: $(1.955m - 1.825m)/(0.415s - 0.377s)$

Vid01	Vid02	Vid03	Vid04	Vid05	Vid06	Vid07	Vid08
3.33	3.01	4.13	0.62	3.60	2.79	2.29	2.35
	2.01	4.96	4.33	5.41	3.34	2.29	3.52
	2.01	4.41	4.33	2.70	2.79	4.13	3.52
	1.67	3.86	0.62	1.80	2.23	4.59	4.69
	1.67	4.41	4.33	0.90	2.23	3.67	4.69
	3.34	4.96	3.10	3.60	2.23	3.67	2.35
	3.34	4.68	3.72	6.31	1.11	2.75	3.52
	2.01	2.20	7.43	4.50	5.01	3.67	4.69
	2.01	2.75	5.57	2.70	5.57	4.59	2.93
	2.68	5.23	3.10	2.70	3.34	2.75	1.76
	2.34	4.13	2.48	2.70	2.79	3.67	2.93
	2.68	3.31	1.86	2.70	2.79	5.05	4.11
	2.01	2.75	3.72	5.41	1.67	4.13	3.52
	2.01	1.65	6.81	1.80		4.13	2.35
	3.01	3.86	4.33	2.70		2.75	1.17
	3.01	4.41	4.33	3.60		2.29	3.52
	3.01	3.03	6.19	3.60		4.13	5.28
	2.68	2.20	4.33	5.41		3.21	3.52
	2.68	3.31	0.62	2.70		4.13	4.69
	3.01	5.51		2.70		5.05	2.93
	3.01	4.68		3.60		2.29	4.11
	2.34	3.03		4.50		3.21	4.69
		1.65				3.67	2.35
		2.75					3.52
		4.13					3.52
		4.13					3.52
		4.68					4.11
		3.86					5.28
		3.58					3.52
		4.68					1.17

Continued on next page

Table A.2 – continued from previous page

Vid01	Vid02	Vid03	Vid04	Vid05	Vid06	Vid07	Vid08
							2.35
							4.11
							3.52
							4.11

Appendix B

Python Code

This code models a standing swinger on a playground swing from William Cases 1996 paper "The pumping of a swing from the standing position."[Case 1996] I coded this to learn the process, and in hopes of developing insight.

B.1 Python Code

```
# Parameters
g = 9.81
s = .2                          # .2 meters = 8 inches
l = 2.44                        # major radius, support beam to seat
R = 0.86                        # support beam to COM
Rprime = np.sqrt((l-s)**2+R**2 )
omega = np.sqrt(g*(l-s))/Rprime
omega_0 = omega
Theta_o = np.pi/12              # 15 degees
C = -(l*s*Theta_o**2)/(2*Rprime**2)
B = (omega*l*s*Theta_o**2)/(Rprime**2)
A = -(omega**2*s*Theta_o**2)/(4*(l-s))
F = (omega**2*l*R**2*Theta_o)/((l-s)*Rprime**2)


def pmp(t, y):                  # define a function where the y arument may be an array
    Phi = y[0]                  # assign each dynamic function needed to its own position in the array
```

```
    PhiDot = y[1]

    dPhi_dt = PhiDot
    dPhiDot_dt = ((B * np.sin(2 * omega * t) * PhiDot) + (A * np.cos(2 * omega * t)*Phi) + (F * np.cos(omega * t)) - np.square(omega_0)*Phi) / (1 - (C * np.cos(2 * omega * t))) #define each of the needed 1st order ODEs

    return np.array([dPhi_dt, dPhiDot_dt]) #return the derivative solutions

t_span_pmp = np.array([0, 15])    # set the times
times_pmp = np.linspace(t_span_pmp[0], t_span_pmp[1], 401)

pmp0 = np.array([0, 0])          # set initial conditions corresponding to each dynamic function

soln_pmp = solve_ivp(pmp, t_span_pmp, pmp0, t_eval=times_pmp)    # use solve idp
t_pmp = soln_pmp.t                                                # an array for the times
Phi_pmp = soln_pmp.y[0]          # and the corresponding x(t) solutions; soln.y[1] would give the v(t) solutions

# plot the solutions
plt.rc("font", size=14)
plt.figure(figsize=(10,6))
plt.plot(t_pmp, Phi_pmp, 'o', label='X')
plt.xlabel("time")
plt.ylabel("position")
plt.legend()
plt.show()
```

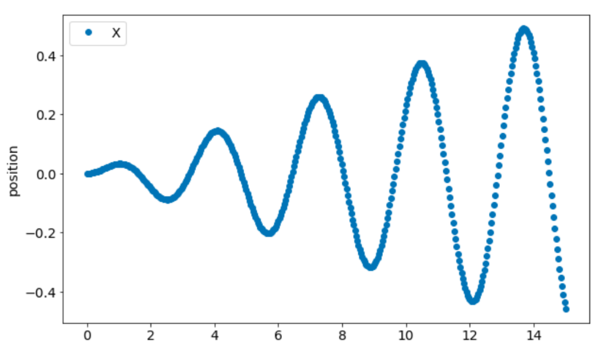

Figure B.1: Output of above code.

The following code calculates velocities based on conservation of energy together with conservation of angular momentum, the approach outlined starting on page 13. The number of iterations, fractional pumps, can be controlled by setting the number of samples in the numpy linspace array named degrees, which generates values between 0 and $\frac{\pi}{2}$. Setting 2 results in one pump at the bottom of the ramp, setting 3 would result in one half pump halfway down, and the second half pump at the bottom, etc.

```
import numpy as np #for more robust numerical calculations
import matplotlib.pyplot as plt #for plots
from scipy.integrate import solve_ivp #for solving ODEs

g = 9.81

Vi = 3.33                          # Average of values from Tracker
Vf = 8.52                          # Average of values from Tracker
m1 = 11.8                          # mass of bike
m2 = 78.5                          # mass of myself
x1 = 0.45                          # center of mass of bike at top of ramp
x2 = 0.925                         # center of mass of myself at top of ramp
x1f = 0.45                         # center of mass of bike at bottom of ramp
x2f = 1.346                 # center of mass of myself at bottom of ramp

COMi = (m1*x1 + m2*x2)/(m1+m2)
COMf = (m1*x1f + m2*x2f)/(m1+m2)

R = 2.482 - COMi            # Radius minus height to COM at top of ramp

p = COMf - COMi             # displacement of COM during pump

V45 = np.sqrt(np.square(Vi) + 2*g*(R*np.sin(np.pi/4))) # no pump at 45
#print("no pump at 45 = ", V45)

V50 = np.sqrt(np.square(Vi) + 2*g*(R*np.sin(-(p-R)/R))) # no pump at 60
#print("no pump at 60 = ", V60)

V75 = np.sqrt(np.square(Vi) + 2*g*(R*np.sin(1.31)))  # no pump at 75
#print("no pump at 75 = ", V75)

V90 = np.sqrt(np.square(Vi) + 2*g*(R))               # No pump @ 90
print("no pump at 90 = ", V90)
```

```
print(" ")

Vp45 = ((R/(R-p)) * V45)                    # V gain from pump at 45
#print("V gain from pump at 45 = ", Vp45)

Vp50 = ((R/(R-p)) * V60)                    # V gain from pump at 60
#print("V gain from pump at height of pump = ", Vp60)

Vp75 = ((R/(R-p)) * V75)                    # V gain from pump at 75
#print("V gain from pump at 75 = ", Vp75)

Vp_45f = np.sqrt(np.square(Vp45) + 2*g*(R - (R-(p*np.sin(np.pi/4))) * np.sin(np.pi/4)) - 2*g*(R - (R-(p*np.sin(np.pi/2))) * np.sin(np.pi/2))) # 1 pump @ 45
print("1 pump at 45 = ", Vp_45f)

Vp_50f = np.sqrt(np.square(Vp50) + 2*g*(R - (R-(p*np.sin(-(p-R)/R))) * np.sin(-(p-R)/R)) - 2*g*(R - (R-(p*np.sin(np.pi/2))) * np.sin(np.pi/2))) # 1 pump @ 45
print("1 pump at height P = ", Vp_50f)

Vp_75f = np.sqrt(np.square(Vp75) + 2*g*(R - (R-(p*np.sin(1.31))) * np.sin(1.31)) - 2*g*(R - (R-(p*np.sin(np.pi/2))) * np.sin(np.pi/2))) # 1 pump @ 45
print("1 pump at 75 = ", Vp_75f)

Vp_90 = (R/(R-p)) * V90            # 1 pump @ 90
print("1 pump at 90 = ", Vp_90)

print(" ")

degrees = np.linspace(0,np.pi/2,200) # array of angles between 0 and 90

P = np.linspace(0,p,len(degrees))    # array of pump displacements

Vp = np.zeros(len(degrees))          #  Array for Velocity values after each pump

Rp = np.zeros(len(degrees))          #  Range of values Rp equivalent to the change in R due to the pump P.

dH = np.zeros(len(degrees))          #  Range of values for the change in height between pumps

Vo=0                                 #  Initial velocity is zero at top of ramp

```

```
for i in range(len(degrees)):        # Calculates Rp values, change in radius due to pump
    Rp[i] = R-P[i]

for i in range(len(degrees)):        # Calcuates dH values, change in height at each point
    if i == 0:
        dH[i] = 0
    else:
        dH[i] = (R - Rp[i-1]*np.sin(degrees[i-1]) - (R - Rp[i-1]*np.sin(degrees[i])))

for i in range(len(degrees)): # Populates array Vp, velocity after each partial pump. Final entry is final velocity at bottom.
    if i == 0:
        Vp[i] = Vi
        Vo = Vi
        #print(Vp[i])
        #print(Vo)
    else:
        Vp[i] = (Rp[i-1]/Rp[i]) * np.sqrt( np.square(Vo) + 2 * g * dH[i])
        Vo = Vp[i]                                                  # Assigns new value to Vo for next iteration
        #print(Vp[i])
        #print(Vo)

print("Initial Velocity = ", Vi)
print("Experimental measured velocity: ",Vf)
print("Vp = theoretical continuous pump = ", Vp[-1]) # continuous pump theoretical value
#print("COMi = ",COMi)                # initial center of mass pre-pump
#print("COMf = ",COMf)                # final center of mass after pump
#print("p = ", p)
#print("R = ", R)
#print("p-R = ",R-p)
#print(1.569/1.249, " vs ", (1.569-.08)/(1.249-.08)) # comparing center of mass precision
#print("Phi = ", np.rad2deg(np.arcsin((p-R)/R)))  # theta at which height of p is reached
```

The following code represents my penultimate approach to model pumping as a parametric oscillator. This derivation is 2 dimensional; however, I'm unclear as to whether or not this approach could me used to model the system.

```

import numpy as np #for more robust numerical calculations
import matplotlib.pyplot as plt #for plots
from scipy.integrate import solve_ivp #for solving ODEs

g = 9.81

Vi = 3.33                             # Average of values from Tracker
Vf = 8.52                             # Average of values from Tracker
m1 = 11.8                             # mass of bike
m2 = 78.5                             # mass of myself
x1 = 0.45                             # center of mass of bike at top of ramp
x2 = 0.925                            # center of mass of myself at top of ramp
x1f = 0.45                            # center of mass of bike at bottom of ramp
x2f = 1.346                 # center of mass of myself at bottom of ramp

COMi = (m1*x1 + m2*x2)/(m1+m2)
COMf = (m1*x1f + m2*x2f)/(m1+m2)

R = 2.482 - COMi            # Radius minus height to COM at top of ramp

p = COMf - COMi             # displacement of COM during pump


def pmp(Phi, y):
    x = y[0]
    v = y[1]
    dx_dPhi = v
    dv_dPhi = ( (-R*P*np.cos(Phi) - R*P*np.sin(Phi) + np.square(P)*np.cos(2*Phi))* np.square(v) - g*R*np.sin(Phi) - g*P*np.cos(Phi)*np.sin(Phi) ) / \
    ( np.square(R) + np.square(P) + 2*R*P*np.sin(Phi) - 2*R*P*np.cos(Phi) - 2*np.square(P)*np.cos(Phi)*np.sin(Phi) )

    return np.array([dx_dPhi, dv_dPhi]) #return the derivative solutions

#set the times
Phi_values = np.linspace(np.pi/2, 0, 100)

```

```
#set initial conditions corresponding to each dynamic function
pmp0 = np.array([np.pi/2, Vi])

#use solve idp
sol = solve_ivp(pmp, [np.pi/2, 0], pmp0, t_eval=Phi_values)

# Plot the solution
plt.plot(sol.t, sol.y[0], label='Position (x)')
plt.plot(sol.t, sol.y[1], label='Velocity (v)')
plt.xlabel('Phi')
plt.ylabel('Angular Velocity')
plt.legend()
plt.show()

print(sol.y[1])
print(" ")
print(P)
print(np.square(P))
print("Tangential Velocity = R_final * ",chr(969), " = ", -sol.y[-1][-1]*R, "m/s")
```

The following code represents my final attempt to model pumping as a parametric oscillator. This is essentially a one dimensional model dependent only on ϕ.

```
import numpy as np #for more robust numerical calculations
import matplotlib.pyplot as plt #for plots
from scipy.integrate import solve_ivp #for solving ODEs

g = 9.81

Vi = 3.33                          # Average of values from Tracker
Vf = 8.52                          # Average of values from Tracker
m1 = 11.8                          # mass of bike
m2 = 78.5                          # mass of myself
x1 = 0.45                          # center of mass of bike at top of ramp
x2 = 0.925                         # center of mass of myself at top of ramp
x1f = 0.45                         # center of mass of bike at bottom of ramp
x2f = 1.346                 # center of mass of myself at bottom of ramp

COMi = (m1*x1 + m2*x2)/(m1+m2)
COMf = (m1*x1f + m2*x2f)/(m1+m2)
```

```python

R = 2.482 - COMi            # Radius minus height to COM at top of ramp

p = COMf - COMi             # displacement of COM during pump


def pmp(Phi, y):
    x = y[0]
    v = y[1]
    dx_dPhi = v
    dv_dPhi = ( R*P*np.sin(Phi)*np.square(v) - g*R*np.sin(Phi) - g*P*np.cos(Phi)*np.sin(Phi) ) / \
    ( np.square(R) - 2*R*P*np.cos(Phi) + np.square(P) )#return the derivative solutions

    return np.array([dx_dPhi, dv_dPhi]) #return the derivative solutions

#set the times
Phi_values = np.linspace(np.pi/2, 0, 100)

#set initial conditions corresponding to each dynamic function
pmp0 = np.array([np.pi/2, Vi])

#use solve idp
sol = solve_ivp(pmp, [np.pi/2, 0], pmp0, t_eval=Phi_values)

# Plot the solution
plt.plot(sol.t, sol.y[0], label='Position (x)')
plt.plot(sol.t, sol.y[1], label='Velocity (v)')
plt.xlabel('Phi')
plt.ylabel('Angular Velocity')
plt.legend()
plt.show()

print(sol.y[1])
print(" ")
print(chr(969))
print("Tangential Velocity = R_final * ",chr(969), " = ", -sol.y[-1][-1]*R, "m/s" )
```