Logistics Planning: Putting Math to Work In a Business Setting

Michael C. Hannan
Bard College

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Michael C. Hannan

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ABSTRACT

The optimization of business procedures benefits all aspects of the product. Maximizing efficiency can lead to more profits for the business, cheaper products for the consumer, and less fuel consumption for the environment. Tracing the history of optimization, we can see that people have always strived for the most efficient way to allocate scarce resources. However, the field of optimization did not blossom until innovations in mathematics allowed us to solve a majority of real world problems. The discovery of linear and nonlinear programming in the 1940s allowed us to optimize problems that were unsolvable before. This paper introduces how linear programming works and then introduces how uncertainty can affect an agent's optimal decision. Expected value and expected utility theory yield different results when accounting for the risk preferences of an individual. In order to optimize a decision for an agent, utility and risk preference must be accounted for.

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CHAPTER 1: INTRODUCTION

What determines the routes that delivery drivers take on a daily basis? What determines which packages belong in each truck? Shipping companies do not randomly assign routes for their deliveries. The route planners take into account each potential factor of the route in order to determine the best route. Things like distance, time, fuel economy, and safety are all factors that matter when route planning. Being able to find the most efficient and safest route allows delivery companies to save money on delivery costs, cut down on delivery time, and keep drivers safe.

The Council of Logistics Management defines logistics as, "Logistics is the combination of transport, storage, and control of material all the way from the supplier, through the various facilities, to the customer, and the collection of all recyclable materials at each step." (Logistics System Design) In order for a business to operate efficiently their logistics must be optimized. Optimization of logistics enables a company to minimize things like cost and production time, while maximizing their profit. The idea of optimization is simple, an agent has certain conditions under which they can make decisions and in order to optimize their decision they must make best use of their conditions. This can either mean minimizing certain things like time, cost of production, risk or labor. It can also mean maximizing things like profit or sales.

With a company like UPS or FedEx being companies that revolve around delivering packages they need to be able to deliver the best product to their customers. In order to be competitive they must be able to offer a service that saves time as well as money. These savings are made from executing on efficient route planning.

Delivery services, like UPS, base their route planning on a version of the truck dispatching problem from Dantzig. This problem uses operations research, or OR, in order to find the most efficient possible routes from a hub to a list of destinations. Not only is figuring out the best routes important for a company like UPS, but also figuring out the best locations to put their distribution centers in. Using OR to solve this problem allows the delivery service to factor in all aspects of the supply chain into making the best decision.

When a delivery service is coming up with an ideal route for their drivers their goal is to make the routes as effective and efficient as possible. Ideally they want to maximize how many deliveries they can make in the shortest amount of time. However there are other conditions that are necessary to consider when route planning. Efficiency on time is ideal but being fuel efficient and safe is also important to companies. All of these factors contribute to how profitable a delivery driver can be. Maximizing the number of deliveries in an allotted amount of time can increase profits from sheer volume of deliveries while saving on the labor costs associated with a driver. On the other hand choosing a fuel efficient route will save on fuel costs that can really add up over the long term. With all these conditions safety is still a factor that can save a company money on lower insurance and minimizing accidents which lead to legal settlements. Linear programming is able to optimize a route by taking into account all of the aspects that can affect the overall profitability.

Logistics and optimization have been happening since before they were something that was studied and sought after. As long as there has been days and time there has been a need to allocate and plan out time. Going back to the inception of the human race there has always been a fight against time. Even in the time where humans were just surviving, there was a necessity to control the time allocated to things like hunting and gathering since it was a matter of life and death.

Time will always be the most scarce resource. This is due to the fact that there is no way to create new time. The only way more time can be created is by taking it away from something else. This leads to a great focus placed on minimized wasted time. When it came down to deciding what to do with their time, a decision had to be made. In the early days of humans that decision was most likely based on what is the best thing I can do to ensure my survival. Once life got more structured and advanced for humans, controlling the time became even more important. Once communities were established along with roles there was more organization with the allocation of time. With more organization came more potential to save time.

Along with time came other important scarce resources. The basics like food and water have needed to be allocated correctly since humans have been alive. Proper allocation could be the difference between life and death of a community. While clever allocation could prevent an outside disaster from wiping out a community. Once civilizations formed there were many ways that the allocation of resources could be influenced. Since resources are always scarce, the decisions that determined what and how they were used were of great importance. The ability to save time and resources on one thing allows them to be allocated towards another, allowing advancements in civilization.

Even before the study of optimal allocation of resources was conceived, humans were striving for the most efficient way to do things. It was not until mathematicians started to play with maximizing aspects of geometry that advancements were made in the study of optimization. Optimization advanced following the advancement of math. From geometry to functions there were new ways to maximize or minimize solutions for these math problems. It was not until the discovery of calculus that optimization became a profound area of interest.

Calculus enabled mathematicians to use math to solve a wider range of problems including problems that pertained to real world applications. Going beyond calculus, algorithms and programs were discovered that were able to take sets of functions and not only find solutions but find optimal solutions. The introduction of linear programming, one of the tools within OR, was the most applicable advancement to real world problems.

Linear programming is used to solve a linear objective given a set of linear constraints. Many logistics problems meet the criteria of linear programming and thus are able to be solved. Linear programming is one of the most widely used methods of optimization in terms of logistics. Linear programming allows an agent to optimize an objective under a set of constraints. Since most real world logistics problems deal with constraints, linear programming becomes a popular method of solution. This method allows us to optimize a linear objective function under a set of linear inequalities which is how linear programming goes beyond other forms of optimization. (Chiang 1967)

This motivation for this senior project was to gain an understanding of how logistics works in a modern sense. This involved taking a look at how decisions are optimized to yield higher profits or lower costs. I was motivated to study this due to an inherent interest in the operation of logistics, specifically in the supply chain and distribution. I have always seen the value hidden in the efficiency of operations. Creating a smooth supply chain does not always bring in the most attention, however I have always appreciated how logistics have worked. In my future I see myself doing work in the operations of a business where optimizing the supply chain can unlock growth for the business.

This senior project is structured around optimization. Included is the history of the development of optimization, starting from ancient times all the way up to the modern day with supercomputers and algorithms. While optimization is being discussed I target linear and nonlinear programming as the avenues of optimization that I will look at. I begin with the discussion of linear programming and the mechanics of solving a linear programming problem. I then provide an example problem and work through the solutions for a simple algebraic method along with the commonly used simplex method. I explain how I can translate a real world production problem into something that linear programming can solve. I then provide the same problem but I solve the problem through MATLAB's program.

After looking at a simple linear programming problem, I introduce ways in which to make the problem harder. I first add an element of uncertainty to a new problem. The uncertainty changes the outcomes to expected outcomes and thus can change how a decision is made. Different forms of decision making under uncertainty are then discussed. First I give a background on how the history of decision making has been studied. I start with the expected value theory and solve a modified linear programming problem under that theory. Then I introduce expected utility theory and explain how that takes the problem from a linear programming problem to a nonlinear programming problem.

Nonlinear programming is then fully introduced and I give a simple example in order to illustrate the mechanics of the solution. I then go off of the simple solution and explain how nonlinear programming could change our actors previous decision in order to optimize expected utility, rather than expected profit.

CHAPTER 2: OPTIMIZATION

Optimization did not start out as such an intensive subject as linear programming. Civilizations were mostly likely optimizing before optimization was even known to be a thing. Decisions on how to plan out a community's agriculture had significant thought go into them. Maximizing yield of crops was crucial for the survival of ancient communities and they likely considered many of the same factors that we consider today when they planted. However these civilizations did not have the mathematical advancements that we are able to depoy today.

The use of math and optimization likely started when ancient Greek mathematicians were figuring out geometry. In 300 BCE, Euclid found axioms that lead into the beginnings of optimization. He found that a square maximizes the area of a rectangle given the length of the edges. He also came up with the Euclidean algorithm, which is able to find the greatest common divisor of two integers. (Artmann, 2023) The next step for optimization would be Dido’s problem. She was allocated as much land as she could encompass with the hide of an ox. With the goal of maximizing the land she received she had to optimize the area with a set perimeter size. Zenodoros found that a circle is the optimal shape for maximizing area with a set perimeter. This is an example of Isoperimetric Inequality but more importantly a way of using math to optimize a sought after output.

After the ancient Greek mathematicians, the next notable innovation was not until Newton and Leibniz both found Calculus in the late 17th century. However, earlier in the 17th Kepler was interested in maximizing the volume of a wine barrel. He approached this by studying areas and volumes and the work he did to maximize the wine barrel ended up being a part of differential calculus. (Cardil, 2012). Calculus made it possible to find local minimum and maximum of functions by using derivatives. This enabled more minimization and maximization issues to be solved, especially staying in the focus of math. Problems like Brachistochrone's problem were able to be solved with calculus. The problem asks on which curve could a bead slide down the fastest from gravity. (Weisstein) Calculus is able to find the curve that minimizes the time it takes for the bead to fall.

Calculus of variation is one of the most profound advancements in the history of optimization. Calculus of variations uses the small changes in functions to find local minima and maxima. Using calculus of variations it is possible to find optimal solutions for objective functions, whether that be maximizing or minimizing outputs. The most commonly used tool in the calculus of variations was discovered by Joseph-Louis Langrange. He found a way to optimize functions within a set of constraints. He came up with the lagrangian multiplier technique which uses systems of equations to optimize functions under certain constraints. Lagrangian optimization is typically used in fields like microeconomics in order to optimize things like profit constrained to a budget. The introduction of lagrangian optimization enabled specific real world problems to be investigated with math.

In the 18th century optimization began to be applied to other problems outside of math. Gaspard Monge was the first to use transportation theory in order to optimize the amount of resources needed for a project. Monge sought to optimize the distribution of resources and transportation. (Bourne, 2018) Monge however did not have the proper methods to fully solve this problem. In the 19th century optimization was further advanced with the introduction of algorithms by a deeper understanding of calculus. Then came the convex production function due to the law of diminishing returns. This made calculus necessary in order to maximize the amount of return on production. Here calculus was being used to solve questions that go beyond mathematics and helped production for communities or businesses.

Joseph Fourier then comes up with a way to manipulate linear inequalities which can then be turned into linear programming. His intent with manipulating the inequalities was not optimization, however he was looking for a solution out of a set of inequalities. While his method is not efficient at solving complex problems, it is easily translated into linear programming and is simpler than the common ways of solving linear programming problems (Williams, 1986).

Optimization really became what is today in the 20th century. In the early 20th century there were still new discoveries dealing with convex functions and the calculus of variations. However it was not until around 1940 that linear programming was founded. Leonid Kantorovich was the first to come up with what is now linear programming. He was working to maximize the distribution of resources and linear programming was the way he solved allocation issues. Independently of Kantorovich, Tjalling C. Koopmans also found linear programming. He was looking for a way to optimize shipping routes in order to minimize costs. His result was a system of equations that took into account the cost of the resources as well as the shipping costs. Both Kantorovich and Koopmans were awarded the Nobel Prize for their contribution to optimization and allocations of scarce resources. (Encyclopedia Britannica)

While Kantorovich and Koopmans were the ones who discovered linear programming, it was really not until after World War II where linear programming became the go to method of solving logistics issues. George Dantzig was a mathematician who was working for the pentagon during the war. He developed the simplex method for solving linear programming problems. This simplex method turned difficult logistic decisions into something that was able to be computable and optimized. Dantzig's contribution to the war efforts enabled the United States to efficiently address the supply chain and allocate resources across the globe.

The scope of influence of Dantzig’s simplex method is truly enormous. The optimization of the entirety of the supply chain had benefits in all aspects. Efficiency of distribution led to lower costs and shorter times. With less costs and more time, productivity was able to increase and so to profits. Since the simplex method has been trickled down into any decision requiring optimization with constraints, the overall value of Dantzig’s work is priceless. It is estimated that linear optimization contributes 5% to overall output, which equates to around $1 trillion, just in the United States alone. (Birge, 2021)

After the breakthrough of linear programming, the optimization of nonlinear functions came next. Dantzig was also on the forefront of working on nonlinear optimization with his simplex method. Probably the most important work however, was done in the aspect of determining optimal solutions from regular solutions. This was done with the Karush-Kuhn-Tucker conditions that created a set of conditions that needed to be met in order for a local solution to be the oprimal solution. Then in the 1960s and 1970s the quasi-Newton method was discovered as a numerical approach to nonlinear optimization. (Su, 2020) This was a generalization of Newton’s method at solving nonlinear equations.

In more modern times optimization has been continuously advancing. With the emergence of stronger algorithms and computers the scope of solvable problems has increased. However they all still share the same bones as the early problems that started from geometry. Being able to maximize or minimize what you are given will always be a valuable tool. Now with the modern advancements of optimization we are able to make better decisions in situations where there are countless factors that need to be accounted for.

CHAPTER 3: LINEAR PROGRAMMING

Examples of linear programming applied to real world logistics issues are seen in, *View of Optimal Fleet Size and Mix for a Rental Car Company* (Saadouli, 2021), where they use linear programming to optimize the fleet size and type of the rental car company. *Routing in Offshore Wind Farms: A Multi-Period Location and Maintenance Problem with Joint Use of a Service Operation Vessel and a Safe Transfer Boat* (Ade Irawan et al., 2023). *Optimization Model for Sustainable Food Supply Chains: An Application to Norwegian Salmon* (De et al. 2022), optimizes the supply chain costs by minimizing fuel consumption in order to lower costs as well as emissions. Linear Programming is versatile in the fact that it does not care what it is optimizing and that most logistics issues are able to fit under its structure.

In order for a problem to be solved using linear programming, the functions used must all be linear. Linear programming uses two different types of functions, an objective function followed by a set of constraint functions. The constraints functions are linear inequalities. I will first start out with a simple example of a logistics problem in order to show the basic mechanics of the solution. The steps in the solution will provide a backbone for more complex problems as well as illustrating what we are really trying to solve for when we look at a similar problem. Here is an example of a simple problem that can be solved easily by plotting inequalities.

A business firm produces two lines of product x and y; the average profit for product x is $40 per ton, and for product y it is $30 per ton. The plant consists of three production departments: cutting, mixing, and packaging - and the equipment in each department can be used 8 hr a day; thus we shall regard 8 hr as the daily capacity in each department. Finally, the process of production can be summarized thus: (1) Product X is first cut, then packaged. Each ton of this product uses up .5 hour of the cutting capacity and ⅓ hour of the packaging capacity. (2) Product Y is first mixed then packaged. Each ton of this product uses up 1 hour of the mixing capacity and ⅔ hour of the packaging capacity.

What combination of products should the firm produce per day in order to maximize total profit?

When first looking at a problem like this we want to construct our objective and constraint functions. In this case our objective function is the profit function since we are looking to maximize profit here. The goal here is to maximize our profit function while staying within the constraints.

We get our profit function of,

$$\pi = 40x + 30y$$

since we are given that product x is \$40 per ton and product y is \$30 per ton.

Our constraint functions are subject to the production limitations explained in the question. Our first constraint is found by product x needing half an hour to be cut for one ton. Since there are 8 possible hours for product x to be cut there can only be a maximum of 16 tons produced per day. Our second constraint expresses the mixing limitations of product y per ton. Since product y uses 1 hour of mixing capacity per ton with only 8 available hours the production of product y is limited to 8 tons per day. Our third constraint expresses the packaging capacity of each product. With product x needing ⅓ hour to be packaged per ton and product y needing ⅔ hour per ton in an 8 hour day we can set up our constraint. The constraint listed is simplified by multiplying the inequality by 3, which turns ⅓ into 1, ⅔ into 2, and 8 into 24. Our last constraint is very simple yet necessary. Since the company cannot produce negative values of either product both x and y must be greater than or equal to zero.

$x \leq 16$

$y \leq 8$

$x + 2y \leq 24$

$x \geq 0$

$y \geq 0$

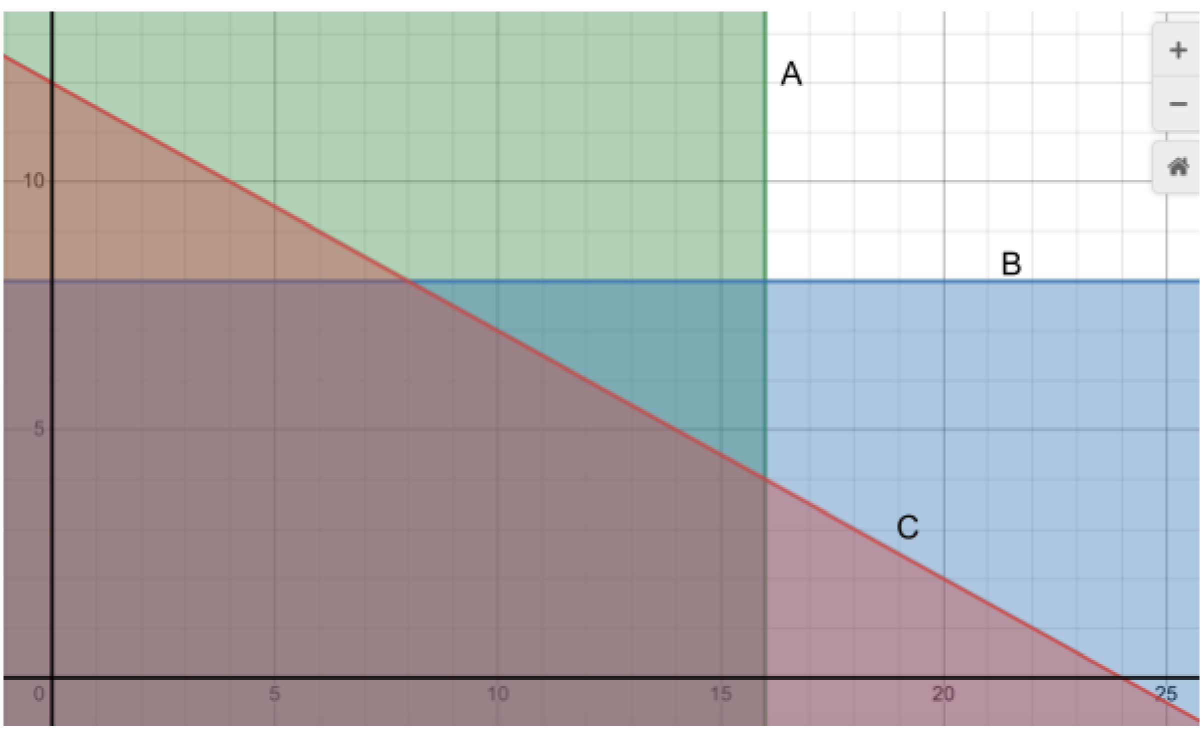
Solving this problem with Linear Programming using inequalities would go as follows:

Graph the constraint functions:

$$A : x \leq 16$$

$$B : y \leq 8$$

$$C : x + 2y \leq 24$$



Our first constraint function A is shown on the graph with the green line. Since the function is an inequality there is a green shading for the values that suit the inequality. So on the graph every x that is less than 16 is shaded green. Our second constraint function B is shown with the blue line that is coupled with the blue shading. Every y less than 8 in this case is shaded in blue. Our third constraint C is shown by the red line and red shading. Every value that satisfies the function is shaded in red. We can use the shading to show us the area of the graph where all our constraints are satisfied. This area is where the green, blue, and red shading all overlap. This area is our feasible region.

The feasible region is where all our inequalities overlap and where our potential solution lies. The vertices of our feasible region give us the points where our constraints are maximized. This allows us to single out viable solutions due to our profit being maximized if the constraints are maximized. The solution always lies on a vertex because a vertex is where multiple constraints are optimized. The extremes of each constraint allow the firm to use up all of their total possible hours, and production equipment. In order to find the maximum amount of profit we must plot our objective function on the graph where it intersects with our feasible region vertices.

Not each vertex of the feasible region is the same however. There are three types of vertices. The first type includes the intersection of two constraint borders, such as (8,8) and (16,4). These points both maximize the limits of two constraints while leaving the third constraint partially filled. This leads to an underutilization of capacity in terms of our question which can be referred to as *slack* in capacity utilization. (Chiang, 1967)

The second type of vertices is where a constraint border intersects an axis. In our example this would be the points (0,8) and (16,0). These points only maximize one constraint and therefore leave slack in the two other constraints. The last type of vertex is the origin, (0,0). This point represents the time where the firm's production is completely idle and not producing anything. Minimization problems, on the other hand, do not have a vertex on the origin. By understanding what each vertex of the feasible region implies we are able to focus on the vertices that lead us to our potential solution.

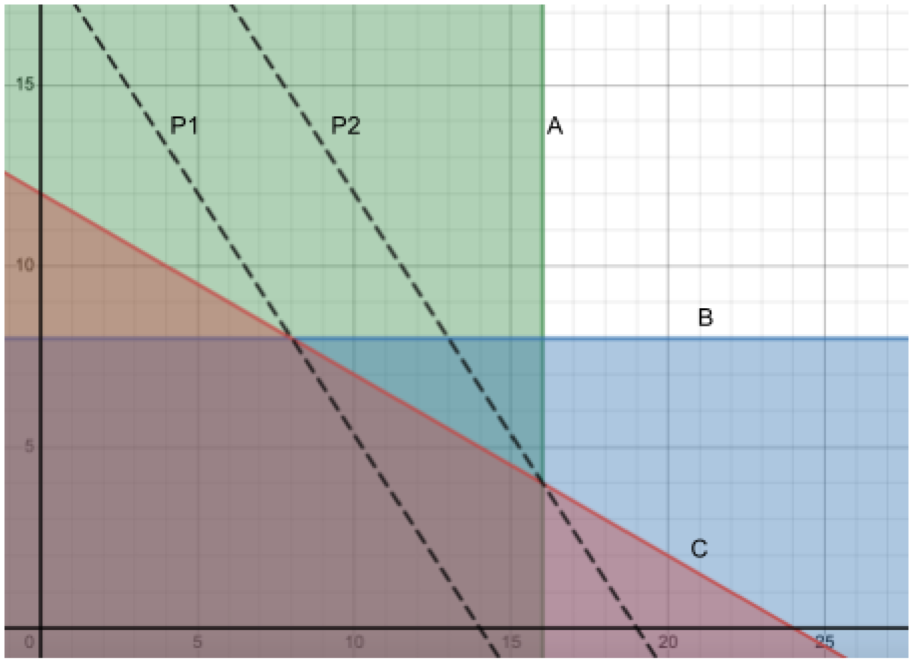
By solving for y we can rewrite our objective function as:

$$y = \frac{\pi}{30} - \frac{4}{3}x$$

The objective functions are shown in the graph by two dotted lines. These lines are isoprofits. Where the overall profit was calculated by inputting the values of the product represented by the respective vertices they intersect with. Our objective function can have infinitely many lines based on the profits. However these isoprofits correspond with our objective function under the given constraints.

$$P1 : 560 = 40x + 30y$$

$$P2 : 760 = 40x + 30y$$



When plugging in the values of the vertices into the profit function we find that the point (16,4) yields more profit ($760) than the point (8,8) which yields $560. For P1 the constraint function B and C are maximized while A is not. P2 on the other hand, maximizes A and C while B is not maximized. This gives us information about which forms of production are more profitable for the firm. Therefore, in order for the firm to optimize their production they should produce 16 tons of product X and 4 tons of product Y.

Going back to the ideal of slacks in the constraints we are able to figure out an algebraic method for finding the extreme points of the feasible region. Since our example has three constraints with two variables, each of the extreme points will have slack in at least one of the constraints. We are able to calculate the magnitude of slack in each constraint. When we find an optimal solution for production we are thus also optimizing the value of the slacks. We will create dummy variables for each slack.

Let's apply the dummy variable technique to our example. Here is our objective function, Maximize:

$$\pi = 40x + 30y + 0s_1 + 0s_2 + 0s_3$$

Subject to:

$$x + s_1 = 16$$

$$y + s_2 = 8$$

$$x + 2y + s_3 = 24$$

$$x, y, s_1, s_2, s_3 \geq 0$$

These constraints can be expressed in matrix notation as well when x = y = 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 24 \end{bmatrix}$$

The leftmost matrix is an identity matrix meaning that it can be taken out of the equation without affecting the other values. Therefore the solution can be interpreted as, s_1=16, s_2=8, s_3=24. This leads us to the point (0,0,16,8,24) in the solution space or in our example, the origin. This method is called the basic feasible solutions method (BFS) and it allows us to find the feasible solutions of a linear programming problem without having to do geometry. This enables us to find feasible solutions when there are too many constraints and variables to plot on a graph.

The simplex method expands on the BFS method by finding the optimal feasible solution. This is necessary when problems become more complex and finding each feasible solution becomes unrealistic. The method works by starting with an extreme point and finding the value of the objective function at that point. From there it will move to an adjacent point to see if the objective function can be improved, if it can be improved then it will keep moving to new extreme points until the objective function cannot be improved upon.

We will start by creating our simplex tableau which includes all our variables for the objective function, constraint functions, as well as our slack variables. Our objective function is transposed to be equal to zero, which is why there are negative values.

Tableau	π	x	y	s_1	s_2	s_3	*Constant*
*Row*0	1	−40	−30	0	0	0	0
*Row*1	0	1	0	1	0	0	16
*Row*2	0	0	1	0	1	0	8
*Row*3	0	1	2	0	0	1	24

From our tableau we then can move to the pivot step. In this problem we are looking to maximize the profit. Our x variable has the highest absolute value and it is a greater profit booster than y. The criterion dictates that the variable which has the negative entry with the largest absolute value in row 0 should be selected. Here this would be -40 in the x column. This will be the pivot column. The pivot element will be in the x column in row 1. We are able to use this value as our pivot element because it does not violate any production constraint. We chose this by following four steps. Step 1: pick out elements in the pivot column (excluding row 0) that are positive. Step 2: divide each element into their constant counterpart. Step 3: compare the resulting quotients, which are the displacement quotients, and choose the row with the smallest quotient. Step 4: select the element that intersects the pivot column and pivot row.

Once we have the pivot element the next step is to transform the picot column into a unit vector by setting the pivot element to 1. Our pivot element is already 1 so nothing has to be done. If it were not zero we would have to divide all the other elements in the row by that number. Our row 2 element is already zero so no action is required. To transform the element -40 in row 0 into a zero we can add 40 times the pivot row to row 0. Likewise, in order to transform the element 1 in row 3 to a zero we can subtract row 1 from row 3. These results give us a new simplex tableau shown here.

Tableau	π	x	y	s_1	s_2	s_3	*Constant*
*Row*0	1	0	−30	40	0	0	640
*Row*1	0	1	0	1	0	0	16
*Row*2	0	0	1	0	1	0	8
*Row*3	0	0	2	−1	0	1	8

Another pivot step is required in order to further maximize profit. A negative entry in row 0 means that the marginal profit rate is positive and improvement on profit is possible. Thus the y column will be the next pivot column. Row 3 will be the pivot row due to it having a smaller displacement quotient. Our pivot element is then 2. In order to turn the row into a unit vector we must first add 15 times row 3 to row 0 in order to turn -30 into zero. Then we must leave row 1 intact since it is already zero. Then we must subtract $\frac{1}{2}$ times row 3 from row 2 to turn 1 into zero. Then we must divide row 3 by 2 to turn 2 into 1. This leaves us with this simplex tableau.

Tableau	π	x	y	s_1	s_2	s_3	*Constant*
*Row*0	1	0	0	25	0	15	760
*Row*1	0	1	0	1	0	0	16
*Row*2	0	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	4
*Row*3	0	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	4

In this tableau there are no more negative entries meaning that pivoting will not be more profitable. Therefore we are able to read the optimal solution out of this tableau. From row 0 we get the value of our profit variable, $\pi = 760$. From row 1 we get the value of our x variable, $x = 16$. From row 3 we get the value of our y variable, $y = 4$. And from row 2 we get the value of our slack variable, $s_2 = 4$. This is the same solution that we got from both the geometric solution as well as the MATLAB solution.

Tracing back our steps we can see that the simplex method took us from the point of origin, which is the initial extreme point, to the next extreme point (16,0). Upon finding that that extreme point was not optimal we then went to the optimal extreme point at (16,4).

Here is how the problem would be solved using MatLab:

For this example, use these linear inequality constraints:

$$x \leq 16$$

$$y \leq 8$$

$$x + 2y \leq 24$$

And the objective function:

$$\pi = 40x + 30y$$

```
A = [1 0

  0 1

  1 2];



b = [16 8 24];
```

Use the objective function $\pi = 40x + 30y$

```
f = [40  30];
```

```
%We set our objective function to negative in order to have MatLab maximize
%our objective function.
f = -f
```

f = *1×2*

-40 -30

Solve the linear program.

```
x = linprog(f,A,b)
```

Optimal solution found.

x = *2×1*

16

4

We can see our solution is the same from the graphical method, simplex method, as well as the MATLAB solution.

Our introduction to linear programming was based on the same simple problem that consisted of a manageable number of constraints and variables, three and two respectively. The first method of solution is the most simple and uses algebra and geometry to locate potential solutions. Those solutions then have to be manually checked and cross referenced in order to find the optimal solution. For simple problems this method is extremely manageable since there are only a few potential solutions. However when more constraints or variables are added to a problem, the amount of potential solutions also increases. On top of having to check more potential solutions for the optimal solution, the geometry becomes un-plottable due to too many dimensions.

For more complex problems, relying on the graphical method is not practical. The simplex method is more effective because it does not rely on graphing at all and instead relies on algebra to optimize a solution. This enables problems with many variables and constraints to be solved. With most real world optimization problems being too complex for the graphical solution, the simplex method becomes even more important. The simplex method took linear programming to the next level by broadening the range of solvable problems. Now going beyond the simplex method, there are computer programs that are able to solve linear programming problems. MATLAB has a built-in command for linear programming and efficiently solves problems that can be extremely complex.

CHAPTER 4: INTRODUCING UNCERTAINTY

Linear programming allows us to maximize an objective function, so long as all the parameters are linear functions. This allows us to use linear programming to maximize more than just value for an agent. Maximizing value is the most straightforward approach at optimization because value is a nominal figure that can be applied universally. When maximizing value or profit under certainty we are easily able to determine the optimal solution. However, what if our agent is making a decision under uncertainty and not fully making their decision based on the nominal value? This leads us into the evolution of ideals on how people deal with uncertainty.

The most simple way to deal with uncertainty in decision making is to find the potential expected values of each decision. Expected value deals with the associated probabilities of each outcome. The expected value of a decision is the potential value of that decision weighted by its probability. In the case of multiple outcomes each outcome would be assigned a value that was the product of its probability multiplied by the value. Rational behavior under the expected value model would be to choose the decision that yields the highest expected value.

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i \cdot p_i$$

Let's look at a simple problem where there is uncertainty in how much yield a farmer can have in a certain year. The yield is uncertain due to uncertainty things like the weather. In this problem we have simplified his range for devoting land to specific crops in order to illustrate a solution that requires many steps per added variable. In the future we will solve a more complicated problem like this with a program.

A farmer has a total of 100 acres of land available for growing wheat and corn. We denote by x1 and x2 the amount of acres devoted to wheat and corn. The farmer must plant the two crops in a 60:40 split. Meaning if 60 acres of wheat are planted, then 40 acres of corn must be planted. The planting costs per acre are 150 and 230 US-Dollars for wheat and corn. The purchase prices of wheat and corn per ton are 238 US-Dollars for wheat and 210 US-Dollars for corn. We denote by w and c the amount in tons of wheat and corn sold. The yield on the farmer's land is a random variable ξ = (ξ1 , ξ2) T which can take on the realizations 3.0 T, 3.6 T (high yield), 2.0 T, 2.4 T, (low yield) per acre for wheat and corn each with probability 1 / 2. The farmer wants to maximize his profits.

We are not assuming that a high yield year for wheat implies a high yield year for corn. Therefore this solution requires us to use expected profits given the probabilities for each given scenario.

Our objective functions are therefore weighted by their respective probabilities.

$$\pi = 238w + 210c$$

w represents tons of wheat sold while c represents tons of corn sold

Probabilities come into play when figuring out how many tons of each crop the farmer is able to sell.

Lets first look at the expected profits given that the farmer chooses to allocate 60 acres to wheat and 40 acres to corn. We will use a bimatrix to show the four different scenarios that are possible under the given allocations of land.

		Wheat	
		Low (.75)	High (.25)
Corn	Low (.25)	Y1 (.1875)	Y2 (.0625)
	High (.75)	Y3 (.5625)	Y4 (.1875)

Given these probabilities our expected yield from corn would be,

$$(2.4\times\ .25)+(3.6\times\ .75)=3.3$$

giving us 3.3 tons per acre.

For wheat this expected yield would be,

$$(2\times\ .75)+(3.0\times\ .25)=2.25$$

giving us 2.25 tons per acre.

Now let's look at the scenario where the farmer allocates 60 acres of land to wheat and 40 to corn.

With an expected yield of 2.25 tons per acre and 60 acres planted, the expected total yield for wheat is 135 tons.

With an expected yield of 3.3 tons per acre and 40 acres planted, the expected total yield for corn is 132 tons.

Now we must find the given profits associated with the expected yields in order to get our expected profit.

Wheat costs $150 per acre to plant and sells for $238 per ton. Corn costs $230 per acre to plant and sells for $210 per ton.

In this scenario the farmer is spending $9,200 to plant corn and $9,000 to plant wheat, totalling $18,200. Therefore we will calculate the revenue from each and subtract the cost to plant.

With an expected total yield for wheat at 135 tons we can multiply by the purchasing price in order to get expected revenue. Our expected revenue is $32,130. Our expected revenue for corn is $27,720.

The expected profit of wheat is our expected revenue minus the cost:

$$\pi = 32,130 - 9,000$$

$$\pi = 23,130$$

The expected profit of corn is:

$$\pi = 27,720 - 9,200$$

$$\pi = 18,520$$

Giving us a total expected profit of $41,650.

Now let's look at the scenario where the farmer allocates 60 acres to corn and 40 to wheat.

With an expected yield of 2.25 tons per acre and 40 acres planted, the expected total yield for wheat is 90 tons.

With an expected yield of 3.3 tons per acre and 60 acres planted, the expected total yield for corn is 198 tons.

Now we must find the given revenues associated with the expected yields in order to get our expected profit.

Wheat costs \$150 per acre to plant and sells for \$238 per ton. Corn costs \$230 per acre to plant and sells for \$210 per ton.

In this scenario the farmer is spending \$13,800 to plant corn and \$6,000 to plant wheat, totalling \$19,800. Therefore we will calculate the revenue from each and subtract the cost to plant.

With an expected total yield for wheat at 90 tons we can multiply by the purchasing price in order to get an expected revenue of \$21,420. Our expected revenue for corn is \$41,580.

The expected profit of wheat is our expected revenue minus the cost:

$$\pi = 21,420 - 6,000$$

$$\pi = 15,420$$

The expected profit of corn is:

$$\pi = 41,580 - 13,800$$

$$\pi = 27,780$$

Giving us a total expected profit of \$43,200.

The farmer will achieve a higher expected profit when he chooses to plant 60 acres of corn and 40 acres of wheat. \$43,200>\$41,650. Thus the optimal decision for the farmer would be to plant 60 acres of corn.

While this is a viable solution to the problem since it follows expected value theory, it might not be applicable to a real human being. This is where expected utility theory takes over expected value.

Bernoulli proved this decision making tendency with the St. Petersburg Paradox. The game goes as follows. A coin is flipped n amount of times until the flip lands heads. When it reaches heads, the player will receive a payout of $\$2^n$. When assessing this through expected value theory, a player should be willing to play this game for an infinite amount of money. This is due to the exponential growth of the payout function, where the payout increases exactly the amount that the odds decrease for the coin to avoid landing on heads. Even though the expected value of playing the game is infinite, people are not willing to pay more than a few dollars to play. This is due to the high probability that the payout is low from the coin landing on heads in the first couple of flips. (Peterson, 2022)

CHAPTER 5: NON-LINEAR PROGRAMMING

It is typical for agents to go against the expected value model because it does not take into account other factors that could influence decision making. This limits the expected value model from being a viable way to approach decisions under uncertainty. The expected value model failed in predicting decisions due to the obvious fact that the particular payoff determined by someone was not always the same as the monetary value. (Prospect Theory, 15-17). Bernoulli proved with the St. Petersburg paradox that expected value was not a rational form of decision making.

The next step past the expected value model would be to look at the utility of outcomes, rather than the nominal value. Expected utility theory is able to focus on what agents actually care about in terms of an outcome. Bernoulli came up with a concave function to illustrate how people value outcomes. It showed a decreasing marginal utility. This means that people gain more utility from gaining $1 when they have a small amount of money than gaining $1 when they have a larger amount of money. The expected utility model also implies risk aversion. Bernoulli argued that almost everyone would prefer a guaranteed $100 over flipping a coin to either get $200 or nothing. (Prospect Theory, 15-17) If we were using the expected value model these two decisions would be equal, even though we know that they are different.

Risk aversion plays an important role in normal human behavior. Risk averse agents are more likely to choose investments or strategies that have a lower potential for loss, even if they also have a lower potential for gain. They take actions that might not maximize expected value in order to prevent the loss of a risky decision not working out. This behavior stems from loss aversion, which is a human tendency to weight losses heavier than gains of equal amounts. The inclusion of risk aversion is important for us to have when solving real world optimization problems. Being able to go beyond profit and expected value with expected utility allows our solution to be tailored to the decision maker.

von Neumann and Morgenstern advanced Bernoulli's utility theory by making preferences define utility. This allows the model to assign a specific utility function to an individual that accurately accounts for their preferences. This is due to the fact that people want to maximize their own subjective expected utility. One person might not have the same utility curve as another, however each strives for their own maximum subjective expected utility. (Prospect Theory, 15-17)

$$u(x) = 1 - e^{-x=R}, R > 0$$

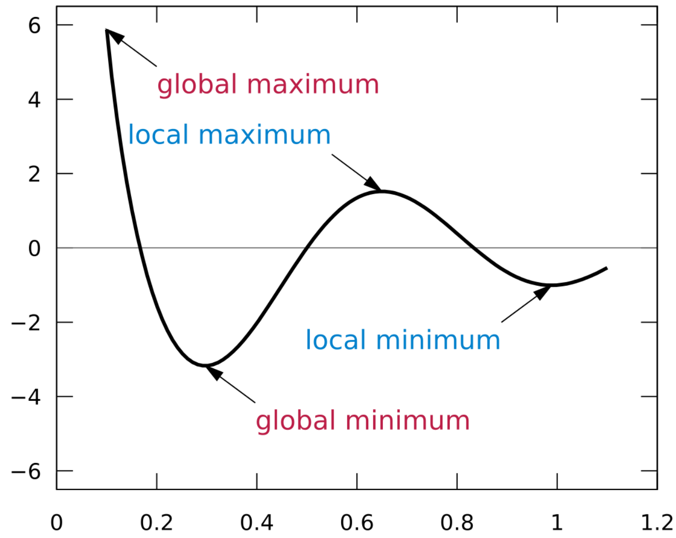
The exponential utility function, shown here, is one of the most widely used utility functions. This function represents the behavior of a risk averse agent whose Arrow-Pratt relative risk aversion measure is constant. It uses a constant, R, to weight the risk tolerance of an agent. As R becomes larger the utility function becomes less risk averse (Kirkwood, 2002).

The Arrow-Pratt measure of risk aversion allows us to measure an agent's risk aversion based off of their utility curve. They needed a measure that did not change during an affine transformation of the utility curve. They did this by taking the negative value of dividing the second derivative of the utility function by the first derivative. They found that when an individual has a relative risk aversion measure that is constant, the individual will hold the same percentage of their wealth in risky assets. (Econport)

This utility function allows us to apply a specific decision making preference to our objective function. This allows us to optimize utility rather than value in a problem with a set of constraints. However since the utility function is nonlinear we are not able to use linear programming to optimize utility. We must shift towards nonlinear programming to be able to maximize utility.

Nonlinear programming is similar to linear programming. Both have an objective function along with a set of constraints. However in nonlinear programming, both the objective function and the constraint functions are able to be non linear. The nonlinearity of the functions causes most nonlinear programming problems to be much more complex than their linear counterparts. Based on the number and types of the constraint functions different methods and algorithms are needed to solve the problem (Hillier and Lieberman). Here we will focus on one of the more simple examples of a nonlinear programming problem in order to show the mechanics of the solution.

Nonlinear programming allows us to solve a much wider scope of problems since we are able to include nonlinear functions in the objective and constraint functions. With a large amount of real world problems being based on nonlinear functions, nonlinear programming gives us the tools to solve them. However, nonlinear programming is much more difficult to solve for a couple of reasons. This can be due to the fact that solutions for nonlinear programming do not always lie on the vertices of the feasible region. Problems with nonlinearities in the constraints might have solutions that lie in the interior of the feasible region. Since the non linearities create curved functions there is also the issue of local minima and maxima which, while they are potential solutions, they are not the optimal solution.



To introduce an example we will look at a problem that has a nonlinear objective function containing two variables. This will be coupled with linear constraint functions. Not only is this a more simple example to solve than some other nonlinear programming problems, but it is also most like the solution we are looking to find in our expected utility problem. This is due to them both having a nonlinear objective function coupled with linear constraint functions. This prevents our solution from being inside the feasible region or the chance that it is a local solution rather than an optimal solution.

Our objective function is given as,

$$Z = 126x - 9x^2 + 182y - 13y^2$$

The nonlinearity is coming from our x and y variables being squared.

Our constraint functions are given as,

$$A : x \leq 4$$

$$B : 2y \leq 12$$

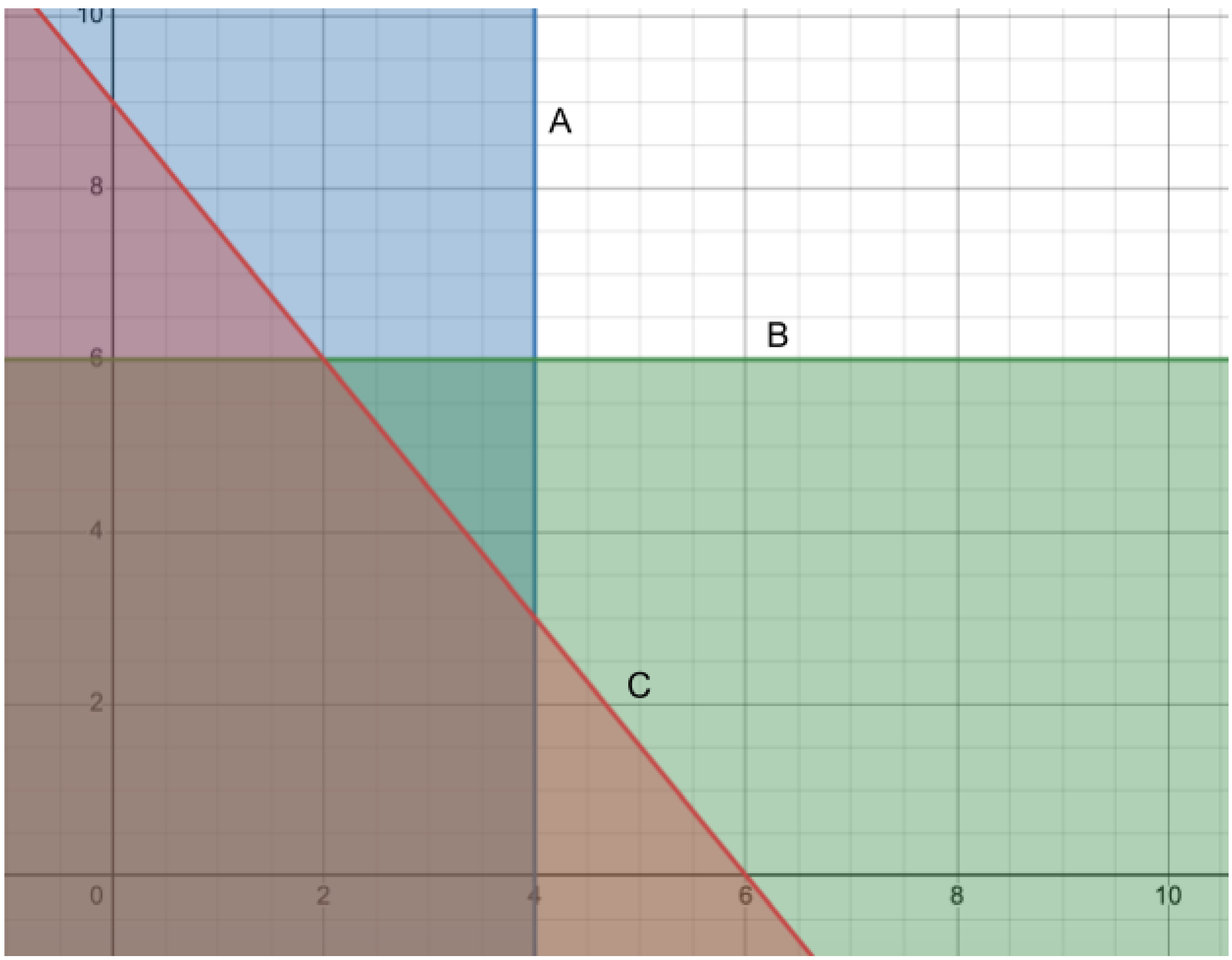
$$C : 3x + 2y \leq 18$$

$$x \geq 0$$

$$y \geq 0$$

In this problem we are looking to maximize Z.

Similar to our linear programming example we will graph the constraints functions in order to find our feasible region.

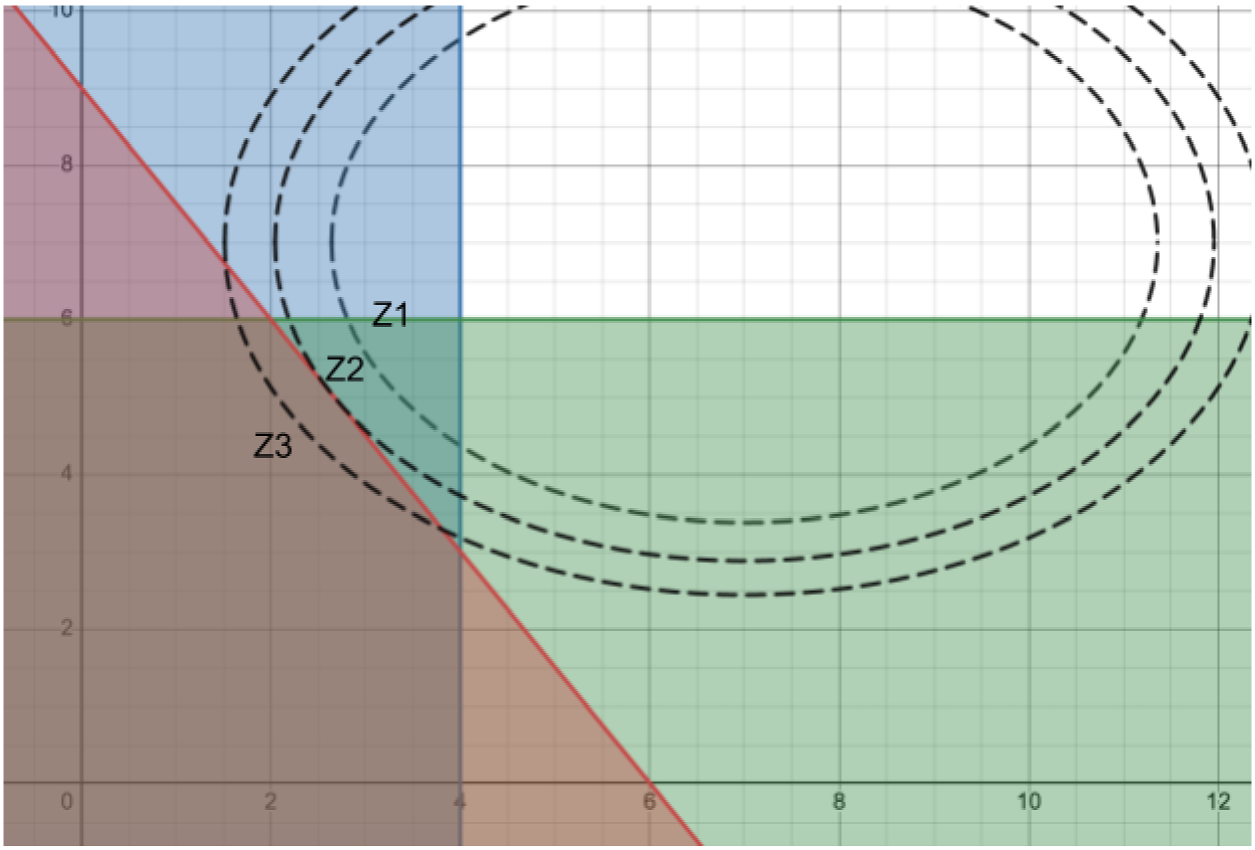


Our feasible region here is where the red, blue, and green all overlap. Now we will draw our isoquants onto the graph to show how different solutions line up with our constraints.

$$Z1 : 907 = 126x - 9x^2 + 182y - 13y^2$$

$$Z2 : 857 = 126x - 9x^2 + 182y - 13y^2$$

$$Z3 : 807 = 126x - 9x^2 + 182y - 13y^2$$



The intersection of our isoquant Z2 and our feasible region is where $x = \frac{8}{3}$ and $y = 5$, this gives us the optimal solution of Z=857.

Another reason why this example does not have some of the common difficulties of a nonlinear programming problem is that the objective function is a convex set. A convex set contains the segment between any two points that lie within the set. This eliminates the potential for any local minima and maxima being a solution due to the fact that a convex set only has one global minima and maxima.

Now let's return to our version of the farmer's problem. We will use the exponential utility function given here, and we will be solving this problem without using nonlinear programming in order to focus on how the riskiness of the different decisions and outcomes gets weighted by the utility function.

$$u(x) = 1 - e^{-x=R}, R > 0$$

I have manipulated the potential yields in this problem in order to exaggerate the potential risk the farmer could take. The new yields for wheat are (2.0, 15.8) and for corn they are (4.8, 6.4). Our expected profit when the farmer plants 60 acres of wheat is $110,026 and when 60 acres of corn is planted it is $107,684. While the decision to plant more wheat has a higher expected profit, it is more risky due to the higher probability that wheat will have a low yield (75%) which is significantly lower than the high yield, as well as the corn low and high yield.

Let us refer to the option of planting 60 acres of wheat as option 1, and planting 60 acres of corn as option 2.

For option 1 let R=3

For option 2 let R=4.5

When plugging in 3 for R in or utility function we get a utility of .95. When plugging in 4.5 we get a utility of .989.

The farmer gets less utility from option 1 than he does from option 2. However this must be weighted by the expected profits from the two options in order for the farmer to make a decision.

For option 1, the farmer would see an expected gain of

$$110,026 \times .95 = 104,524.7$$

For option 2, the farmer would see an expected gain of

$$107,684 \times .989 = 106,499.5$$

Here we can see that our risk averse farmer should choose option 2 over option 1. Although option 1 has a higher expected value, the farmer would prefer option 2 due to more certainty of a higher profit. The farmer is willing to sacrifice losing potential expected value in order to lower the risk of a bad yield year.

Here we were able to apply the expected utility function to our expected value problem in order to best optimize the farmers decision.

CHAPTER 6: CONCLUSIONS

The versatility of linear programming has allowed it to maintain the status of the most common solution method for optimization under constraints since the 1940s. Evolution has added a nonlinear counterpart that is able to encompass almost all issues that deal with optimization. While the optimization of profit is important, especially for businesses, it is not always the best solution. Allowing room for the incorporation of utility maximization is able to give a unique perspective on what agents are really trying to achieve from a decision. Individuals who are risk averse are able to quantify that aversion and apply it to the solution for their optimization problem. While this senior project has looked at how risk can be a factor in optimization under uncertainty, it does not include prospect theory. Prospect theory has been the new frontier for decision making theory and there is more to be explored there.

Prospect theory was created by Tversky and Kahneman in order to most accurately represent realistic human decision making. Prospect theory predicts individuals to be risk averse when they are gaining, but relatively risk seeking when they are losing. In the future the work I have done here can be extended by account of loss aversion as well as probability weighting to make the analyses more accurate.

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