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Classifying Derived Voltage Graphs

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Abstract

Gross and Tucker's voltage graph construction assigns group elements as weights to the edges of an oriented graph. This construction provides a blueprint for inducing graph covers. Thomas Zaslavsky studies the criteria for balance in voltage graphs. This project primarily examines the relationship between the group structure of the set of all possible assignments of a group to a graph, including the balanced subgroup, and the isomorphism classes of covering graphs. We examine connectedness, planarity, and chromatic number in the derived graph. Lastly we explain the future research possibilities involving the fundamental group.

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Dedication

To my family.

Acknowledgments

My advisor, Lauren Rose, gave me so much support and confidence during the course of this project. Greg Landweber helped me see the larger context that my research fit into. Lauren, Greg and Bob McGrail all gave me great feedback during my board meeting. Every single professor I've taken a math class with has been fantastic. Jules Albertini taught me that math was beautiful in Calculus I. All of the professors after that—Mary Krembs, John Cullinan, Sam Hsiao, Jim Belk, Gidon Eshel, Greg Landweber—have heightened my love for mathematics. My friends have listened while I tried to explain my project to them, despite the fact that they mostly major in the arts and humanities and didn't understand anything I was saying. My family has given me so much love, and I miss them very much.

1

Introduction

This project began in September of 2010 as an investigation of the combinatorial properties of the voltage graph. A voltage graph is a triple, $(G, \mathcal{V}, \varphi)$ where G is an oriented graph, $\mathcal{V}$ is a group, and $\varphi : E(G) \longrightarrow \mathcal{V}$ is an assignment of group elements to edges. This construction is called a voltage graph because of its similarities to circuit diagrams drawn by physicists. In a circuit diagram, the voltage around any loop the current transverses must sum to 0. Mathematical voltage graphs do not have this restriction, cycles can sum to any group element. If all cycles in a voltage graph sum to zero, we have a condition called balance.

The most interesting thing about voltage graphs is their use in topological graph theory. It turns out that if you assign group elements as weights to the edges of a graph, you have a blueprint for the construction of a covering graph of your first graph. These induced covering graphs are called derived graphs. The topological graph theorists are interested in voltage graphs as a means to solve one of the main problems in topological graph theory: what surface does a given graph imbed in? One of the ways that they do that is to take a quotient of the graph in question and see if it imbeds in some quotient of the given surface. It is useful then to know systematic ways of creating quotients which can then be reconstructed using voltage graphs.

This project does not delve very deeply into the larger problems of topological graph theory. We do not discuss imbedding graphs or wonder about their genus. There are two general questions we ask about the derived graph of a voltage graph. The first is, given a group and a graph, how many derived graphs up to isomorphism can we induce? The second is, what graph characteristics are preserved in the derived graph?

In Chapter 2 we define basic graph theory terminology that we use throughout the rest of the paper. In Chapter 3 we discuss the concept of a covering graph. We then introduce the voltage graph and its induced derived graph. We show that the derived graph is a covering graph.

In Chapter 4 we present our results concerning the number of derived graphs induced by a given base graph and group. The number of derived graphs depends on the set T. We define T as the set of all assignments of group elements to edges of G. It turns out that T is a group under an operation similar to the operation in the voltage group $\mathcal{V}$. When $\mathcal{V}$ is abelian, the set B of balanced assignments (every cycle sums to zero) form a subgroup. We prove that all voltage assignments in a given coset of B in T induce the same derived graph. This result does not fully explain the number of derived graphs, however. There are actually fewer derived graphs up to isomorphism than the number of cosets of B for many graphs. So in Section 5.3 we define voltage isomorphisms. We show that if the voltage assignment φ_1 on a graph G can be rearranged via a graph automorphism into an assignment in a different coset, φ_2, the two assignments will induce the same derived graph.

Chapter 5 contains our results concerning which graph characteristics are preserved in the derived graph. We start in Section 5.1 with connectedness. Certain voltage assignments induce disconnected derived graphs of connected voltage graphs. This occurs when the base graph is a tree, the voltage assignment is completely balanced, or when all cycles in the voltage graph sum to some element of subgroup of the voltage group. In Section 5.2 we show that the derived graph of a planar graph is planar, and in Section 5.3 we give an upper bound for the chromatic number of the derived graph.

Chapter 6 describes possible areas of further research. This project has far reaching implications into algebraic topology. Near the end of the time alloted for senior project we found a relationship of the fundamental group of the base graph to the quotient group T/B. For a general topological space, there exists a relationship between the fundamental group and the isomorphism classes of covering spaces. An interesting project for further research would be to extend that relationship to the voltage graph derived graph method of covering spaces. Other further research that does not depend on knowledge of topology is possible as well. What other graph characteristics are preserved or not preserved in derived graphs?

Finally, the Appendix displays all the derived graphs up to isomorphism induced by the assignment of $\mathbb{Z}_2, \mathbb{Z}_3$, and $\mathbb{Z}_4$ on K_4.

2

Graph Theory Preliminaries

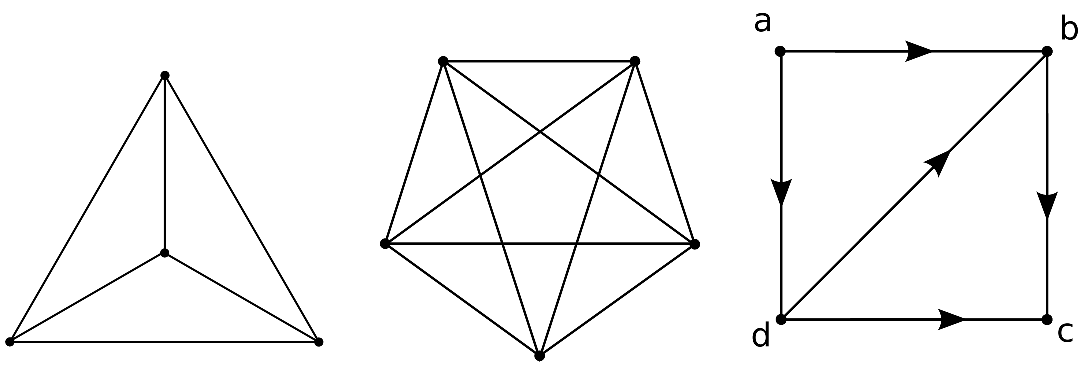
In this chapter we provide the basic graph theory background needed for this paper. All of the following definitions can be found in West's *Introduction to Graph Theory* [5].

First we'll define a graph, an oriented graph, and a subgraph, and then provide basic terminology.

Definition 2.0.1. A **graph** is an ordered pair $G = (V, E)$ consisting of a set V of vertices together with a set E of edges such that each edge e has vertex set $\{u, v\}$ with $u, v \in V$. We denote the set of edges of a graph by $E(G)$ and the set of vertices of a graph by $V(G)$.

Definition 2.0.2. An **orientation** of a graph is obtained by assigning a direction to each edge, denoted by an arrow along the edge. Any graph constructed this way is called an **oriented graph**. We denote the set of vertices of an oriented edge e as the ordered pair (u, v). We call the first vertex in the pair the **head** and the second the **tail** of the edge.

Definition 2.0.3. A graph is **simple** if it has no more than one edge between two distinct vertices, and no loops (edges from a vertex to itself).



(a) K_4 (b) K_5 (c) An oriented graph

Figure 2.0.1: Examples of Graphs and Oriented Graphs

Definition 2.0.4. Two vertices v, w of a graph are called **adjacent** if there exists an edge e such that v and w are both endpoints of e. We denote adjacency between two vertices by $v \sim w$.

For example, in the graph in Figure 2.0.1c, the vertices a and b are adjacent, but a and c are not.

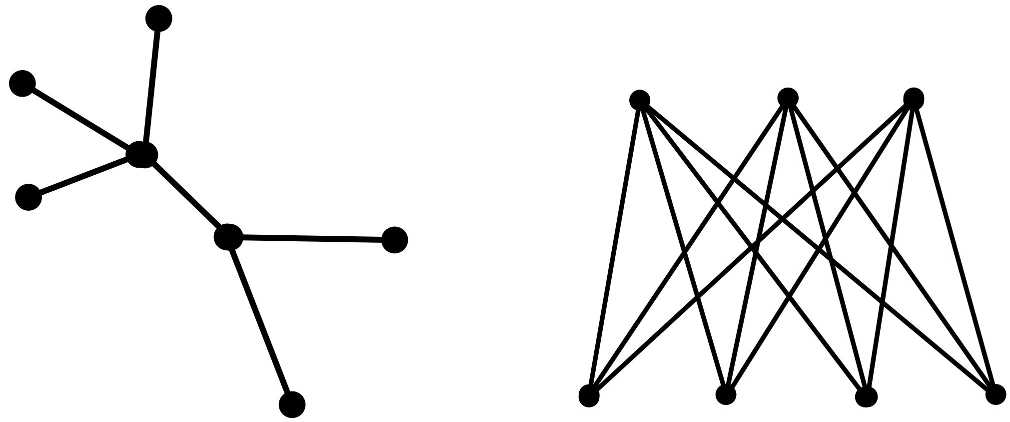
Definition 2.0.5. The **neighborhood** of a vertex v, written $N(v)$, is the set of all vertices adjacent to v.

For example, in Figure 2.0.1c, $N(a) = \{b, d\}$.

Definition 2.0.6. A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. The **length** of a walk is its number of edges. A walk is **closed** if $v_0 = v_k$. We call a closed walk a **cycle**. A **trail** is a walk in which no edges are repeated.

Definition 2.0.7. In a graph G, two vertices u and v are **connected** if there exists a walk from u to v.

Definition 2.0.8. A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and if every edge of H is also an edge of G. We call H **induced** if whenever $e \in E(G)$ involves vertices



(a) A tree (b) $K_{3,4}$

Figure 2.0.2: A tree and a complete bipartite graph

from H, $e \in E(H)$. A **spanning subgraph** is a subgraph that includes all of the vertices in G.

Now we will define specific classes of graphs and their properties.

Definition 2.0.9. A **tree** is a simple graph with no cycles.

Example 2.0.10. The graph in Figure 3.3.1a is a tree.

Trees have many equivalent characterizations. We present only two of these from a longer theorem in Section 2.1 in [5]:

Theorem 2.0.11. *For a graph G, the following are equivalent and characterize a tree:*

i) For $u, v \in V(G)$, G has exactly one u, v trail.

ii) Adding one edge to a tree forms exactly one cycle.

Definition 2.0.12. A **complete graph** is a simple graph in which every pair of distinct vertices is connected by an unique edge.

We denote a complete graph with n vertices by K_n. The graph in Figure 2.0.1a is K_4, the complete graph on four vertices, and the graph in Figure 2.0.1b is K_5.

Definition 2.0.13. A **complete bipartite graph** is a graph whose vertices can be divided into two sets, U and V such that there exists an edge between any $u \in U$ and $v \in V$.

We denote a complete bipartite graph with m vertices in U and n vertices in V by $K_{m,n}$. The graph in Figure 3.3.1b is $K_{3,4}$.

The notion of isomorphism extends to graphs.

Definition 2.0.14. A **graph isomorphism** between G and H is a bijection $f : V(G) \longrightarrow V(H)$ such that adjacency is preserved, i.e. $(a, b) \in E(G)$ if and only if edge $(f(a), f(b)) \in E(H)$.

Similarly, we can define the concept of automorphism for graphs.

Definition 2.0.15. An **automorphism** of G is an isomorphism from G to G.

We define the concept of graph coloring and planarity, which we use in Chapter 5.

Definition 2.0.16. A k-**coloring** of a graph G is a labeling $f : V(G) \longrightarrow S$, where S is a set and $|S| = k$. The labels are **colors**. A k-coloring is **proper** if adjacent vertices have different labels. A graph is called k-**colorable** if it has a proper k-coloring. The **chromatic number** of G, $\chi(G)$, is the least k such that G is k-colorable.

Definition 2.0.17. A **planar graph** is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

For example, K_4 is planar but K_5 is not as can be observed in Figure 2.0.1. We use Kuratowski's Theorem, Theorem 2.0.19, to determine whether or not a graph is planar. First we need to define subdivision.

Definition 2.0.18. A **subdivision** of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge $e = \{u, v\}$ yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, $\{u, w\}$ and $\{w, v\}$.

Theorem 2.0.19. *(Kuratowski [1930])* [5] *A graph is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$.

3

Voltage Graphs and their Derived Graphs

Now we will define the objects that we will discuss for the rest of this paper. First we will define a covering graph using some topological concepts. Then we will define the voltage graph, which is an oriented graph weighted with elements of an abelian group. Next we will show how the voltage graph is a blueprint for the construction of a derived graph, which is a covering graph. Lastly, we will present some theorems from Gross and Tucker [1] about lifting walks to the derived graph that we will use the next few chapters.

3.1 Covering Graphs

We will first give an intuitive description of the general theory of covering spaces in topology. Point-set topology is concerned with the study of spaces. For our purposes, think of a space as a region of $\mathbb{R}^2$ or $\mathbb{R}^3$ that has some nice properties. Now consider two spaces, X and $\tilde{X}$. We say $\tilde{X}$ covers X if it is in some way larger than X but with a similiar local structure. So the area around any point in X looks the same as the area around a similiar point in $\tilde{X}$. In Figure 3.1.1 there are two subsets of $\mathbb{R}^3$. The larger circle is wrapped around itself to demonstrate how it looks the same as the smaller circle locally.

A more rigorous and complete discussion of covering spaces can be found in Section 53 of [3].

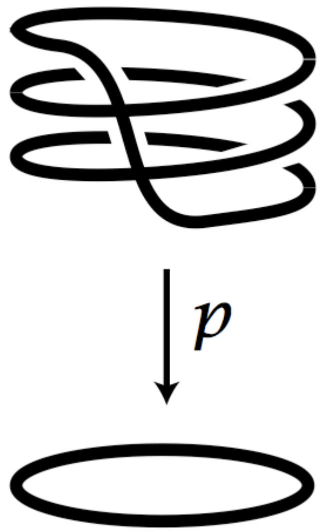


Figure 3.1.1: A covering space of a circle

It turns out that all covering spaces of graphs are graphs. We now give a formal definition for a covering graph that maintains the same intrinsic idea of a covering space. Essentially, a graph covers another if it is larger and has the same local structure.

Definition 3.1.1. A graph C is a **covering graph** of a graph G if there exists a surjection $p : V(C) \longrightarrow V(G)$ and a bijection $g : \text{nbhd}(v) \longrightarrow \text{nbhd}(p(v))$ for $v \in V(C)$.

Example 3.1.2. Consider the graph in Figure 3.1.2. The larger graph C is a covering graph of the smaller graph G. To check, define $p : V(C) \longrightarrow V(G)$ by $p(v) = w$ if v and w have the same color. Clearly this is surjective. Now observe the neighborhood around one of the red vertices in C. This neighborhood is a yellow vertex and a blue vertex. The red vertex in G has the same neighborhood.

The larger theory of covering spaces can be specified to discuss covering graphs. For more about this topic, see Chapter 6.

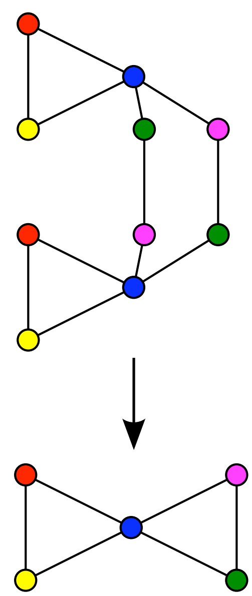


Figure 3.1.2: A covering graph of a smaller graph

3.2 Voltage Graphs and Balance

Voltage graphs are oriented graphs whose edges are weighted with group elements. We are primarily interested in them because their derived graphs (which we introduce in the next section) are covering graphs. Voltage graphs are also studied for their own merit, however. In the more combinatorial setting they are often called *gain graphs*, and have been studied extensively by Thomas Zaslavsky [4].

Definition 3.2.1. A **voltage graph** is a triple $\Lambda = (G, \mathcal{V}, \varphi\ )$ where G is a finite oriented graph, $\mathcal{V}$ is a finite abelian group, and $\varphi : E(G) \longrightarrow \mathcal{V}$ is an assignment of group elements to the oriented edges of G, where if (u, v) is an oriented edge, we define $\varphi(v, u) = -\varphi(u, v)$. The voltage on some walk W in G is defined by $\varphi(W) = \sum_{i=1}^{k} \varphi(e_i)$. We call the graph G the **base graph** of the voltage graph and $\mathcal{V}$ the **voltage group**.

We define all graphs to be finite and the groups to be finite and abelian for the purposes of this paper. Many of our findings and theorems would not hold for non-abelian voltage groups. However, non-abelian groups can indeed be used to weight the edges. A discussion of voltage graphs weighted with permutation groups can be found in Chapter 2.4 of [1].

This paper includes many voltage graphs where the voltage group is $\mathbb{Z}_2$. In that case we will omit the orientation on the edges of the graph, because the direction of transversal does not affect the voltage. When we examine voltage graphs, we are most interested in graphs with cycles. It turns out that the sum around each cycle will determine the derived graph, which we introduce in the next section.

Definition 3.2.2. We define a **balanced cycle** to be a cycle of G in which the the edge labels sum to the identity. A **balanced assignment** is an assignment of edges φ for which every cycle is balanced.

The following example demonstrates the balanced and unbalanced case.

Figure 3.2.1: A balanced and an balanced assignment of K_4 to $\mathbb{Z}_5$

Example 3.2.3. Figure 3.2.1 is an image of two different voltage graphs such that $G = K_4$ and $\mathcal{V} = \mathbb{Z}_5$. The left graph has a balanced assignment, since all cycles sum to zero, while the right does not. We can check that this holds. In the left graph, in cycle a, 4+4+2=0, in cycle b, 4+3+3=0, and in cycle c, 2+1-3=0. In fact, every cycle in this graph sums to zero. We claim that the right graph is unbalanced. Cycle c sums 2+2-3=4, so the graph is unbalanced. An assignment needs only one unbalanced cycle to be an unbalanced assignment, but every cycle must sum to zero for the graph to have a balanced assignment.

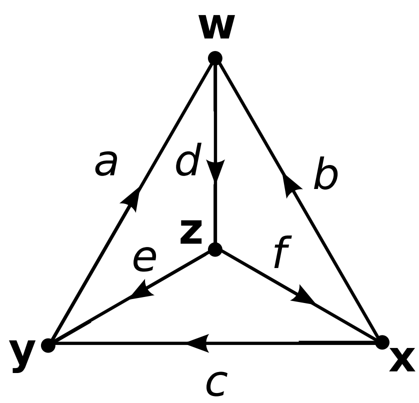
3.3 Derived Graphs

In 1974 Gross and Tucker [1] developed the ordinary voltage graph construction to create covering graphs using a voltage graph. These induced covering graphs are called **derived graphs** and are denoted $\tilde{G}$. It turns out that a voltage graph is a blueprint for the construction of a covering graph.

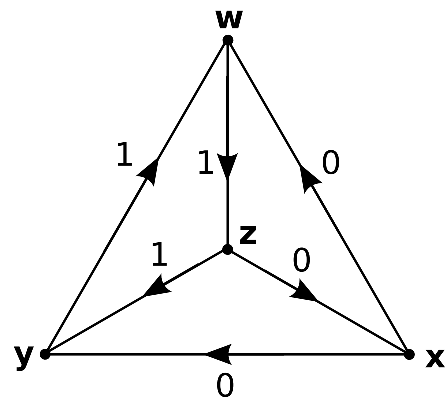
We know that any graph has an infinite number of covers. By forming a voltage graph with the group $\mathcal{V}$,we can construct a subset of their covers. These covering graphs turn out to be $|\mathcal{V}|$-fold covers of the original graph.

The derived graph is created by taking the cartesian product of the edge set with the voltage group and the vertex set with the voltage group. These ordered pairs form the vertices and edges of a graph. We will see that the derived graph forms a cover of G.

Definition 3.3.1. Let $\Lambda = (G, \mathcal{V}, \varphi)$ be a voltage graph. We define the **derived graph** $\tilde{G}$ as follows. Let $V(\tilde{G}) = V \times \mathcal{V}$ and $E(\tilde{G}) = E \times \mathcal{V}$. For $v \in V(G)$ and $k \in \mathcal{V}$ we denote the vertex $(v, k) \in V(\tilde{G})$ by v_k. Similarly we denote edges in $E(\tilde{G})$ by e_k. Let $e = (u, v) \in E(G)$, and let $\varphi(e) = k \in \mathcal{V}$. Then the head and tail of $e_k \in E(\tilde{G})$ are u_k and $v_{k+\varphi(e)}$, i.e., $e_k = (u_k, v_{k+\varphi(e)})$.



(a) Voltage graph with edges labeled



(b) Same graph with edge assignments labeled

Figure 3.3.1: Edge assignments of a voltage graph

Example 3.3.2. We will create a derived graph of the voltage graph $G = K_4$, $\mathcal{V} = \mathbb{Z}_2$ with the following edge assignment:

$$\varphi(a) = 1 \quad \varphi(b) = 0 \quad \varphi(c) = 0 \quad \varphi(d) = 1 \quad \varphi(e) = 1 \quad \varphi(f) = 0$$

We then make a table of the edges of the derived graph to find the head and tail of each edge:

Edge	(Tail, Head)
a_0	(y_0, w_1)
a_1	(y_1, w_0)
b_0	(x_0, w_0)
b_1	(x_1, w_1)
c_0	(y_0, x_0)
c_1	(y_1, x_1)
d_0	(w_0, z_1)
d_1	(w_1, z_0)
e_0	(z_0, y_1)
e_1	(z_1, y_0)
f_0	(z_0, x_0)
f_1	(z_1, x_1)

Connecting this produces the graph in Figure 3.3.2. Note that the edge labels are italicized and the vertex labels are bolded. The structure of the derived graph varies according to

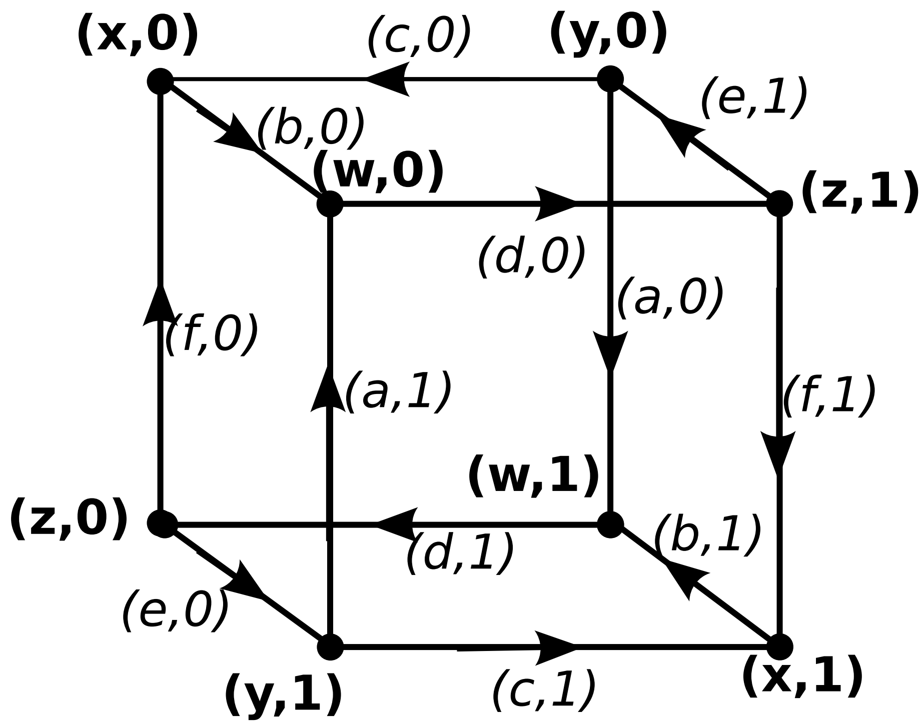


Figure 3.3.2: A derived graph of K_4

the chosen φ. However, multiple φ's can give the same derived graph up to isomorphism. The complete list of derived graphs of K_4 where $\mathcal{V} = \mathbb{Z}_2$ is given in Appendix A.

Lets check to see if this derived graph does indeed cover the original graph. Let $p : V(\tilde{G}) \longrightarrow V(G)$ such that $p(v, a) = v$ for $v \in V(G)$ and $a \in \mathcal{V}$. This function is clearly surjective. Then by construction, for any vertex w such that $v \sim w$ in G, there exists a vertex $w_b \in \tilde{G}$ such that $v_a \sim w_b$. So the derived graph is a covering graph.

We now prove that all derived graphs are covering graphs.

Theorem 3.3.3. *Let G be a graph with derived graph $\tilde{G}$ induced by voltage graph $\Lambda = (G, \mathcal{V}, \varphi)$. Then $\tilde{G}$ covers G with covering function $p : V(\tilde{G}) \longrightarrow V(G)$ defined by $p(v_k) = v$.*

Proof. Let $p : V(\tilde{G}) \longrightarrow V(G)$ be defined by $p(v_k) = v$ for $v \in V(G)$ and $k \in \mathcal{V}$. We claim this is surjective. Let $v \in V(G)$. Then for any $k \in \mathcal{V}$, $v_k \in V(\tilde{G})$ and $p(v_k) = v$.

Now let any $v_k \in V(\tilde{G})$. Then define g by $g = p|_{nbhd(v_k)}$. Claim this function is bijective. Let $u \in nbhd(p(v_k))$. Then $u \in nbhd(v)$ so there exists an edge $e = (v, u) \in E(G)$. Then $e_k = (v_k, u_{k+\varphi(e)})$. Then $u_{k+\varphi(e)} \in nbhd((v_k))$ and $g(u_{k+\varphi(e)}) = u$. So g is surjective.

Let $u_j, w_l \in nbhd(v_k)$. So $j = k + \varphi(e_1)$ where $e_1 = (v, u)$ and $l = k + \varphi(e_2)$, where $e_2 = (w, u)$. Let $g(u_j) = g(w_l)$. Then $u = w$, which implies $e_1 = e_2$. Therefore $j = l$ and $u_j = w_l$. So g is bijective and $\tilde{G}$ covers G.

□

This previous example is the only example that we will construct explicitly. For more examples of derived graphs, see the Appendices.

3.4 Lifting Walks to Derived Graphs

Another important concept in the general theory of topological covering spaces is the lift. The idea is that any path in the base space has some inverse image in the covering space: some collection of copies of the original path. But since we are studying graphs, the paths we lift are walks. Before we define a lift, consider the case where the walk has only one vertex. We call the inverse image of a single vertex a fiber.

Definition 3.4.1. Let G be a graph with derived graph $\tilde{G}$. Then the **fiber** of $v \in V(G)$ is $p^{-1}(\{v\}) = \{v_k \mid k \in \mathcal{V}\}$ where p is the covering map.

In topology, the lift of a path beginning at a given point in the covering space is unique. We have the following similiar fact about lifts from voltage graphs to their derived graphs.

Theorem 3.4.2. [1] *Let W be a walk in a voltage graph with initial vertex u. Then for each vertex u_a in the fiber over u there is a unique lift of W that starts at u_a.*

Proof. Consider the first oriented edge of W, $e = (u, v)$ or $e = (v, u)$. Assume first $e = (u, v)$. Then the lift of e is $\tilde{e} = (u_a, v_{a+\varphi(e)})$. The vertex $v_{a+\varphi(e)}$ is the only vertex in the fiber of v adjacent to u_a because $v_{b+\varphi(e)}$ is the tail of the edge $\tilde{e}' = (u_b, v_{b+\varphi(e)})$ for all $a \neq b$. If the edge is $e = (v, u)$, then $\tilde{e} = (v_{a-\varphi(e)}, u_a)$. The vertex $v_{a-\varphi(e)}$ is again uniquely determined.

Similarly, there is only one possible choice for a second edge of the lift of W, since the initial point of that second edge must be the terminal point of the first edge, and since that second edge of the lift must lie in the fiber over the second edge of the base walk W. This uniqueness holds, of course, for all the remaining edges as well. □

One of the great features of the voltage-graph construction is that you can predict where the lift of a walk terminates in the derived graph. Consider the following example:

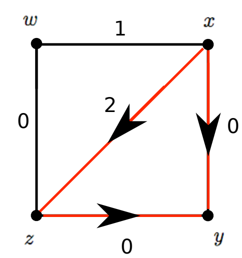
Example 3.4.3. Let G and $\tilde{G}$ be the voltage graph and derived graph in Figure 3.4.1 where $\mathcal{V} = \mathbb{Z}_3$ The walk $\{x, z, y, z\}$ has net voltage 2, and the lift of this walk starting at x_0 terminates at the vertex $x_{0+2} = x_2$, as predicted. The following theorem makes this explicit.

Theorem 3.4.4. [1] *Let W be a walk from u to v in a voltage graph , and let b be the net voltage on W. Then the lift W_a starting at u_a terminates at the vertex v_{a+b}.* [1]

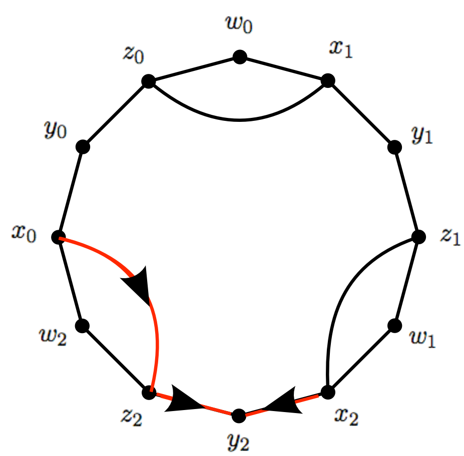
Proof. Let $b_1, \ldots, b_n$, such that $\sum_1^n b_i = b$, be the successive voltages encountered on the walk W. Then the subscripts of the vertices of W_a are

$$a, a + b_1, a + b_1 + b_2, \ldots, a + b_1 + \cdots + b_n = a + b.$$

Since W_a terminates in the fiber over v, its final vertex is v_{a+b}. □



(a)



(b)

Figure 3.4.1: G and $\tilde{G}$ with a walk lifted in red

4

Classification of Derived Graphs

Given a graph G and a group $\mathcal{V}$ how many non-isomorphic derived graphs can be induced by the set of all possible assignments of φ? Immediate intuition might say that every voltage assignment lifts the graph to a unique derived graph. This is not the case. For example, there are only three non-isomorphic derived graphs when G is K_4 and $\mathcal{V}$ is $\mathbb{Z}_2$ and 2^6 voltage assignments. In this chapter we explain how to determine when two assignments will give the same derived graph. In Section 4.1 we will describe the group structure of the voltage assignments, including the balanced subgroup. In Section 4.2 will demonstrate that the cosets of the balanced subgroup correspond to equivalence classes of derived graphs. Lastly, in Section 4.3, we present an additional case where two voltage graphs in different cosets can induce the same derived graph.

4.1 Group Structure of Voltage Assignments

Consider an abelian group $\mathcal{V}$ and a graph G. It turns out that set of all assignments, $T : \{\varphi : E(G) \longrightarrow \mathcal{V}\}$ forms a group under the induced group operation of $\mathcal{V}$, i.e., $\varphi_1 + \varphi_2(e) = \varphi_1(e) + \varphi_2(e)$. Since $\mathcal{V}$ is abelian, we use $+$ to denote its operation. Since we are free to assign edges with whichever group element we please, $T \approx \mathcal{V}^k$ where k is the number of edges of G. We will also show that the subset of balanced assignments B forms a subgroup of T.

Theorem 4.1.1. *Let G be an oriented graph and $\mathcal{V}$ be an abelian group. Take the set T such that*

$$T = \{\varphi : E(G) \mid \varphi : E(G) \longrightarrow \mathcal{V}\}.$$

Then T is an abelian group.

Proof. We claim T is a group with group operation $+$ defined by $(\varphi_1 + \varphi_2)(e) = \varphi_1(e) + \varphi_2(e)$.

Commutativity. Let $\varphi_1, \varphi_2 \in T$. Then

$$(\varphi_1 + \varphi_2)(e) = \varphi_1(e) + \varphi_2(e) = \varphi_2(e) + \varphi_1(e) = (\varphi_2 + \varphi_1)(e)$$

Closure. Let $\varphi_1, \varphi_2 \in T$. Then $(\varphi_1 + \varphi_2)(e) = \varphi_1(e) + \varphi_2(e) \in \mathcal{V}$ so $\varphi_1 + \varphi_2 \in T$.

Associativity. Let $\varphi_1, \varphi_2, \varphi_3 \in T$. Then

$$\begin{aligned} ((\varphi_1 + \varphi_2) + \varphi_3)(e) &= (\varphi_1 + \varphi_2)(e) + \varphi_3(e) = \\ \varphi_1(e) + \varphi_2(e) + \varphi_3(e) &= \varphi_1(e) + \varphi_1(e) + \varphi_2(e) = (\varphi_1 + (\varphi_2 + \varphi_3))(e) \end{aligned}$$

Identity. Define $\varphi_0 \in T$ by $\varphi_0(e) = 0$ for all $e \in E(G)$. Then take $\varphi_1 \in T$. So

$$(\varphi_0 + \varphi_1)(e) = \varphi_0(e) + \varphi_1(e) = 0 + \varphi_1(e) = \varphi_1(e)$$

and by commutativity, $\varphi_1 + \varphi_0 = \varphi_1$.

Inverse. Take $\varphi \in T$. Then define $-\varphi$ by $(-\varphi)(e) = -(\varphi(e))$. Then $-\varphi \in T$ and

$$(\varphi + (-\varphi))(e) = \varphi(e) - \varphi(e) = 0$$

which implies $\varphi + (-\varphi) = \varphi_0$, and by commutativity, $(-\varphi) + \varphi = \varphi_0$. □

Theorem 4.1.2. *Let*

$$B = \{\varphi \in T \mid \textit{ such that } \varphi \textit{ is balanced}\}.$$

Then B is a subgroup of T.

Proof. Let $\varphi_1, \varphi_2 \in B$. Let C be a cycle. Then $\varphi_1(C) = \varphi_2(C) = 0$ and hence:

$$-(\varphi_1 + \varphi_2)(C) = -((\varphi_1 + \varphi_2)(C)) = -\varphi_1(C) - \varphi_2(C) = 0$$

So $-(\varphi_1 + \varphi_2) \in B$ and B is a subgroup by the one-step subgroup test. □

We would like to be able to predict which group B is isomorphic to. In fact, in all simple planar graphs we examined, $B \approx \mathcal{V}^m$ for some $m \leq k$. This makes sense if we notice that for every graph we can select a proper subset of edge assignment that uniquely determine the remaining assignments if we want all cycles to sum to zero. For example, consider the partial assignment of $\mathbb{Z}_2$ to K_4 shown in Figure 4.1.1

Given three edge assignments, there are unique values for x, y, z such that the voltage graph is balanced. For instance, $x = 0 + 1 = 1$. Therefore $z = 1 + 0 = 1$ and $y = 1 + 1 = 0$. So for K_4, three edges are free variables while three edges are dependent variables. It turns out that any assignment of voltages onto a maximal spanning tree will uniquely determine the remaining values of a balanced assignment. This works because each edge added onto a spanning tree will create a distinct fundamental cycle. Each fundamental cycle gives us a simple equation where we solve for the missing edge assignment.

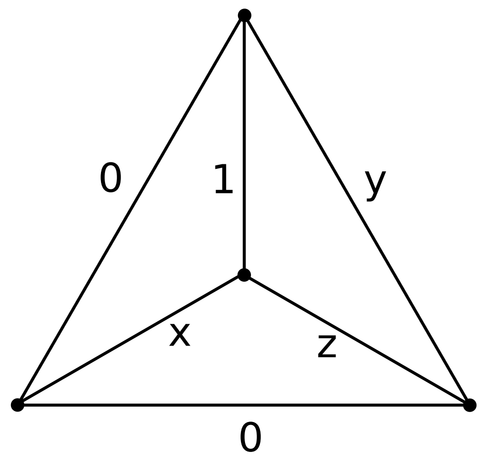


Figure 4.1.1: Partial edge assignment of K_4

Theorem 4.1.3. *Let $\mathcal{V}$ and G be given. Then $T \approx \mathcal{V}^k$ where $k = |E(G)|$.*

Proof. Arbitrarily number edges of G to be $e_1, \ldots e_k$. We define $f : T \longrightarrow \mathcal{V}^k$ by $f(\varphi) = (\varphi(e_1), \ldots, \varphi(e_r))$. We claim this is a group isomorphism.

Surjective. Let $(a_1, \ldots a_k) \in \mathcal{V}^k$. Then define a function $\varphi : E(G) \longrightarrow \mathcal{V}$ where $\varphi(e_i) = a_i$ for all $1 \geq i \geq k$. Then $\varphi \in T$ and $f(\varphi) = (\varphi(e_1), \ldots, \varphi(e_k)) = (a_1, \ldots, a_k)$.

Injective. Let $\varphi_1, \varphi_2 \in T$ such that $f(\varphi_1) = f(\varphi_2)$. Then $(\varphi_1(e_1), \ldots, \varphi_1(e_k)) = (\varphi_2(e_1), \ldots, \varphi_2(e_k))$ implies $\varphi_1(e_i) = \varphi(e_i)$ for all i so $\varphi_1 = \varphi_2$.

Homomorphism. Let $\varphi_1, \varphi_2 \in B$. Then

$$f(\varphi_1) + f(\varphi_2) = (\varphi_1(e_1), \ldots, \varphi_1(e_r)) + (\varphi_2(e_1), \ldots, \varphi_2(e_r)) =$$

$$(\varphi_1(e_1) + \varphi_2(e_1), \ldots, \varphi_1(e_k) + \varphi_2(e_k)) = ((\varphi_1 + \varphi_2)(e_1), \ldots, (\varphi_1 + \varphi_2)(e_k)) = f(\varphi_1 + \varphi_2)$$

$\square$

Theorem 4.1.4. *Let $\mathcal{V}$ and G be given. Then $B \approx \mathcal{V}^r$ where $r = |E(R)|$ where R is a maximal spanning tree of G.*

Proof. Choose some maximal spanning tree R of G. Then arbitrarily number the edges of R by $e_1, \ldots e_r$. Now define $f : B \longrightarrow \mathcal{V}^r$ by $f(\varphi) = (\varphi(e_1), \ldots, \varphi(e_r))$. We claim this is an isomorphism of groups.

Surjective. Let $(a_1, \ldots, a_r) \in \mathcal{V}^r$. Then let $\varphi : E(R) \longrightarrow \mathcal{V}$ be defined by $\varphi(e_i) = a_i$ for $e_i \in R$. We claim that this can be extended to a function $\varphi : E(G) \longrightarrow \mathcal{V}$ such that $\varphi \in B$.

We present a proof by induction on $n = |E(G) - E(R)|$. First observe the assignment $\varphi : E(R) \longrightarrow \mathcal{V}$. Now add an ege $e \in G - R$ to R. This connects to some walk W to create a cycle C by Theorem 2.0.11. Let $\varphi(e) = -\varphi(W)$, so $\varphi(e) + \varphi(W) = \varphi(C) = 0$ and the cycle is balanced.

Now assume we have added n edges to R from $G - R$ and assigned voltages such that all cycles are balanced. Next add a $(n+1)$st edge from $G - R$ called e. Adding this edge can create multiple cycles. Say e connects to W and W' to create cycles C and C' respectively. Assign $\varphi(e) = -\varphi(W)$ and $\varphi(e)' = -\varphi(W')$. We claim that $\varphi(e) = \varphi(e)'$. Since $W - W'$ is a cycle, $\varphi(W - W') = 0$ and $\varphi(W) = \varphi(W')$ and $\varphi(e) = \varphi(e)'$. So the function is surjective.

Injective. Assume $f(\varphi_1) = f(\varphi_2)$. Then $\varphi_1(e_i) = \varphi_2(e_i)$ for all $e_i \in E(R)$. Now consider $e \in E(G - R)$. Adding this edge creates some cycle with walk $W \in R$. So

$$\varphi_1(W) + \varphi_1(e) = 0 = \varphi_2(W) + \varphi_2(e).$$

Therefore $\varphi_1(e) = \varphi_2(e)$ for all $e \in E(G)$ and $\varphi_1 = \varphi_2$.

Homomorphism. Let $\varphi_1, \varphi_2 \in B$. Then

$$f(\varphi_1) + f(\varphi_2) = (\varphi_1(e_1), \ldots, \varphi_1(e_r)) + (\varphi_2(e_1), \ldots, \varphi_2(e_r)) =$$

$$(\varphi_1(e_1) + \varphi_2(e_1), \ldots, \varphi_1(e_r) + \varphi_2(e_r)) = (\varphi_1 + \varphi_2(e_1), \ldots, \varphi_1 + \varphi_2(e_r)) = f(\varphi_1 + \varphi_2)$$

□

4.2 Coset Classification

Consider the balanced subgroup B. Because the voltage groups we are dealing with are abelian, B is a normal subgroup of T. It turns out that derived graphs of assignments in the same coset of B are in the same isomorphism class. We have to be careful with this, because two assignments that induce the same derived graph are not necessarily in the same coset of B. As Appendix A shows, K_4 has only three non-isomorphic derived graphs with $\mathbb{Z}_2$, even though $|T|/|B| = 8$. However, this number is an upper bound on the number of derived graph isomorphism classes. We will prove the fact that assignments in the same coset induce the same derived graph with the help of the following lemma. More implications of this lemma are discussed in Section 5.1.

Lemma 4.2.1. *Let W_1 and W_2 be walks from v to w in a balanced voltage graph with assignment $\varphi : E \longrightarrow \mathcal{V}$. Then $\varphi(W_1) = \varphi(W_2)$.*

Proof. Let W_1 be walks from v to w with voltage a, and let W_2 be another walk from v to w. Because G is totally balanced, $\varphi(W_1) - \varphi(W_2) = 0$, so $\varphi(W_1) = \varphi(W_2) = a$. □

Theorem 4.2.2. *Let $\varphi_1, \varphi_2 : E \longrightarrow \mathcal{V}$ be two assignments of voltages on some graph G that are in the same coset of B. Then the derived graphs $\tilde{G}_1$ and $\tilde{G}_2$ are isomorphic.*

Proof. Since φ_1 and φ_2 are in the same coset of B, there exists some $\varphi_0 \in B$ such that $\varphi_0 + \varphi_1 = \varphi_2$. Let $v \in G$. Then define $f : V(\tilde{G_1}) \longrightarrow V(\tilde{G_2})$ by $f(w_b) = w_{c+b}$ where $b \in \mathcal{V}$ and the walk W from v to w has $\varphi_0 = c$.

We claim f is bijective. Let $f(u_a) = f(w_b)$. Then $x_{c+a} = x_{c+b}$ so $u = w$ and $c + a = c + b$ and $a = b$. We know c is unique by Lemma 4.2.1. Therefore f is injective. Let $w_a \in \tilde{G_2}$ Then $f(w_{a-c}) = w_{a-c+c} = w_a$. So f is bijective.

Now we show that f preserves adjacency. Let $\tilde{e} = (x_a, y_{a+b}) \in \tilde{G_1}$. So $e = (x, y) \in G$. Then $f(x_a) = x_{a+c}$ and $f(y_{a+b}) = y_{a+b+c+\varphi_0(e)}$. Since $b + \varphi_0(e) = (\varphi_1 + \varphi_2)(e) = \varphi_2(e)$, $(x_{a+c}, y_{a+c+\varphi_0(e)}) \in \tilde{G_2}$. Define $f^{-1} : V(\tilde{G_2}) \longrightarrow V(\tilde{G_1})$ by $f^{-1}(w_a) = w_{a-c}$ where the walk W from v to w has assignment $\varphi(W) = c$. The proof for adjacency in this direction is similar. $\square$

This result easily gives us the following corollary:

Corollary 4.2.3. *Let k be the number of distinct derived graphs of G induced by voltages from a group $\mathcal{V}$. Then $k \leq |T|/|B|$ where T is the set of all possible voltage assignments and B is the set of all possible balanced voltage assignments.*

For some graphs, $k = |T|/|B|$, as seen in the following example.

Example 4.2.4. Let G be the graph in Figure 4.2.1, and let $\mathcal{V} = \mathbb{Z}_2$. Since G has six edges, there are 2^6 possible assignments $\varphi : E(G) \longrightarrow \mathbb{Z}_2$, so $|T| = 64$. The bolded spanning tree has four edges, so 2^4 φ's such that the voltage graph is balanced, and $|B| = 16$. Therefore there are 4 cosets of B in T. We will describe the derived graph induced by each of the four cosets.

In Figure 4.2.2 we show an assignment (a) and the resulting derived graph (b). We do the

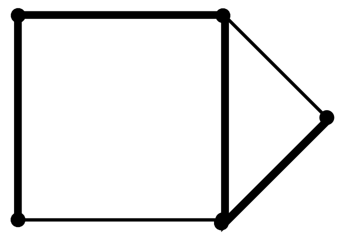
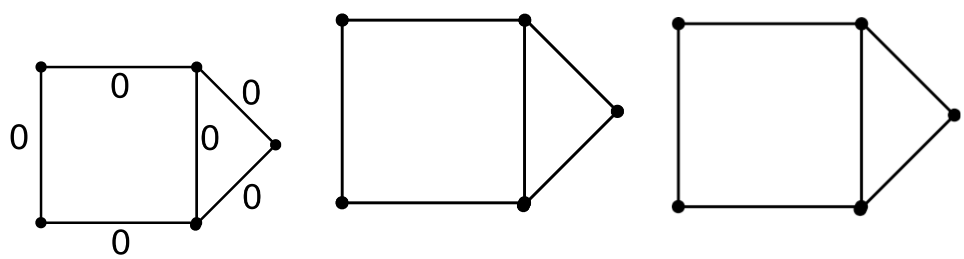


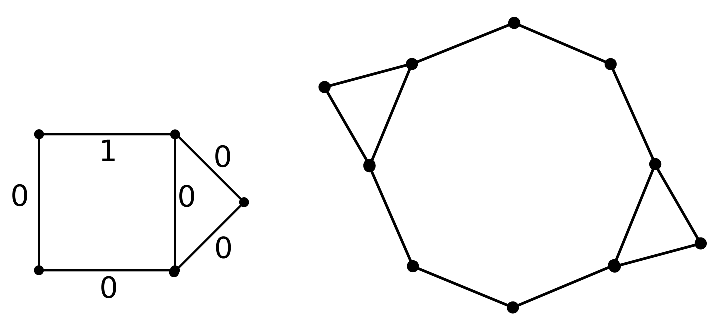
Figure 4.2.1: Non-symmetric base graph with bolded spanning tree

same for Figures 4.2.3, 4.2.4, and 4.2.5. In each case we have take the assignment φ in (a) to be one element of the coset $B + \varphi$.



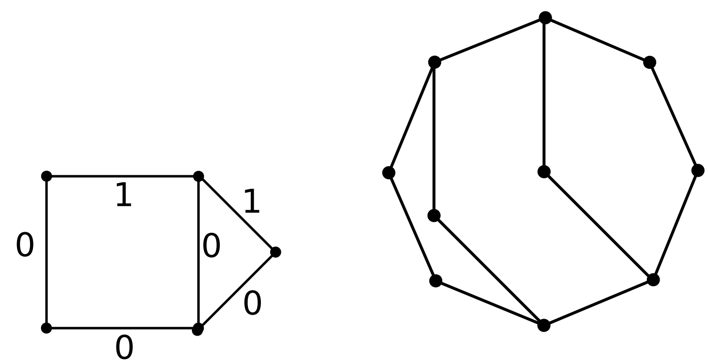
(a) (b)

Figure 4.2.2: Voltage graph and its derived graph I



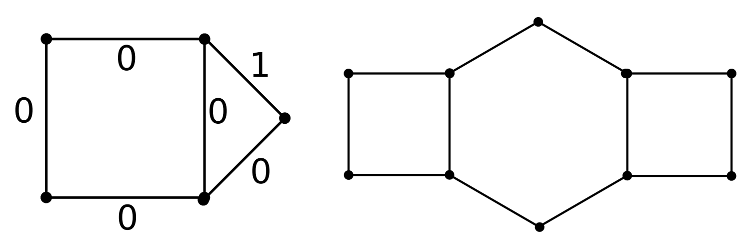
(a) (b)

Figure 4.2.3: Voltage graph and its derived graph II



(a) (b)

Figure 4.2.4: Voltage graph and its derived graph III



(a) (b)

Figure 4.2.5: Voltage graph and its derived graph IV

We see that there are four non-isomorphic derived graphs corresponding to each of the four cosets of B. The four cosets correspond to four cases: both cycles of G are balanced (Figure 4.2.2a), only the square cycle is balanced (Figure 4.2.3a), only the triangle cycle is balanced (Figure 4.2.4a), or no cycles are balanced (Figure 4.2.5a). The two cycles are a basis for the cycle space, so they determine all other cycle assignments.

This example gives us another way to think about the criteria for voltage graphs to have isomorphic derived graphs. A coset of B in T has the property that for any pair $\varphi_i, \varphi_j \in \varphi + B$, $\varphi_i(C) = \varphi_j(C)$ for all cycles C in G. We prove this fact below:

Theorem 4.2.5. *The assignments φ_1, φ_2 are in the same coset $\varphi_k + B$ if and only if $\varphi_1(C) = \varphi_2(C)$ for all cycles $C \in G$.*

Proof. Let $\varphi_1, \varphi_2 \in \varphi_k + B$. Then there exists $\varphi_o \in B$ such that $\varphi_1 = \varphi_2 + \varphi_0$. Let C be a cycle.

$$\varphi_1(C_i) = (\varphi_2 + \varphi_o)(C_i) = \varphi_2(C_i) + \varphi_o(C_i) = \varphi_2(C_i) + 0.$$

Let $\varphi_1, \varphi_2 \in T$ such that $\varphi_1(C) = \varphi_2(C)$ for all $C \in G$. Then define $\varphi = \varphi_1 - \varphi_2$. Then $\varphi(C) = 0$ for all $C \in G$. So $\varphi \in B$. So φ_1, φ_2 are in the same coset of B. $\square$

4.3 Voltage Isomorphism Classification

In Example 4.2.4 in the previous section we showed a graph with a one-to-one correspondence between isomorphism classes of derived graphs and cosets of the balanced subgroup.

We can find examples of graphs, however, where two voltage graphs in different cosets can induce the same derived graph. For example, consider the following two assignments of $\mathbb{Z}_2$ on K_4 in Figure 4.3.1. They both induce the derived graph in Figure 4.3.1c. Clearly, the two assignments are not in the same coset because they don't have the same sums on each cycle, but they have the same derived graph.

This is not surprising because the voltage graphs in Figure 4.3.1 are symmetric with the voltages flipped over the central axis of the graph. We expand on this idea and define an automorphism for voltage graphs.

Definition 4.3.1. Let $\Lambda_1 = (G, \mathcal{V}, \varphi_1)$ and $\Lambda_2 = (G, \mathcal{V}, \varphi_2)$ be voltage graphs with the same base graph and voltage group, and let $\sigma : V(G) \longrightarrow V(G)$ be a graph isomorphism.

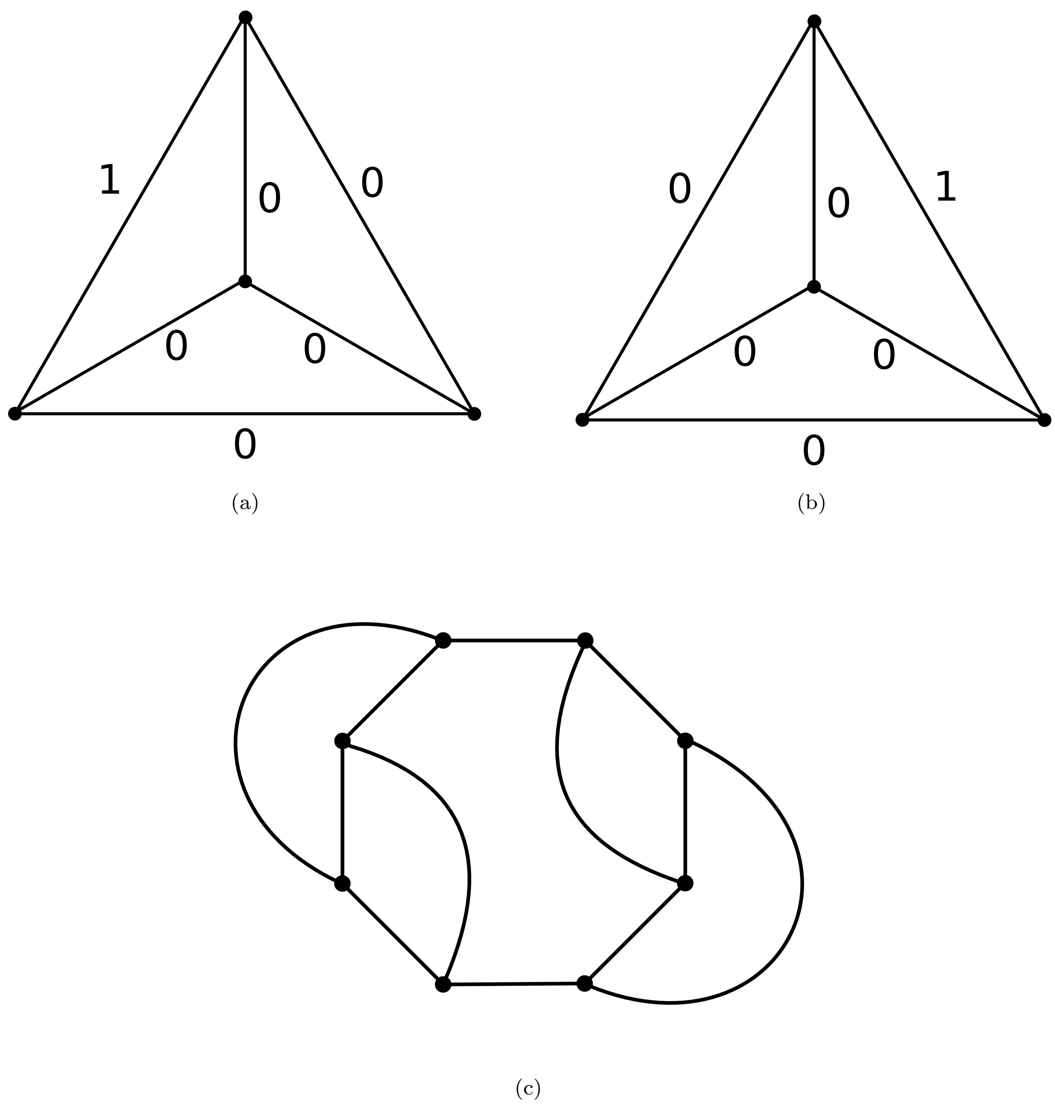


Figure 4.3.1: Two assignments of K_4 with the same derived graph

The voltage graphs are called **voltage isomorphic** if for all $(u,v) \in E(G)$, $\varphi_1(u,v) = \varphi_2(\sigma(u),\sigma(v))$.

Theorem 4.3.2. *Let Λ_1 and Λ_2 be voltage isomorphic. Then their derived graphs $\tilde{G}_1$ and $\tilde{G}_2$ are isomorphic.*

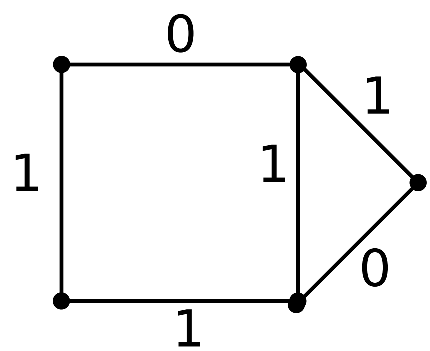
Proof. Let $\Lambda_1 = (G, \mathcal{V}, \varphi_1)$ be a voltage graph with derived graph $\tilde{G}_1$ and $\Lambda_2 = (G, \mathcal{V}, \varphi_2)$ be a voltage graph with derived graph $\tilde{G}_2$. Define $f : V(\tilde{G}_1) \longrightarrow V(\tilde{G}_2)$ by $f(v_a) = (\sigma(v))_a$. Because σ is bijective, f is bijective. We claim that f preserves adjacency.

Let $(u_a, v_{a+b}) \in \tilde{G}_1$. Then $e = (u,v) \in G$ such that $\varphi_1(u,v) = b$. Now consider $f(u_a) = (\sigma(u))_a$ and $f(v_{a+b}) = (\sigma(u))_{a+b}$. Since Λ_1 and Λ_2 are voltage isomorphic, $b = \varphi_1(u,v) = \varphi_2(\sigma(u),\sigma(v))$ and thus $((\sigma(u))_a, (\sigma(v))_{a+b}) \in \tilde{G}_2$.

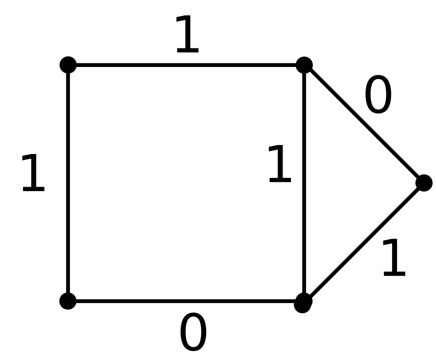
The inverse function $f^{-1} : V(\tilde{G}_2) \longrightarrow V(\tilde{G}_1)$ is $f(v_a) = (\sigma^{-1}(v))_a$. Because σ^{-1} is an automorphism, the proof of adjacency for f^{-1} is similar. Therefore f preserves adjacency and $\tilde{G}_1$ and $\tilde{G}_2$ are isomorphic.

□

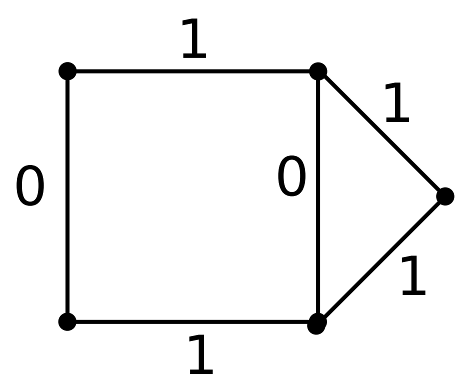
Now let's go back to an earlier example. Clearly, the graph in Figure 4.2.1 has some symmetry. We can flip the graph over its horizontal axis. So why do none of the cosets of B induce graphs isomorphic to graphs derived from assignments from other cosets? Take the example of the following two voltage assignments on the graph G shown in Figure 4.3.2. They are voltage isomorphic but they are also in the same coset, since their difference is the balanced assignment shown in Figure 4.3.2c. Additionally, in this case there are no cycles that are symmetric to one another.



(a)



(b)



(c)

Figure 4.3.2: Three voltage graphs with assignments φ_1, φ_2, and $\varphi_1 - \varphi_2$

5
The Structure of Derived Graphs

In this chapter we investigate the structure of the derived graph. In Section 4.1, we examine the case where the derived graph is disjoint. It turns out that this depends entirely on the specific voltage assignment and the notion of balance is again crucial. In Section 4.2, we prove that the derived graph of a planar graph is planar. In Section 4.3 we find an upper bound for the chromatic number of the derived graph. These theorems presented in the last two sections will not depend of the particular voltage assignment at all, but hold for all derived graphs.

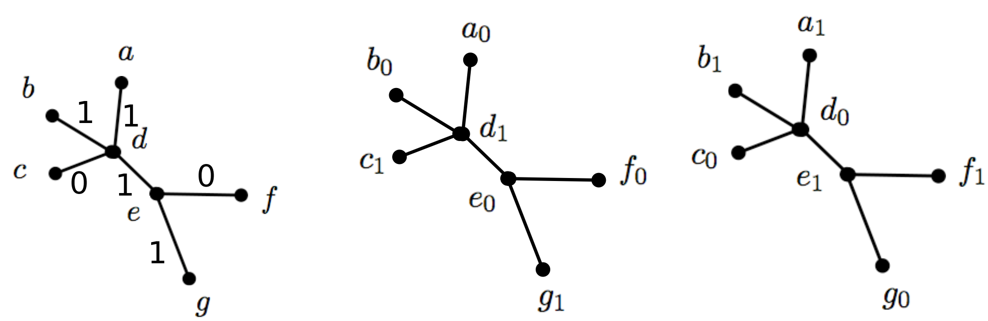
5.1 Connectivity

Sometimes, a derived graph of a connected graph is disconnected. In this section we explore three cases where this occurs: when the base graph G is a tree, when the voltage graph Λ is balanced, and when every cycle of Λ sums to an element in a subgroup of $\mathcal{V}$.

First, let's think about how two vertices in a derived graph can be connected. If $v, w \in V(G)$ and $a, b \in \mathcal{V}$, then $v_a, w_b \in \tilde{G}$ are connected if and only if there exists a walk between v and w in G with net voltage $b - a$. So it is perhaps more surprising that so many derived graphs are connected. For a graph to be connected, there must exist a walk from v_a to w_i for all $w_i \in p^{-1}(\{w\})$. This means that there must exists at least as many possible walks from v to w in G with distinct net voltages as there are elements of $\mathcal{V}$.

With this in mind, our first case, that of the derived graph of a tree, is easy.

Example 5.1.1. Consider the voltage graph and derived graph with $\mathcal{V} = \mathbb{Z}_2$ shown in Figure 5.1.1a and Figure 5.1.1b. The fact that $\tilde{G}$ consists of two disjoint copies of G is



(a) (b)

Figure 5.1.1: Voltage graph and derived graph of a tree

not surprising. There exists a unique walk between any two vertices of a tree with some net voltage. So every walk from u to v lifted to u_a terminates at the same $v_b \in p^{-1}(\{v\})$. Therefore each u_i is connected to only one v_j in $\tilde{G}$. So the derived graph will always be $|\mathcal{V}|$ disjoint copies of the tree.

Theorem 5.1.2. *Let R be a tree, and let $\Lambda = (R, \varphi, \mathcal{V})$ be a voltage graph for some assignment φ and group $\mathcal{V}$. Then the derived graph $\tilde{R}$ consists of $|\mathcal{V}|$ disjoint copies of R.*

Proof. First we will prove that $\tilde{R}$ is disjoint. Let $u_a, u_b \in \tilde{R}$ such that $a \neq b$. Define two subgraphs induced by the following sets of vertices:

$$S_a = \{v_c \mid \text{ such that there exists a walk from } u_a \text{ to } v_c\}$$

$$S_b = \{v_d \mid \text{ such that there exists a walk from } u_b \text{ to } v_d\}$$

We claim S_a and S_b are disjoint. First assume contrary. Then there exists a v_c such that $v_c \in S_a \cap S_b$. Then there exists a walk from u_a to v_c and a walk from u_b to v_c in $\tilde{R}$. However, Theorem 2.0.10 states that there exists a unique trail L from u to v in R. So $\varphi(L) = d$ and the lift of L_a terminates at v_{a+d} and the lift L_b terminates at v_{b+d} by Theorem 3.4.4. So the subgraphs are disjoint by contradiction and are size $|R|$ because no two vertices in the same fiber can be connected.

Now we claim the subgraph induced by S_a is isomorphic to R. Define $f : V(R) \longrightarrow S_a$ by $f(v) = v_{a+b}$ where $\varphi(L) = b$ for the unique trail L from u to v in R.

Let $v_c \in S_a$. Then there exists a trail L from u to v in R with $\varphi(L) = c - a$. So $f(v) = v_{a+c-a} = v_c$ and the function is surjective. Let $f(v) = f(w)$. Then by the definition of the derived graph, $v = w$. So the function is bijective.

Lastly, we claim f preserves adjacency. Let $e = (v, w) \in V(R)$. So the walk L from u to v has assignment $\varphi(: +L) = b$ and $\varphi(e) = c$. So $f(v) = v_{a+b}$ and $f(w) = w_{a+b+c}$ and (v_{a+b}, w_{a+b+c}) is an edge in $\tilde{R}$.

Now define $f^{-1} : S_a \longrightarrow V(R)$ by $f^{-1}(v_a) = v$. Let $(v_a, w_{a+b}) \in S_a$. Then $(v, w) \in R$ by the definition of the derived graph. So f preserves adjacency.

□

The derived graph of any totally balanced voltage graph is $|\mathcal{V}|$ copies of the base graph. Here is an example.

Example 5.1.3. Consider the balanced assignment of edges on K_4 where $\mathcal{V} = \mathbb{Z}_2$ shown in Figure 5.1.2.

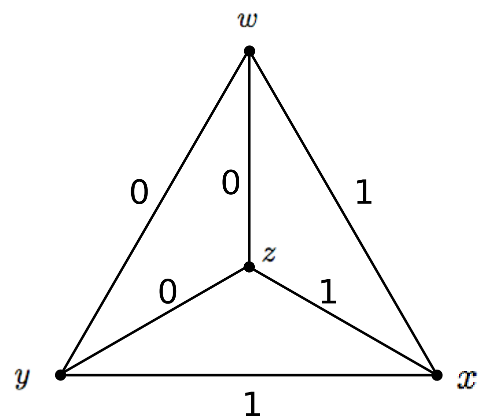


Figure 5.1.2: Balanced voltage assignment on K_4

This produces the derived graph shown in Figure 5.1.3.

The fact that the derived graph is disjoint follows from Lemma 4.2.1. Lemma 4.2.1 states that any two walks between vertices u and v in a blanced voltage graph will have the same net voltage. so if u and v in a blanced voltage graph are connected by a walk with net voltage b, the lift of any walk W_a from u to v will terminate at v_{a+b}. In fact, for a vertex v_c where $c \neq a + b$, there is now walk in the derived graph between u_a and v_c. We will use this notion to prove the net theorem.

Theorem 5.1.4. *Let $\Lambda = (G, \mathcal{V}, \varphi)$ be a totally balanced voltage graph.. Then $\tilde{G}$ is $|\mathcal{V}|$ disjoint subgraphs isomorphic to G.*

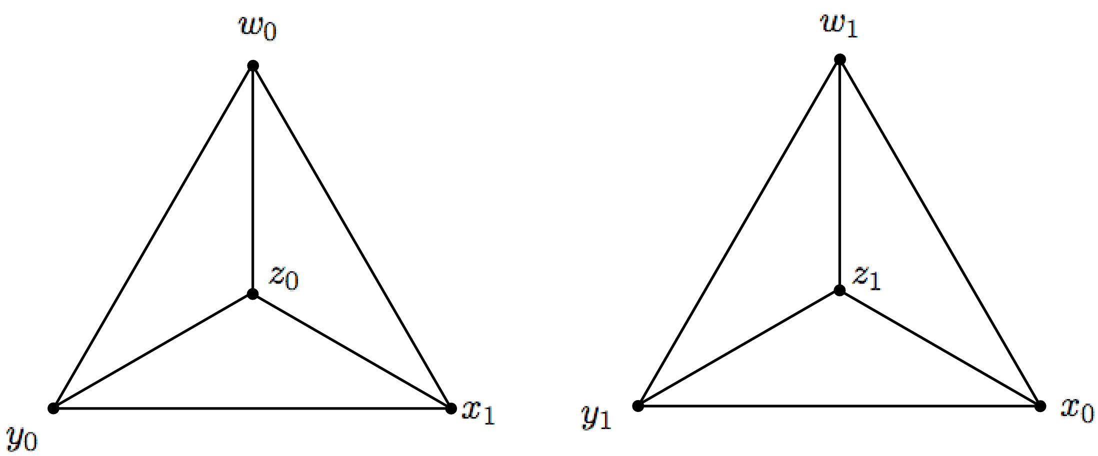


Figure 5.1.3: The derived graph of a balanced voltage graph

Proof. Define S_a to be the induced subgraph of $\tilde{G}$ such that

$$V(S_a) = \{v_b \in \tilde{G} \mid \text{ there exists a path } u_a \text{ to } v_b\}$$

We claim that S_a is isomorphic to G.

Let $f : V(S_a) \longrightarrow V(G)$ such that $f(v_b) = v$. We claim f is bijective. Let $v \in V(G)$. Any walk W from u to v in G has a unique assignment $\varphi(W) = c$ by Lemma 5.1.4. So the lift W_a terminates at v_{a+c}. Therefore $f(v_{a+c}) = v$, and f is surjective.

Assume $f(v_c) = f(v_d) \in G$. By Lemma 5.1.4., every walk from u to v has the same voltage, namely b. So every path from u to v lifted to u_a terminates at v_{a+b}, and v_{a+b} is the only lift of the vertex v in S_a. Therefore $f(w_c) = f(w_d)$ implies that $c = d$. Therefore f is bijective. It is also clearly adjacency preserving because the derived graph preserves adjacency.

We claim that $f^{-1} : V(G) \longrightarrow V(S_a)$ define by $f^{-1}(v) = v_{a+b}$ where the walk W from u to v has assignment $\varphi(W) = b$. So let $e = (v, w) \in G$. Then $f(v) = v_{a+b}$ and $f(w) = v_{a+b+\varphi(e)}$.

So $(v_{a+b}, w_{a+b+\varphi(e)}) \in S_a$.

So there exists an induced subgraph S_b isomorphic to G for each $b \in \mathcal{V}$ such that

$$V(S_i = \{v_b \in \tilde{G} \mid \text{ there exists a path } u_i \text{ to } v_c\}$$

Lastly we show these S_bs are disjoint. First assume $v_c \in V(S_a)$ and $v_c \in V(S_b)$. Then there exists paths $v_c \longrightarrow u_a$ and $v_c \longrightarrow u_b$. By our argument for the injectivity of f we find $u_a = u_b$. So the S_is are disjoint. □

Now consider a voltage graph Λ where $\varphi : E(G) \longrightarrow \mathcal{V}$ such that $\varphi(C) \in \mathcal{W}$ for all cycles $C \in G$ where $\mathcal{W} \subsetneq \mathcal{V}$ is a subgroup. Similarily to the case where Λ is totally balanced, there are certain pairs of vertices in the derived graph that do not have a walk between them. For instance, u_a and u_{a+b} where $b \notin \mathcal{W}$ cannot be connected. Observe the next example.

Example 5.1.5. First observe the voltage graph with $\varphi : E(G) \longrightarrow Z_4$ and corresponding derived graph in Figure 5.1.4. Note that all cycles sum to an element in $\{0, 2\} = \mathcal{W} \subsetneq \mathcal{V}$ and that the derived graph is disjoint.

Theorem 5.1.6. *Let $\Lambda = (G, \mathcal{V}, \varphi)$ be a voltage graph such that $\varphi(C) \in \mathcal{W}$ for all cycles $C \in G$ where $\mathcal{W} \subseteq \mathcal{V}$ is a subgroup. Then the derived graph $\tilde{G}$ has $|\mathcal{V}|/|\mathcal{W}|$ disjoint components.*

Proof. Choose some $u_a \in \tilde{G}$. Then define the subset of vertices S_a that induces the subgraph H_a by

$$S_a = \{v_b \mid \text{ there exists a walk from } u_a \text{ to } v_b\}.$$

First we show there are $|\mathcal{V}|/|\mathcal{W}|$ such components. Examine the fiber $p^{-1}(\{u\}) \cap S_a$. There exists a walk from the vertex u_a to v_b in $\tilde{G}$ if $b - a \in \mathcal{W}$. So $u_b \in S_a$ if b is in a coset of $\mathcal{W}$ generated by a. So $|\mathcal{W}| = |p^{-1}(\{u\}) \cap S_a|$.

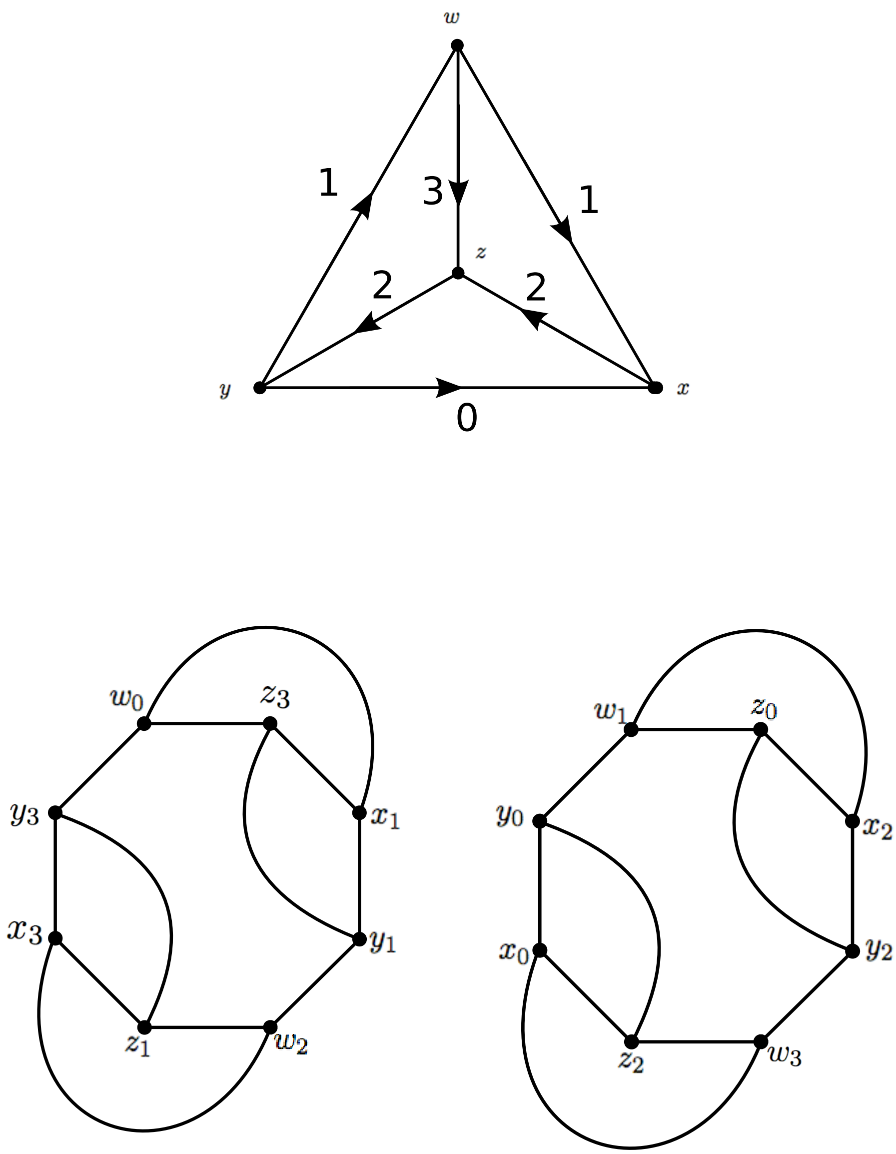


Figure 5.1.4: A voltage graph and its disjoint derived graph

Now examine $p^{-1}(\{v\}) \cap S_a$. The walk u to v in G has net voltage c. So $v_{a+c} \in S_a$. Additionally, v_{a+c} is connctd to v_{a+c+d} for all $d \in \mathcal{W}$. So $v_b \in S_a$ if b is in the coset of $\mathcal{W}$ generated by $a + c$. So $|\mathcal{W}| = |p^{-1}(\{v\}) \cap S_a|$. This follows for all vertices in G. Therefore $|S_a| = |V(G)| \cdot |\mathcal{W}|$, and there are $|\mathcal{V}|/|\mathcal{W}|$ such components in $\tilde{G}$.

Now let S_a and S_b with $a \neq b$ be defined by

$$S_a = \{v_c \mid \text{ there exists a walk from } u_a \text{ to } v_c\}$$

$$S_b = \{v_d \mid \text{ there exists a walk from } u_b \text{ to } v_d\}$$

We claim the subgraphs H_a and H_b induced are disjoint. Let $v_c \in S_a$ and $v_c \in S_b$. So c is in the coset of $\mathcal{W}$ generated by $c - a$ and in the coset of $\mathcal{W}$ generated by $c - b$. We have reached a contradiction, and the H_i's are disjoint.

□

The fact that a totally balanced voltage graph induces a derived graph with $|G|$ disjoint components as proved in Theorem 5.1.5 actually follows from Theorem 5.1.7.

5.2 Planarity

The derived graph, along with quotient graphs which we have not discussed, is often used to determine the surface a given graph can be imbedded into In general, the **genus** of a derived graph—the minimum number of handles in the plane required for the graph to have a drawing without crossings—can be either higher or lower than the base graph. [DO I WANT AN EXAMPLE?] We consider the case when G is planar, and thus G has genus 0. The derived graph cannot have smaller genus than the base graph, so it must have genus the same or larger. For more information on genus and imbedding see Chapter 3 of [5]. In the next theorem, we show that a planar graph must have planar derived graphs. To prove this, we use Kuratowski's theorem, introduced in Chapter 2.

Theorem 5.2.1. *The derived graph $\tilde{G}$ of a simple planar graph G is planar.*

Proof. Recall Theorem 2.0.19, which states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Suppose $\tilde{G}$ is not planar. Assume there exists an induced subgraph of $\tilde{G}$ that is homeomorphic to $K_{3,3}$. So for some $v_1, \ldots, v_6 \in V_G$ and for some $\pi_1, \ldots \pi_6 \in \mathcal{V}$ there exists the following subgraph of $\tilde{G}$.

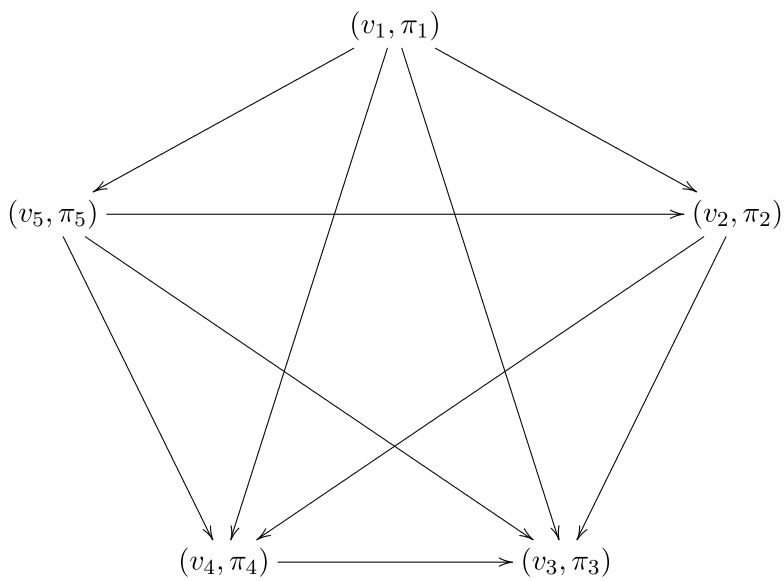
$$\begin{tikzcd}
(v_1,\pi_1) \arrow[d] \arrow[dr] \arrow[drr] & (v_2,\pi_2) \arrow[dl] \arrow[d] \arrow[dr] & (v_3,\pi_3) \arrow[dll] \arrow[dl] \arrow[d] \\
(v_4,\pi_4) & (v_5,\pi_5) & (v_6,\pi_6)
\end{tikzcd}$$

We claim that $v_i \neq v_j$ for all $i, j \in \{1, 6\}$. For example, consider v_1. Since $(v_1, \pi_1) \sim (v_j, \pi_j)$ for all $4 \leq j \leq 6$, $v_1 \neq v_j$ for $4 \leq j \leq 6$. Also, $(v_1, \pi_1) \sim (v_4, \pi_4) \sim (v_2, \pi_2)$. Since G is simple, there exists a unique lift of the edge $e = (v_1, v_4)$ in $\tilde{G}$. So $v_1 \neq v_2$. By similar reasoning, $v_1 \neq v_3$. The same argument can be made to show that no vertex in the induced subgraph of $\tilde{G}$ has the same base vertex in G.

Because $(v_1, \pi_1) \sim (v_4, \pi_4) \in \tilde{G}$, $v_1 \sim v_4 \in G$. By similar logic we find the following subgraph in G:

$$\begin{tikzcd}
v_1 \arrow[d] \arrow[dr] \arrow[drr] & v_2 \arrow[dl] \arrow[d] \arrow[dr] & v_3 \arrow[dll] \arrow[dl] \arrow[d] \\
v_4 & v_5 & v_6
\end{tikzcd}$$

Therefore G is not planar and we have reached a contradiction. Now assume there exists an induced subgraph isomorphic to K_5 in $\tilde{G}$ as follows:



with $v_i \in V(G)$ and $\pi_i \in \Pi$. So $(v_i, \pi_i) \sim (v_j, \pi_j)$ for all vertices in the subgraph. So we reach a contradiction by the argument above. Thus $\tilde{G}$ is planar. $\square$

5.3 Coloring the Derived Graph

Recall the definitions of proper coloring and chromatic number from Chapter 2. One might first assume that the derived graph would have the same chromatic number as the base graph. The next example demonstrates otherwise.

Example 5.3.1. Recall that the cubic graph is a derived graph of K_4. The chromatic number of K_4 is 4, while the chromatic number of the cubic graph is 2 as you can see from Figure 5.3.1.

The next theorem shows that the chromatic number of the base graph is an upper bound for the chromatic number of the derived graph.

Theorem 5.3.2. *Let G be a graph with chromatic number $\chi(G)$ and let $\tilde{G}$ be a derived graph of G with chromatic number $\chi(\tilde{G})$. Then $\chi(\tilde{G}) \leq \chi(G)$.*

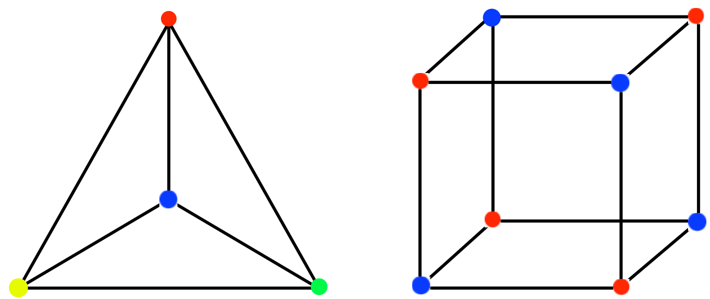


Figure 5.3.1: Proper coloring of K_4 and its derived graph

Proof. Let $V(G) = v_1, \ldots, v_n$ and let the set $c_1, \ldots c_n$ be a minimal coloring of $V(G)$ such that the color of any vertex v_i is c_i

Now induce a derived graph $\tilde{G}$ using some group $\mathcal{V}$ and assignent φ. Create a coloring of $\tilde{G}$ such that any vertex $(v_i, \pi_j) \in \tilde{G}$ is assigned color c_i. We claim that this is a proper coloring of $\tilde{G}$.

Take any two adjacent points (v_a, π_i) and (v_b, π_j) in $\tilde{G}$ with the given color assignments c_a and c_b respectively. Because $v_a \sim v_b$ in G, $c_a \neq c_b$. Therefore there exists a proper coloring of $\tilde{G}$ of size $\chi(G)$. So $\chi(\tilde{G}) \leq \chi(G)$. $\square$

6
Further Research

Theorem 4.2.5 states that graphs with the same net voltage around each cycle will induce the same derived graphs. The dependence on the voltage around the cycle rather than the voltage on individual edges creates an intuitive connection between the set of possible cycle assignments and the fundamental group of the graph. In algebraic topological there are many known connections between the fundamental group of a space and the isomorphism classes of covering spaces. In this section we will define the fundamental group of a graph and show the relationship between the quotient group T/B and the fundamental group. By doing this we will demonstrate areas of possible future research.

We will define the **fundamental group**, called π_1, of a graph via an example. Consider oriented K_4 in Figure 6.0.1. The fundamental group is the set of all possible cycles of a graph. Choose x as a base point. Now observe the cycles based at x. All these cycles form the cycle space which is generated by the three minimal cycles which form the basis of the cycle space.

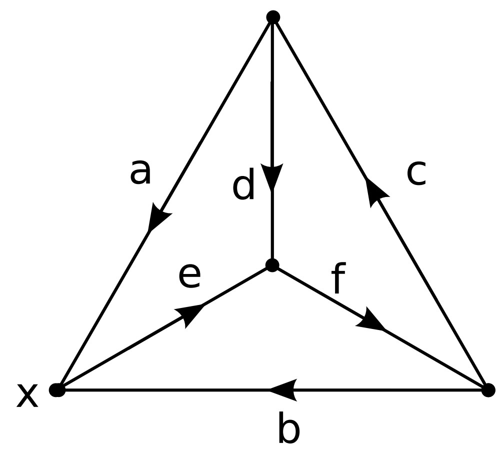


Figure 6.0.1: Proper colorings of K_4 and its covering graph

The fundamental group of a graph is always a **free group**. This free group is the **free product** of any number of copies of $\mathbb{Z}$, and is denoted $*$. Here is the definition from Hatcher's *Algebraic Topology* [2]:

Definition 6.0.3. As a set, the **free product** of α copies of $\mathbb{Z}$, $*_\alpha\mathbb{Z}$, consists of all words $z_1z_2\cdots z_m$ of arbitrary finite length $m \geq 0$, where each letter z_i belongs to a copy of $\mathbb{Z}$ and is not 0, and adjacent letters z_i and z_{i+1} belong to different copies of $\mathbb{Z}$, that is, $\alpha_i \neq \alpha_{i+1}$.

The fundamental group of a single cycle is $\mathbb{Z}$, because it can be transversed as many times as desired in either direction. When we attach cycles together, we have two cycles to transverse. Since our basis of K_4 is three cycles, $\pi_1(K_4) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Hatcher gives us a theorem to predict the structure of the derived graph for any group:

Theorem 6.0.4. *(Hatcher [2]) For a connect graph G with maximal tree R, $\pi_1(X)$ is a free group with basis corresponding to the edges of $X - R$.*

Recall that the subgroup of balanced assignments is isomorphic to $\mathcal{V}^r$ and the group of all assignments of cycles with group elements is isomorphic to $\mathcal{V}^k$ where $k = IX - R|$. So this group looks like a finite abelian version of the fundamental group of the space. We claim that there exists the following relation:

Theorem 6.0.5. *Let G be a graph and $\mathcal{V}$ be an abelian group. Then $T/B \simeq Hom(\pi_1, \mathcal{V})$.*

There are several theorems in Hatcher's *Algebraic Topology* about the connection between the fundamental group and covering spaces. If T/B is this finite subset of π_1, what relationships between covering spaces and the fundamental group will be preserved in the relationship between T/B and the derived voltage graphs? For example, take the following theorem, which explains that every subgroup of $\pi_1(X)$ corresponds to a covering space of X:

Theorem 6.0.6. *(Hatcher) Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the sbgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \longrightarrow X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

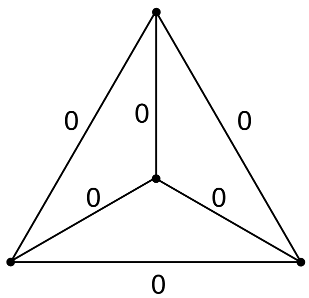
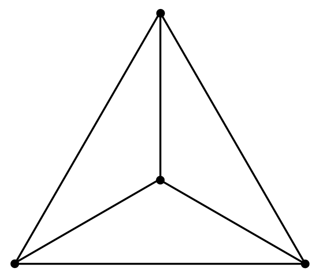
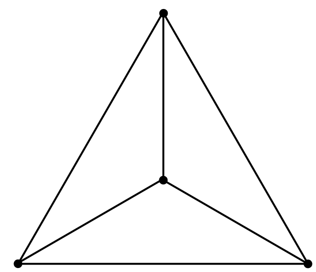
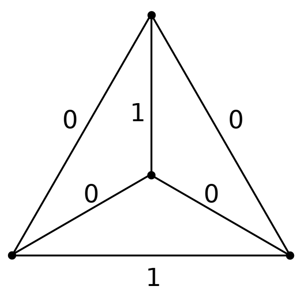
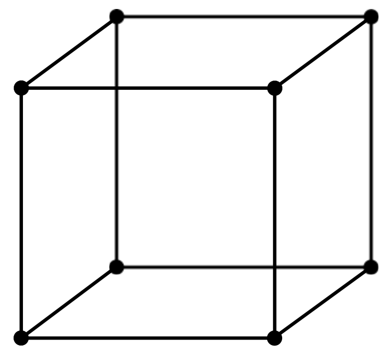
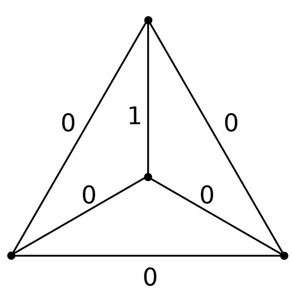
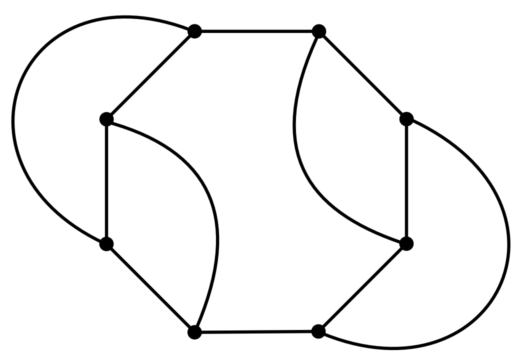
This theorem had promise to answer more fully our question about number of derived graphs. However, it doesn't take into account voltage isomorphism anymore than the coset classification does. For example, take K_4. In this case, $T/B \approx (\mathbb{Z}_2)^3$. This group has eight subgroups, but we know that the voltages induce only three non-isomorphic graphs. The fundamental group only cares about number of cycles, not about size of cycles.

After discovering the relation between π_1 and T/B, I found a brief discussion of it in Chapter 2.5. of [1]. The authors state that "From the correspondence just described, one may obtain for graphs all of the standard topological theorems on the relationship between fundamental groups and covering spaces." They do not go on to obtain all the topological theorems. So although a future senior project student would not be breaking any new ground by investigating this question, we can guarantee that it is true.

Appendix A
Derived Graphs of K_4

A.1 Derived Graphs of K_4 with $\mathbb{Z}_2$

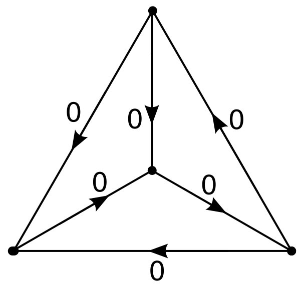
There are 64 assignments of $\mathbb{Z}_2$ to K_4. The group of balanced assignments, B, is of order 8. . In this appendix we have one assignment φ that generates the derived graph.

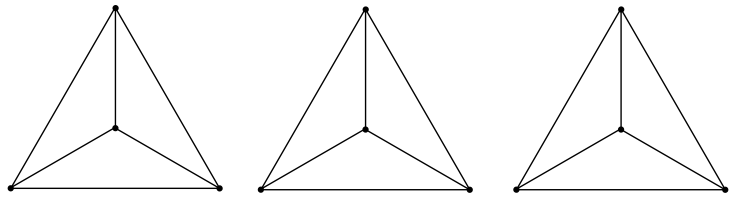
φ	$\tilde{G}$
	 
	
	

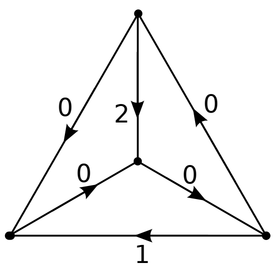
A.2 Derived Graphs of K_4 with $\mathbb{Z}_3$

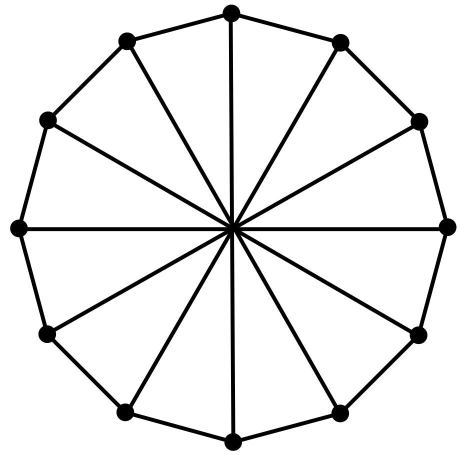
There are 729 assignments of $\mathbb{Z}_3$ to K_4. The group of balanced assignments, B, is of order 27. In this appendix we have an assignment φ that generates each derived graph

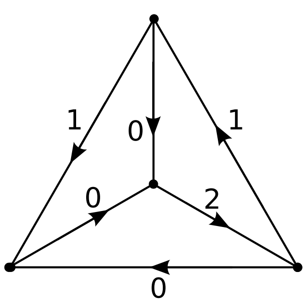
φ	$\tilde{G}$

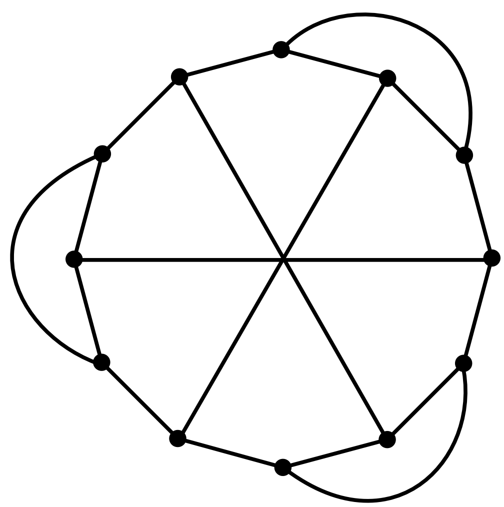


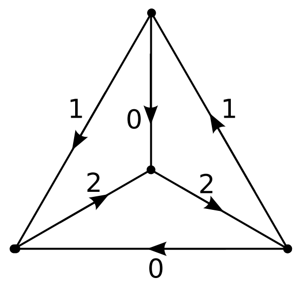


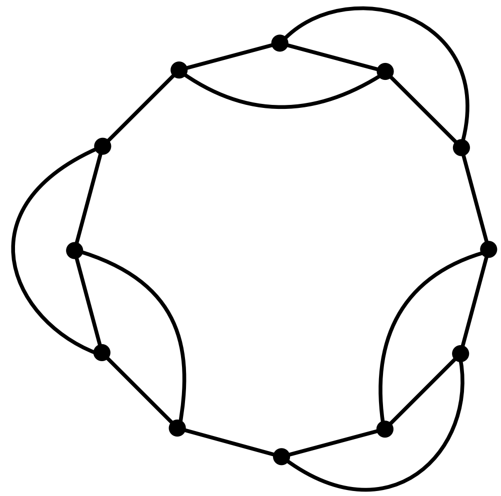








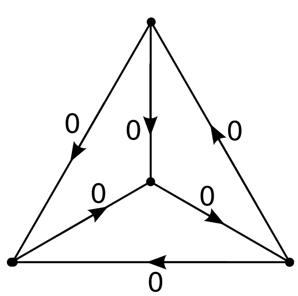


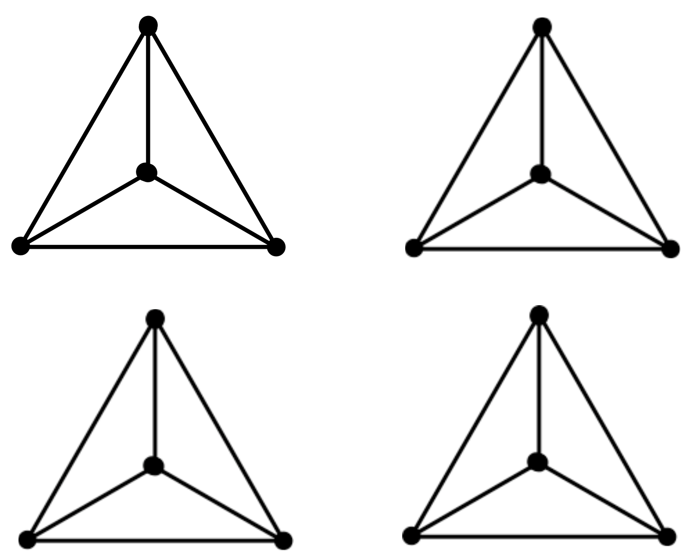


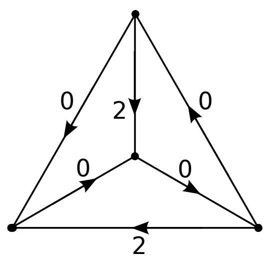
A.3 Derived Graphs of K_4 with $\mathbb{Z}_4$

There are 4069 assignments of $\mathbb{Z}_3$ to K_4. The group of balanced assignments, B, is of order 64. In this appendix we have an assignment φ that generates each derived graph

φ	$\tilde{G}$



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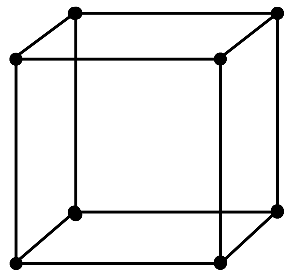

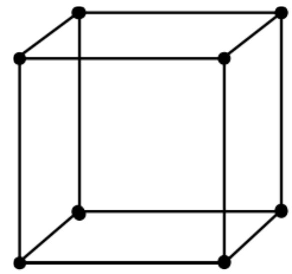

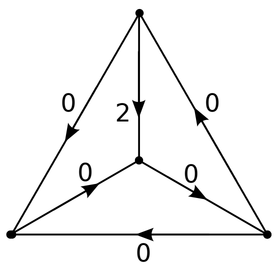
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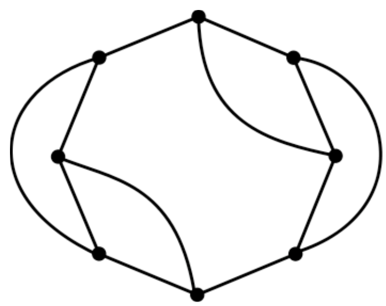

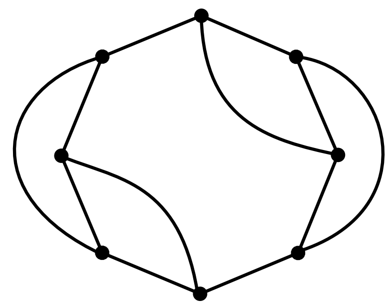

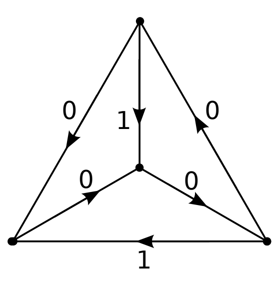
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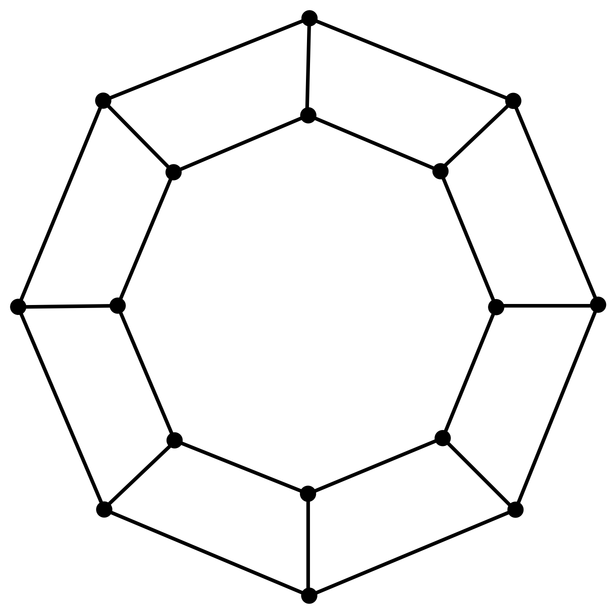

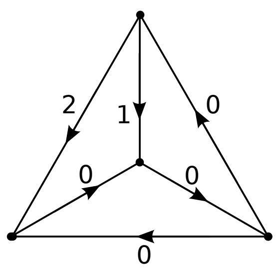
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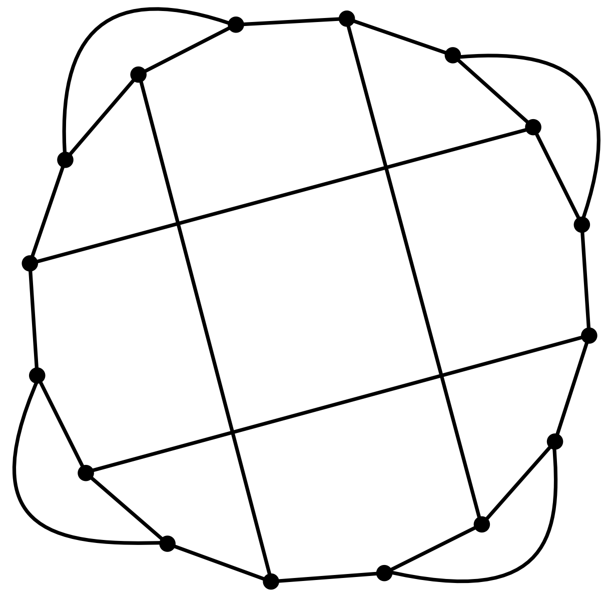

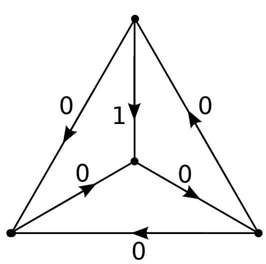
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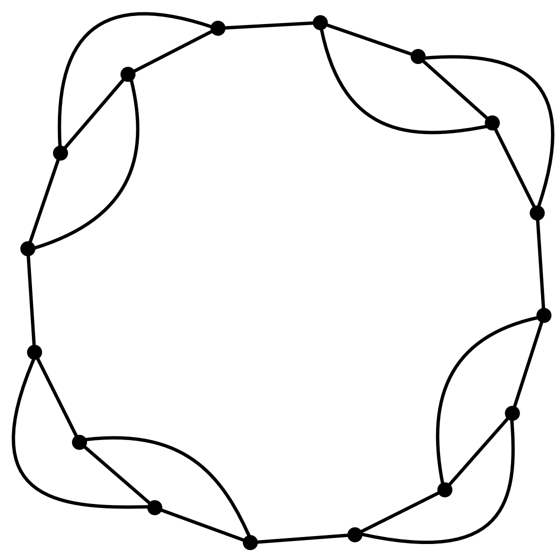

Bibliography

[1] Jonathan L. Gross and Thomas W. Tucker, *Topological Graph Theory*, John Wiley and Sons, New York, 1987.

[2] Allen Hatcher, *Algebraic Topology*, Electronic, 2000.

[3] James R. Munkres, *Topology*, Prentice Hall, Upper Saddle River, NJ, 2000.

[4] Konstantin Rybnikov and Thomas Zaslavsky, *Criteria for Balance in Abelian Gain Graphs, with Applications to Piecewise-Linear Geometry* (1982), 417-423.

[5] Douglas B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 2001.