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Winning Strategies in The Board Game Nowhere To Go

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Abstract

Nowhere To Go is a two player board game played on a graph. The players take turns placing blockers on edges, and moving from vertex to vertex using unblocked edges and unoccupied vertices. A player wins by ensuring their opponent is on a vertex with all blocked edges. This project goes over winning strategies for Player 1 for Nowhere To Go on the standard board and other potential boards.

Contents

Dedication

For my grandmother Brenda who gave me candy while I did math homework in elementary school.

Acknowledgments

I'd like to thank most of my math teachers during my undergraduate career, my mom for making me apply for BHSECQ, and my puppy Cole.

1

Prerequiestite Graph Theory and Definitions

1.1 Definitions

The following definitions are from [1] unless specified.

Definition 1.1.1. A **Graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not neccesarilly distinct) called endpoints. **Multiple edges** are edges that have the same pair of endpoints. **Loops** are edges whose endpoints are the same vertex. △

In this project graphs may have multiple edges but not loops.

Definition 1.1.2. Let G be a graph. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to the edges in H is the same as in G. We then write $H \subseteq G$ and say that "G **contains** H". △

Definition 1.1.3. Let G be a graph and let $u \in V(G)$ and $v \in V(G)$. A **walk** in $G$is a list v_0, e_1, v_1, ..., e_k, v_k of vertices and edges such that, for all $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edge. A u, v-**walk** or a u, v-**trail** has the first vertex u and the last vertex v. △

Definition 1.1.4. Let G be a graph and let $u \in V(G)$ and $v \in V(G)$. The graph G is **connected** if it has a u, v-path for any $u, v \in V(G)$ (otherwise, G is **disconnected**). If G has a u, v-path, then u is **connected to** v in G. △

Definition 1.1.5. Let G be a graph, $v, w \in V(G)$. The vertices v and w are **adjacent** if there is an edge in $e \in E(G)$ such that both v and w are the endpoints of e. △

Definition 1.1.6. Let G be a graph and let $v, w \in V(G)$, and $e \in E(G)$. Let v, w be the two endpoints to the edge e. We then say e is **incident** to both v and w. △

The following definitions are not from [1].

Definition 1.1.7. Let V be a set. Let $|V|$ will denote the number of elements in that set. △

Definition 1.1.8. Let V and W be sets. The sets V, W are **disjoint** if $V \cap W = \emptyset$. △

Definition 1.1.9. Let G be a graph and let $v \in V(G)$. The **degree** of a vertex v of G is the number of edges incident with v, and is denoted $deg(v)$. △

Definition 1.1.10. Let G be graph and ket $v \in V(G)$. The **link of** v is the set of vertices adjacent to v and the set edges in G such that both endpoints are adjacent to v denoted $link(v)$. The **linkset of** v is the set edges such that both endpoints are adjacent to v. △

Definition 1.1.11. Let G be a graph and let $u \in V(G)$ and $v \in V(G)$. A **trailset** in G is a trail in which vertices are removed from the list. A u, v-**trailset** is a list of edges whose endpoints are u and v. △

2

Game Rules and Example Game

2.1 Introduction

In this section we will go over the game rules for Nowhere To Go and show an example game. In all of the graphs for this project let 1 and 2 denote the positions of Player 1 and Player 2 repsectively and let NTG denote the standard Nowhere To Go Board which is a graph.

2.2 Game Rules

Player 1 begins located on vertex H and Player 2 begins located on vertex L in $V(NTG)$ (see Figure 2.2.1). The Players then take turns placing up to 4 blockers per player on any of the edges in the graph as long as the edges the players block are not incident to the vertices that are the starting vertices of the graph; this is the setup phase of the game. These blockers will be denoted by small black squares on the edges that are being blocked. Blockers cannot be placed on top of other blockers. After the players are done placing their blockers the players decide whose turn is first, for all of these proofs the player who moves first will be considered Player 1. The players then alternate turns using trails to move from the vertex the player began their turn on to any of the vertices in the graph that are connected to the vertex player began their turn on. The trails cannot contain edges that have blockers on them and cannot contain vertices that either of the players are currently positioned on. When a player can no longer create a trail that player has lost the game and their opponent has won the game.

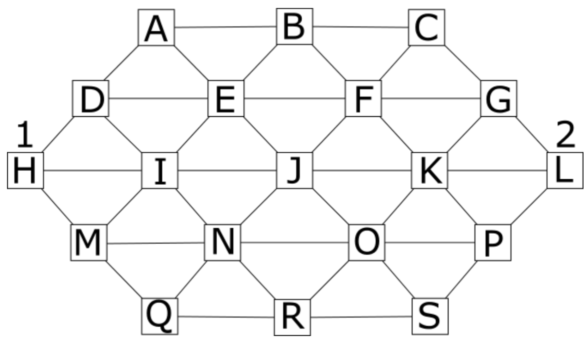

Figure 2.2.1. Starting Positions

2.3 Example Game

We will now see an example game where Player 1 wins. We will begin this example game from the setup phase of the game where both players have already placed their blockers (as seen in Figure 2.3.1), notice that none of the initial blockers are placed on edges incident to the starting vertices as that is against the game rules.

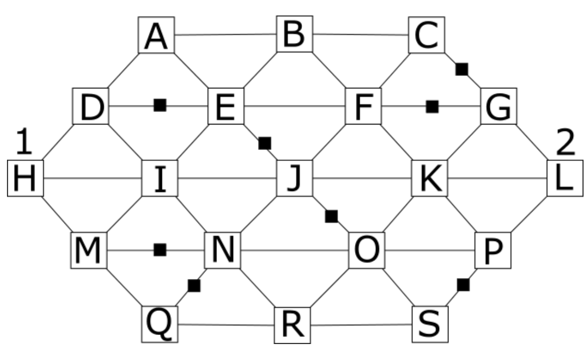

Figure 2.3.1. Example Game setup phase

Player 1 then moves from vertex H to vertex N as Player 1 has a trail that isn't being obstructed by either the blockers or Player 2 by using vertex I. Player 1 then places a blocker on the edge incident to both vertex L and vertex G (see Figure 2.3.2).

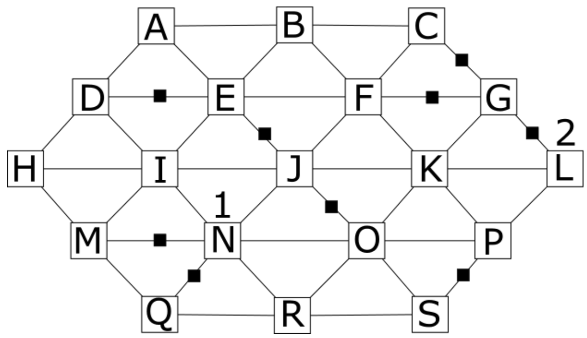

Figure 2.3.2. Example Game Player 1 Turn 1

It is now Player 2's first turn. Player 2 has a trail to vertex G from vertex L but this wouldn't be a good move as G only has one incident edge that isn't being blocked by a blocker, meaning it would be easy for Player 2 to lose on Player 1's next turn. Instead Player 2 moves from vertex L to vertex Q, Player 2 can use vertex L to vertex K to vertex O to vertex R to vertex Q. Player 2 then places a blocker on the edge incident to vertex N and vertex R (see Figure 2.3.3).

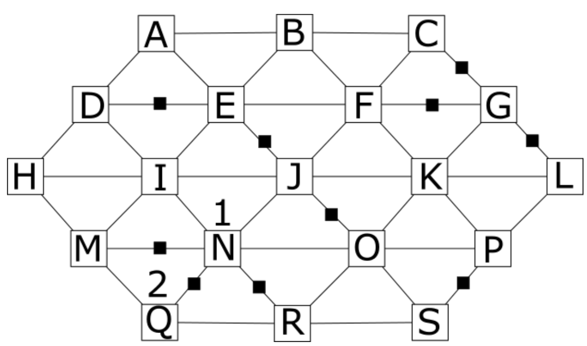

Figure 2.3.3. Example Game Player 2 Turn 1

It is now Player 1's second turn. Player 1 has a trail from vertex N to vertex M by going to vertex I and then vertex M, as there are no blockers or players in the trail. Then

Player 1 places their blocker on the edge incident to vertex Q and vertex R (see Figure 2.3.4).

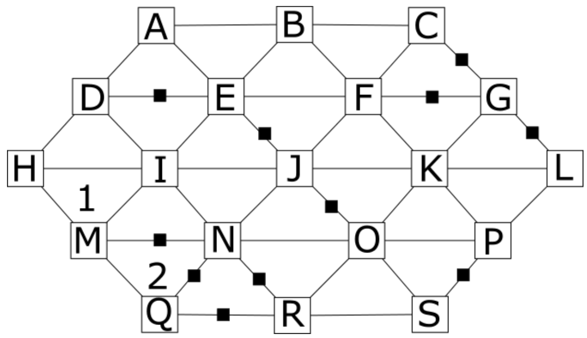

Figure 2.3.4. Example Game Player 1 Turn 2

It is now Player 2's second turn. There is only one edge incident to vertex Q that isn't blocked by a blocker (which is the edge also incident to vertex M), but Player 1 is currently positioned on vertex M so Player 2 cannot use any trails that use vertex M. Therefore Player 2 has no paths to any other vertex from their current vertex and has lost the game, and Player 1 has won the game.

3

Nowhere To Go with Standard Board

3.1 Introduction

We will now discuss a winning strategy for the standard Nowhere To Go Board, in which Player 1 always wins by using 2 of their initial blockers. We will first use a few lemmas to make the strategy and understanding the proof easier.

3.2 General Lemmas

The following are general lemmas that will be used throughout the entire project.

Lemma 3.2.1. *Let G be a graph. Let H be a subgraph of G such that due to the placement of blockers there are no a, b-walks for all $a \in V(G-H)$ and for all $b \in V(H)$. Let $v \in V(H)$. Suppose all of the edges contained in the linkset of v have blockers placed on them, Player A is located on a vertex in $V(H)$, Player B is on a vertex in $V(G-H)$, and Player B can complete two turns regardless of where Player A places blockers, Then Player B can win.*

Proof. If Player A is located on a vertex $a \in V(H - \{v\})$ then that vertex has $deg(a) = 1$ therefore on Player B's first turn Player B can place a blocker on the edge incident to a

and then Player B has won the game. If Player A is located on v then on Player B's first turn Player B places a blocker on any edge incident to v. If there are more than two edges incident to v then Player A has a path to a vertex a that is only adjacent to v. Then on Player B's second turn Player B places a blocker on the edge incident to a and Player B has won the game. □

Lemma 3.2.2. *Let G be a graph. Let H be a subgraph of G such that due to the placement of blockers there are no a, b-walks for all $a \in V(G - H)$ and for all $v \in V(H)$. Let $u \in V(G-H)$ .Let $v \in V(H)$ and let n be the number of edges in the linkset of v. Suppose Player A is on a vertex in $V(H)$, Player B is on a vertex in $V(G-H)$, and Player B can make complete $n+2$ turns regardless of where Player A places blockers then Player B can win.*

Proof. Suppose that Player B can complete $n + 2$ turns regardless of where Player A places blockers. Player B performs n of their turns and for each turn Player B blocks an edge in the linkset of v. Player B can then still perform two more turns when there are no edges in the linkset of v that are not blocked by edges then by Lemma 3.2.1 Player B can win the game. □

Lemma 3.2.3. *Let G be a graph. Let u and v be vertices in $V(G)$. If there are at least n disjoint u, v-trailsets, the n disjoint u, v-trailsets cannot be blocked with $n - 1$ or fewer blockers.*

Proof. This proof is trivial. □

3.3 Player 1's Winning Strategy with 2 Initial Blockers

Theorem 3.3.1. *On NTG, Player 1 has a winning strategy using 2 initial blockers.*

Proof. During the setup phase of the game Player 1 places blockers on the edges CG and FG (see Figure 3.3.1) .

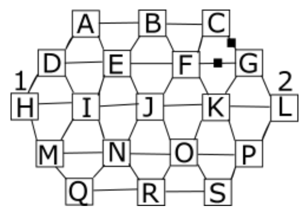

Figure 3.3.1.

Case 1: Suppose there is a trail from vertex H to vertex K that is not blocked on Player 1's first turn (see Figure 3.3.2). Player 1 then takes a trail from vertex H to vertex

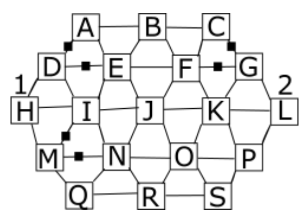

Figure 3.3.2.

K and then places a blocker on the edge LP (see Figure 3.3.3). Player 2 can then only

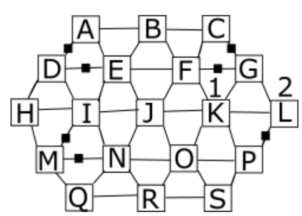

Figure 3.3.3.

take a trail from vertex L to vertex G. Player 2 takes this trail and then Player 2 places a blocker anywhere on the board.

Subcase 1.1: Suppose Player 1 has a trail from vertex K to vertex L (see Figure 3.3.4). Player 1 uses that trail from vertex K to vertex L and then places a blocker on the edge KL (see Figure 3.3.5). Player 2 then has no other trails and has lost the game, therefore Player 1 has won the game.

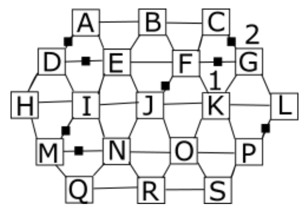

Figure 3.3.4.

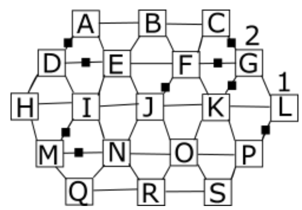

Figure 3.3.5.

Subcase 1.2: Suppose Player 1 does not have a trail from vertex K to vertex L, then Player 2 has placed a blocker on the edge KL after their first turn. Since Player 1 had a H, K-trail on their first turn and Player 2 has not used a blocker to block that K, H-trail then Player 1 has a trail back to their starting vertex H from vertex K. Player 1 uses this K, H-trail and then places a blocker on the edge KG. Player 2 then only has a trail from vertex G to vertex L . Player 2 uses this G, L-trail (see Figure 3.3.6) and then has one blocker Player 2 can place on the board. By the game rules a player is not allowed

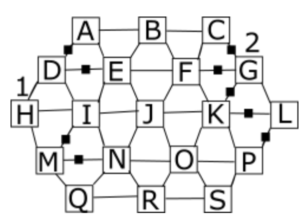

Figure 3.3.6.

to place an initial blocker on the edges incident to the starting vertices, therefore before Player 2 places their blocker there are at least two vertices adjacent to vertex H that are not blocked (see Figure 3.3.7). On Player 1's turn Player 1 then take a trail from vertex

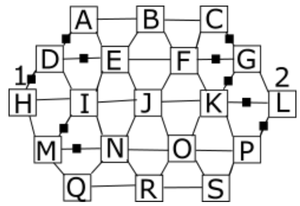

Figure 3.3.7.

H to whichever vertex adjacent to vertex H that Player 2 has not blocked. Then Player 1 can place a blocker on the edge GL and has won the game (see Figure 3.3.8).

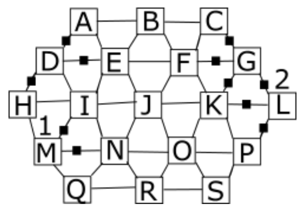

Figure 3.3.8.

Case 2: Suppose there is not a trail from vertex H to vertex K. For this to happen the edges FK, JK, OK, PK must be blocked by Player 2 as blocking these edges prevents Player 1 from using a trail from vertex H to vertex K either because the edges incident to vertex K are either have blockers placed on them or Player 2 is blocking one of the endpoints with their position (see Figure 3.3.9). Player 1 then takes a trail from vertex

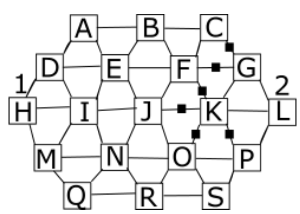

Figure 3.3.9.

H to vertex I and places a blocker on the edge PL (see Figure 3.3.10). Player 2 is now stuck in a subgraph that contains the vertices K, L and G, with only one edge in the linkset of L. Therefore by Lemma 3.2.2 if Player 1 can complete 3 more turns regardless of where Player 2 places blockers Player 1 can win. Player 2 then takes their turn and

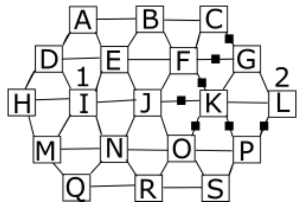

Figure 3.3.10.

places a blocker in the graph. Player 1 then takes a trail from vertex I to vertex J, either through the trailset $\{IN, NJ\}$ or $\{IJ\}$ (see Figure 3.3.11). Since Player 2 has only placed

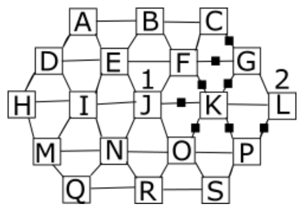

Figure 3.3.11.

one blocker outside of what Player 2 used during the setup phase of the game then by Lemma 3.2.3 Player 1 can use one of the $\{IN, NJ\}$ or $\{IJ\}$ trailsets. Player 2 then takes their turn and places a blocker. Player 1 then has a trail from vertex J to vertex E one of the following trailsets $\{JE\}$, $\{JF, FE\}$, or the same I, J-trailset that Player 1 used as their previous trailset and then using the edge IE (see Figure 3.3.12). Since Player 2 has

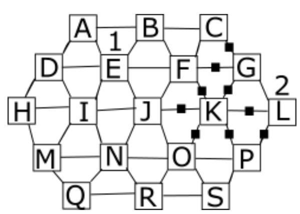

Figure 3.3.12.

two blockers outside of what Player 2 used during the setup phase of the game then by Lemma 3.2.3 Player 1 can use the $\{JE\}$, $\{JF, FE\}$, or the same I, J-trailset that Player 1 used as their previous trailset and then using the edge IE trailsets. Player 2 then takes their turn and places a blocker. Player 1 then has a trail from vertex E to vertex N one of the following trailsets $\{EI, IN\}$, $\{ED, DH, DM, MN\}$, $\{EJ, JN\}$, or $\{EF, FJ, JO, ON\}$ (see Figure 3.3.13). Since Player 2 has at most three blockers outside of what Player

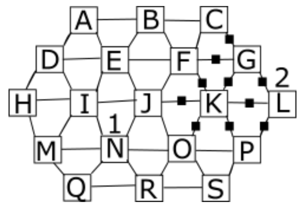

Figure 3.3.13.

2 used during the setup phase of the game then by Lemma 3.2.3 Player 1 can use one of the $\{EI, IN\}$, $\{ED, DH, DM, MN\}$, $\{EJ, JN\}$, or $\{EF, FJ, JO, ON\}$ trailsets. Player 1 has then completed three turns then by Lemma 3.2.2 can win the game. □

As we move on to looking at other graphs that can be used as boards and showing where Player 1 still has a winning strategy because Player 1 goes first it is important to note that none of thegraphs we will be discussing for the rest of the project will include NTG, which is why standard NTG graph received it's own separate proof.

4

Other Graphs as Boards

4.1 Introduction

We can look at the graph for the board game as three subgraphs. The first subgraph includes Player 1's starting vertex and all of the vertices adjacent to Player 1's starting vertex, and all of the edges whose endpoints are either the starting vertex or one of it's adjacent vertices. The second subgraph includes Player 2's starting vertex and all of the vertices adjacent to Player 2's starting vertex, and all of the edges whose endpoints are either the starting vertex or one of it's adjacent vertices. The third subgraph you contains by the rest of the vertices in the graph not included in the first subgraph. Just like in the default board: the two starting vertices are not chosen by the players, the two subgraphs that contain the players starting positions are separated from being adjacent to each other by the middle subgraph. We are also dealing with planar graphs, as we are looking at things that can phyiscally be 2 dimensional boards.

We are trying to find more graphs in which Player 1 has a winning strategy due to their advantage of being able to move first. Since by design of a "fair game" the standard board game is symmetrical, and we will assume all of the following graphs will also be symmetrical. This is to ensure that Player 1's winning strategy is reliant on the advantage of going first, not because the board is weighed in their favor from the beginning. We will need some more definitions before moving ahead.

4.2 More Definitions

Definition 4.2.1. Let G be a graph, let $v \in V(G)$. The **star of** v, denoted $s(v)$ is the subgraph that contains v and all of the vertices adjacent to v, and all of the edges whose endpoints are either v or the vertices adjacent to v. △

When we say "Player A's star" we are not refering to the star of the vertex the player is currently on but are actually refering to the star of the vertex that is their predetermined starting vertex.

Definition 4.2.2. Let G be a graph with starting vertices $v \in V(G)$ and $w \in V(G)$. We say G is a **game graph** if the following hold:

1. The graph G is isomorphic if the vertices v and w are swapped.

2. The subgraphs $s(v)$ and $s(w)$ are disjoint.

△

Note that NTG is a game graph with starting vertex H and vertex L for Player 1 and Player 2 repsectively.

Definition 4.2.3. Let G be a game graph and let $v \in V(G)$. Let $e \in E(G)$. The edge e is a **bridge** of $s(v)$ if one of the endpoints of e is contained in $V(G - s(v))$ and the other endpoint is contained in $V(s(v))$. △

Definition 4.2.4. Let G be a game graph and let $v \in V(G)$. Let $u \in V(s(v))$. The vertex u is an **exit** of $s(v)$ if u is incident to a bridge of $s(v)$. △

For the following figures in which we are only showing vertcies within the star of a vertex assume that the edges that are missing an endpoint are bridges.

4.3 Types of Vertices in The Link Of A Starting Vertex

When looking at the vertices contained inside of the link of a starting vertex there are three attributes that can be looked at to categorize the vertices into types. The first attribute is the number of edges incident to other vertices contained in the link of the vertex. The second attribute is how many bridges that vertex is incident to. The third is how many exits are adjacent to the vertex. Note that you can only have at most 2 adjacent exits because the board planar. Also note that the number of incident edges a vertex has to other vertices in the star will always be greater than or equal to the number of adjacent exits a vertex. Let G be a game graph with a as one of the starting vertices and let $v \in link(a)$. The vertex v is a **Type A** vertex if it has 0 incident bridges, 0 adjacent exits, and 0 incident edges with endpoints within $link(a)$ (see Figure 4.3.1).

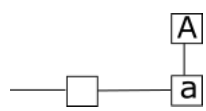

Figure 4.3.1. Type A

The vertex v is a **Type B** vertex if it has 0 incident bridges, 0 adjacent exits, and 1 incident edges with endpoints within $link(a)$ (see Figure 4.3.2).

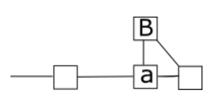

Figure 4.3.2. Type B

The vertex v is a **Type C** vertex if it has 0 incident bridges, 0 adjacent exits, and at least 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.3).

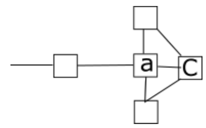

Figure 4.3.3. Type C

The vertex v is a **Type D** vertex if it has 0 incident bridges, 1 adjacent exits, and 1 incident edges with endpoints within $link(a)$ (see Figure 4.3.4).

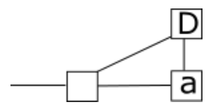

Figure 4.3.4. Type D

The vertex v is a **Type E** vertex if it has 0 incident bridges, 2 adjacent exits, and 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.5).

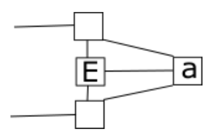

Figure 4.3.5. Type E

The vertex v is a **Type F** vertex if it has 0 incident bridges, 1 adjacent exits, and 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.6).

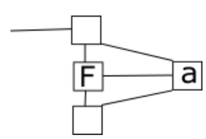

Figure 4.3.6. Type F

The vertex v is a **Type G** vertex if it has 0 incident bridges, greater than 2 adjacent exits, and at least 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.7).

The vertex v is a **Type H** vertex if it has at least 1 incident bridge, 0 adjacent exits, and 0 incident edges with endpoints within $link(a)$ (see Figure 4.3.8).

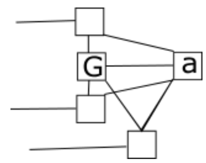

Figure 4.3.7. Type G

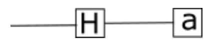

Figure 4.3.8. Type H

The vertex v is a **Type I** vertex if it has at least 1 incident bridge, 0 adjacent exits, and 1 incident edge with endpoints within $link(a)$ (see Figure 4.3.9).

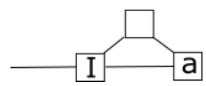

Figure 4.3.9. Type I

The vertex v is a **Type J** vertex if it has at least 1 incident bridge, 0 adjacent exits, and 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.10).

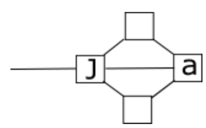

Figure 4.3.10. Type J

The vertex v is a **Type K** vertex if it has at least 1 incident bridge, 1 adjacent exits, and 1 incident edge with endpoints within $link(a)$ (see Figure 4.3.11).

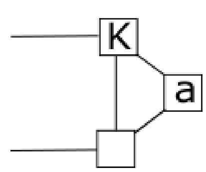

Figure 4.3.11. Type K

The vertex v is a **Type L** vertex if it has at least 1 incident bridge, 2 adjacent exits, and at least 3 incident edges with endpoints within $link(a)$ (see Figure 4.3.12).

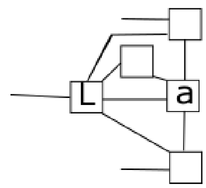

Figure 4.3.12. Type L

The vertex v is a **Type M** vertex if it has at least 1 incident bridge, 2 adjacent exits, and 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.13).

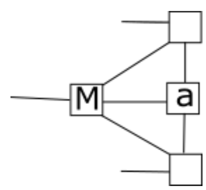

Figure 4.3.13. Type M

The vertex v is a **Type N** vertex if it has at least 1 incident bridge, 1 adjacent exits, and 2 incident edges with endpoints within $link(a)$ (see Figure 4.3.14).

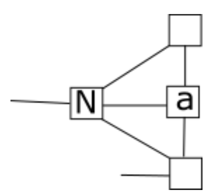

Figure 4.3.14. Type N

The vertex v is a **Type O** vertex if it has at least 1 incident bridge, at least 3 adjacent exits, and at least 3 incident edges with endpoints within $link(a)$ (see Figure 4.3.15).

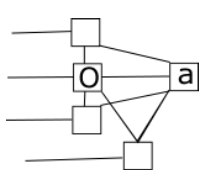

Figure 4.3.15. Type O

The vertex v is a **Type P** vertex if it has at least 1 incident bridge, 1 adjacent exits, and at least 3 incident edges with endpoints within $link(a)$ (see Figure 4.3.16).

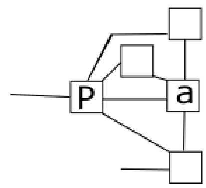

Figure 4.3.16. Type P

The following table will list all of the types of vertices contained within the link of a (see Figure 4.3.17).

Type of Verices within the link of a			
Vertex Type	Number of Incident Bridges (n where $n \geq 1$)	Number of Adjacent Exits (n where $n \geq 3$)	Number of Incident Edges with endpoints within the link of a(n where $n \geq 3$)
A	0	0	0
B	0	0	1
C	0	0	n
D	0	1	1
E	0	2	2
F	0	1	2
G	0	n	n
H	n	0	0
I	n	0	1
J	n	0	2
K	n	1	1
L	n	2	n
M	n	2	2
N	n	1	2
O	n	n	n
P	n	1	n

Figure 4.3.17.

Definition 4.3.1. Let G be a game graph. Let $w \in V(G)$ be a starting vertex. Let $v \in link(w)$. The vertex v is a **internal vertex** if v is a vertex type A, B, or C. $\triangle$

Definition 4.3.2. Let G be a game graph with starting vertices a and b. Let $v \in V(s(a))$. The vertex v is a **exit** if v is a vertex type H, I, J, K, L, M, N, P, or Q then the vertex v is an **exit**. Let v, x, and y be exits in $V(s(a))$ where n, m, and o equal the number corresponding number of bridges. If $n \geq m$ for all $x \in V(s(a))$ then v is the **main exit**. If $n \geq m \geq o$ for all $y \in link(v)$ then x is a **secondary exit**. $\triangle$

Note that for a star to be connected to the rest of a game graph there must exit a main exit.

5
Winning Strategies on Other Game Graphs

5.1 Introduction

In on the standard NTG graph the strategy that Player 1 uses is a two-part strategy that relies on keeping Player 2 in the star of Player 2's starting vertex. During the setup phase Player 1 wants to find which exit a the main exit of Player 2's star. If Player 1 could block Player 2 from leaving Player 2's star by just using their initial blockers, then since the board is symmetrical Player 2 could also block Player 1 from leaving Player 1's star by just using their initial blockers. Player 1 wants their first move to be taking a trail from their starting vertex to the main exit Player 1 finds in Player 2's star so that Player 2 cannot leave the star of Player 2's starting vertex given that Player 1 does not have enough initial blockers to stop Player 2. If there are other exits in Player 2's star Player 1 wants to make sure Player 2 cannot leave Player 2's star which is done with a mix of either blocking the bridges incident those other exits in Player 2's star have or blocking the trails Player 2 has to those exits. From this point onward Player 1 wants to force Player 2 into vertices with fewer and fewer trails without having to leave Player 2's star. This first part of the method is calling trapping, as by doing this Player 1 traps Player 2 in their star and then consequently onto vertices with fewer and fewer options using their own piece to remove paths Player 2 has. For some graphs it is possible for Player 1 to win with just trapping. In the event that Player 1 interacts with vertices not within Player 2's star for any reason Player 1 then goes into the second part of their winning strategy, deletion. After not being in Player 2's star, Player 1's goal is to keep Player 2 contained within the star of Player 2's starting vertex and delete all of the edges in the linkset of Player 2's starting position. After Player 1 has made enough moves to delete all of the edges Player 1 can win in two turns depending by Lemma 3.2.1. For some graphs it is possbile for Player 1 to with solely using deletion. The main idea of the strategy works under the assumption that the reason Player 1 has an advantage because Player 1 moves first and not because the graph is skewed in their favor. Player 1 tries and use this advantage to then force Player 2 to vertices with fewer and fewer trails.

This section will go over some graphs in which Player 1 has a winning strategy on game graphs other than the standard Nowhere To Go board. We will start by looking at graphs in which Player 1 can win solely by using the trapping aspect of the winning strategy, as in neither Player 1 or Player 2 leave Player 2's star. We will then look into graphs in which Player 1 has a winning strategy after being forced out of Player 2's strategy and having to to continue into the deletion method, or scenarios in which Player 2 uses their blockers to keep Player 1 out of their star. We will then look into graphs where Player 1 wins with just deletion.

5.2 Winning Strategy with Just Trapping

In this section we will look at graphs in which Player 1 has a winning strategy using only the trapping portion of the winning strategy. This requires Player 1's first turn to be a trail from their starting vertex to the main exit in Player 2's start, and then from that turn onward neither Player 1 nor Player 2 leaves Player 2's star.

Definition 5.2.1. Let G be a game graph with starting vertices a and b for Player 1 and Player 2 repsectively and let $v \in V(s(b))$ such that v a main exit of $s(b)$, let m be the number of bridges incident to v, and let n be the total number of bridges incident to vertices in $V(s(b) - \{v\})$. The graph G is a **type 1 game graph** if $m, n > 4$, there are at least 5 disjoint a, v-trailsets and G satisfies one of the following conditions:

1. The vertex v is a type H vertex and there are no vertices in $V(s(b) - \{v, b\})$ (see Figure 5.2.1).

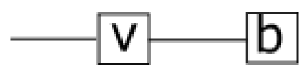

Figure 5.2.1.

2. The vertex v is a type I vertex is the only exit in $s(b)$ the there are any number of type A, type B, or type C vertices within $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.2).

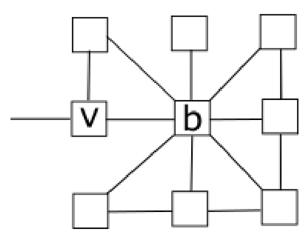

Figure 5.2.2.

3. The vertex v is a type J vertex and one of the following hold:

   (a) The vertex v is the only exit in $s(b)$, the number of internal vertices adjacent to v is greater than 2, and there are any number of type A, type B, or type C vertices in $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.3).

   (b) There are only 2 exits in $s(b)$ and one of the following hold:

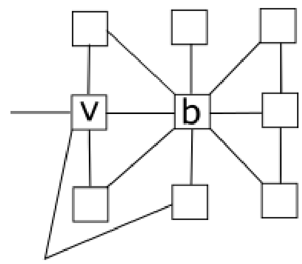

Figure 5.2.3.

i. The secondary exit is a type H (see Figure 5.2.4) or type I (see Figure 5.2.5).

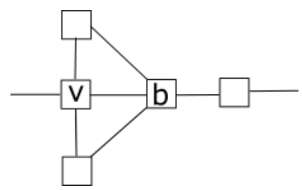

Figure 5.2.4.

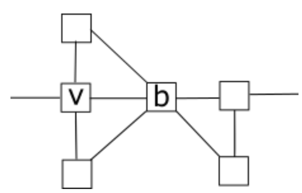

Figure 5.2.5.

ii. The secondary exit is a type J and the number of adjacent internal vertices is less than 5 (see Figure 5.2.6).

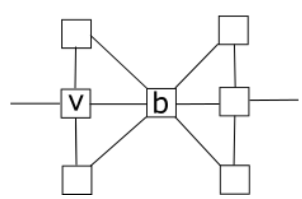

Figure 5.2.6.

(c) There are only 3 exits in $s(b)$ and one of the following hold:

i. A secondary exit in $s(b)$ is a Type H vertex and if the third exit in $s(b)$ is a type H vertex (see Figure 5.2.7) or type I vertex (see Figure 5.2.8).

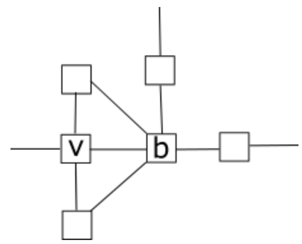

Figure 5.2.7.

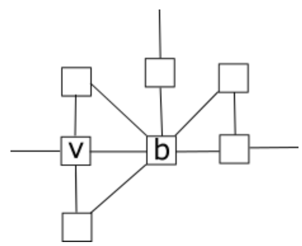

Figure 5.2.8.

ii. A secondary exit in $s(b)$ is a Type H vertex and if the third exit in $s(b)$ is a type J vertex with less than 5 incident bridges (see Figure 5.2.9).

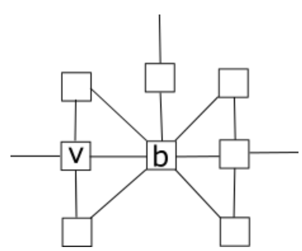

Figure 5.2.9.

iii. The secondary exit in $s(b)$ is a type I vertex and one of the following hold:

A. The third exit in $s(b)$ is a type H with less than 4 incident bridges (see Figure 5.2.10).

B. The third exit in $s(b)$ is a type I with less than 4 incident bridges (see Figure 5.2.11).

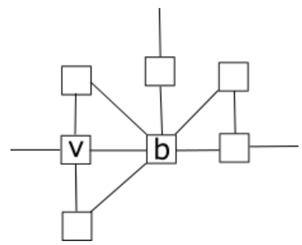

Figure 5.2.10.

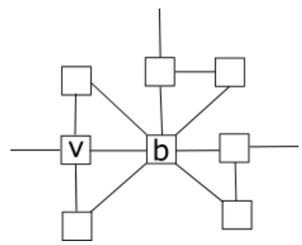

Figure 5.2.11.

C. The third exit in $s(b)$ is a type J with 1 incident bridge and is only adjacent to type D vertices(see Figure 5.2.12).

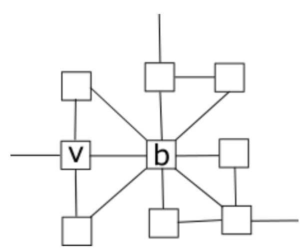

Figure 5.2.12.

iv. Both the secondary exit in $s(b)$ and third exit in $s(b)$ are type K vertices, and the third exit has less than 4 incident bridges (see Figure 5.2.13).

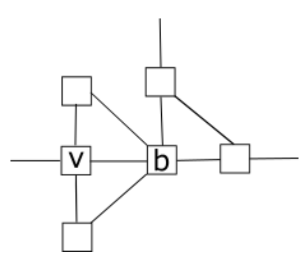

Figure 5.2.13.

4. The vertex v is a type K vertex and one of the following hold:

   (a) There are 2 exits in $s(b)$ and there are any number of type A, type B, or type C vertices in $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.14).

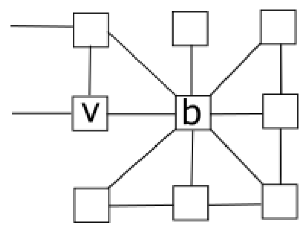

Figure 5.2.14.

   (b) There are 3 exits in $s(b)$ and if a secondary exit in $s(b)$ is a type K and the third exit in $s(b)$ is a type H vertex (see Figure 5.2.15) or type I vertex with one adjacent type D vertex and with less than 5 incident bridges (see Figure 5.2.16).

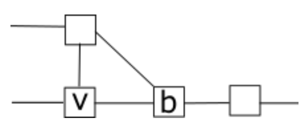

Figure 5.2.15.

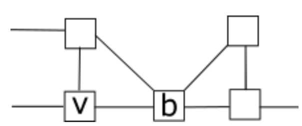

Figure 5.2.16.

5. The vertex v is a type M vertex and the number of bridges incident to the exit in $s(b)$ that isn't a secondary exit in $s(b)$ is less than 4 (see Figure 5.2.17).

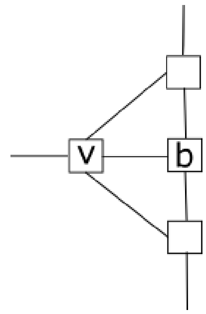

Figure 5.2.17.

6. The vertex v is a type P vertex and the number of bridges incident to the exit in $s(b)$ that isn't a secondary exit is less than 4 and the only adjacent internal vertices to v are type D vertices (see Figure 5.2.18).

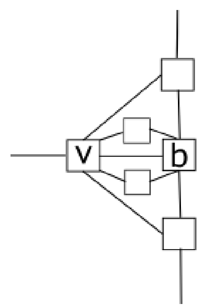

Figure 5.2.18.

$\triangle$

Theorem 5.2.2. *Let G be a type 1 game graph. Then Player 1 has a winning strategy.*

Proof. Suppose G has starting vertices a and b for Player 1 and Player 2 respectively. Let $v \in s(b)$ such that v is a main exit in $s(b)$.

Case 1: Suppose v is a type H vertex and there are no other vertices in $V(s(b) - \{b\})$ (see Figure 5.2.1). On Player 1's first turn Player 1 takes a a, v-trail. On Player 2's first turn since v is the only other vertex in $V(s(b))$ and Player 1 is currently positioned on v, Player 2 does not have any trails to any other vertices and has lost the game and Player 1 has won (see Figure 5.2.19).

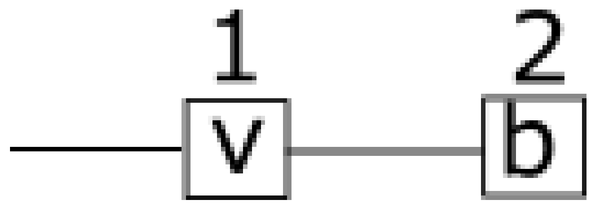

Figure 5.2.19.

Case 2: Suppose v is a type I vertex and is only one exit in $V(s(b))$ and there are any number of type A, type B, or type C vertices in $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.2).During the setup phase of the game Player 1 places their initial blockers on 4 of the edges in the linkset of b, specifically the edges connecting the type B and type C vertices (see Figure 5.2.20). On Player 1's first turn

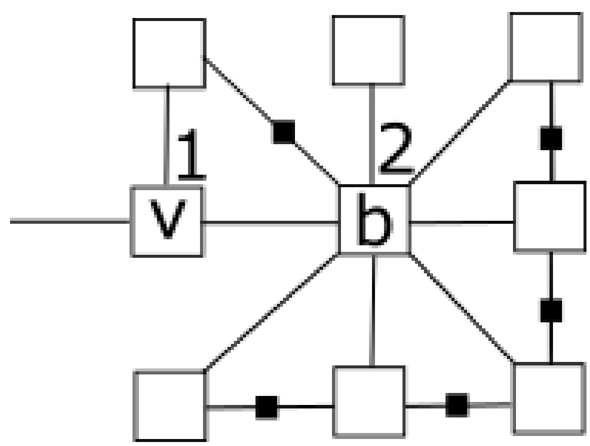

Figure 5.2.20.

Player 1 takes take a a, v-trail and place their first blocker on the edge incident to b and the vertex adjacent to v that is contained in $V(s(b) - \{b, v\})$. If there are no vertices in the Player 2's star other than v and the vertex adjacent to both v and b then Player 2 has lost the game and Player 1 has won. If there are type A vertices in the $s(b)$ and Player 2 moves to a type A vertex on their first turn, Player 1 then has a path either to b or the vertex adjacent to both b and v, and can then block the edge incident to Player 2's vertex and has won the game. If there are type B or C vertices in $s(b)$ and Player 2 moves to a type B vertex or type C vertex on their first turn, Player 1 then has a path to either b or the vertex adjacent to both b and v. Since Player 1 used their initial blockers to so that the degree of the type B and type C vertices on the first turn is 1, then Player 1 can place their blocker on the edge incident to Player 2's position and Player 1 has won the game.

Case 3: Suppose v is a type J vertex.

Subcase 3.1: Suppose v is only one exit in $s(b)$, the number of internal vertices adjacent to v is greater than 2, and there are any number of type A, type B, or type C vertices in $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.3). Player 1 places their initial blockers on the edges incident to the type B and type C vertices if there are any on the setup phase of the game. On Player 1's first turn Player 1 takes a a, v-trail and block another edge in the linkset of type B's and type C (see Figure 5.2.21). If Player 2 moves to a type A vertex in their star, on Player 1's next turn Player

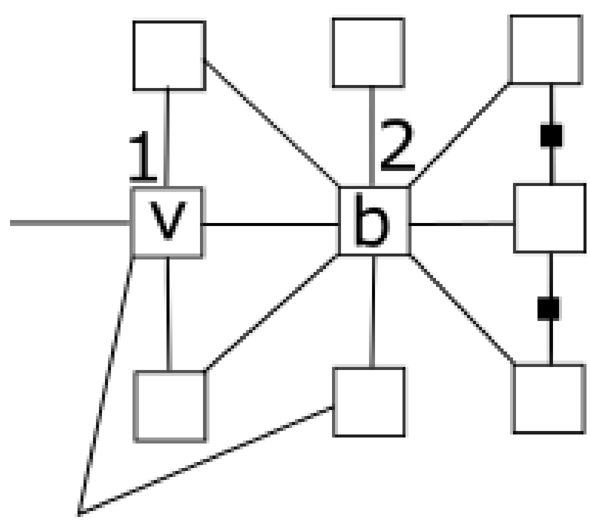

Figure 5.2.21.

1 has a trail from v to at least one of the internal vertices adjacent to v and can place a blocker incident to Player 2's vertex and thus has won the game. If Player 2 moves to a type B or type C vertex in their $s(b)$, on Player 1's next turn Player 1 has a trail from v

to at least one of the internal vertices adjacent to v and can place a blocker on the edge incident to Player 2's vertex and Player 1 has won.

Subcase 3.2: Suppose there is a secondary exit that is a type H vertex (see Figure 5.2.4). On Player 1's first turn Player 1 takes a a, v-trail and block the edge incident to b and and the secondary exit (see Figure 5.2.22). Player 2 then has to move to one of the

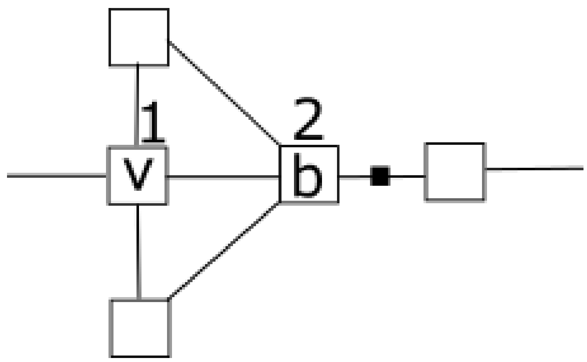

Figure 5.2.22.

internal vertices adjacent to v. On Player 1's next turn Player 1 has a v, b-trail, either directly from v to b or by going through one of the internal vertices adjacent to v (see Figure 5.2.22). Then Player 1 places a blocker on the edge incident to v and Player 2's current vertex and Player 1 has won.

Subcase 3.3: Suppose there is a secondary exit that is type I vertex (see Figure 5.2.5). During the setup phase of the game Player 1 places an initial blocker on the edge incident to the secondary exit and it's adjacent internal exit. On Player 1's first turn Player 1 takes a a, v-trail and blocks the edge incident to b and and the secondary exit (see Figure 5.2.23). Player 2 then has to move to one of the internal vertices adjacent to v or the internal vertex that was adjacent to the secondary exit. On Player 1's next turn Player 1 has a v, b-trail, either directly from v to the b or by going through one of the internal vertices adjacent to v. If Player 2 moved to a vertex incident to v then Player 1 can place

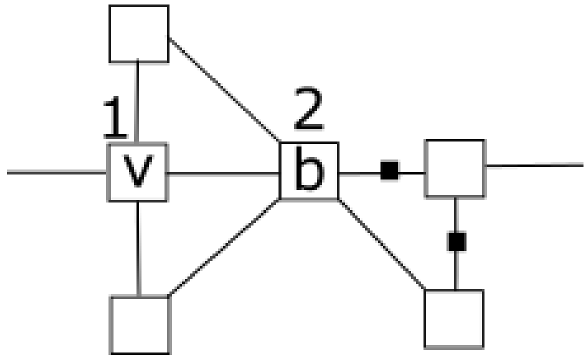

Figure 5.2.23.

a blocker on the edge incident between those two vertices and has won. If Player 2 has moved to the vertex adjacent to the secondary exit, then Player 1 has won.

Subcase 3.4: Suppose there is a secondary exit that is a type J vertex whose number of adjacent internal vertices is less than 5 (see Figure 5.2.6). On the setup phase of the game Player 1 places an initial blocker on all of the edges incident to both the secondary exit and it's adjacent internal vertices. On Player 1's first turn Player 1 takes a a, v-trail and blocks the edge incident to b and and the secondary exit (see Figure 5.2.24). Player 2

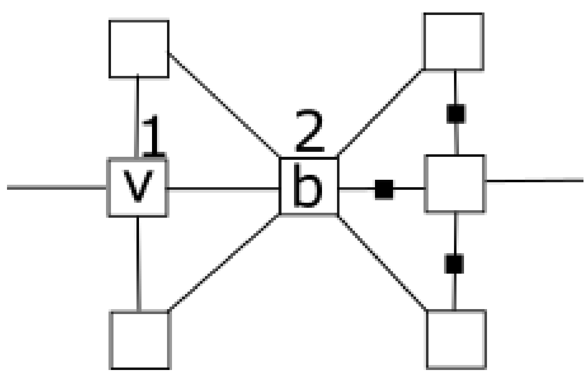

Figure 5.2.24.

then has to move from b to one of the internal vertices adjacent to v. On Player 1's next turn Player 1 have a v, b-trail, either directly from v to the b or by going through one of the internal vertices adjacent to v. If Player 2 moved to a vertex incident to v then Player

1 can place a blocker on the edge incident between those two vertices and has won. If Player 2 has moved to the vertex adjacent to the secondary exit, then Player 1 has won.

Subcase 3.5: Suppose there is a secondary exit that is a type H vertex and a third exit that is type H with less than 5 incident bridges (see Figure 5.2.7). During the setup phase of the game Player 1 places their initial blockers on the bridges incident to the third exit. On Player 1's first turn Player 1 takes a a, v and place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.25). If Player 2 moves to a vertex adjacent to v

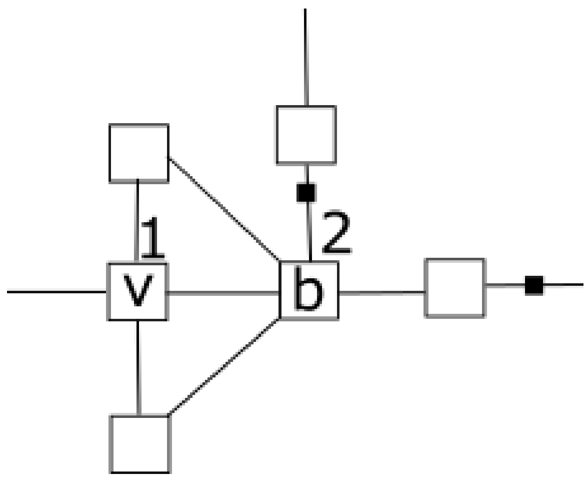

Figure 5.2.25.

then on Player 1's next turn Player 1 has a v, b-trail and can place a blocker incident to v and Player 2's vertex and win. If Player 2 moves to the third exit then Player 1 has still has a v, b-trail and can place a blocker incident to Player 2's vertex and win.

Subcase 3.6: Suppose there is a secondary exit that is a type H vertex and a third exit that is type I with less than 5 incident bridges (see Figure 5.2.8). On the setup phase of the game Player 1 places their initial blockers on the bridges incident to the third exit. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.26). If Player 2 moves to a vertex adjacent to v then on Player 1's next turn Player 1 has a v, b-trail Player 1 can take and can place a

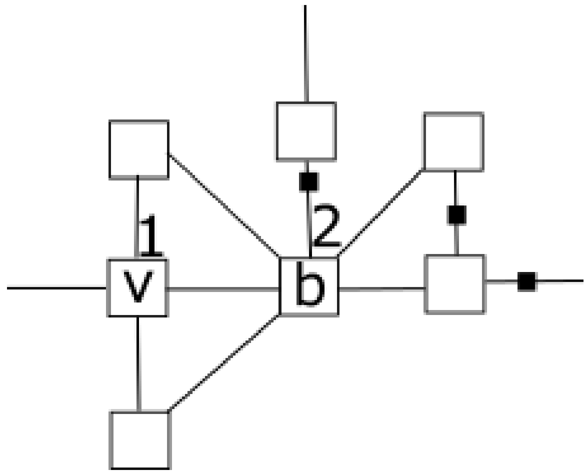

Figure 5.2.26.

blocker incident to v and Player 2's vertex and can won. If Player 2 moves to the third exit or it's adjacent vertex then Player 1 has still has a v, b-trail and can place a blocker on the edge incident to the third exit and it's adjacent internal vertex and has won.

Subcase 3.7: Suppose there is a secondary exit that is a type H vertex and a third exit that is type J vertex with 1 incident bridge (see Figure 5.2.9). During the setup phase of the game Player 1 places their initial blockers on the bridge incident to the third exit. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.27). If Player 2 moves to a vertex adjacent to v then on Player 1's next turn Player 1 has a v, b-trail and can place a blocker incident to v and Player 2's vertex and has won. If Player 2 moves to the third exit or one of it's adjacent vertices then Player 1 has still has a v, b-trail and can place a blocker on the edge incident to the third exit and it's adjacent internal vertex and has won.

Subcase 3.7: Suppose there is a secondary exit that is type I vertex and a third exit that is type H vertex with less than 4 incident bridges (see Figure 5.2.10). During the setup phase of the game Player 1 places their initial blockers on all of the bridges incident to the third exit, and on the edge incident to the secondary exit and it's adjacent internal

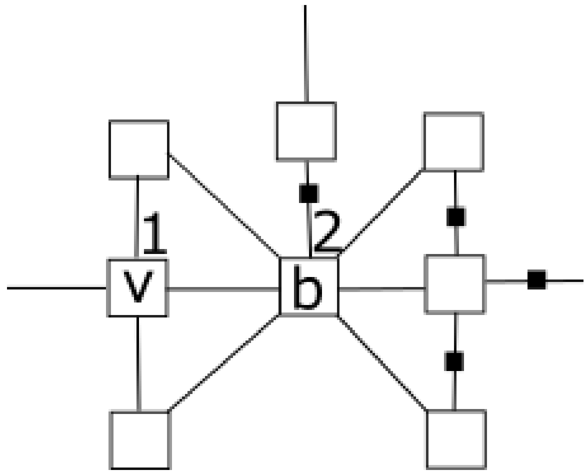

Figure 5.2.27.

vertex. On Player 1's first turn Player 1 takes a a, v-trail and then place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.28). Suppose on Player 2's first

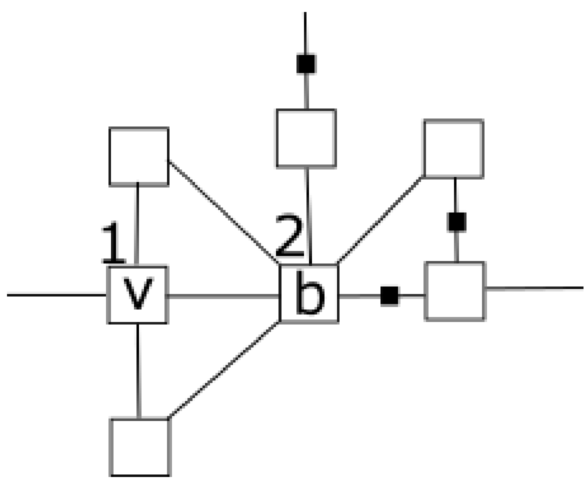

Figure 5.2.28.

turn Player 2 moves to one of the vertices adjacent to v then Player 1 can take a v, b-trail and then place a blocker on the edge incident to v and Player 2's current vertex and has won. Suppose on Player 2's first turn Player 2 takes a trail to the internal vertex adjacent to the secondary exit, then Player 1 has a v, b-trail and has won. Suppose Player 2 takes a trail to the third exit, then Player 1 has a v, b-trail and has won.

Subcase 3.8: Suppose there is a secondary exit that is type I vertex and a third exit that is type I vertex with less than4 incident bridges (see Figure 5.2.11). During the setup phase of the game Player 1 places their initial blockers on all of the bridges incident to the third exit, and on the edge incident to the secondary exit and it's adjacent internal vertex. On Player 1's first turn Player 1 takes a a, v-trail and then place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.29). Suppose if on Player 2's first

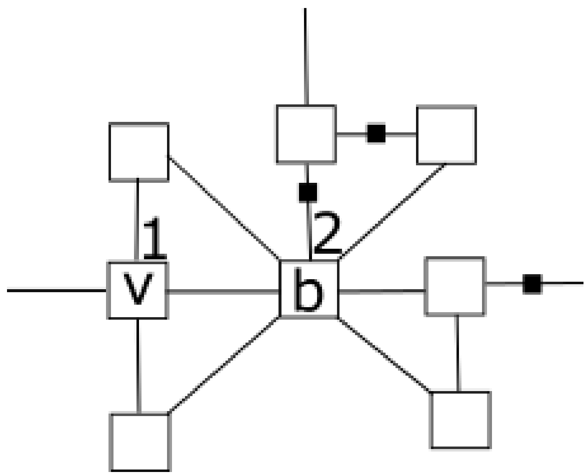

Figure 5.2.29.

turn Player 2 moves to one of the vertices adjacent to v Player 1 can take a v, b-trail and can then place a blocker on the edge incident to v and Player 2's current vertex and has won. Suppose if on Player 2's first turn Player 2 moves to the internal vertex adjacent to the secondary exit, then Player 1 has a v, b-trail and has won. Suppose Player 2 moves to the third exit or it's adjacent vertex, then Player 1 has a v, b-trail and can place a blocker on the edge incident to the third exit and it's adjacent vertex and has won.

Subcase 3.9: Suppose there is a secondary exit that is type I and a third exit that is type J with 1 incident bridge and that is only adjacent to type D vertices (see Figure 5.2.12). During the setup phase of the game Player 1 places their initial blockers on the bridge incident to the third exit and the edges incident to the type D vertices and the third exit, and on the edge incident to the secondary exit and it's adjacent internal vertex (see Figure 5.2.30). On Player 1's first turn Player 1 takes a a, v-trail and then place a

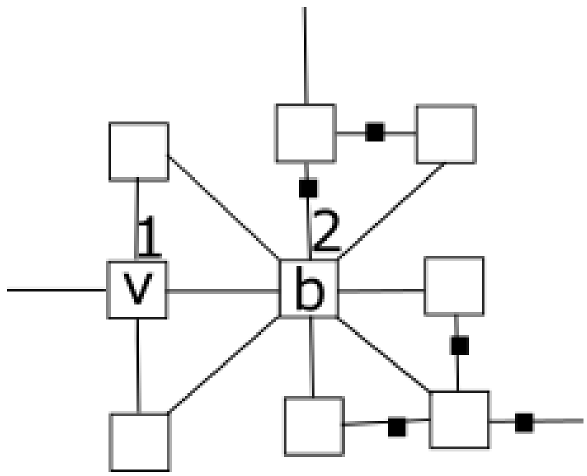

Figure 5.2.30.

blocker on the edge incident to Player 2's starting vertex and the secondary exit. Suppose if on Player 2's first turn Player 2 moves to one of the vertices adjacent to v Player 1 can take a v, b-trail and then place a blocker on the edge incident to v and Player 2's current position. Suppose Player 2's moves to an internal vertex adjacent to the secondary exit, then Player 1 has a v, b-trail Player 1 can take and has won. Suppose Player 2 moves to the third exit, then Player 1 has a v, b-trail Player 1 can take and has won.

Subcase 3.10: Suppose both the second and third exit are type K vertices and the third exit has less than 4 incident bridges (see Figure 5.2.13). During the setup phase of the game Player 1 places blockers on the bridges incident to the third exit and on the edge incident to the two type K vertices. On Player 1's first turn Player 1 takes a a, v-trail and places a blocker on the edge incident to the secondary exit and b (see Figure 5.2.31). Suppose on Player 2's first turn if Player 2 moves to a vertex adjacent to v Player 1 can take a v, b-trail and can then place a blocker on the edge incident to Player 2's current

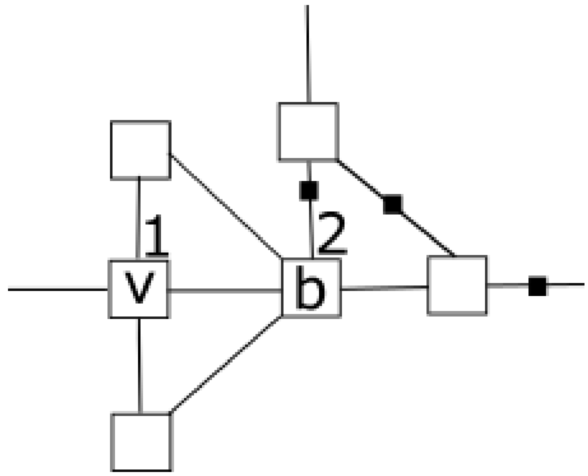

Figure 5.2.31.

vertex and v and has won. Suppose on Player 2 moves to the third exit, then Player 1 can take a v, b-trail and has won.

Case 4: Suppose$v$ is a type K vertex.

Subcase 4.1: Suppose there are 2 exits and there are any number of type A, type B, or type C vertices in $link(b)$ such that the number of edges in the linkset of b is less than 5 (see Figure 5.2.14). During the setup phase Player 1 places their initial blockers on the edges in the linkset of b connecting the type B and C vertices. On Player 1's first turn Player 1 takes a a, v-trrail and block the edge incident to b and the secondary exit (see Figure 5.2.32). If there are no vertices in $V(s(b) - \{v, b\})$ Player 1 has won. If Player 2 moves to a type A, type B, or type C vertex then Player 1 either takes a trail from v to the secondary exit or v, b-trail and has won.

Subcase 4.2: Suppose there is a secondary exit that is a type K vertex and the third exit is a type H with less than 5 incident bridges (see Figure 5.2.15). During the setup phase of the game Player 1 places their blockers on the bridges incident to the third exit. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to the secondary exit and b (see Figure 5.2.33). Player 2's only usable trail is to move to

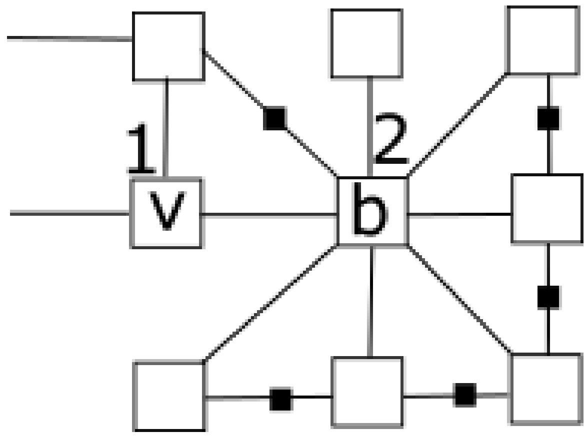

Figure 5.2.32.

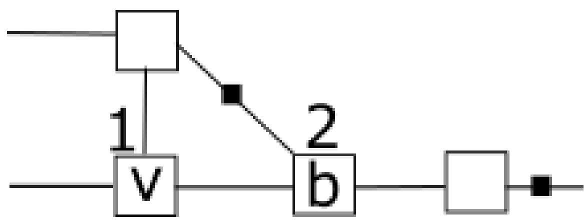

Figure 5.2.33.

the third exit. Player 1 then can take a trail from v to either b or the secondary exit, and can place a blocker on the edge incident to b and the third exit and has won.

Subcase 4.3: Suppose there is a secondary exit that is type K vertex and the third exit is a type I with less than 4 incident bridges and one adjacent type D vertex (see Figure 5.2.16). During the setup phase of the game Player 1 places their blockers on the bridges

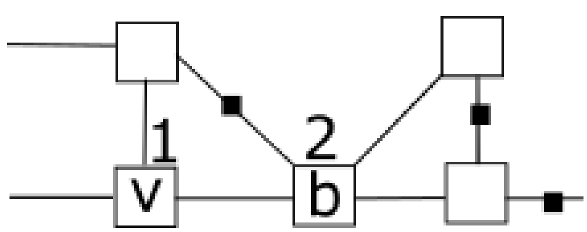

Figure 5.2.34.

incident to the third exit, and the edge incident to the third exit and it's adjacent internal vertex. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to the secondary exit and b (see Figure 5.2.34). Player 2's only trail is from b to the third exit or it's adjacent vertex. Player 1 then takes a trail from v to either b or the secondary exit, and can place a blocker on the edge incident to b and Player 2's current vertex and has won.

Case 5: Suppose v is a type M vertex and the number of bridges incident to the exit that isn't the secondary exit is less than 4 (see Figure 5.2.17). During the setup phase Player 1 places their initial blockers on the bridges incident to the third exit and the edge incident to v and the third exit. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.35). Player

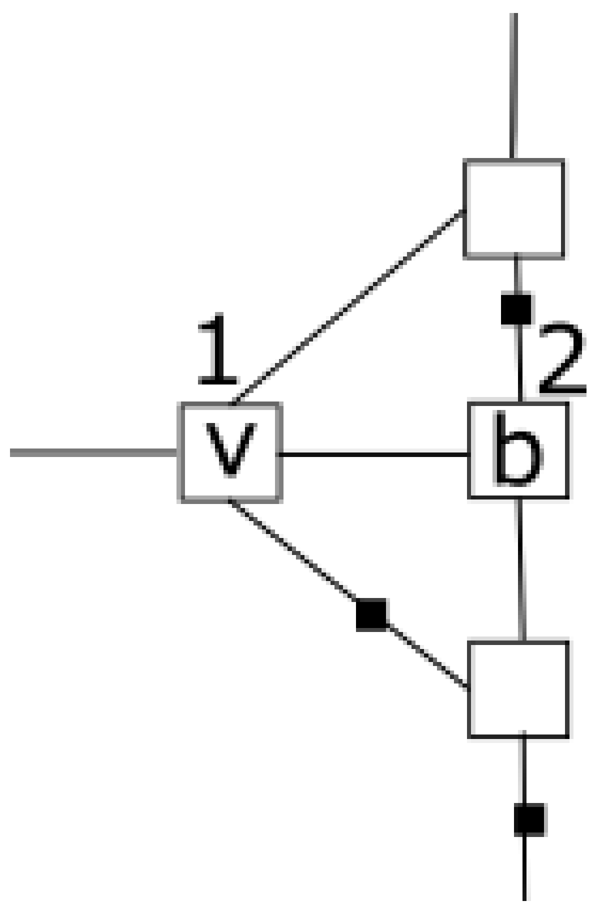

Figure 5.2.35.

2's only trail is from b to the third exit. Player 1 then takes a trail from v to either the secondary exit or b and can place a blocker on the edge incident to b and the third exit and has won.

Case 6: Suppose v is a type P vertex, the number of bridges incident to the exit that isn't the secondary exit is less than 4 and the only adjacent internal vertices to v are type D (see Figure 5.2.18). During the setup phase of the game Player 1 places their blockers on the bridges incident to the third exit. On Player 1's first turn Player 1 takes a a, v-trail and place a blocker on the edge incident to b and the secondary exit (see Figure 5.2.36). If Player 2 moves to a type D vertex Player 1 has a can take a v, b-trail and can place a

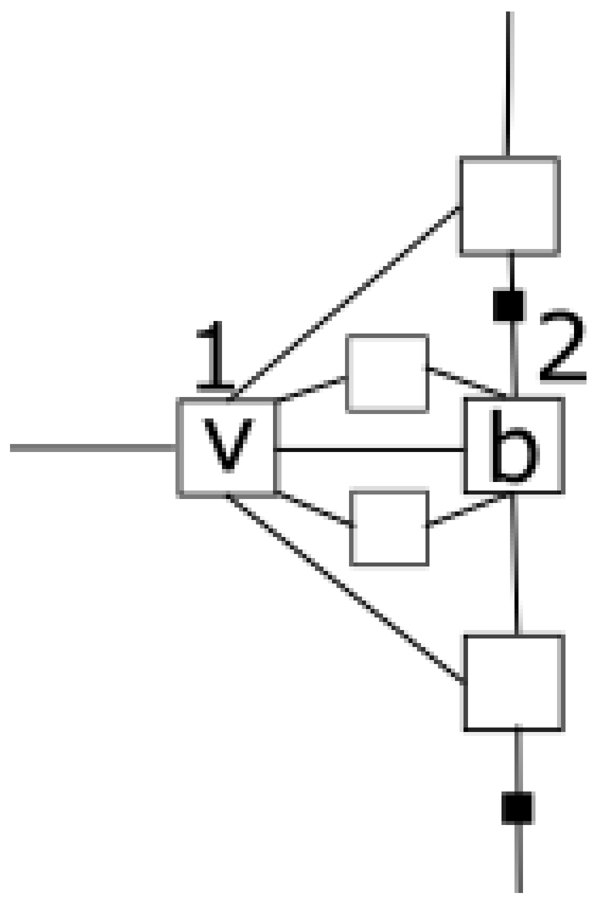

Figure 5.2.36.

blocker on the edge incident to Player 2's vertex and v and has won. If Player 2 moves to the third exit Player 1 can take a v, b-trail and can place a blocker on the edge incident to v and the third exit and has won. $\square$

5.3 Winning Strategy with Using Trapping and Then Deletion

In this section we will discuss graphs in which Player 1 has a winning strategy using trapping and then deletion. On these graphs Player 1 focuses on blocking the bridges of exits in Player 2's star that aren't incident to the main exit in Player 2's star. Player 1 takes a trail from Player 1's starting vertex to the main exit in Player 2's star so that Player 2 cannot leave Player 2's star. Player 1 then closes Player 2 off withins Player 2's star and begins to complete enough turns so that Player 1 can delete enough edges within Player 2's star regardless of where Player 2 places blockers.

Definition 5.3.1. Let G be a game graph with starting vertices x and y for Player 1 and Player 2 respectively, let $v \in V(s(y))$ such that the vertex v is a main exit in $s(y)$, let n be the number of edges in the linkset of y and let m be the total number of bridges incident to vertices contained in $V(s(y) - \{v\})$ and let q be the number of edges incident to v and a vertex contained in $V(s(y) - \{v, y\})$. The graph G is a **type 2 game graph** if there exists $u \in V(G - s(y))$ such that there are at least $n + 6$ disjoint v, u-trailsets, there are at least 5 disjoint x, v-trailsets, and one of the following conditions hold:

1. The vertex v is a type H vertex and $m = 5$ (see Figure 5.3.1).

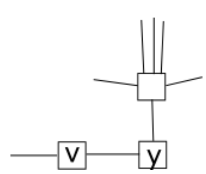

Figure 5.3.1.

2. The vertex v is a type I vertex and $m = 4$ (see Figure 5.3.2).

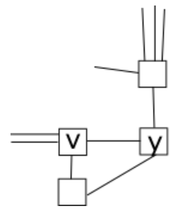

Figure 5.3.2.

3. The vertex v is a type J (see Figure 5.3.3), type K (see Figure 5.3.4), type M (see Figure 5.3.5), or type N (see Figure 5.3.6) and $m = 3$.

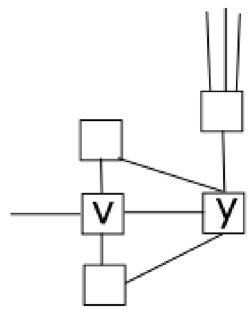

Figure 5.3.3.

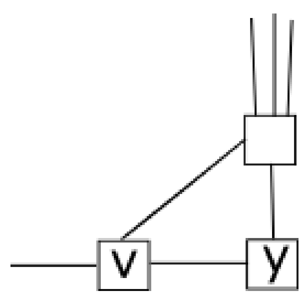

Figure 5.3.4.

4. The vertex v is a type L vertex, $m = 2$, and a total of 5 vertices in $link(y)$ that are adjacent to v (see Figure 5.3.7).

5. The vertex v is a type P vertex and $m + q = 5$ where $m \leq 2$ and $3 \leq q \leq 4$ (see Figure 5.3.8).

$\triangle$

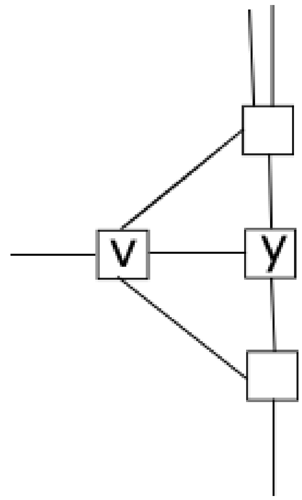

Figure 5.3.5.

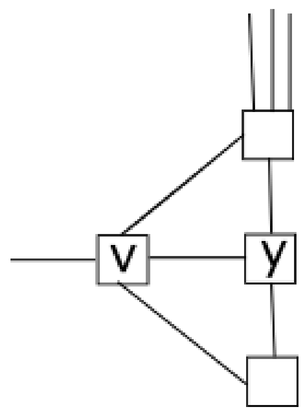

Figure 5.3.6.

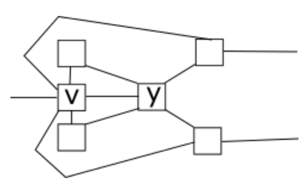

Figure 5.3.7.

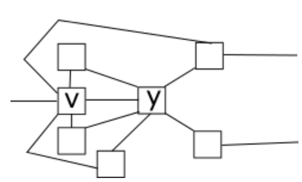

Figure 5.3.8.

Note that the difficulty for Player 1 to identify if a game graph is a type 2 game graph comes from having to find a trail such that there are $n+6$ disjoint trailsets, and doing so would potentially take a person a very unreasonable amount of time to find and in practice the following proof would be difficult to implement.

Theorem 5.3.2. *Let G be a type 2 game graph. Then Player 1 has a winning strategy.*

Proof. Suppose G has starting vertices x and y for Player 1 and Player 2 repsectively. Let $v \in V(s(y))$ such that v is a main exit in $s(y)$ and let n be the number of edges in the linkset of y. Let r be the number of total bridges incident to vertices contained in $V(s(y)-\{v\})$ and let t be the number of edges incident to both v and a vertex contained in $V(s(y)-\{v,y\})$.

Case 1: Suppose v is a type H vertex and $r=5$ (see Figure 5.3.1). Player 1 uses their initial blockers on 4 of the 5 bridges incident to the vertices in $V(s(y)-\{v\})$. On Player 1's first turn Player 1 takes a x,v-trail and place their first blocker on the 5th bridge incident to the exits that are in $V(s(y)-\{v\})$ (see Figure 5.3.9). Player 2 is now

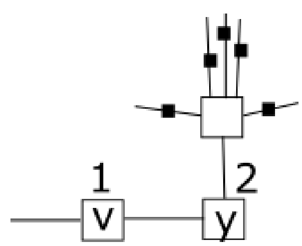

Figure 5.3.9.

trapped inside of $V(s(y))$ and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y))-\{v,y\}$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G-s(y))$ such that there are $n+6$ disjoint v,u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v,u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Case 2: Suppose v is a type I vertex and $r = 4$ (see Figure 5.3.2). Player 1 uses their initial blockers to block the 4 bridges incident to the vertices in $V(s(y))$ that are not v. On Player 1's first turn Player 1 takes a x, v-trail and Player 1 places their blocker on the edge whose endpoints are v and the vertex within $V(s(y)) - \{y\}$ (see Figure 5.3.10). Player

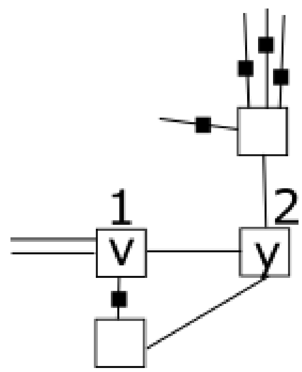

Figure 5.3.10.

2 is now trapped inside of $V(s(y))$ and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y)) - \{v, y\}$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n + 6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n + 6$ disjoint v, u-trailsets. After Player 1 has completed $n + 1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Case 3: Suppose v is a type J, type K, type M, or type N vertex and $r = 3$.

Subcase 3.1: Suppose v is a type J (see Figure 5.3.3). Player 1 places 3 of their initial blockers on the 3 bridges incident to the vertices in $V(s(y) - \{v\})$ that are not v and places their 4th blocker on one of the edge whose endpoints are v and the vertex within

$V(s(y))-\{y\})$, note that there are two of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x,v-trail and block the edge e (see Figure 5.3.11). Player 2 is now trapped inside of $V(s(y))$ and only has trails

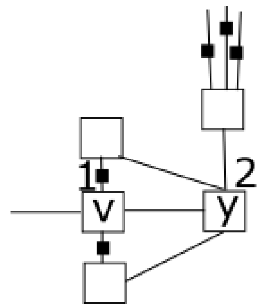

Figure 5.3.11.

to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y))-\{v,y\})$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G-s(y))$ such that there are $n+6$ disjoint v,u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v,u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Subcase 3.2: Suppose v is a type K (see Figure 5.3.4). Player 1 places 3 of their initial blockers on the 3 bridges incident to the vertices in $V(s(y)-\{v\})$ that are not v and places their 4th blocker on one of the edge whose endpoints are v and the vertex within $V(s(y))-\{y\})$, note that there are two of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x,v-trail and block the edge e (see Figure 5.3.12). Player 2 is now trapped inside of $V(s(y))$ and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y))-\{v,y\})$ and Player 2 then places a blocker anywhere on the board.

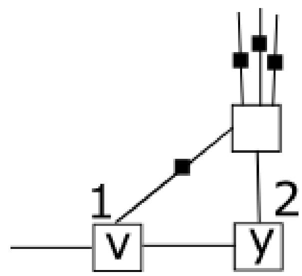

Figure 5.3.12.

Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n+6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v, u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Subcase 3.3: Suppose v is a type M (see Figure 5.3.5). Player 1 places 3 of their initial blockers on the 3 bridges incident to the vertices in $V(s(y) - \{v\})$ that are not v and places their 4th blocker on one of the edge whose endpoints are v and the vertex within $V(s(y)) - \{y\})$, note that there are two of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x, v-trail and block the edge e (see Figure 5.3.13). Player 2 is now trapped inside of $V(s(y))$ and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y)) - \{v, y\})$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n+6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v, u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

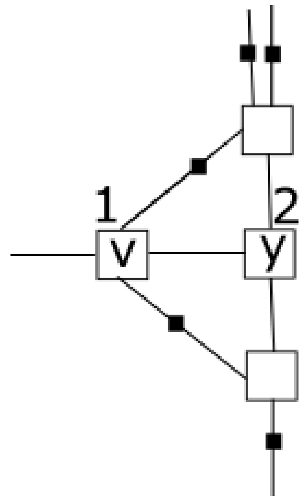

Figure 5.3.13.

Subcase 3.4: Suppose v is a type N (see Figure 5.3.6). Player 1 places 3 of their initial blockers on the 3 bridges incident to the vertices in $V(s(y) - \{v\})$ that are not v and places their 4th blocker on one of the edge whose endpoints are v and the vertex within $V(s(y)) - \{y\})$, note that there are two of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x, v-trail and block the edge e (see Figure 5.3.14). Player 2 is now trapped inside of $V(s(y))$ and only has trails

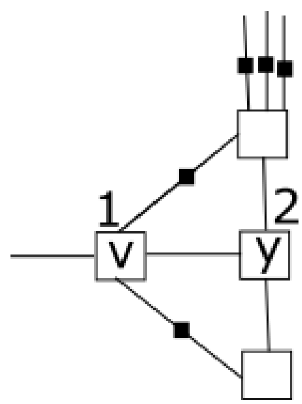

Figure 5.3.14.

to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y)) - \{v, y\})$ and Player 2 then places a blocker anywhere on the board.

Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n+6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v, u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Case 4: Suppose v is a type L vertex, and $r = 2$ incident bridges and there are exactly 5 vertices in $link(y)$ that are adjacent to v (see Figure 5.3.7). Player 1 places 2 of their initial blockers on the 2 bridges incident to the vertices in $V(s(y) - \{v\})$ that are not v and places 2 of their blockers on the edges whose endpoints are v and the vertex within $V(s(y)) - \{y\})$, note that there are three of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x, v-trail and block the edge e (see Figure 5.3.15). Player 2 is now trapped inside of $V(s(y))$ and only has trails

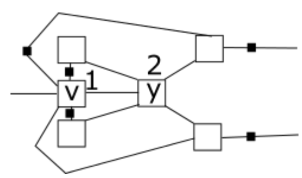

Figure 5.3.15.

to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y)) - \{v, y\})$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n+6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v, u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win.

Case 5: Suppose v is a type P vertex and $r + t = 5$, where $r \leq 2$ and $3 \leq t \leq 4$ (see Figure 5.3.8). Player 1 places r of their initial blockers on the bridges incident to the vertices in $V(s(y) - \{v\})$ and places $t - 1$ of their blockers on the edge whose endpoints are v and the vertex within $V(s(y)) - \{y\})$, note that there are two of the edges and one of them will still be unblocked we will call this edge e. On Player 1's first turn Player 1 takes a x, v-trail and block the edge e (see Figure 5.3.16). Player 2 is now trapped inside of $V(s(y))$

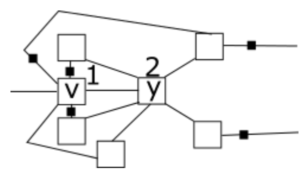

Figure 5.3.16.

and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y)) - \{v, y\})$ and Player 2 then places a blocker anywhere on the board. Since G is a type 2 game graph there exists $u \in V(G - s(y))$ such that there are $n+6$ disjoint v, u-trialsets. Player 1 begins alternating position between the vertices v and u using one of the $n+6$ disjoint v, u-trailsets. After Player 1 has completed $n+1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win. □

5.4 Winning Strategy with Just Deletion

While using only the deletion part of the strategy using a combation of their initial blockers and the blocker obtained from the first turn Player1 wants to trap Player 2 inside of Player 2's star, and then complete two more turns than needed to delete the edges contained in linkset of Player 2's starting vertex. We will then modify a previously proven lemma to find graphs such that Player 1 has a winning strategy if Player 1 can complete enough turns to block the edges contained in the linkset of Player 2's starting vertex.

Definition 5.4.1. Let G be a game graph with starting vertices v and w for Player 1 and Player 2 repsectively and let n be the number of edges in the linkset of w. The graph G is a **type 3 game graph** if there exists a $u \in V(G)$ such that there are at least $n+6$ disjoint v, u-trailsets and the total number of bridges incident to vertices contained in $link(w)$ is exactly 5. △

Similarly to the strategy used for a type 2 game graph the difficulty is once again Player 1 finding a trail such that there are a large amount of disjoint trailsets and thus the following proof would be difficult to implement in practice.

Theorem 5.4.2. *Let G be a type 3 game graph. Then Player 1 has a winning strategy.*

Proof. Suppose G has starting vertices x and y for Player 1 and Player 2 repsectively. Let n be the number of edges in the linkset of y. During the setup phase of Player 1 places 4 blockers on 4 of the bridges incident to vertices in $V(s(y))$. Let $u \in V(G - s(y))$ such that there are at least $n + 6$ disjoint x, u-trailsets. On Player 1's first turn Player 1 uses a x, u-trailsets and blocks the final bridge of $s(y)$. Player 2 is now trapped inside of $V(s(y))$ and only has trails to other vertices contained within $V(s(y))$. Player 2 takes a trail from vertex y to another vertex within $V(s(y))$ and Player 2 then places a blocker anywhere on the board. Since G is a type 3 game graph there exists $u \in V(G - s(y))$ such that there are $n + 6$ disjoint x, u-trialsets. Player 1 begins alternating position between the vertices x and u using one of the $n + 6$ disjoint x, u-trailsets. After Player 1 has completed $n + 1$ turns Player 1 has completed enough turns so that Player 1 can block all of the edges within $linkset(y)$. Since Player 1 can then continue to complete two move turns then by Lemma 3.2.2 Player 1 can win. □

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