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Mathematical Structure of Musical Tuning Systems

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Abstract

Over the course of history, western music has created a unique mathematical problem for itself. From acoustics, we know that two notes sound good together when they are related by simple ratios consisting of low primes. The problem arises when we try to build a finite set of pitches, like the 12 notes on a piano, that are all related by such ratios. We approach the problem by laying out definitions and axioms that seek to identify and generalize desirable properties. We can then apply these ideas to a broadened algebraic framework. Rings in which low prime integers can be factored are of particular interest. Unique viable solutions can be found in various Euclidean domains, including the Gaussian integers and the Eisenstein integers.

Contents

Dedication

For my Mum and Pa.

Acknowledgments

I would like to express my gratitude to my advisor John for teaching me so much and for giving me space to explore my own ideas. I would also like to thank my partner Cameron, my brother Blue, and my best friend Pat for existing, my friends Zeph, Medrano, Isabelle, and Aleda for making things fun, and Suitcase and T for being good guys.

1
Introduction

The problem of tuning has been a dynamic part of Western musical thought for centuries. The set of notes available to musicians necessarily informs how they play, and continually changing preferences for certain intervals and harmonies reflect developments in how the structure of music is envisioned. The piano can be seen as the focal point of issues with tuning, as its strict construction represents the effort to condense the infinite continuum of sound into a 12 note interface. Indeed, there exist possibilities for notes in between the keys on the piano and notes in between those notes. This is evidenced by the fact that the modern piano contains an entirely different scheme of pitches than pianos at other points throughout history. The ongoing nature of this problem shows how tricky it is to pin down exactly what we want from a finite scale.

Part of the reason for the difficulty in devising a tuning system is the degree of subjectivity that surrounds it. Each musician might have drastically different taste in what sorts of sounds should be producible on an instrument. Part of the intrigue in approaching the problem from a mathematical perspective is the prospect of objectivity. While it can be argued that there is no objectively good music, we can create a mathematical framework in which it is possible to make objectively justifiable musical decisions.

The goal of this paper is to lay out an axiomatic framework in which to devise tuning systems and discuss the significance of doing so. First, we go over a number of definitions that will be necessary to our understanding of the problem. Once we are precise about what we mean by notes, intervals, and scales, we can use this as a foundation on which to construct our framework. We also review historical approaches to tuning in order to get a sense for the ideals used in the past that we should consider including in our axioms. Since this is not a study primarily concerned with history, this review will be brief. The goal is only to pinpoint some common ideas that can be generalized and reconstructed.

Next, we move outside the realm of historical approaches and introduce some concepts to help us move in new directions. Here we are concerned with ideas from ring theory. Some basic definitions quickly lead us to quadratic integer rings, where novel factorizations of integers present new possibilities. We also ensure that the rings we are working with are Euclidean domains, where unique factorization is possible (it would be difficult to make sense of this framework otherwise).

Once equipped with these new concepts, we look at how they can be applied to tuning. We begin by reconstructing the work of Boland and Hughston[1] in this rigorous algebraic framework. Here we introduce the idea of using complex valued quadratic integer rings as a basis for devising a tuning system. We review Boland and Hughston's approach using the Gaussian integers before applying the same methodology using the Eisenstein integers. We also go over what it looks like to use this approach in the context of real valued quadratic integer rings.

After having devised several new scales our framework, we examine some ways in which we could consider comparing these scales to one another. Here we use some ideas from various areas of mathematics including field theory.

Lastly, we look at what it means to begin adjusting our definitions and axioms in pursuit of different results. Namely, we look at just a few of the possibilities that emerge when we remove the restriction on the number of notes that a scale should contain. We can then take a step back in order to discuss more generally the significance of relying on certain axioms to devise tuning systems. The current widespread tuning system used in Western music can be seen as relying upon some of the strictest axioms ever used. But if we can obtain such a wide variety of results via new axioms, what stops us from doing so? The methods used in this paper are not meant to impose additional restrictions on the discussion. In fact, the intention is the opposite. It will be seen that an axiomatic approach can help us to widen the scope of solutions to the problem of tuning.

2
History of Tuning Systems in Western Music

We begin by establishing definitions for some musical notions that we will need later. There is a degree of subjectivity regarding what is meant by certain musical terms that largely depends on context. Since this paper is concerned with mathematics, we will set out to develop definitions that are conducive to a rigorous framework in which to operate. Once some definitions are established, we can see the ideas in context by reviewing historical approaches to constructing scales.

2.1 Foundational Terminology

A great deal of the musical language that we will be using deals with the perception and organization of frequencies of sounds. Almost all sounds that occur naturally, including those produced by instruments, are more complicated than a single sine wave or frequency. However, based on the way that we perceive these sounds, it is easy to designate a single value to them.

Definition 2.1.1: A **note** is the perceived frequency of a sound. It is denoted by a letter, which indicates its relationship to other notes.

For example, we write $A = 440$hz to refer to any sound that we hear as oscillating at a frequency of 440 cycles per second. The note A sounds noticeably different when produced by a piano than when it is produced by, say, a trumpet. This is because the acoustic profiles of the sounds produced by these instruments are more complicated than one frequency, but since we hear both as the same dominant note we call them both by A.

We also need language to describe the relationship between different notes.

Definition 2.1.2: An **interval** is the multiplicative relationship between the frequencies of two notes.

For example, the interval between $A = 440$hz and $E = 660$hz is $\frac{3}{2}$. This notion will be important throughout the paper, as we use it to construct sets of notes with specific relationships to one another. There is one interval that is worth naming for simplicity. Though many intervals have historical names, this is the only one we will use.

Definition 2.1.3: The interval $\frac{2}{1}$ is called an **octave**.

The octave will be important to our notion of a scale, which we can now define.

Definition 2.1.4: A **scale** is an ordered set of 12 notes, each denoted by a letter and the interval between it and the first note in the scale.

The first note in the scale is denoted trivially by the interval $\frac{1}{1}$. By convention, when discussing 12 note scales, we often take the first note to be the note C. The note denoted by the octave is aurally interpreted as the “same” note but at double the frequency. For this reason, when defining a scale we need only consider notes that lie between the note indicated by $\frac{1}{1}$ and the note indicated by $\frac{2}{1}$. The scale can then be extended by simply multiplying any note by 2 as many times as we would like. We will rely on these basic concepts to discuss and create scales throughout this study.

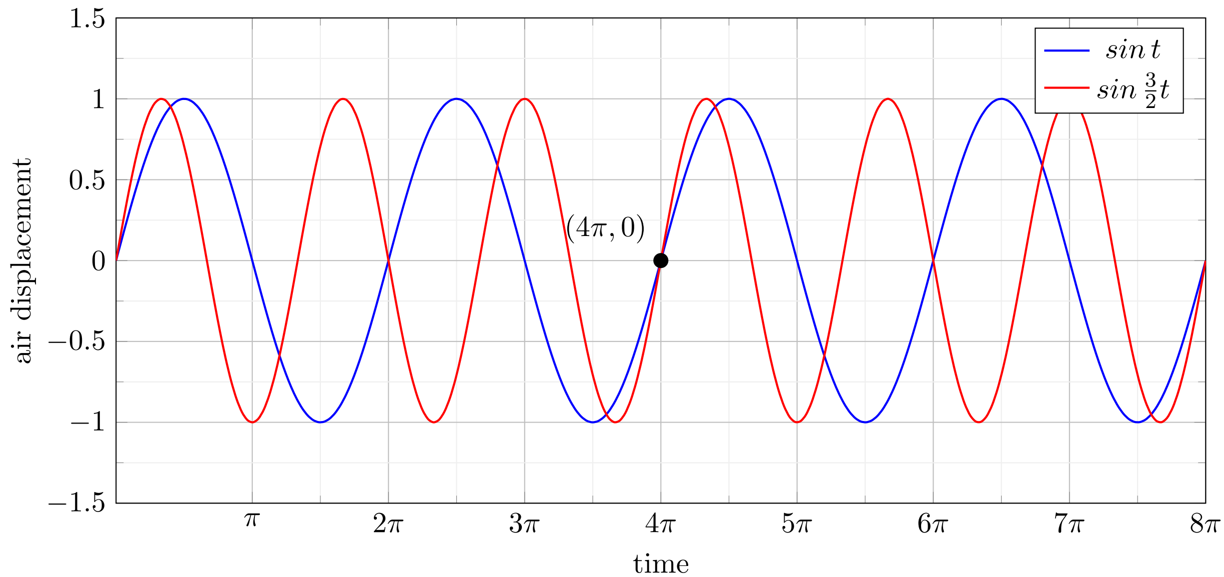
2.2 Acoustics Background

We have now established what we mean by a scale, as well as what our basic intention is in creating a scale. We seek to fill the space occupied by an octave with additional notes. But which intervals should we choose to describe these notes? The answer to this question involves the pursuit of a rather elusive musical ideal, which is commonly called consonance.

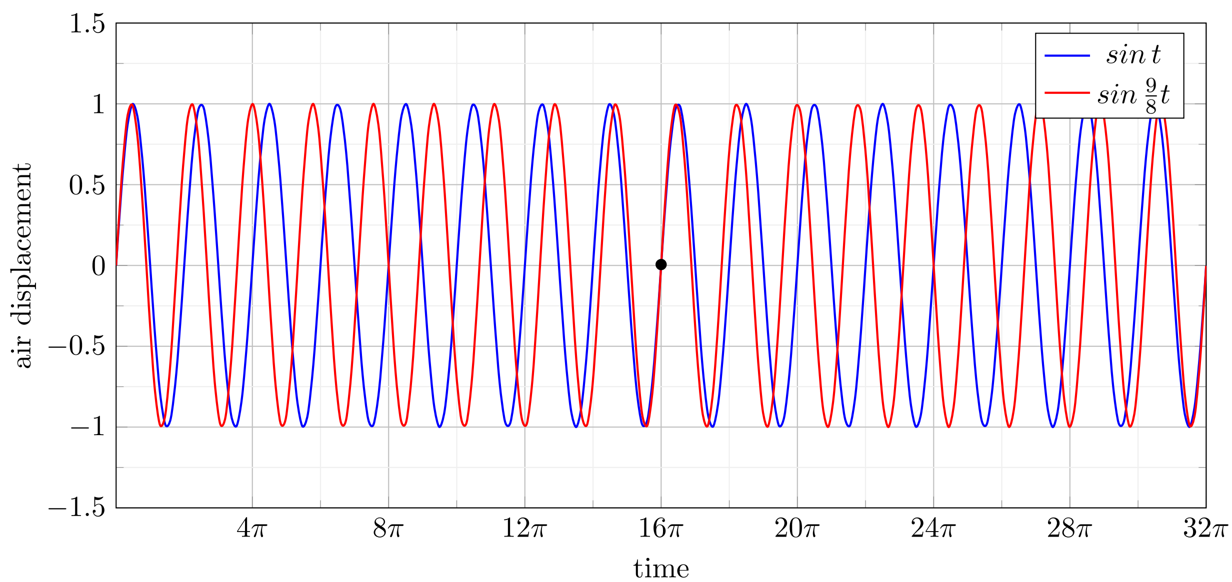
Definition 2.2.1: Two notes are said to be **consonant** with one another if the interval that relates them consists of low primes or low powers of primes.

There are many definitions for this word and many of them evade precision. Even here there is potential for ambiguity. We would like for the interval that relates two notes to consist of the lowest primes or powers of primes possible in a given framework. The implications of this idea will become clearer with examples. But generally, there is an argument in acoustics for this definition.

The reasoning for using notes related by certain intervals comes from the physical properties of sound. On a very simplistic level, we can view a note as being associated with a sine wave of a certain frequency. Again, an actual note produced by, say, a piano, is more complicated acoustically. But the sine wave model will provide us with a lot of the information necessary to make an argument for certain intervals. The reason that we want intervals consisting of low primes is that such intervals represent a certain physical relationship between the sound waves of two notes. The figure below shows the graphs of two arbitrary frequencies related by the interval $\frac{3}{2}$.



While the relationship between these functions may look messy initially, observe that it is a pattern that repeats every 4π units of time. In the next graph, look at the relationship between two arbitrary frequencies related by the interval $\frac{9}{8} = \frac{3^2}{2^3}$.



The pattern here takes 16π units of time to repeat itself. So a ratio consisting of higher powers of the same prime integers describes a much more complicated physical interference between sound waves. Since this is a tangible phenomenon, our ear picks up on the difference in complexity that can be seen in these graphs. Since this is not the subject of this study, we will not go further in explaining these concepts or justifying their validity. Hopefully, it suffices to see that there is a physical phenomenon that corresponds to our definition of consonance.

Another idea from acoustics that informs our preferences for certain intervals is the harmonic series.

Definition 2.2.2: The **harmonic series** for a note with frequency $\lambda \in \mathbb{R}$ is the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n\lambda$.

So the harmonic series includes all frequencies that are integer multiples of a given frequency. The harmonic series is a naturally occurring phenomenon, as some of its elements are audible when a note is produced by certain materials. The first element in the harmonic series of any note, referred to as the fundamental, is trivially the dominant frequency of the note itself. The second element is the frequency of the note one octave higher than the fundamental; elements indexed by subsequent powers of 2 are the frequencies of notes several octaves above the fundamental. The harmonic series also gives us some other useful values. In the following section, we use the note that is one octave below the 3rd note in the harmonic series, denoted by the interval $\frac{3}{2}$. This interval consists of the lowest possible pair of primes (besides, trivially, $\frac{2}{2} = 1$).

The harmonic series serves as a strong basis for concepts in tuning since it can be observed physically. Indeed, it is observable that when a note is produced by an acoustic instrument, resonance occurs at the frequencies given by the harmonic series. Particular emphasis on certain elements of the harmonic series is what gives us the unique color of different instruments. However, our use for the harmonic series will be to derive intervals for scales.

2.3 Pythagorean Tuning

Some of the first approaches to tuning within the paradigm that we are examining have been attributed to the Pythagoreans. Wading through a plethora of myth surrounding Pythagoras' discoveries leads to the central idea that desirable scale notes are related by intervals consisting of low primes. One of the simplest intervals, which we will examine here, is $\frac{3}{2}$.

Definition 2.3.1: Any scale consisting of notes related by intervals whose numerator and denominator can be factored into powers of 2 and 3 is called a **Pythagorean tuning**.

The idea is that we can generate a scale simply by repeatedly applying the ratio $\frac{3}{2}$ and dividing by powers of 2 to stay within one octave when necessary. Following this line of reasoning, we would first get the interval $\frac{3}{2}$, then $\frac{9}{4}$, which is greater than 2 so we divide it by 2 to get $\frac{9}{8}$. Continuing in this fashion, the next few intervals we would find are $\frac{27}{16}, \frac{81}{64}, \frac{243}{128}, \frac{729}{512}, \ldots$ and herein the essential problem of tuning begins to become apparent. Ideally, we would like these values to eventually cycle around to the "same" note that we started with several octaves up, the ratio $\frac{2}{1}$. We would then have a finite and repeatable scale. But by simply using $\frac{3}{2}$ as a generator, this will not happen. According to music theory and our intuition for what a piano should look like, applying this interval to a note 12 times should give us the same value 7 octaves up. But $\left(\frac{3}{2}\right)^{12} = \frac{531441}{4096}$ whereas $2^7 = 128 = \frac{524288}{4096}$ [3, p. 15]. The discrepancy is small but certainly a problem. It shows that the symmetry and continuity desired cannot be achieved this simply. So though this approach initially provides us with some nice intervals consisting of low primes, its practicality quickly diminishes as it generates intervals of increasing complexity.

In terms of assigning these intervals to notes, the key idea for now is that we should pick 12 of them and order them from least to greatest, preferring lower powers of 2 and 3. There is also a great deal of historical precedent that can be relied upon to assign each ratio to a note. For example, the idea that $G = \frac{3}{2}$ is present in most scales that use the ideas we have gone over so far. A common version of a Pythagorean scale is [3, p. 49]

$$C = 1, \quad Db = \frac{256}{243}, \quad D = \frac{9}{8}, \quad Eb = \frac{32}{27}, \quad E = \frac{81}{64}, \quad F = \frac{4}{3}, \quad Gb = \frac{729}{512}, \\ G = \frac{3}{2}, \quad Ab = \frac{128}{81}, \quad A = \frac{27}{16}, \quad Bb = \frac{16}{9}, \quad B = \frac{243}{128} \tag{2.3.1}$$

While some of the intervals match the description of what we have deemed desirable, others include very high powers of primes, which is not ideal. For example, the note $Gb = \frac{729}{512}$ is particularly troublesome. We would like for this interval and others to consist of smaller numbers.

We could try the same approach with other generators. Following this line of thinking, we would still want our generator to be a ratio of low primes. Some other possibilities are $\frac{5}{4}$ or $\frac{5}{3}$. However, we would quickly run into similar issues as with $\frac{3}{2}$. Later, we will see that we need a different approach to selecting a single generator if that is the method we wish to use. However, for now we will continue examining the concept that motivated this initial approach, which is that intervals consisting of low primes are desirable.

2.4 Just Intonation

What other methods can we use to implement this idea, that we want the notes in our scale to be related by ratios of low primes? As one might imagine, there are many angles from which we can approach this problem. After the less than ideal result of our approach using a single rational generator, one might think to instead use several of these generators. This would yield a different scale with added dimension, since we can quickly derive more values.

Here we introduce more terminology, used to differentiate between scales with different rational generators. First,

Definition 2.4.1: Just intonation refers to any scale containing notes related by rational intervals.

The example of Pythagorean tuning (Fig. 2.3.1) is a just intonation scale. But other examples of such a scale can include any number of rational generators. Here we establish the notion of a limit as a signification of which generators are being used.

Definition 2.4.2: For a prime integer p, a ***p*-limit scale** is a scale containing notes related by intervals whose numerator and denominator can be factored into powers of prime integers no greater than p.

For example, Pythagorean tuning (Fig. 2.3.1) could be referred to as 3-limit tuning (though this terminology is rarely used in this specific case as it is rather trivial). More interesting perhaps is the example of 5-limit tuning. This case is notable because it quickly gives us a very reasonable 12-note scale which has been used in various forms throughout history. The scale is as follows [3, p. 56]:

$$\begin{gathered} C=1, \quad Db=\frac{16}{15}, \quad D=\frac{9}{8}, \quad Eb=\frac{6}{5}, \quad E=\frac{5}{4}, \quad F=\frac{4}{3}, \quad Gb=\frac{45}{32}, \\ G=\frac{3}{2}, \quad Ab=\frac{8}{5}, \quad A=\frac{5}{3}, \quad Bb=\frac{9}{5}, \quad B=\frac{15}{8} \end{gathered} \tag{2.4.1}$$

The derivation of this scale outlines a process that we will use throughout this study. As we can see, the idea is that we want our scale to include certain simple intervals, like $\frac{3}{2}$ and $\frac{5}{4}$. Both of the notes represented by these intervals are consonant with $C = \frac{1}{1}$ and central to the Western musical form. Hence, all other notes in the scale above are denoted by multiplicative combinations of these two intervals. For example, we have $B = \frac{15}{8} = \frac{3}{2} \cdot \frac{5}{4}$. It is left to the reader to observe that each note in this scale can be derived in this manner.

But how do we decide on which notes to include in the scale? In an ideal world, we would like for each note in the scale to be related to some other notes in the scale by the simple intervals that we have been discussing. This has to do with the idea of playing effectively in different keys. We have so far considered everything in relationship to C. When we play in the key of C using the scale above, we have simple ratios at our disposal. But what if we want to play in the key of D using this same scale? We would then want D to be related to other notes in the scale by the nice intervals that we are interested in. For example, we would want the note D to be related to some other note in the scale by the interval $\frac{3}{2}$. This would be the note $\frac{27}{16}$ (since $\frac{9}{8} \cdot \frac{3}{2} = \frac{27}{16}$), which is not found in our scale. The closest we have is $A = \frac{5}{3}$, which is pretty close, but not exact. From a musical perspective, we would want the note A to be consonant with the note D, but this is not the case in this scale. So when deciding which notes to include in our scale, it quickly becomes apparent that we must make decisions about which notes we prefer to have consonant relationships with other notes. This comes at the cost of other notes in the scale not boasting the same consonant relationships. For example, if we were to change the note A to be consonant with D, then it would no longer be consonant with C.

The discrepancies we see here do not necessarily invalidate the scale. Perfect acoustic consonance in all keys is not necessarily the only thing that can be considered valuable from a musical perspective. Much of Western music's classical repertoire was composed within the paradigm of just intonation with its idiosyncrasies in mind. There is some intrigue in the idea that different key centers can sound different. It arguably adds a certain dimension to composition. However, to discuss this further would become a matter of subjectivity, which is not the intent of this study. For now, studying just intonation scales leads us to the conclusion that depending on our ideals, starting only with ratios of prime integers may not get us where we need to go.

Example 2.4.1 (Syntonic Comma):

As an aside, we will briefly examine an interesting value that appears in a variety of situations when studying just intonation scales. In the previous section, we discussed the following problem: our just intonation value for A is $\frac{5}{3}$, but for A to be consonant with $D = \frac{9}{8}$, we would want it to be $\frac{27}{16}$. One might think to ask exactly how far away these two intervals are from one another. Since we are dealing strictly in multiplicative relationships, the answer is found by dividing $\frac{27}{16}$ by $\frac{5}{3}$ to get $\frac{81}{80}$. This value of $\frac{81}{80}$ emerges so commonly in problems similar to this one that it has been termed the "syntonic comma." A comma in this context refers to a margin of error, a discrepancy. We often see the syntonic comma appear when we are comparing viable options for the value of a note based on its consonances with other notes. We can view the syntonic comma as a representation of the inexactitudes of various just intonation systems. Building a scale based on rational values is perceptually very close to but mathematically slightly off from a perfect solution.

2.5 Equal Temperament

The modern solution to the problems outlined in our discussion of just intonation aims to create a scale that is equally viable in all keys. It abandons the idea of using rational values in pursuit of this new ideal.

Definition 2.5.1: Equal Temperament refers to a scale that contains only equal divisions of the octave.

Under our current definition of a scale, there is only one equal tempered scale. That is, there is only one way to divide an octave equally into 12 parts in a multiplicative sense. The system works as follows. Let $\alpha = \sqrt[12]{2}$. Then

$$\begin{gathered} C = 1, \quad Db = \alpha, \quad D = \alpha^2, \quad Eb = \alpha^3, \quad E = \alpha^4, \quad F = \alpha^5, \quad Gb = \alpha^6, \\ G = \alpha^7, \quad Ab = \alpha^8, \quad A = \alpha^9, \quad Bb = \alpha^{10}, \quad B = \alpha^{11}, \quad C' = \alpha^{12} = 2 \end{gathered} \tag{2.5.1}$$

At first glance, this may appear to be an extremely viable solution. Its mathematical elegance lies primarily in the fact that it is generated by a single interval α. It is no wonder that this solution is now used commonly throughout Western musical practice. However, if we look more closely we can quickly see the primary issue. This is that none of our intervals, besides $\frac{1}{1}$ and $\frac{2}{1}$, can be represented as ratios of integers, let alone low prime integers. For example, let's compare our new equal temperament interval for the note G to the just intonation value. Our new value is

$$G = \alpha^7 = 2^{\frac{7}{12}} \approx 1.498307077$$

whereas our just intonation value was $G = \frac{3}{2} = 1.5$. So we have not achieved perfection. In fact, none of the intervals in an equal tempered scale are truly consonant except for the octave. They are often imperceptibly close to consonant relationships, as in the case of the interval between C and G. The average ear cannot hear the difference between a G in equal temperament and a G in just intonation.

Essentially, the equal temperament system sets out to solve the problem by averaging out discrepancies across the entire scale. In a just intonation scale, some relationships are perfectly consonant, while others are noticeably off. In equal temperament, no relationships are perfectly consonant, but most are passable to the average listener. In terms of the goal of uniformity and playability in all keys, equal temperament is successful. However, it presents neither a musically nor mathematically perfect solution under our working definitions.

3
Quadratic Integer Rings

3.1 Euclidean Domains

The previous chapter outlines historical approaches to tuning that use both rational and irrational values for intervals. In both cases, perfection was left to be desired. In order to find new viable solutions to the problem of tuning, we turn to Euclidean domains. Euclidean domains are essentially a type of ring with a number of nice properties that are conducive to our study of the structure of tuning systems. We proceed with an overview of these properties before discussing how these mathematical objects apply to tuning.

Definition 3.1.1: Let R be ring. Let $a \in R$ and $a \neq 0$. The element a is called a **zero divisor** if there exists some nonzero element $b \in R$ such that $ab = 0$ or $ba = 0$ [2, p. 226].

Definition 3.1.2: A commutative ring with identity $1 \neq 0$ and with no zero divisors is called an **integral domain** [2, p. 228].

Integral domains generally share some properties with the integers and are therefore more familiar than rings without these parameters. For example, cancellation is possible in integral domains [2, p. 228]. Because of these desirable properties, we are able to define certain types of functions on integral domains.

Definition 3.1.3: A **norm function** N on an integral domain R is a function $N : R \to \mathbb{Z}^+ \cup \{0\}$ with $N(0) = 0$ [2, p. 270].

This is clearly a loose definition, and there are several functions that could conceivably satisfy it for a given integral domain. But we can achieve more precision by adding more conditions on our notion of a norm.

Definition 3.1.4: An integral domain is called a **Euclidean domain** if there is a norm function N on R that satisfies the following condition: for $a, b \in R$ with $b \neq 0$, there exist some $q, r \in R$ such that

$$a = qb + r, \text{ with } r = 0 \text{ or } N(r) < N(b)$$

[2, p. 270].

So a Euclidean domain satisfies our notion of the Euclidean algorithm for the non-negative integers. This will become useful in that it will allow us to decompose elements uniquely into products of prime elements. We should take a moment to be precise about what we mean by "unique" factorization, since this concept is central to the study. We will soon be dealing with rings where implementing this concept will require more than our intuition for factorization of integers.

Definition 3.1.5: Let R be a ring with identity $1 \neq 0$ and let $u \in R$. The element u is called a **unit** of R if it is invertible; that is, there exists some $v \in R$ such that $uv = vu = 1$. The set of units of R is denoted $R^\times$ [2, p. 226].

Definition 3.1.6: Let R be an integral domain.

(a) Let $r \in R$ such that r is nonzero and r is not a unit. Then r is called **irreducible** if whenever $r = ab$ for some $a, b \in R$, at least one of a or b is a unit of R. Otherwise r is said to be **reducible**.

(b) Let $p \in R$ such that p is nonzero and p is not a unit. The element p is said to be **prime** if whenever $p|ab$ for some $a, b \in R$, then either $p|a$ or $p|b$.

(c) Let $a, b \in R$. The elements a and b are said to be **associate** in R if $a = ub$ for some unit $u \in R$ [2, p. 284].

From this point forward, whenever we use this terminology without further specification, we mean it as defined here. For example, when talking about a unique factorization, we mean a factorization that is unique up to associates. That is, if an element x of a ring R factors only as $x = ab$ and $x = cd$ for $a, b, c, d \in R$ where a and c are associate and b and d are associate, we still call the factorization unique. The concept of irreducibility and the idea that not every irreducible element is prime is important in ring theory. However, since we will seek certain nice characteristics in the rings we look at here, this distinction will not arise. We will largely refer to elements as prime and not irreducible since it connotes our intuition for prime integers.

For the purpose of this study, we are primarily interested in quadratic integer rings that are Euclidean domains.

Definition 3.1.7: Let D be a squarefree integer, so that $\sqrt{D}$ is the root of some quadratic polynomial. A **quadratic integer ring** $\mathcal{O}_D$ is defined by

$$\mathcal{O}_D = \mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$$

where

$$\omega = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \bmod 4 \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \bmod 4 \end{cases}$$

[2, p. 229].

For example, the quadratic integer ring $\mathbb{Z}[\sqrt{2}]$ adjoins the element $\sqrt{2}$, the root of the polynomial $x^2 - 2 = 0$, to $\mathbb{Z}$. In particular, we are interested in quadratic integer rings that satisfy the properties of Euclidean domains, the most important being the unique factorization of elements of the ring into primes. This means that we will be interested in norm functions on quadratic integer rings. We are interested in these objects because they are simple and have a norm function that gives a notion of the magnitude of an element, an idea that we will elaborate upon shortly. In order to take a closer look, we will need the notion of a conjugate.

Definition 3.1.8: Let D be a squarefree integer so that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a quadratic integer ring. A **conjugate** $\overline{\omega}$ is

$$\overline{\omega} = \begin{cases} -\sqrt{D} & \text{if } D \equiv 2,3 \bmod 4 \\ \frac{1-\sqrt{D}}{2} & \text{if } D \equiv 1 \bmod 4 \end{cases}$$

[2, p. 230].

Simply put, the conjugate is the "other" root of the quadratic polynomial in question. For example, in $\mathbb{Z}[\sqrt{2}]$, the conjugate of the element $\omega = \sqrt{2}$ is $\overline{\omega} = -\sqrt{2}$.

Corollary 3.1.1: The following identities involving the conjugate will be used in subsequent proofs.

(a)

$$\omega + \overline{\omega} = \begin{cases} 0 & \text{if } D \equiv 2,3 \bmod 4 \\ 1 & \text{if } D \equiv 1 \bmod 4 \end{cases}$$

(b)

$$\omega\overline{\omega} = \begin{cases} -D & \text{if } D \equiv 2,3 \bmod 4 \\ \frac{1-D}{4} & \text{if } D \equiv 1 \bmod 4 \end{cases}$$

Proof: The proofs of both of these identities are quickly apparent from arithmetic using Definitions 3.1.7 and 3.1.8 and are therefore omitted.

With the notion of a conjugate in mind, we can now determine an effective norm function for our quadratic integer rings.

Theorem 3.1.1: Let D be a squarefree integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a quadratic integer ring. Define the function $N : \mathbb{Z}[\omega] \to \mathbb{Z}$ by

$$N(a+b\omega) = \begin{cases} (a+b\omega)(a+b\overline{\omega}) & \text{if } D<0 \\ |(a+b\omega)(a+b\overline{\omega})| & \text{if } D>0 \end{cases}$$

Then N is a norm function on $\mathbb{Z}[\omega]$.

Proof: In any case, it is clear that $N(0) = 0$. So we are left to show that the domain of N is restricted to the non-negative integers.

In the first case, suppose that $D < 0$. Let $a + b\omega \in \mathbb{Z}[\omega]$. Then

$$\begin{aligned} N(a+b\omega) = (a+b\omega)(a+b\overline{\omega}) &= a^2 + ab(\omega+\overline{\omega}) + b^2\omega\overline{\omega} \\ &= \begin{cases} a^2 - Db^2 & \text{if } D \equiv 2,3 \bmod 4 \\ a^2 + ab + \frac{1-D}{4}b^2 & \text{if } D \equiv 1 \bmod 4. \end{cases} \end{aligned}$$

In the case where $D \equiv 2, 3 \bmod 4$, clearly $a^2 - Db^2 \geq 0$ since $-D > 0$ and $a^2, b^2 \geq 0$. The case where $D \equiv 1 \bmod 4$ is not as immediate. First, we can quickly observe that $\frac{1-D}{4}$ is a positive integer, since $1 - D \equiv 0 \bmod 4$. Additionally, we know $a^2, b^2 \geq 0$. However, it is possible that one of a, b is negative, so then ab would be negative. We examine this possibility in two additional cases.

In the case where $|a| = |b|$, we have

$$\left| a^2 + ab + \frac{1-D}{4}b^2 \right| \geq a^2 + \frac{1-D}{4}b^2 - |ab| \geq 2a^2 - a^2 = a^2 \geq 0.$$

In the case where $|a| \neq |b|$, suppose without loss of generality that $|a| > |b|$. Then

$$\left| a^2 + ab + \frac{1-D}{4}b^2 \right| \geq a^2 + \frac{1-D}{4}b^2 - |ab| \geq a^2 - |ab| > a^2 - a^2 = 0.$$

Now, in the case where $D > 0$, it is trivial that $N(a + b\omega) > 0$ from the definition of N.

So this function N is a norm function on $\mathbb{Z}[\omega]$. In the following corollaries, we explore the properties of this particular function that make it useful and interesting.

Corollary 3.1.2: The function N described in Theorem 3.1.1 is multiplicative.

Proof: Let $x, y \in \mathbb{Z}[\omega]$. Then $x = a + b\omega$ and $x = c + d\omega$ for some $a, b, c, d \in \mathbb{Z}$. Then

$$N(xy) = N((a + b\omega)(c + d\omega)) = ((a + b\omega)(c + d\omega)(a + b\overline{\omega})(c + d\overline{\omega})) = N(x)N(y).$$

There are other properties of this norm function that will prove helpful to us. For example, it gives us some information about whether a given element is prime.

Corollary 3.1.3: Let N denote the norm function described in Theorem 3.1.1. Suppose that $\mathbb{Z}[\omega]$ is a Euclidean domain. Let $x \in \mathbb{Z}[\omega]$. If $N(x)$ is a prime integer, then x is prime in $\mathbb{Z}[\omega]$.

Proof: By way of contradiction, suppose that $N(x)$ is a prime integer and x is not prime in $\mathbb{Z}[\omega]$. This means that we can write $x = x_1x_2$ for some $x_1, x_2 \in \mathbb{Z}[\omega]$ where neither x_1 nor x_2 is a unit. Then

$$N(x) = N(x_1x_2) = N(x_1)N(x_2)$$

by Corollary 3.1.2. We know that $N(x_1)$ and $N(x_2)$ are both integers and neither is equal to 1. Then their product is a composite integer, which is a contradiction.

We can use this property to help us factor integers in new Euclidean domains and determine whether these factors are prime. We immediately know that a positive integer x factors in a Euclidean domain $\mathbb{Z}[\omega]$ if there exists some $a+b\omega \in \mathbb{Z}[\omega]$ such that $N(a+b\omega) = (a+b\omega)(a+b\overline{\omega}) = x$, since then $a + b\omega$ and $a + b\overline{\omega}$ are factors of x. Furthermore, if x is prime, then $a + b\omega$ and $a + b\overline{\omega}$ are prime. We will use this line of thinking often in the next section.

We will use the norm function in Theorem 3.1.1 to show that the quadratic integer rings we consider satisfy the conditions of Euclidean domains and are therefore the type of mathematical objects that we seek. However, not all quadratic integer rings meet these standards.

Example 3.1.1: Consider $\mathbb{Z}[\sqrt{-5}]$. Since $-5 \equiv 3 \bmod 4$, this is a quadratic integer ring according to Definition 3.1.7. However, we quickly notice that elements of $\mathbb{Z}[\sqrt{-5}]$ do not factor uniquely. For example, we have $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ where 2, 3, $(1+\sqrt{-5})$, and $(1-\sqrt{-5})$ are all irreducible and not associate.

This is not meant to show that $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain with any degree of rigor, though this is a known proof. The point is to see why a ring such as this would be difficult to work with in this context, which is clear in the example above. So we will only be concerned with Euclidean domains.

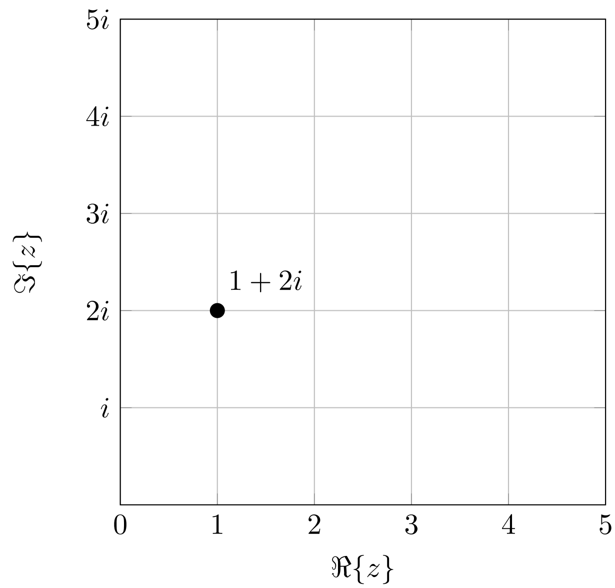
The additional importance of the norm function in Theorem 3.1.1 is that it gives us a notion of magnitude or size for the elements in the ring. This is especially helpful in the case where elements are complex and therefore unordered. When we are unable to order elements of a ring themselves, we will be able to order them based on their norms. This norm function comes into play a great deal when trying to apply the abstract and unordered structure of the Gaussian and Eisenstein integers to the concept of a real, physically implementable scale. Before discussing this in detail, however, we will briefly take a closer look at the structure of some of the rings in question.

3.2 Gaussian Integers

We begin by looking at the Gaussian integers, denoted by $\mathbb{Z}[i]$ where i is the root of the quadratic polynomial $x^2+1=0$. The Gaussian integers are the set

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\},$$

the complex numbers whose real and imaginary parts are both integers. They form a square lattice in the complex plane:



A Gaussian integer is located at each intersection of any two grid lines. The norm function for the Gaussian integers that we will look at is

$$N(a+bi) = (a+bi)(a-bi) = a^2 + b^2.$$

This norm function in particular is where the idea of size or magnitude becomes apparent. Taking the norm of a Gaussian integer is equivalent to computing its modulus. The modulus of a complex number is its distance from the origin, or its length when viewed as a 2-dimensional vector. The norm function for any complex element of a quadratic integer ring will give us a similar notion of magnitude.

We briefly go over the units for the Gaussian integers, of which there are 4.

Theorem 3.2.1: The units of the Gaussian integers are $\{\pm 1, \pm i\}$.

Proof: Let $a+bi \in \mathbb{Z}[i]$ and suppose that $a+bi$ is a unit. Then there exists some $c+di \in \mathbb{Z}[i]$ such that $(a+bi)(c+di) = 1$. Then we should also have $N((a+bi)(c+di)) = N(1)$, or $(a^2+b^2)(c^2+d^2) = 1$. So then necessarily $a^2 + b^2 = 1$, so either $a = \pm 1$ and $b = 0$, or $a = 0$

and $b = \pm 1$. Then $a + bi$ is equal to one of $\{\pm 1, \pm i\}$.

We now can set out to prove that the Gaussian integers form a Euclidean domain and are therefore of the form that we seek.

Theorem 3.2.2: The Gaussian integers form a Euclidean domain with respect to the norm function $N(a + bi) = a^2 + b^2$.

Proof: Let $\alpha, \beta \in \mathbb{Z}[i]$ such that $\alpha = a + bi$ and $\beta = c + di$ with $\beta \neq 0$. We need to find some $q, r \in \mathbb{Z}[i]$ such that $\alpha = \beta q + r$ with $N(r) < N(\beta)$ or, equivalently, such that

$$\frac{\alpha}{\beta} = q + \frac{r}{\beta} \text{ with } N\left(\frac{r}{\beta}\right) < 1.$$

So first observe that

$$\frac{\alpha}{\beta} = \frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\left(\frac{bc - ad}{c^2 + d^2}\right).$$

Call $x = \frac{ac+bd}{c^2+d^2}$ and $y = \frac{bc-ad}{c^2+d^2}$, so that $\frac{\alpha}{\beta} = x + iy$. If $x + iy \in \mathbb{Z}[i]$, then we simply set $r = 0$ and $q = \frac{\alpha}{\beta}$ and we are done. In the case that $x + iy \notin \mathbb{Z}[i]$, then $x + iy \in \mathbb{Q}(i)$, since $a, b, c,$ and d are integers. Then we can write

$$\frac{\alpha}{\beta} = x + iy = (z_1 + iz_2) + (\mu_1 + i\mu_2),$$

where $z_1 + iz_2 \in \mathbb{Z}[i]$ and $\mu_1 + i\mu_2 \in \mathbb{Q}(i)$ with $N(\mu_1 + i\mu_2) \leq \frac{1}{2}$ due to the geometry of the Gaussian integers in the complex plane. So we can choose $q = z_1 + iz_2$ and $r = \beta(\mu_1 + i\mu_2)$. Then

$$N(r) = N(\beta(\mu_1 + i\mu_2)) = N(\beta)N(\mu_1 + i\mu_2) \leq \frac{1}{2}N(\beta),$$

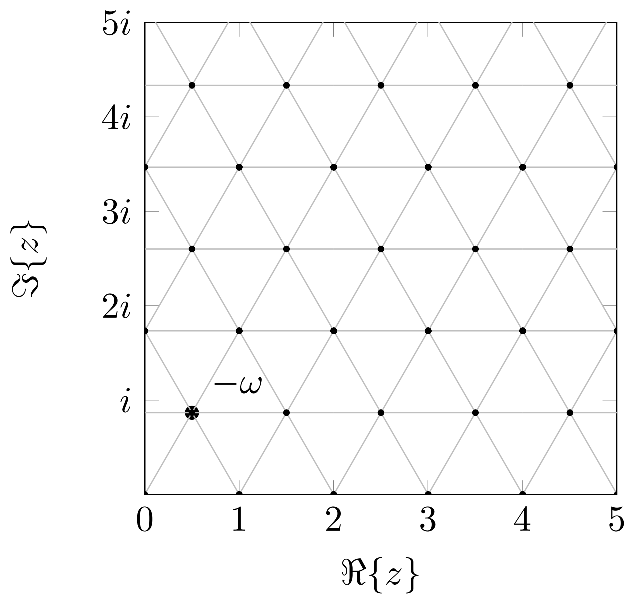
which is even stronger than necessary.

3.3 Eisenstein Integers

The Eisenstein integers are denoted by $\mathbb{Z}[\omega]$ where $\omega = \frac{-1+i\sqrt{3}}{2}$ is the root of the quadratic polynomial $x^2 + x + 1 = 0$. So they are the set

$$\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}.$$

They form a triangular lattice in the complex plane.



An Eisenstein integer is located at each node. Though from Definition 3.1.7 we would expect that we would have $\omega = \frac{1+i\sqrt{3}}{2}$, we can easily show that it is equivalent to write $\omega = \frac{-1+i\sqrt{3}}{2}$ since it simply involves offsetting by a constant.

Lemma 3.3.1: Let $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\alpha = \frac{1+i\sqrt{3}}{2}$. Then $\mathbb{Z}[\omega] = \mathbb{Z}[\alpha]$.

Proof: Let $z \in \mathbb{Z}[\alpha]$ so that $z = a + b\alpha$ for some $a, b \in \mathbb{Z}$. Then

$$a + b\alpha = a + b\left(\frac{1+i\sqrt{3}}{2}\right) = a + b - b + b\left(\frac{1+i\sqrt{3}}{2}\right) = (a+b) - b + \frac{b}{2} + i\frac{b\sqrt{3}}{2}$$
$$= (a+b) - \frac{b}{2} + i\frac{b\sqrt{3}}{2} = (a+b) + b\left(\frac{-1+i\sqrt{3}}{2}\right) = (a+b) + b\omega \in \mathbb{Z}[\omega].$$

So $\mathbb{Z}[\alpha] \subseteq \mathbb{Z}[\omega]$. Now let $z \in \mathbb{Z}[\omega]$ so that $z = a + b\omega$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} a + b\omega &= a + b\left(\frac{-1+i\sqrt{3}}{2}\right) = a - b + b + b\left(\frac{-1+i\sqrt{3}}{2}\right) = (a-b) + b - \frac{b}{2} + i\frac{b\sqrt{3}}{2} \\ &= (a-b) + \frac{b}{2} + i\frac{b\sqrt{3}}{2} = (a-b) + b\left(\frac{1+i\sqrt{3}}{2}\right) = (a-b) + b\alpha \in \mathbb{Z}[\alpha]. \end{aligned}$$

So $\mathbb{Z}[\omega] = \mathbb{Z}[\alpha]$.

By convention, we will proceed examining $\mathbb{Z}[\omega]$ with $\omega = \frac{-1+i\sqrt{3}}{2}$ as above. The Eisenstein integers have 6 units: ± 1, ω, $\overline{\omega}$, $\omega + 1$, and $\overline{\omega} + 1$. The proof of this statement is similar conceptually to determining the units for the Gaussian integers and is therefore omitted.

Remember that the conjugate of the element ω is $\overline{\omega} = \frac{-1-i\sqrt{3}}{2}$. Knowing this, we can quickly derive the following properties that will be useful in future computations.

$$\begin{aligned} \omega\overline{\omega} &= \left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right) = \frac{1-\left(i\sqrt{3}\right)^2}{4} = 1 \\ \omega + \overline{\omega} &= \left(\frac{-1+i\sqrt{3}}{2}\right) + \left(\frac{-1-i\sqrt{3}}{2}\right) = \frac{-2}{2} = -1 \\ \omega^2 &= \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{-2-2i\sqrt{3}}{4} = \frac{2\left(-1-i\sqrt{3}\right)}{4} = \overline{\omega} \\ -1 - \omega &= -\frac{2}{2} - \frac{-1+i\sqrt{3}}{2} = \frac{-1-i\sqrt{3}}{2} = \overline{\omega} \end{aligned}$$

Using this information, we can simplify the norm function to gain insight. Let $x \in \mathbb{Z}[\omega]$ then $x = a + b\omega$ for some $a, b \in \mathbb{Z}$. Then

$$N(x) = (a + b\omega)(a + b\overline{\omega}) = a^2 + ab(\omega + \overline{\omega}) + b^2(\omega\overline{\omega}) = a^2 - ab + b^2$$

We can now show that the Eisenstein integers form a Euclidean domain.

Theorem 3.3.1: The Eisenstein integers form a Euclidean domain with respect to the norm function $N(a + b\omega) = a^2 - ab + b^2$.

Proof: Let $\alpha, \beta \in \mathbb{Z}[\omega]$ such that $\alpha = a + b\omega$ and $\beta = c + d\omega$ with $\beta \neq 0$. We need to find some $q, r \in \mathbb{Z}[\omega]$ such that $\alpha = \beta q + r$ with $N(r) < N(\beta)$ or, equivalently, such that

$$\frac{\alpha}{\beta} = q + \frac{r}{\beta} \text{ with } N\left(\frac{r}{\beta}\right) < 1.$$

So first observe that

$$\frac{\alpha}{\beta} = \frac{a+b\omega}{c+d\omega} = \left(\frac{a+b\omega}{c+d\omega}\right)\left(\frac{c+d\overline{\omega}}{c+d\overline{\omega}}\right) = \frac{ac+ad\overline{\omega}+bc\omega+bd(\omega\overline{\omega})}{c^2-cd+d^2} = \frac{ac+bd+ad(-1-\omega)+bc\omega}{c^2-cd+d^2}$$
$$= \frac{ac+bd-ad-ad\omega+bc\omega}{c^2-cd+d^2} = \frac{ac+bd-ad}{c^2-cd+d^2} + \left(\frac{bc-ad}{c^2-cd+d^2}\right)\omega$$

Call $x = \frac{ac+bd-ad}{c^2-cd+d^2}$ and $y = \frac{bc-ad}{c^2-cd+d^2}$, so that $\frac{\alpha}{\beta} = x + y\omega$. If $x + y\omega \in \mathbb{Z}[\omega]$, then we simply set $r = 0$ and $q = \frac{\alpha}{\beta}$ and we are done. In the case that $x + y\omega \notin \mathbb{Z}[\omega]$, then $x + y\omega \in \mathbb{Q}(\omega)$, since $a, b, c,$ and d are integers. Then we can write

$$\frac{\alpha}{\beta} = x + y\omega = (z_1 + z_2\omega) + (\mu_1 + \mu_2\omega)$$

where $z_1 + z_2\omega \in \mathbb{Z}[\omega]$ and $\mu_1 + \mu_2\omega \in \mathbb{Q}(\omega)$ with $N(\mu_1 + \mu_2\omega) \leq \frac{\sqrt{3}}{3}$ due to the geometry of the Eisenstein integers in the complex plane. So we can choose $q = z_1 + z_2\omega$ and $r = \beta(\mu_1 + \mu_2\omega)$. Then

$$N(r) = N(\beta(\mu_1 + \mu_2\omega)) = N(\beta)N(\mu_1 + \mu_2\omega) \leq \frac{\sqrt{3}}{3}N(\beta),$$

which is even stronger than necessary.

3.4 Real Quadratic Integer Rings

We will also briefly review the properties of some real-valued quadratic integer rings. Though they can behave differently from the rings that we have looked at so far, several are known to be Euclidean domains and are therefore the type of mathematical objects that we are interested in. There is one key attribute that we will address in order to avoid confusion when discussing factorization in the future.

Theorem 3.4.1: Let D be a square free positive integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a quadratic integer ring. Then $\mathbb{Z}[\omega]$ has infinitely many units.

The proof of this theorem is related to Pell's equation, which is of the form $x^2 - dy^2 = 1$ for nonsquare integer d. It is beyond the scope of this paper to prove that these equations have infinitely many solutions, and that in turn our real quadratic integer rings have infinitely many units. So we will take this theorem as given.

The two real quadratic integer rings that we will be dealing with in the following chapter are $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\varphi]$ with $\varphi = \frac{1+\sqrt{5}}{2}$. It is well known that both of these are Euclidean domains, so we omit these proofs. Having established that we can factor elements of these rings, our only difficulty will arise in reconciling the idea of infinitely many units in the framework that we construct.

4
Quadratic Tuning

We have now laid the groundwork to begin implementing the structure of quadratic integer rings in the context of the problem of tuning. As a reminder from Section 2.1, we are interested in relating notes in a scale by intervals consisting of low primes or low powers of primes. Historical approaches have exhausted the possibilities of using the lowest prime integers 2, 3, and 5. We can move forward in the realm of certain quadratic integer rings in which these numbers are no longer prime. In general, we will begin the derivation of a scale by identifying prime integers that can be factored in a given ring and constructing intervals based on their factors.

4.1 Constructing a Scale

Before getting into the new possibilities that quadratic integer rings present, we will lay out an axiomatic framework in which to construct scales. The methods and concepts outlined here draw upon historical approaches and conventions and seek to implement them in this new context. The process of constructing a scale leads to infinitely many possibilities and can involve subjective choices, so our goal here is to create parameters in which we can produce consistent desirable results. We begin with some new definitions that will form the foundation of our axiomatic framework.

Definition 4.1.1: Let $S = \{n_1, ..., n_{12}\}$ be a scale. We measure the distance between two consecutive notes n_i and n_{i+1} with $1 \leq i \leq 12$ as 1 **scale step**.

For example, the notes C and Db in any of the scales we have examined are separated by 1 scale step. This is an alternative measurement to intervals, and both will be necessary in this section. Scale steps are additive and provide a much more general framework for how notes in a scale ought to be related.

Next, we notice that Definition 2.4.2 for a p-limit scale is no longer applicable, since we are now dealing with complex numbers which are not ordered. So we can no longer set the parameter of a "greatest" prime that we will include in our scale. Instead, we proceed under a new definition.

Definition 4.1.2: Let R be a Euclidean domain. Let P be the set

$$P = \{p \in R \,|\, p \text{ is prime}\}.$$

Let $Q \subseteq P$. A ***Q*-limit scale** is a scale containing notes related by intervals whose numerator and denominator can be factored into powers of elements of Q.

When constructing each scale, we will restrict ourselves to working with a finite set of primes from which to derive intervals. We use these primes to define all notes in a scale as follows.

Axiom 4.1.1:

(a) By convention, each scale begins with $C = 1$ and ends with $C' = 2$, with all other intervals between these values.

(b) The notes in a scale are designated in order as follows:

$$C,\ Db,\ D,\ Eb,\ E,\ F,\ Gb,\ G,\ Ab,\ A,\ Bb,\ B,\ C'$$

(c) Each scale contains the following intervals from 5-limit just intonation (Fig. 2.4.1):

$$E = \frac{5}{4}, \quad F = \frac{4}{3}, \quad G = \frac{3}{2}, \quad A = \frac{5}{3}$$

Axiom 4.1.2: Let D be a square free integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a Euclidean domain. Let N denote the norm function described in Theorem 3.1.1. Let $p \in \mathbb{Z}$ such that p is a prime integer but p is not a prime element of $\mathbb{Z}[\omega]$, such that $p = (a + b\omega)(a + b\overline{\omega})$ for some $a, b \in \mathbb{Z}$. We can derive a new interval based on this novel factorization as follows:

(i) Identify the note n indicated by p. Since p is an integer, we can refer to the harmonic series. If $p > 3$, rewrite it as $\frac{p}{m}$ for some $m \in \mathbb{Z}$ such that m is a power of 2 and $\frac{p}{m} < 2$.

(ii) Identify the number of scale steps k between C and $n = p$ (or $n = \frac{p}{m}$). If k is even, identify the note n_2 that is $\frac{k}{2}$ scale steps above C. Set $n_2 = \frac{a+b\omega}{r}$, with $r \in \mathbb{Z}$ such that r is a power of 2 and $N\left(\frac{a+b\omega}{r}\right) < 2$.

(iii) In the case that n_2 is already assigned to another value via Axiom 4.1.1, do not reassign it. However, it is possible to consider using $\frac{a+b\omega}{r}$ in future calculations as outlined in the next axiom.

Axiom 4.1.3: Let R be a Euclidean domain. Let P be the set

$$P = \{p \in R \mid p \text{ is prime}\}.$$

Let $Q = \{p_1, ..., p_n\}$ such that $Q \subseteq P$. Let S be a Q-limit scale. Any notes in S, besides those given in Axiom 4.1.1 and 4.1.2, must be found by taking products of elements of Q using the following guidelines.

(i) Consider the case where we are deriving possible values for a new note X that is m scale steps away from an established note Y. We find an interval α that already relates two notes that are m scale steps apart, then we multiply the interval that denotes Y by the interval α. There may be multiple potential intervals that relate two notes that are m scale steps apart.

(ii) From the intervals that we find using part (i), we will choose one with a preference for low primes or low powers of primes and multiple relationships to other notes.

In other words, each note in the scale must be related to at least one other note in the scale by an established interval. Ideally, a given note will be related to several other notes in the scale by established intervals.

Example 4.1.1: Say we are looking for a value for the note D. We notice that D is 5 scale steps below G. Similarly, C is 5 scale steps below F. By Axiom 4.1.1, C is related to F by $\frac{4}{3}$ (with inverse $\frac{3}{4}$) . So we can multiply the interval for the note G by $\frac{3}{4}$ to get $\frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$ as a possible value for D. Similarly, we would notice that D is 7 scale steps below A, just as C is 7 scale steps below G. Since C is related to G by $\frac{3}{2}$ (with inverse $\frac{2}{3}$), we can use $\frac{5}{3} \cdot \frac{2}{3} = \frac{10}{9}$. If we stopped here, by Axiom 4.1.2(iii), we would choose $\frac{9}{8}$ as our value for D, since it consists of powers of lower primes (the numerator and denominator of $\frac{9}{8}$ can be factored into powers of 2 and 3 whereas the numerator and denominator of $\frac{10}{9}$ are factored into powers of 2, 3, and 5).

We will proceed by implementing these axioms to make sense of values from quadratic integer rings. We will make further observations within this framework when necessary.

4.2 Gaussian Tuning

We begin by summarizing the work of Boland and Hughston [1] concerning devising a tuning system in the context of the Gaussian integers. We set out to reach their results using the axioms outlined in the previous section. As a reminder, the Gaussian integers are the set

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

Here we can implement the norm function from Theorem 3.1.1 in conjunction with Corollary 3.1.3 in order to factor elements of $\mathbb{Z}[i]$. Namely, we know that a prime integer p can be factored if there exist positive integers a and b such that $N(a+bi) = a^2 + b^2 = p$. Then $p = (a+bi)(a-bi)$, where $a + bi$ and $a - bi$ are prime Gaussian integers. First and foremost, we notice that 2 is not a prime Gaussian integer. It factors as $2 = (1 + i)(1 - i)$. Next, we can notice that

$5 = (1+2i)(1-2i)$, as well as that 3 is a prime Gaussian integer. We could keep checking different prime integers for novel factorizations, but it as just as well to pause here and see what we can do with what we know so far.

Let P be the set

$$P = \{p \in \mathbb{Z}[i] \mid p \text{ is prime}\}.$$

Let $Q = \{1+i, 1+2i, 3\}$ so that $Q \subseteq P$. We will proceed to devise a Q-limit scale. We get our first few intervals from Axiom 4.1.1(c). In order to proceed, we examine the utility of the other primes in S. We can find a use for $1+i$ via Axiom 4.1.2. The interval $\frac{2}{1}$ denotes C', which is 12 scale steps above C. Since the note Gb is 6 scale steps above C, we set $Gb = 1+i$. We can proceed from here by applying the just intonation intervals to our new complex interval. We use the approach outlined in Axiom 4.1.3. Since Db is 5 scale steps below Gb, we can set $Db = \frac{3}{4}(1+i)$. Similarly, we choose $B = \frac{4}{3}(1+i)$, $Bb = \frac{5}{4}(1+i)$, and $Eb = \frac{4}{5}(1+i)$.

These values exhaust the possibilities of combining our given intervals with the new complex interval $1+i$ without taking higher powers of elements of $1+i$ and 3. However, we have yet to use the last element of S, namely $1+2i$. We have $E = \frac{5}{4}$, and E is 4 scale steps above C. So since D is 2 scale steps above C, we can reasonably set $D = \frac{1+2i}{2}$. With this new value, we only need an interval to describe Ab. We observe that Ab is 6 scale steps away from D, just as Gb is 6 scale steps away from C. So we can multiply $\left(\frac{1+2i}{2}\right)(1+i)$ to get $Ab = \frac{-1+3i}{2}$. We now have intervals for all 12 notes, so we can write the full scale as follows:

$$\begin{gathered} C = 1, \quad Db = \frac{3}{4}(1+i), \quad D = \frac{1}{2}(1+2i), \quad Eb = \frac{5}{6}(1+i), \quad E = \frac{5}{4}, \\ F = \frac{4}{3}, \quad Gb = 1+i, \quad G = \frac{3}{2}, \quad Ab = \frac{1}{2}(-1+3i), \quad A = \frac{5}{3}, \\ Bb = \frac{5}{4}(1+i), \quad B = \frac{4}{3}(1+i), \quad C' = 2 \end{gathered} \tag{4.2.1}$$

[1, p. 11]

On the surface, this scale satisfies our desire for ratios of low primes or powers of primes. The differences between this scale and 5-limit just intonation (Fig. 2.4.1) are immediately apparent. For example, the intervals for D are $D = \frac{9}{8} = \frac{3^2}{2^3}$ in just intonation and $D = \frac{1}{2}(1+2i) = \frac{1+2i}{(1+i)^2}$ in Gaussian tuning. So if we look at the degree of the numerator and the degree of the denominator individually, the Gaussian value for D can be viewed as simpler.

4.3 Eisenstein Tuning

Since the results of devising a tuning system within the framework of the Gaussian integers prove interesting, it is reasonable to extend the idea to other quadratic integer rings. Recall that the Eisenstein integers are the set

$$\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$. We can reason using Theorem 3.1.1 and Corollary 3.1.3 as in the previous section in order to find prime integers that factor as Eisenstein integers. Here, a prime integer p can be factored if there exist positive integers a and b such that $N(a+b\omega) = a^2 - ab + b^2 = p$. Then $p = (a + b\omega)(a + b\overline{\omega})$, where $(a + b\omega)$ and $(a + b\overline{\omega})$ are prime Eisenstein integers. So we can notice that 3 factors as $3 = (1 + 2\omega)(1 + 2\overline{\omega})$ and 7 factors as $7 = (1 + 3\omega)(1 + 3\overline{\omega})$. These factorizations are not as immediately helpful for scale construction as the factorization of 2 in the Gaussian integers. However, we have still found a framework which contains a new prime elements and so we proceed with interest.

The factorization of 3, while interesting, produces results that quickly begin to diverge from our primary goals. We can see the problem by comparing to our approach in the derivation of Gaussian tuning. In that example, we used Axiom 4.1.2 in order to determine that Gb was the note to which $1 + i$ should be assigned. This approach works out nicely in the context of a 12 note scale since $C' = \frac{2}{1}$ is 12 scale steps above C and so Gb is 6 scale steps above C. Using the same methodology for 3, we quickly run into problems. From the harmonic series, we know that the interval $\frac{3}{1}$ describes the note one octave above $G = \frac{3}{2}$, call it $G' = \frac{3}{1}$ for simplicity. In a 12 note scale, there are 19 scale steps between C and G'. So the most straightforward way to interpret the meaning of $1 + 2\omega$ would be to assign it to the middle note in this interval, which would be 9.5 scale steps above C. This doesn't make any sense in our framework. Making anything of it would clearly require us to rework some of our axioms and definitions. Since this is a possibility, we will return to the concept later. But for now, we move on in hopes that the Eisenstein integers will yield some result under the current axioms.

So we instead turn first to the factorization of 7. We have not come across the number 7 so far in this study, since we have only looked at 5-limit just intonation scales. In 7-limit just intonation, the value $Bb = \frac{7}{4}$ is often used. So this is a possibility for Eisenstein tuning. However, we can now factor this ratio as $\frac{7}{4} = \left(\frac{1+3\omega}{2}\right)\left(\frac{1+3\overline{\omega}}{2}\right)$. In terms of scale steps, the note F is halfway between C and Bb, so we could reason that $F = \left(\frac{1+3\omega}{2}\right)$. However, from Axiom 4.1.1(c) we already have that $F = \frac{4}{3}$. But we can convince ourselves that it is reasonable to include $\left(\frac{1+3\omega}{2}\right)$ as a potential interval to relate two notes that are separated by 5 scale steps. So we proceed using this factorization.

Let P be the set

$$P = \{p \in \mathbb{Z}[\omega] \bigm| p \text{ is prime}\}.$$

Let $Q = \{2, 1+3\omega\ , 5\}$, so that $Q \subseteq P$. We set out to devise a Q-limit tuning. We begin with the intervals given by Axiom 4.1.1(c). Next we notice that as we determined above, it would be reasonable to use the interval $\left(\frac{1+3\omega}{2}\right)$ to relate two notes that are 5 scale steps apart (besides C and F, since we already have values for these). But we observe that Bb is 5 scale steps above F, so we can try multiplying $\left(\frac{4}{3}\right)\left(\frac{1+3\omega}{2}\right)$ to get $Bb = \frac{2}{3}(1+3\omega)$. One might notice that this seems to be a somewhat roundabout way of getting a value for Bb. We started with one idea for a value with $Bb = \frac{7}{4}$ from just intonation, then we broke it up into factors and recombined it with other intervals available to us. But through the lens of our desire for lower powers of primes, we have made a slight improvement. Compare $Bb = \frac{7}{4} = \frac{(1+3\omega)(1+3\overline{\omega})}{2^2}$ to our new value $Bb = \frac{2(1+3\omega)}{3}$. The latter is certainly simpler if we are only considering primes in Q. If we were to expand our frame of reference to the entirety of the Eisenstein integers, where the 3 in the denominator of our new Bb factors as well, we would still break even in terms of simplicity. In any case, the value $Bb = \frac{2}{3}(1+3\omega)$ is tenable under our axioms.

From here, we can proceed via Axiom 4.1.3 in order to determine more intervals. The process is similar to the example of the Gaussian integers, so we will not review it in detail. One point worth noting is that the just intonation value $B = \frac{15}{8}$ ends up being our simplest possibility for B. We get the scale

$$\begin{gathered} C = 1, \quad Db = \frac{2}{5}(1+3\omega), \quad D = \frac{5}{12}(1+3\omega), \quad Eb = \frac{4}{9}(1+3\omega), \quad E = \frac{5}{4}, \\ F = \frac{4}{3}, \quad Gb = \frac{8}{15}(1+3\omega), \quad G = \frac{3}{2}, \quad Ab = \frac{3}{5}(1+3\omega), \quad A = \frac{5}{3}, \\ Bb = \frac{2}{3}(1+3\omega), \quad B = \frac{15}{8}, \quad C' = 2 \end{gathered} \tag{4.3.1}$$

Similarly to Gaussian tuning, this scale makes some noticeable improvements on just intonation in terms of our current notions of simplicity and consonance. While some ratios are more complicated than those in Gaussian tuning, others are simpler. It is not immediate what it would mean to compare these two scales, whether to each other or to just intonation. In future chapters, we will propose some ideas for what this might mean. But for now, we take a brief detour from the derivation and analysis of new scales in order to make some sense of the complex intervals that we have found so far.

4.4 Application of Complex Intervals

Extending our reach to the complex numbers yields a variety of interesting results. We are able to find new scales whose notes are related by intervals comprised of low powers of primes. However, it is not immediately apparent how we would tune any sort of instrument, electronic or acoustic, to a complex value. So we turn again to what we know about quadratic integer rings in order to make sense of these new tuning systems.

One way in which we can proceed is to continue to draw upon the properties of Euclidean domains. As mentioned before, norm functions for Euclidean domains are useful here because they map complex numbers to positive integers, thereby giving us a notion of the magnitude of a number.

Definition 4.4.1: Let D be a square free integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a Euclidean domain. Let $S \subseteq \mathbb{Z}[\omega]$ denote a scale that includes notes related by complex intervals. Let n be a note in S defined by a complex interval, so that $n = a + b\omega$ for some $a, b \in \mathbb{Q}$. The **note value** of n is the output of the function $M : \mathbb{Q}(\omega) \to \mathbb{R}_+ \cup \{0\}$ defined by

$$M(n) = \sqrt{|(a + b\omega)(a + b\overline{\omega})|},$$

the square root of the norm function N described in Theorem 3.1.1.

For example, the note value of $Gb = 1 + i$ is $M(1 + i) = \sqrt{2}$. Evaluating the note value of complex notes moves us back into the real numbers but into the realm of irrationals. So in the process we lose the outward appearance of ratios of low prime integers. But since the viable framework of complex values still underlies the process, it seems worth exploring. We can find the note value of each note in a complex scale. Note that if x is a non-negative rational number, we have that $M(x) = x$.

Definition 4.4.2: Let D be a square free integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a Euclidean domain. Let $S \subseteq \mathbb{Z}[\omega]$ denote a scale that includes notes related by complex intervals. The **real-valued scale** for S is $M(S)$, the note value of each note in S.

When we take the note value of each note in Gaussian tuning (Fig. 4.2.1), we get the following real-valued scale:

$$\begin{gathered} C = 1, \quad Db = \frac{3}{4}\sqrt{2}, \quad D = \frac{1}{2}\sqrt{5}, \quad Eb = \frac{5}{6}\sqrt{2}, \quad E = \frac{5}{4}, \\ F = \frac{4}{3}, \quad Gb = \sqrt{2}, \quad G = \frac{3}{2}, \quad Ab = \frac{1}{2}\sqrt{10}, \quad A = \frac{5}{3}, \\ Bb = \frac{5}{4}\sqrt{2}, \quad B = \frac{4}{3}\sqrt{2}, \quad C' = 2 \end{gathered} \tag{4.4.1}$$

When we take the note value of each note in Eisenstein tuning (Fig. 4.3.1), we get the following real-valued scale:

$$\begin{gathered} C = 1, \quad Db = \frac{2}{5}\sqrt{7}, \quad D = \frac{5}{12}\sqrt{7}, \quad Eb = \frac{4}{9}\sqrt{7}, \quad E = \frac{5}{4}, \\ F = \frac{4}{3}, \quad Gb = \frac{8}{15}\sqrt{7}, \quad G = \frac{3}{2}, \quad Ab = \frac{3}{5}\sqrt{7}, \quad A = \frac{5}{3}, \\ Bb = \frac{2}{3}\sqrt{7}, \quad B = \frac{15}{8}, \quad C' = 2 \end{gathered} \tag{4.4.2}$$

Each of these are described strictly by real intervals, which we could use to tune an actual instrument. It is true that many of these intervals no longer satisfy our definition of consonance, since their values are irrational. Yet it also remains true that they were derived from intervals consisting of prime elements of corresponding quadratic integer rings.

4.5 Real Quadratic Number Rings

Before moving on, we can briefly look at some examples where we use a real-valued quadratic integer rings in order to build a scale. Consider the quadratic integer ring

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Z}\}.$$

Since $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain, we can factor its elements uniquely into primes. The most immediate new (albeit somewhat trivial) factorization available to us is $2 = \left(\sqrt{2}\right)\left(\sqrt{2}\right)$. So let P be the set

$$P = \{p \in \mathbb{Z}[\sqrt{2}] \,|\, p \text{ is prime}\}.$$

Let $Q = \{\sqrt{2}, 3, 5\}$ so that $Q \subseteq P$. We set out to devise a Q-limit tuning in $\mathbb{Z}[\sqrt{2}]$. First, we observe that since we can factor 2 and we have $C = 1$ and $C' = 2$ from Axiom 4.1.1(a), we can write $Gb = \sqrt{2}$. By following the methodology outlined in Axiom 4.1.3, we can fill out almost all of the scale. It ends up being justifiable to set $Ab = \frac{8}{5}$ here, which is an additional value from 5-limit just intonation (Fig. 2.4.1). This gives us the last few values that we need, and the resulting scale is:

$$\begin{gathered} C = 1, \quad Db = \frac{3}{4}\sqrt{2}, \quad D = \frac{4}{5}\sqrt{2}, \quad Eb = \frac{5}{6}\sqrt{2}, \quad E = \frac{5}{4}, \\ F = \frac{4}{3}, \quad Gb = \sqrt{2}, \quad G = \frac{3}{2}, \quad Ab = \frac{8}{5}, \quad A = \frac{5}{3}, \\ Bb = \frac{5}{4}\sqrt{2}, \quad B = \frac{4}{3}\sqrt{2}, \quad C' = 2 \end{gathered} \tag{4.5.1}$$

which is described entirely by real-valued intervals. One can check that we need not use the norm function in order to determine alternative values for these notes since for any note n in this scale we have that $M(n) = n$.

Something to notice is that some of the notes here are similar if not identical to those that we derived for Gaussian tuning, due to the fact that we relied on a factorization of 2 in both cases.

Interestingly enough, there is another real quadratic integer ring that has historically come up in the theory of tuning as well as in other modes of art. This is the ring

$$\mathbb{Z}[\varphi] = \{a + b\varphi \mid a, b \in \mathbb{Z}\} \text{ where } \varphi = \frac{1+\sqrt{5}}{2},$$

which has the norm function

$$N(a + b\varphi) = (a + b\varphi)(a + b\overline{\varphi}) = a^2 + ab - b^2.$$

The number φ has arisen in the context of so many subjects that it has been termed the "golden ratio" among other names. We have that

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033989,$$

which is between $G = \frac{3}{2}$ and $A = \frac{5}{3}$. So it is possible to set $Ab = \varphi$ and proceed from there. This is the most straightforward approach. It is unlikely that artists who have used this ratio in the past did so through the lens of ring theory, but it is interesting that so much use has been found for a number that fits nicely into our framework for quadratic integer rings. As such, we will proceed not by using the value of φ itself, but by treating it as an element of $\mathbb{Z}[\varphi]$ and proceeding via the axioms we have established. This is certainly more roundabout than just using the number itself, but the goal here is to exhaust the potential of our current axiomatic framework.

For now, we will use that fact that a prime integer p can be factored in $\mathbb{Z}[\varphi]$ if there exist integers a and b such that $N(a + b\varphi) = a^2 + ab - b^2 = p$. Then $p = (a + b\varphi)(a + b\overline{\varphi})$, where $a + b\varphi$ and $a + b\overline{\varphi}$ are prime elements of $\mathbb{Z}[\varphi]$. We can notice that $5 = (2 + \varphi)(2 + \overline{\varphi})$. So let P be the set

$$P = \{p \in \mathbb{Z}[\varphi] \bigm| p \text{ is prime}\}.$$

Let $Q = \{2, 2 + \varphi, 3\}$ so that $Q \subseteq P$. We set out to devise an Q limit tuning system.

By Axiom 4.1.1(c), we have that $E = \frac{5}{4} = \left(\frac{2+\varphi}{2}\right)\left(\frac{2+\overline{\varphi}}{2}\right)$, so it would reasonable to set $D = \frac{2+\varphi}{2}$. From here, we proceed once again using the methodology in Axiom 4.1.3. For example, we notice that D and Gb are separated by 4 scale steps, just as C and E are separated by 4 scale steps. Se we can set $Gb = \left(\frac{2+\varphi}{2}\right)\left(\frac{5}{4}\right) = \frac{5}{8}(2+\varphi)$. Proceeding in this manner, we can find values for all 12 notes. However, it is apparent that for several notes, the best option is to use values from just intonation in order to achieve ratios of low numbers. For example, let us consider values for Db based on what we know so far. Since Db is 5 scale steps below Gb, just as C is 5 scale steps below F, we could set $Db = \left(\frac{3}{4}\right)\left(\frac{5}{8}(2+\varphi)\right) = \frac{15}{32}(2+\varphi)$. By the criteria of low numbers, we would prefer to simply use that just intonation value $Db = \frac{16}{15}$ as in Figure 2.4.1. The rest of the computations are analogous to these examples. We get the scale

$$\begin{gathered} C=1, \quad Db=\frac{16}{15} \quad D=\frac{1}{2}(2+\varphi), \quad Eb=\frac{6}{5}, \quad E=\frac{5}{4}, \\ F=\frac{4}{3}, \quad Gb=\frac{5}{8}(2+\varphi), \quad G=\frac{3}{2}, \quad Ab=\frac{8}{5}, \quad A=\frac{5}{3}, \\ Bb=\frac{4}{5}(2+\varphi), \quad B=\frac{5}{6}(2+\varphi), \quad C'=2. \end{gathered} \tag{4.5.2}$$

However, since we have relied so far on the framework of quadratic integer rings, upon closer examination this scale is not applicable as is. We can see that every note in this scale is described by a real valued interval, since φ is real. The notes should be in order from least to greatest for the scale to make sense. However, we have that $D = \frac{1}{2}(2+\varphi) \approx 1.809016994$ and $E = \frac{5}{4} = 1.25$. So D would represent a higher note than E. In order to make sense of this scale, we need to treat values in $\mathbb{Q}(\varphi)$ as we did the complex values in Section 4.4. So we will take the square root of the norm of each of these values. We get the new scale

$$\begin{gathered} C=1, \quad Db=\frac{16}{15} \quad D=\frac{1}{2}\sqrt{5}, \quad Eb=\frac{6}{5}, \quad E=\frac{5}{4}, \\ F=\frac{4}{3}, \quad Gb=\frac{5}{8}\sqrt{5}, \quad G=\frac{3}{2}, \quad Ab=\frac{8}{5}, \quad A=\frac{5}{3}, \\ Bb=\frac{4}{5}\sqrt{5}, \quad B=\frac{5}{6}\sqrt{5}, \quad C'=2 \end{gathered} \tag{4.5.3}$$

which notably shares the value note $D = \frac{1}{2}\sqrt{5}$ with the real valued scale for Gaussian tuning (Figure 4.4.1), due to our reliance on the factorization of 5.

This is where it becomes apparent that it is much less straightforward to implement our axioms in the context of real quadratic integer rings. For example, let us compare how we factored 5 in the Gaussian integers and in $\mathbb{Z}[\varphi]$. In the Gaussian integers, we had that $5 = (1+2i)(1-2i)$. It would be sensible to work with either $1+2i$ or $1-2i$ in a scale. This is because the only way that we have of making sense of these values is via the norm function, and one can check that $N(1+2i) = N(1-2i) = 5$. We chose to use $1+2i$ in the scale and did not run into any complications. So one might think that we could take an analogous approach using φ. As above, we factor 5 as $5 = (2+\varphi)(2+\overline{\varphi})$. But here $(2+\varphi)$ and $(2+\overline{\varphi})$ are both real numbers with different values, neither of which fit nicely in the scale as is. Additionally, recall from Section 3.4 that real quadratic integer rings have infinitely many units. Therefore, both $(2+\varphi)$ and $(2+\overline{\varphi})$ have infinitely many associates, and so we could have chosen to write the factorization of 5 in infinitely many ways. This does not pose any issues algebraically, since we have already said that we are only concerned with unique factorization up to associates. However, if we view each of these associates as real numbers instead of elements of $\mathbb{Z}[\varphi]$, they each have distinct values. This is why we end up needing the norm function here, even for real intervals. The norm function is the only method we have of translating our algebraic argument for a scale into an applicable form.

However, with infinitely many possibilities for factorization come infinitely many possible approaches. Perhaps there are other factorizations that would suggest different ways of making sense of the results other than using this norm function. So real quadratic integer rings, while not as straightforward in their results as their complex counterparts, pose many opportunities for creative manipulation of their algebraic properties.

This is not how the golden ratio has been used historically (note that φ is not even present in the final scale), but it shows how the quadratic integer ring associated with φ fits into our framework. There are more quadratic integer rings that are Euclidean domains with $D > 0$ than those with $D < 0$. There are many conceivable scales that could be found by using the approach outlined here in the context of different quadratic integer rings. Furthermore, the possibilities quickly become infinite with even minor tweaks to the axioms.

5
Comparing Scales

Having examined a few methods of derivation, it is interesting to begin to think about how the resulting scales relate to one another as mathematical objects. Comparing scales is usually an extremely subjective practice, but perhaps devising a mathematical framework in which to do so will be helpful. There are a couple ways to go about this.

5.1 An Analytic Perspective

One of the most straightforward ways to assess the difference between two finite sets of values is by viewing them as vectors and comparing elements. So we can assign each of our scales to a vector in 12-dimensional space. We can then define a few ways of computing distance in order to proceed.

Definition 5.1.1: Let $\mathbf{a} = \{a_1, a_2, ..., a_n\}$ and $\mathbf{b} = \{b_1, b_2, ..., b_n\}$ be n-dimensional vectors. The **Euclidean distance** between a and b is given by the function

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2}$$

This definition of distance gives us a clear understanding of how much the notes in two scales differ from one another on average. Below is a table of the Euclidean distance between pairs of some scales that we have looked at so far. For the golden ratio tuning (Fig. 4.5.3) and scales that include complex values, the corresponding real valued scales were used for computation. The resulting values have been truncated at five decimal points.

Euclidean Distance	Just (Fig. 2.4.1)	Equal (Fig. 2.5.1)	Gaussian (Fig. 4.4.1)	Eisenstein (Fig. 4.4.2)	$\mathbb{Z}[\sqrt{2}]$ (Fig. 4.5.1)	Golden (Fig. 4.5.3)
Just (Fig. 2.4.1)	0	0.03497	0.04601	0.05148	0.04187	0.01957
Equal (Fig. 2.5.1)	0.03497	0	0.02661	0.03753	0.02979	0.04000
Gaussian (Fig. 4.4.1)	0.04601	0.02661	0	0.02085	0.02310	0.04550
Eisenstein (Fig. 4.4.2)	0.05148	0.03753	0.02085	0	0.03387	0.04468
$\mathbb{Z}[\sqrt{2}]$ (Fig. 4.5.1)	0.04187	0.02979	0.02310	0.03387	0	0.04350
Golden (Fig. 4.5.3)	0.01957	0.04000	0.04550	0.04468	0.04350	0
Average	0.03878	0.03378				

There are many ways to interpret the results displayed here. Perhaps of particular interest are the first two columns (equivalently the first two rows). These show us to what extent each scale deviates from just intonation and equal temperament. Since just intonation epitomizes many of the pure acoustics arguments that we have for certain intervals, the distance of each scale from just intonation shows how well it approximates this particular ideal. The idea with our new scales is to replicate aspects of just intonation while making adjustments that seek to solve some of the issues inherent in the system. The golden ratio scale is the closest to just intonation by this measure. This makes sense, as many of the notes are the same with only a few minor adjustments based on novel factorizations. In the other extreme, Eisenstein tuning is the farthest from just intonation by this measure. This likely results from the fact that a factorization of 7 was used in this scale. The just intonation scale used here is 5-limit, and none of the other scales make use of prime integers greater than 5. So while Eisenstein tuning imitates some aspects of 5-limit just intonation, its derivation quickly makes it distinct.

Since modern pianos are tuned to equal temperament, the second column can be seen as representing how unusual or unfamiliar a scale might sound to the average Western music listener. Gaussian tuning is the closest to equal temperament and would therefore sound the most familiar. In this regard, it achieves the idea of incorporating elements of just intonation without deviating far from what we are used to hearing. On the other hand, the golden ratio tuning is the farthest from equal temperament by this measure (even farther, notably, than just intonation). It would sound the most unfamiliar to most listeners. We will return to the average values shortly.

Euclidean distance gives us an understanding of how scales differ from each other as a whole. We can define another function to give us more information about individual notes.

Definition 5.1.2: Let $\mathbf{a} = \{a_1, a_2, ..., a_n\}$ and $\mathbf{b} = \{b_1, b_2, ..., b_n\}$ be n-dimensional vectors. The **maximum variance** between a and b is given by the function

$$m(\mathbf{a}, \mathbf{b}) = \max(|a_1 - b_1|, |a_2 - b_2|, ..., |a_n - b_n|).$$

Given two scales, the function m tells us about the note whose values differ the most between the scales. Below is a table of the maximum variance between pairs of some scales that we have looked at so far, as well as the note at which the maximum variance occurs. For the golden ratio tuning and scales that include complex values, the corresponding real valued scales were used for computation. The resulting values have been truncated at five decimal points.

Maximum Variance	Just (Fig. 2.4.1)	Equal (Fig. 2.5.1)	Gaussian (Fig. 4.4.1)	Eisenstein (Fig. 4.4.2)	$\mathbb{Z}[\sqrt{2}]$ (Fig. 4.5.1)	Golden (Fig. 4.5.3)
Just (Fig. 2.4.1)	0	0.01820 (Bb)	0.03223 (Bb)	0.03616 (Bb)	0.03223 (Bb)	0.01161 (B)
Equal (Fig. 2.5.1)	0.01820 (Bb)	0	0.01512 (A)	0.02006 (D)	0.01512 (A)	0.02435 (B)
Gaussian (Fig. 4.4.1)	0.03223 (Bb)	0.01512 (A)	0	0.01563 (D)	0.01886 (Ab)	0.02222 (B)
Eisenstein (Fig. 4.4.2)	0.03616 (Bb)	0.02006 (D)	0.01563 (D)	0	0.02897 (D)	0.02502 (Bb)
$\mathbb{Z}[\sqrt{2}]$ (Fig. 4.5.1)	0.03223 (Bb)	0.01512 (A)	0.01886 (Ab)	0.02897 (D)	0	0.02222 (B)
Golden (Fig. 4.5.3)	0.01161 (B)	0.02435 (B)	0.02222 (B)	0.02502 (Bb)	0.02222 (B)	0
Average	0.026086	0.01857				

There are several interesting entries in the table. Once again, it will be especially informative to take a closer look at the first two columns. Something to notice is that the maximum variance between just intonation and almost every other scale occurs at the value for Bb, which is $Bb = \frac{9}{5}$ in 5-limit just intonation. The value $Bb = \frac{4}{3}\sqrt{7}$ in Eisenstein tuning represents the highest variance of any note from a note in just intonation. Recall that Eisenstein tuning was the farthest from just intonation by the measure of Euclidean distance as well. We suggested that this is likely related to our factorization of 7 for Eisenstein tuning. Our observations about the significance of the value for Bb corroborate this idea. The value $Bb = \frac{7}{4}$, a value derived from the harmonic series, is often used in other just intonation systems. It makes sense that 5-limit just intonation would do a poor job of estimating this value, and that Eisenstein tuning with its use of the factorization of 7 would therefore deviate from the 5-limit value for Bb. The difference between the interval relating C and Bb in just intonation and the same interval in Eisenstein tuning would likely be noticeable to listeners.

The situation is clearly different when comparing other scales to equal temperament. Different scales vary from equal temperament at several different notes. The greatest maximum variance in this column is between the golden ratio tuning value $B = \frac{5}{6}\sqrt{5}$ and the equal temperament value $B = \left(\sqrt[12]{2}\right)^{11}$. We can also look at the averages for Euclidean distance and maximum variance in order to corroborate what we know about the structure of just intonation and equal temperament. Notice that, on average, the scales we have looked at are farther from just intonation than equal temperament by both measures. Specifically, the average Euclidean distance between just intonation and other scales is approximately 1.148 times greater than the same value for equal temperament. However, the average maximum variance between just intonation and other scales is approximately 1.404 times greater than the same value for equal temperament. There is clearly a greater difference between the latter two averages. This shows that equal temperament is very good at distributing discrepancies throughout the scale. Just intonation varies from other scales only slightly more than equal temperament in terms of Euclidean distance. But it does so at a few points of significant disagreement as indicated by the higher average maximum variance.

Measuring the difference between scales is an extremely subjective endeavor. The beauty of a strictly analytical approach is that it gives no information beyond what it claims to provide. It simply presents the result of framing scales in a certain way. Saying that two scales are different from one another is not to say that either is better than the other. No judgement is inherently passed, and the results can be interpreted in any number of ways by the reader.

5.2 Assessing Algebraic Complexity

In the derivation of each scale, we have begun by making decisions about which intervals are considered acceptable by imposing a limit of some kind. The rationals are always acceptable, and we consider additional elements when working in quadratic integer rings. We can begin to ask questions about the algebraic complexity involved in moving outside the rationals in search of new intervals.

Definition 5.2.1: The **interval field** of a scale is the smallest field $\mathbb{Q}(\alpha)$ containing all intervals that relate notes in the scale.

For example, the interval field for Gaussian tuning is $\mathbb{Q}(i)$. The interval field for any just intonation scale is $\mathbb{Q}$.

Definition 5.2.2: If F is a subfield of field K, then K is said to be an **extension field** of F. This is denoted by K/F or the subfield diagram

$$\begin{array}{c} K \\ \Big| \\ F \end{array}$$

[2, p. 511]

For example, $\mathbb{Q}(i)$ is an extension field of $\mathbb{Q}$. We can view the interval field for each scale as an extension field of $\mathbb{Q}$. With this notion in mind, it is possible to assess the complexity of a given scale through other concepts in field theory.

Definition 5.2.3: If F is a subfield of K, then K is an F-vector space. If this vector space has finite dimension, then we define the **degree** of the field extension K/F, denoted $[K : F]$, to be the dimension of this vector space. It is indicated in a subfield diagram as follows,

$$\begin{array}{c} K \\ \Big| \; d \\ F \end{array}$$

where $d = [K : F]$ [2, p. 511].

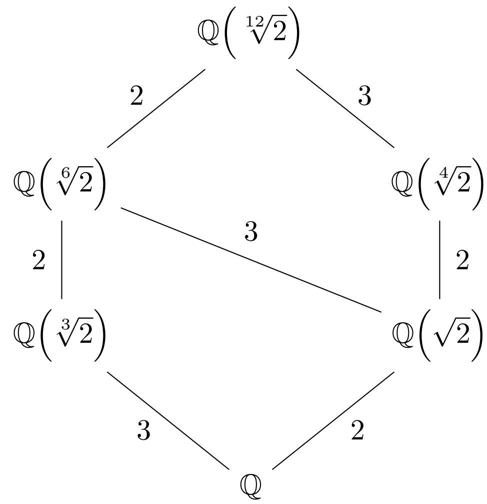
Definition 5.2.4: The **field complexity** of a scale with interval field $\mathbb{Q}(\alpha)$ is the degree of the field extension

$$\begin{array}{c} \mathbb{Q}(\alpha) \\ \Big| \\ \mathbb{Q} \end{array}$$

By this measure, just intonation (Fig. 2.4.1) trivially has the lowest field complexity of 1 since it requires no field extensions. Equal temperament (Fig. 2.5.1) has the highest field complexity of the scales we have seen so far. The interval field for equal temperament is $\mathbb{Q}\left(\sqrt[12]{2}\right)$. Elements of $\mathbb{Q}\left(\sqrt[12]{2}\right)$ are of the form

$$\{a + b\sqrt[12]{2} + c\left(\sqrt[12]{2}\right)^2 + d\left(\sqrt[12]{2}\right)^3 + ... + l\left(\sqrt[12]{2}\right)^{11} \;\Big|\; a, ..., l \in \mathbb{Q}\}$$

so $\mathbb{Q}\left(\sqrt[12]{2}\right)$ is a 12-dimensional vector space over $\mathbb{Q}$. This is also observable in the subfield diagram of $\mathbb{Q}\left(\sqrt[12]{2}\right)/\mathbb{Q}$:

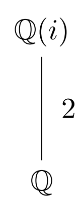


Gaussian and Eisenstein tuning lie between the extremes represented by just intonation and equal temperament in this context. But with the added ambiguity of interpreting complex numbers, there are two ways in which we can define the field complexity of these scales.

First, we look at Gaussian tuning (Fig. 4.2.1). The interval field for Gaussian tuning is $\mathbb{Q}(i)$, whose elements are of the form

$$\{a + bi \mid a, b \in \mathbb{Q}\}.$$

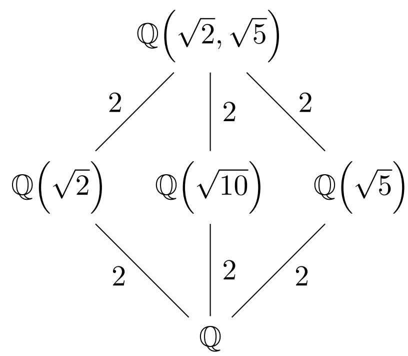
So $\mathbb{Q}(i)$ is a 2-dimensional vector space over $\mathbb{Q}$. The subfield diagram for $\mathbb{Q}(i)/\mathbb{Q}$ is



so we can conclude that the field complexity of Gaussian tuning is equal to 2. However, as we saw in Section 4.4, any conceivable application of Gaussian tuning is necessarily comprised of real intervals determined by the note values of each complex note. The interval field of the real-valued scale for Gaussian tuning (Fig. 4.4.1) is $\mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$, whose elements are of the form

$$\{a + b\sqrt{2} + c\sqrt{5} + d\sqrt{10} \mid a, b, c, d \in \mathbb{Q}\}.$$

So $\mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$ is a 4-dimensional vector space over $\mathbb{Q}$. The subfield diagram for $\mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)/\mathbb{Q}$ is

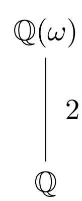


so we can conclude that the field complexity of this scale is equal to 4.

The field complexity of Eisenstein tuning (Fig. 4.3.1) can be computed similarly. The interval field for Eisenstein tuning is $\mathbb{Q}(\omega)$ (with $\omega = \frac{-1-i\sqrt{3}}{2}$ as in previous sections), whose elements are of the form

$$\{a + b\omega \mid a, b \in \mathbb{Q}\}.$$

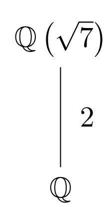
So $\mathbb{Q}(\omega)$ is a 2-dimensional vector space over $\mathbb{Q}$. The subfield diagram for $\mathbb{Q}(\omega)/\mathbb{Q}$ is



so we can conclude that the field complexity of Eisenstein tuning is equal to 2. However, once again, it may be informative to also consider the real-valued scale for Eisenstein tuning (Fig. 4.4.2). The interval field for this scale is $\mathbb{Q}\left(\sqrt{7}\right)$, whose elements are of the form

$$\{a + b\sqrt{7} \mid a, b \in \mathbb{Q}\},$$

so $\mathbb{Q}\left(\sqrt{7}\right)$ is a 2-dimensional vector space over $\mathbb{Q}$. The subfield diagram for $\mathbb{Q}\left(\sqrt{7}\right)/\mathbb{Q}$ is



so we can conclude that the field complexity of this scale is equal to 2.

We can find the field complexity of the scales that incorporate real quadratic integer rings similarly. We will review these briefly. The interval field for the scale devised using $\mathbb{Z}[\sqrt{2}]$ (Fig. 4.5.1) is $\mathbb{Q}\left(\sqrt{2}\right)$, whose elements are of the form

$$\{a + b\sqrt{2} \,|\, a, b \in \mathbb{Q}\}.$$

So we can conclude that the field complexity of this scale is equal to 2.

The interval field for the golden ratio tuning (Fig. 4.5.3) is $\mathbb{Q}(\varphi)$ (with $\varphi = \frac{1+\sqrt{5}}{2}$ as in previous sections), whose elements are of the form

$$\{a + b\varphi \,|\, a, b \in \mathbb{Q}\}.$$

So we can conclude that the field complexity of this scale is equal to 2.

Once again, it is possible to draw many subjective conclusions from this method of comparing scales. The most noticeable result is that equal temperament is a particularly complicated scale when evaluated algebraically. Between this result and its failure to meet our definition of consonance, there is very little argument for equal temperament under our current axiomatic framework. It requires a massive complication the rationals, resulting in intervals that are measurably far from those with any basis in acoustics or the arithmetic of rational numbers.

Many of the other scales we have looked at raise similar issues, but not to the extent that equal temperament does. Each of Gaussian, Eisenstein, $\mathbb{Z}[\sqrt{2}]$, and golden ratio tuning involve moving outside the rationals to some extent. The real-valued scale for Gaussian tuning is the most complicated of these, as it requires the use of three irrational values. Note that this resembles some degree of consistency with our Euclidean distance comparison of the scales, where Gaussian tuning was the closest to equal temperament. Here, it is the closest to equal temperament in terms of algebraic complexity. Each of the other scales only involve one additional irrational value, and are therefore less complicated by this measure.

Again, this is not to say whether any of these scales are better than the others. To say that a scale is algebraically complex is not to say that it is bad, it is only to say that it is algebraically complex. These measurements are only as helpful in evaluating a scale as the interpreter makes them out to be. But they do give us ideas about how these different scales are related to one another through concepts commonly used to learn about similar mathematical objects. Of course, most of our notions about comparing scales only really make sense in the context of the definitions and axioms that we have laid out in previous chapters. But it is also precisely these definitions and axioms that allow us to realize a framework in which we can go as far as to make such comparisons.

6
Beyond Twelve Notes

Up until this point, we have operated under the restriction that a scale contains 12 notes by definition. This is not true of all scales in existence. In fact, it is only true generally in the history of Western musical practice. Redefining our notion of a scale to include more values opens up an even greater range of musical and mathematical possibilities. In this chapter we will explore just some of the possibilities available to us when we change the definitions in Chapter 2 and the axioms in Chapter 4. In fact, we will start more or less entirely from scratch by redefining our notion of a scale and proceeding from there.

6.1 Generalized Pythagorean Tuning

We begin with a new definition

Definition 6.1.1 A **scale** $S = \{n_1, n_2, ..., n_m\}$ with $m \in \mathbb{N}$ is an ordered set of notes.

A scale can now consist of 2, 12, 100, or infinitely many notes. We abandon the note names from previous sections, since they are a convention for 12 note scales. We continue to describe each note by the interval between it and the first note in the scale. In this section, we will expand upon the idea of Pythagorean tuning from Section 2.3.

Definition 6.1.2: Let D be a square free integer such that $\mathcal{O}_D = \mathbb{Z}[\omega]$ is a Euclidean domain. Let P be the set

$$P = \{p \in \mathbb{Z}[\omega] \mid p \text{ is prime}\}$$

and let $Q = \{p_1, p_2\}$ be a subset of P containing any two elements p_1, p_2 of P. Any Q-limit scale is a **generalized Pythagorean tuning**.

Though our definition of a scale has changed, our notion of a limit need not be adjusted. Here we will look at the possibilities of making a scale using only products and powers of some two primes. Pythagorean tuning (Fig. 2.3.1) is a generalized Pythagorean tuning with $Q = \{2, 3\}$. There are infinitely many ways we could go about developing generalized Pythagorean tunings.

As an example, we return to the Eisenstein integers $\mathbb{Z}[\omega]$ with $\omega = \frac{-1+i\sqrt{3}}{2}$. Recall from Section 4.3 that we can factor 3 and 7 in this ring. However, in our axiomatic framework we found little use for the factorization of 3. Perhaps we can find a place for it within our expanded parameters. Recall that $3 = (1 + 2\omega)(1 + 2\overline{\omega})$ and $7 = (1 + 3\omega)(1 + 3\overline{\omega})$. So let P be the set

$$P = \{p \in \mathbb{Z}[\omega] \mid p \text{ is prime}\}$$

and let $Q = \{1 + 2\omega, 1 + 3\omega\}$ so that $Q \subseteq P$. Now let R be the infinite set

$$R = \{(1 + 2\omega)^j (1 + 3\omega)^k \mid j, k \in \mathbb{Z}\}.$$

So R and any non-empty subset of R are generalized Pythagorean tunings. Notice that Q does not contain 2, so R must not contain any multiples of 2 or any ratios with denominator 2. The key idea being explored here is the absence of pure octaves. The following is an example of a

Q-limit scale:

$$\begin{gathered}
n_1 = 1, \quad n_2 = \frac{(1+3\omega)^4}{(1+2\omega)^7}, \quad n_3 = \frac{(1+2\omega)^2}{1+3\omega}, \quad n_4 = \frac{(1+3\omega)^3}{(1+2\omega)^5}, \quad n_5 = \frac{(1+2\omega)^4}{(1+3\omega)^2}, \\
n_6 = \frac{(1+3\omega)^2}{(1+2\omega)^3}, \quad n_7 = \frac{(1+2\omega)^6}{(1+3\omega)^3}, \quad n_8 = \frac{1+3\omega}{1+2\omega}, \quad n_9 = \frac{(1+2\omega)^8}{(1+3\omega)^4}, \\
n_{10} = 1+2\omega, \quad n_{11} = \frac{(1+3\omega)^4}{(1+2\omega)^6}, \quad n_{12} = \frac{(1+2\omega)^3}{1+3\omega}, \quad n_{13} = \frac{(1+3\omega)^3}{(1+2\omega)^4}, \\
n_{14} = \frac{(1+2\omega)^5}{(1+3\omega)^2}, \quad n_{15} = \frac{(1+3\omega)^2}{(1+2\omega)^2}, \quad n_{16} = \frac{(1+2\omega)^7}{(1+3\omega)^3}, \quad n_{17} = 1+3\omega, \\
n_{18} = \frac{(1+3\omega)^5}{(1+2\omega)^7}, \quad n_{19} = 3
\end{gathered} \tag{6.1.1}$$

In the absence of octaves, the interval from our previous scales that is preserved is $\frac{3}{1}$. As mentioned previously, the interval $\frac{3}{1}$ is one octave above $\frac{3}{2}$ (though of course $\frac{3}{2}$ is not included in this scale). Here, we take n_{19} to be the "same" as n_1 in the same sense that C' was the "same" note as C in previous scales. The scale can be extended by multiplying any intervals by powers of 3. For example, n_{20} would be found by multiplying n_2 by 3. In general, we have $n_{k+18} = 3n_k$ for $k \in \mathbb{N}$.

The reasoning behind choosing these particular intervals is not immediately apparent. First observe that $n_{10} = 1 + 2\omega$, a factor of 3, is halfway between $n_1 = 1$ and $n_{19} = 3$ in terms of scale steps, which is how we have made use of new factorizations so far. The rest of the notes in this scale are largely the simplest elements of the set R. That is, they are all elements $(1+2\omega)^j(1+3\omega)^k$ with $|j|$ and $|k|$ minimized. We thereby preserve the notion that we seek low powers of primes in our ratios. Recall that the example of Pythagorean tuning in Chapter 2 contained the value $Gb = \frac{729}{512} = \frac{3^6}{2^9}$. So by this measure of complexity, we are not far off from what we might expect from a system that mimics Pythagorean tuning.

The notes are ordered according to their note values, found by taking the square root of the norm of each. There is no other reliable way to make sense of these messy complex intervals otherwise. However, the note value conversion here is pretty straightforward, since $M(1 + 2\omega) = \sqrt{3}$ and $M(1 + 3\omega) = \sqrt{7}$ as one might expect, and the norm function we are using is multiplicative. So we quickly get the real valued scale

$$\begin{gathered} n_1 = 1, \quad n_2 = \frac{(\sqrt{7})^4}{(\sqrt{3})^7}, \quad n_3 = \frac{(\sqrt{3})^2}{\sqrt{7}}, \quad n_4 = \frac{(\sqrt{7})^3}{(\sqrt{3})^5}, \quad n_5 = \frac{(\sqrt{3})^4}{(\sqrt{7})^2}, \\ n_6 = \frac{(\sqrt{7})^2}{(\sqrt{3})^3}, \quad n_7 = \frac{(\sqrt{3})^6}{(\sqrt{7})^3}, \quad n_8 = \frac{\sqrt{7}}{\sqrt{3}}, \quad n_9 = \frac{(\sqrt{3})^8}{(\sqrt{7})^4}, \\ n_{10} = \sqrt{3}, \quad n_{11} = \frac{(\sqrt{7})^4}{(\sqrt{3})^6}, \quad n_{12} = \frac{(\sqrt{3})^3}{\sqrt{7}}, \quad n_{13} = \frac{(\sqrt{7})^3}{(\sqrt{3})^4}, \\ n_{14} = \frac{(\sqrt{3})^5}{(\sqrt{7})^2}, \quad n_{15} = \frac{(\sqrt{7})^2}{(\sqrt{3})^2}, \quad n_{16} = \frac{(\sqrt{3})^7}{(\sqrt{7})^3}, \quad n_{17} = \sqrt{7}, \\ n_{18} = \frac{(\sqrt{7})^5}{(\sqrt{3})^7}, \quad n_{19} = 3 \end{gathered} \tag{6.1.2}$$

We can imagine following this process with any of the other quadratic integer rings that we have looked at. We need not restrict ourselves to any particular number of notes, as surely each pair of primes would lend itself to certain ways of dividing up a given space.

6.2 Remarks on Other Equal Temperaments

In light of Definition 6.1.1, we can revisit the notion of equal temperament from Section 2.5. Since a scale no longer necessarily contains 12 notes, there are now many more ways in which we can divide up an octave (infinitely many, in fact). In general, equal temperaments work as follows. Let $\alpha = \sqrt[m]{2}$ for some positive integer m. Then

$$n_1 = 1,\ n_2 = \alpha,\ n_3 = \alpha^2,\ \ldots,\ n_m = \alpha^{m-1},\ n_{m+1} = \alpha^m = 2 \tag{6.2.1}$$

is an equal temperament where the octave is divided into m equal parts. We call this equal temperament m-EDO to abbreviate m equal divisions of the octave (for example, the equal temperament from Section 2.5 (Fig. 2.5.1) is called 12-EDO). When asking what in particular differentiates each equal temperament, we quickly realize that our methods of comparing scales from Chapter 5 are not as informative. We cannot compare vectors of different lengths as in Section 5.1. The field complexity of a m-EDO is just m, which is interesting in and of itself but very straightforward.

A common approach for evaluating the benefits of certain equal temperaments is examining how closely they approximate certain elements of the harmonic series. So far, we have not made extensive use of the harmonic series in any aspect of our discussions. We mentioned how it is the primary source of our desire for ratios consisting of low primes. We saw how the 5-limit just intonation values $G = \frac{3}{2}$ and $E = \frac{5}{4}$ can be seen as elements of the harmonic series scaled back into the range of a single octave. We can now look at the next few unique values in the harmonic series. These are given by odd integers, since even integers represent scaling a known value up by octaves in this context. The values $\frac{7}{4}$, $\frac{9}{8}$ and $\frac{11}{8}$ represent the next few odd elements of the harmonic series scaled back into the range of an octave. Since we have mainly looked at 5-limit just intonation, some of these values are unfamiliar. But $\frac{7}{4}$ and $\frac{11}{8}$ are known intervals in 7-limit and 11-limit just intonation, respectively. We establish a new definition in order to take a closer look at how well these values are approximated by various equal temperaments.

Definition 6.2.1: Let h be an element of the harmonic series and let $r \in \mathbb{Z}$ such that $1 < \frac{h}{2^r} < 2$. Consider an m-EDO scale $\{n_1, ..., n_m\}$. for some $m \in \mathbb{N}$. The ***h*-approximation** for m-EDO is given by the function

$$\eta(h) = \min\left\{\left|n_1 - \frac{h}{2^r}\right|, ..., \left|n_m - \frac{h}{2^r}\right|\right\}$$

Any equal temperament trivially has a 2^k-approximation of 0 for any $k \in \mathbb{N}$ based on our current definition. When considering different equal temperaments, we could impose the axiom that an equal temperament should admit a low h-approximation for some element h.

The following table shows the three lowest h-approximations for $h \in \{3, 5, 7, 9, 11\}$ admitted by m-EDO scales with $5 \leq m \leq 100$ with the first row containing the lowest values of $\eta(h)$. The brackets containing some integers $[k, m]$ indicate that $\left|n_k - \frac{h}{2^r}\right| = \eta(h)$ for m-EDO. The value underneath bracketed integers $[k, m]$ is the h-approximation for m-EDO.

***h*-approximations**	3	5	7	9	11
First	[31, 57] 5.9096×10^{-5}	[28, 87] 7.7122×10^{-5}	[67, 83] 1.5284×10^{-4}	[17, 100] 5.8484×10^{-5}	[17, 37] 2.6534×10^{-5}
Second	[55, 94] 1.4960×10^{-4}	[19, 59] 9.1675×10^{-5}	[46, 57] 4.0919×10^{-4}	[9, 53] 8.8643×10^{-5}	[45, 98] 2.3629×10^{-4}
Third	[38, 65] 3.6086×10^{-4}	[29, 90] 2.5486×10^{-4}	[21, 26] 4.0929×10^{-4}	[8, 47] 2.2441×10^{-4}	[40, 87] 3.2265×10^{-4}

There are a number of observations to be made about this data. It makes sense intuitively that dividing the octave in to more parts would help us approximate rational values more closely. However, this correlation is certainly not strict. There are several instances where a lower m-EDO admits the lowest h-approximation. For example, 57-EDO has a lower h-approximation than 94-EDO. It also happens to be true that 57-EDO is the only scale that appears twice in the table. It has a very low 3-approximation and 7-approximation. Though it is not shown in the table, 57-EDO also has a lower 5-approximation than 12-EDO. We also see that 12-EDO does not appear in the table.

The point of this discussion is not to throw all of our hard work developing axioms out the window. If anything, it is meant as an argument for taking an axiomatic approach. By being clear about any adjustments made to axioms, we can fully contextualize all results. This can help us develop new methods of comparing and thereby making sense of new scales. In this way, we can continue to see what it means to approach the problem of tuning from different perspectives and using different methods.

7
Conclusion

This study outlines one approach to developing a mathematical framework in which to contextualize the problem of tuning using ideas from algebra. In Chapter 6, we see that we can just as easily develop an entirely different framework and produce entirely different results. We can also choose different methods of comparing scales and thereby convince ourselves that scales are similar or different from one another by other standards. If we were to develop several distinct but equally rigorous frameworks for this problem, it would be largely meaningless to discuss whether one was preferable from a mathematical perspective.

This is what makes the effort to apply mathematical thinking to a topic fraught with subjectivity interesting. Ultimately, the musical usefulness of any of these ideas is determined strictly by the preferences of the individual. This shows that it is rather silly to say that any one scale or methodology for generating scales is correct, or more correct than some other scale. Any such notions are inherently based on assumptions made ahead of time. This may seem obvious to say in the context of a mathematics study. Of course, any results we achieved here can only be said to be rigorous in the context of our axiomatic framework. It would be meaningless to say that these scales are "good" or "correct" in virtually any senses of the words. Yet these subjective descriptors delineate exactly the extent to which Western music has dug itself into a conceptual trench. We as a culture have decided upon a single 12-note scale as the correct tuning in which to play music. Equal temperament as defined in Section 2.5 can be seen arguable and rigorous in the context of two axioms: that there should be 12 notes in a scale and that the octave should be divided up evenly in a multiplicative sense. In fact, it is the only possible scale under these axioms. But if we choose different axioms, we can make an entirely different scale that is equally arguable and rigorous!

With all of this in mind, it seems almost unreasonable to restrict ourselves to the 12 notes that we have decided are correct. There are infinitely many other ways in which we could adjust our axioms in order to incorporate new ideas, which is both exciting and daunting. When attempting to make sense of the infinite spectrum of sounds available to us, we should not restrict ourselves with arbitrary or even well-founded assumptions about what is correct. Instead, by operating within the context of axioms, we can more easily recognize both the power and the limitations of our framework and continue to explore effectively.

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