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Analysing Flow Free with One Pair of Dots

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by
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Abstract

Flow Free is a smartphone puzzle game where the player is presented with an $m \times m$ grid containing multiple pairs of colored dots. In order to solve the puzzle, the player must draw a path connecting each pair of points so that the following conditions are met: each pair of dots is connected by a path, each square of the grid is crossed by a path, and no paths intersect. Based on these puzzles, this project looks at grids of size $m \times n$ with only one pair of dots to determine for which configurations of dots a solution exists.

Contents

Dedication

I dedicate this to my mother, Melissa Marie Vorhies.

Acknowledgments

Thank you to Ethan Bloch, for advising me on this project; the other members of my board, Steven Simon and Japheth Wood; and Sam Baumgartner, for his work on this topic off of which I have built.

1
Introduction

This project is based off of the game Flow Free. Flow Free is an smartphone puzzle game in which the player is presented with a grid containing pairs of colored dots. In order to solve the puzzle, the player must create paths between each pair of dots so that the following conditions are met: each pair of dots is connected by a path, each square of the grid has a path travelling through it, and no paths intersect, as shown in Figures 1.0.1 and 1.0.2. The ultimate goal is to find a process to determine if a solution exists for any given $n \times m$ grid with any configuration of pairs of dots. To start, the first goal is to determine when there is a solution if there is only one pair of dots.

For the purposes of this project, the puzzles will be depicted as grid graphs. Each square in the original puzzle will be depicted as and referred to as vertices on the grid, and the paths will go along edges of the grid as opposed to through the squares. The colored dots will then be on vertices, and will be referred to as endpoints. This method of depiction is shown in Figures 1.0.3 through 1.0.6.

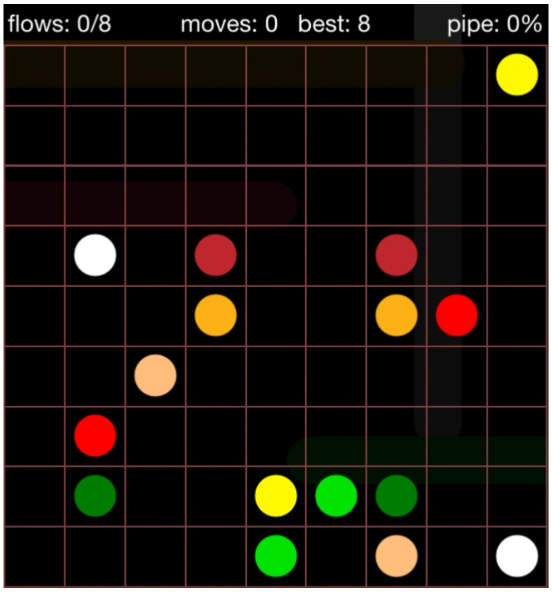

Figure 1.0.1.

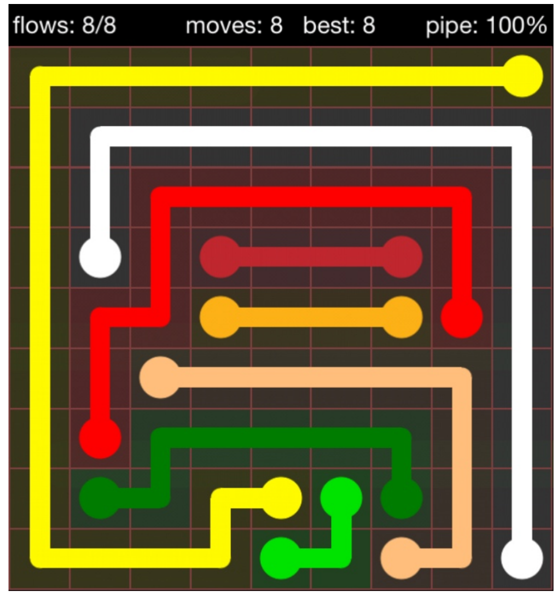

Figure 1.0.2.

1.1 Previous Work

This project is a continuation of a project attempted by Sam Baumgartner in the 2017-2018 academic year, *Hamilton Graphs in the Game of Flow*. In his project, he accomplished several goals.

1. He proved several small lemmas regarding graphs with only one pair of endpoints, such as the lemma stating that the endpoints of a path which solves an $n \times m$ grid graph have the same parity if and only if nm is even.

2. He defined many useful terms, such as parity.

3. He proved that for a $3 \times n$ grid graph where n is odd and $n \geq 3$ there exists a solution if both endpoints are on even vertices.

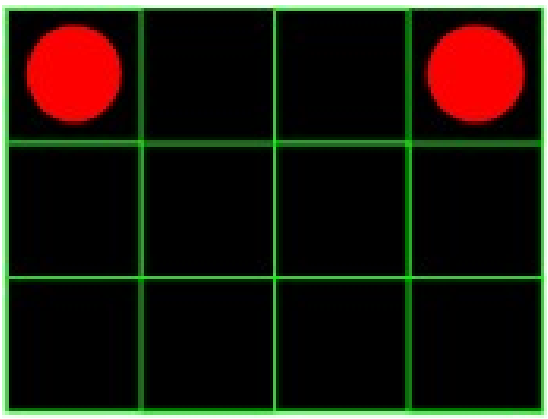

Figure 1.0.3.

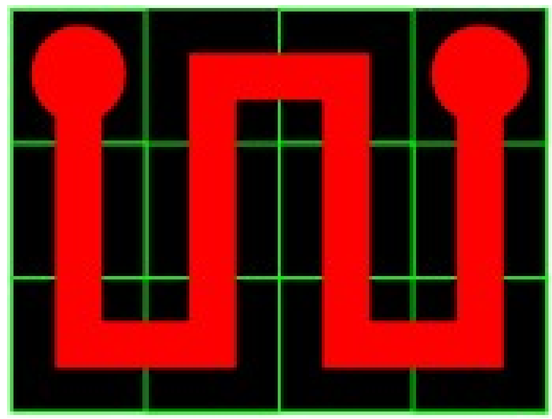

Figure 1.0.4.

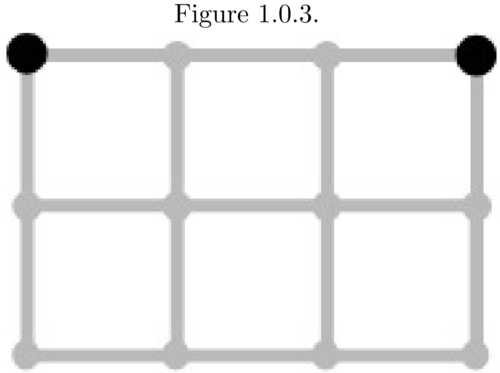

Figure 1.0.5.

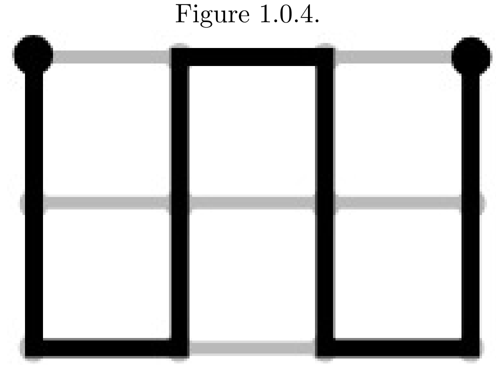

Figure 1.0.6.

2
Preliminary Lemmas

2.1 Definitions

The following definitions will be used throughout this project. We use the standard definitions of graph, path, and Hamilton path defined in [1].

Definition 2.1.1.

1. Let $m, n \in \mathbb{N}$. An $m \times n$ **grid graph** is a graph that is an $m \times n$ lattice with m rows and n columns, where each intersection between a row and a column is a vertex.

2. A **grid graph** is an $a \times b$ grid graph for some $a, b \in \mathbb{N}$.

3. The rows of a grid graph are numbered from bottom to top and the columns are numbered from left to right.

4. A vertex in the column numbered x and the row numbered y has coordinates (x, y). $\triangle$

A numbered 3×4 grid graph is shown in Figure 2.1.1.

By a **path** in a grid graph G, we refer to a path according to standard graph theory terminology; see [1].

See [1, Chapter 10] for the definition of a Hamilton path in a graph.

Definition 2.1.2. Let G be a grid graph.

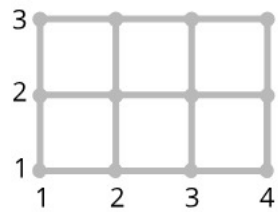

Figure 2.1.1.

1. Suppose we are given two distinct vertices in G. A **solution** to the Flow Free problem (referred to just as a solution) is a Hamilton path starting at one of the two vertices and ending at the other.

2. Any two distinct vertices in G are called **endpoints** whether or not there is a solution for them. △

The above definition of an endpoint is unconventional because a given endpoint may not in fact be the end of any path. However, it is used for the purposes of this project nonetheless.

On a grid graph with more than one pair of endpoints, as in the original game of Flow Free, a solution would not fall under the definition of a Hamilton path.

Definition 2.1.3. Let G be a grid graph.

1. A vertex in G with coordinates (x, y) has **even parity** if $x + y$ is even and **odd parity** if $x + y$ is odd.

2. An **even column** in G is a column numbered x where x is an even number. An **odd column** in G is defined similarly. An **even row** and an **odd row** are also defined similarly. △

If an endpoint is on an odd row of an odd column or an even row of an even column, then it has even parity; similarly, if an endpoint is on an odd row of an even column or an even row of an odd column, then it has odd parity.

Definition 2.1.4. Let G be a grid graph.

1. Two endpoints on G are **co-columned** if they are both in the same column, and they are **non-co-columned** if they are not in the same column.

2. An **endzone** of G is the two columns on one end of G.

3. An **almost-endzone** of G is the column next to an endzone of G.

4. A **vertical endzone** of G is the top two rows or bottom to rows of G.

5. An **vertical almost-endzone** of G is the row directly below or above a vertical endzone of G. △

2.2 Lemmas

The following lemma is from [2], where a proof may be found.

Lemma 2.2.1. *Let $n, m \in \mathbb{N}$, and let G be an $m \times n$ grid graph. Suppose G has a solution. Then the endpoints have the same parity if and only if at least one of n or m is even.*

The following are a series of lemmas for $3 \times n$ grid graphs which will be useful for theorems in the next two chapters. This first lemma is a special case that must be proven in order to find only if criteria for both $3 \times n$ grid graphs where $n \in \mathbb{N}$ is an even number and $3 \times n$ grid graphs where $n \in \mathbb{N}$ is an odd number.

Lemma 2.2.2. *Let $n \in \mathbb{N}$, and let G be a $3 \times n$ grid graph. Suppose the endpoints are non-co-columned and one of the endpoints is in the center of one of the end columns. Then there is no solution.*

Proof. We use proof by contradiction. Suppose a solution to G exists. Without loss of generality, let the endpoint be in the center of the leftmost column, as shown in Figure 2.2.1.

From this endpoint, the path could only go in one of three directions: up, down, or right. If it starts up, there is no way to get back to the bottom-left vertex, because that would require the path to reach a dead end at that vertex, so there is no path in that case. If the path starts down, the same problem occurs, this time with the top-left vertex. If the path starts to the right, then

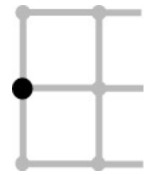

Figure 2.2.1.

the same problem occurs yet again but this time with both the top-left and bottom-left vertices. Therefore there is no path in this case, which is a contradiction. □

The following lemma is the general case of the previous lemma, however the proof for the special case above is quite different from the proof here.

Lemma 2.2.3. *Let $n \in \mathbb{N}$, and let G be a $3 \times n$ grid graph. Suppose the endpoints are non-co-columned and the left endpoint is in the center of an odd column that is not an end column. Then there is no solution.*

Proof. The situation in this lemma is shown in Figure 2.2.2.

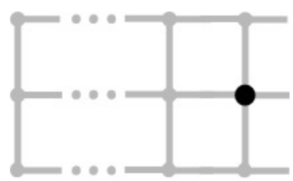

Figure 2.2.2.

We use proof by contradiction. Suppose there is a solution. Then there are four initial directions in which the path could go, which creates four separate cases. Two of these are the same due to vertical symmetry.

Case 1: Suppose the path starts by going to the left. The path can only go back to the right by going either above or below the endpoint, but not both. In this case either the vertex directly above or the vertex directly below the endpoint will be untouched, as in the specific example shown in Figure 2.2.3, where the path can never reach the circled vertex. Therefore, since there will always be a vertex which is never reached, there is no solution in this case.

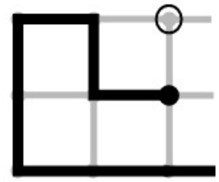

Figure 2.2.3.

Case 2: Suppose the path starts by going up. If the path then goes to the right, it will not be able to go back to the left to reach the vertices to the left of the endpoint and then go back to the right. Thus, the path must go up and then to the left. Then, it must go back right via the vertex directly under the endpoint, after which it can never go back to the left. Thus, all of the vertices to the left of the endpoint must be reached, with the path first reaching the one at the top and last reaching the one at the bottom. Thus, everything to the left of the left endpoint becomes a smaller self-contained $3 \times m$ grid H where $m \in \mathbb{N}$ is an even number as shown in Figure 2.2.4.

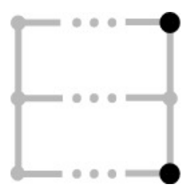

Figure 2.2.4.

Since the two new endpoints in graph H have the same parity, by Lemma 2.2.1, graph H has no solution, which is a contradiction.

Case 3: The path starts by going down. This is the mirror image of Case 2.

Case 4: The path starts by going to the right. Then, the path must go eventually go back to the left by going either above or below the endpoint, then go back to the right again via the opposite side of the endpoint. Thus, this becomes the same smaller grid graph as in Case 2.

Therefore, there is no solution. □

This next lemma is another special case that must be proven in order to find only if criteria for both $3 \times n$ grid graphs where $n \in \mathbb{N}$ is an even number and $3 \times n$ grid graphs where $n \in \mathbb{N}$ is an odd number. The previous lemmas were both regarding endpoints in the center row, whereas the next two lemmas are regarding endpoints on the top or bottom rows.

Lemma 2.2.4. *Let $n \in \mathbb{N}$, and let G be a $3 \times n$ grid graph. Suppose an endpoint is on the top or bottom of the column next to an end and the other endpoint is not in the same column or in an adjacent column. Then there is no solution.*

Proof. Without loss of generality, suppose the endpoint is on the left side on the top, as shown in Figure 2.2.5.

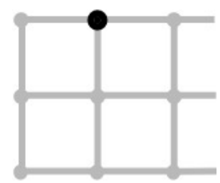

Figure 2.2.5.

We use proof by contradiction. Suppose a solution to G exists. The only situation in which the top left vertex will be reached is if the path starts to the left in order to reach that vertex. Then, the path must go down from there. From that position, the path must go down again to the bottom left corner, or that corner will never be reached. From there, the path can only go right, then the path must go up to the vertex directly below the endpoint, or that vertex will never be reached. From there, the only direction available to go is right. At this point, the path is by necessity as shown in Figure 2.2.6.

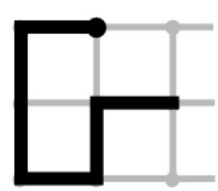

Figure 2.2.6.

Then, the path could theoretically go in any of three directions. However, if it goes up, then the vertex at the bottom will never be reached. If it goes down, then the vertex at the top will never be reached. If it goes right, then both the vertex at the bottom and the vertex at the top will never be reached. Therefore, there is no solution, since at least one of the vertices circled in Figure 2.2.7 will never be reached. □

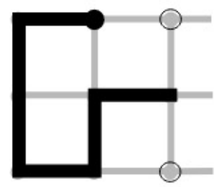

Figure 2.2.7.

The following lemma is the general case of the previous lemma, and the two lemmas together give the full story for why an endpoint cannot be on the top or bottom of every other row.

Lemma 2.2.5. *Let $n \in \mathbb{N}$, and let G be a $3 \times n$ grid graph. Suppose the endpoints are not co-columned, the left endpoint is on the top or bottom of an even column that is not next to an end column and the other endpoint is not in the adjacent column to the right. Then there is no solution.*

Proof. We use proof by contradiction. Suppose there exists a path which is a solution to G. Without loss of generality, suppose the left endpoint is on the top. Let the left endpoint be x and the number of the column with x be p. Let H be a $3 \times p$ sub-graph within G such that such that x is the top right corner of H. There are three initial directions in which the path could go from x: left, down, or right.

Case 1: Suppose the path starts by going to the left from x. If the path then goes back to the right of x and then back again to the left of x, then the path will be stuck to the left of x and will not reach the other endpoint. Thus the only way for every vertex to the left of x to be reached is if the path does not go back to the right of x until every vertex to the left of or under x, that is, every vertex in H, is reached. Thus there must be a self-contained solution to H where one endpoint is x and the other endpoint is directly below x either in the middle or on the bottom as shown in Figure 2.2.8 and Figure 2.2.9. If it is as shown in Figure 2.2.9, then the two endpoints have the same parity, so there is no solution by Lemma 2.2.1. Thus, by necessity the endpoints in H are as shown in Figure 2.2.8. Then there is a solution to H. This means that after solving H the path in G must go right from the middle point in column p. However, when the path goes right from the middle point, it can only reach one of the top and bottom points in column $p + 1$ but not both. Thus there is no solution to G in this case.

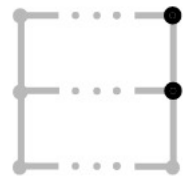

Figure 2.2.8.

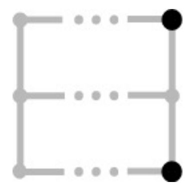

Figure 2.2.9.

Case 2: Suppose the path starts by going down from x. The path must then go left to fill in the vertices to the left of column p. This forms the same sub-graph H again, but this time the path has already reached the the middle vertex in column p, so the second endpoint must be on the bottom of column p, as in Figure 2.2.9. As previously stated, H has no solution when the endpoints are in this arrangement. Thus, there is no solution to G in this case.

Case 3: Suppose the path starts by going to the right from x. No matter what the path does next, it must come back left through column p to fill in the vertices to the left of and in column p, or rather subgraph H. Either the path will reenter H through the middle vertex in column p, completely fill H, then exit through the bottom vertex in column p, or vice versa. If the path reenters H through the middle vertex of column p, then that is functionally the same on graph H as if the path had started by going down from x directly to the middle vertex in p, just as in Case 2. As we know from Case 2, there is no solution in that arrangement. Thus the path must reenter H through the bottom vertex of column p. However, this is the same path as before, since the direction in which the path is going does not effect whether the path can hit every vertex in H. Thus there is no solution in this case.

Therefore, there is no solution to G. $\square$

3

Grid Graphs of the form $3 \times n$ where n is Even

3.1 Goal

Theorem 3.1.1 is the goal of this entire chapter. Although the theorem is stated here, it will not be proven until the very end of this chapter, due to its many separate sections.

Theorem 3.1.1. *Let $n \in \mathbb{N}$ be an even number, and suppose $n \geq 4$. Let G be a $3 \times n$ grid graph. Suppose there are two endpoints in G. Then there is a solution if and only if both of the following hold.*

1. *The endpoints have opposite parity.*
2. *At least one of the following is true.*
   (a) *The two endpoints are in the same column.*
   (b) *The two endpoints are not in the same column, and the left endpoint has even parity and the right endpoint has odd parity.*
   (c) *The endpoints are in adjacent columns and neither is in the center.*

To prove Theorem 3.1.1, it must be broken down into parts, each one of which can be proved by one or more individual lemmas, and which, together, account for every possibility of pairs of endpoints. For everything, opposite parity may be assumed by Lemma 2.2.1, since there is never a solution for a $3 \times n$ grid graph where $n \in \mathbb{N}$ is an even number when the two endpoints have the same parity.

3.2 Definitions

For any $3 \times n$ grid graph such that $n \in \mathbb{N}$ is an even number, there is a useful way to create a loop that will hit every single vertex, shown in Figure 3.2.1. Depending on how many 3×2 segments of the type shown in Figure 3.2.2 are in the standard graph, this graph could be any even length.

Definition 3.2.1. Let $n \in \mathbb{N}$ be an even number, and let G be a $3 \times n$ grid graph. The path shown in Figure 3.2.1 is called the **Standard Graph** for G. △

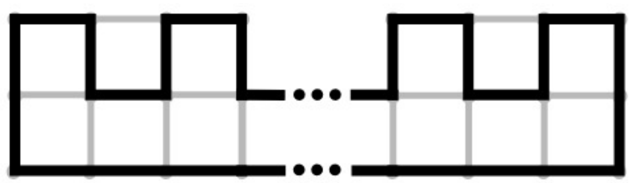

Figure 3.2.1.

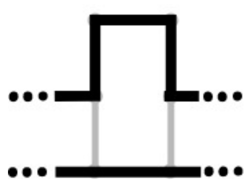

Figure 3.2.2.

3.3 Lemmas

The standard graph will be a method to prove the following series of three lemmas, all of which are proven by simply removing one edge from the standard graph.

Lemma 3.3.1. *Let $n \in \mathbb{N}$ be an even number, and let G be a $3 \times n$ grid graph. Suppose the two endpoints are in the same column and have opposite parity. Then there is a solution.*

Proof. The only way for there to be two endpoints in the same column with opposite parity is if there is an endpoint in the middle and the other one is either on top or on bottom. Without loss of generality, suppose the point that is not in the middle is on top.

Start with the Standard Graph shown in Definition 3.2.1. Observe that the two endpoints are connected by an edge in the Standard Graph, no matter where they are along the graph. Thus, a solution to G is the standard graph with the edge between the two endpoints removed, as in the example shown in Figure 3.3.1.

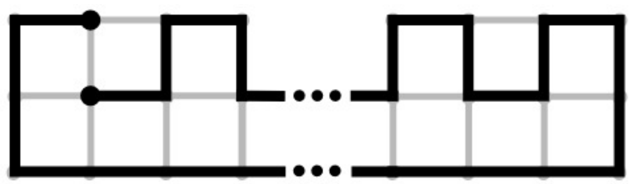

Figure 3.3.1.

Thus, this is a valid solution to any grid graph with two endpoints with opposite parity in the same column. □

Lemma 3.3.2. *Let $n \in \mathbb{N}$ be an even number, and let G be a $3 \times n$ grid graph. Suppose the two endpoints are adjacent in the center row, and the left endpoint has even parity and the right endpoint has odd parity. Then there is a solution.*

Proof. This Lemma can be proven using the same logic as in Lemma 3.3.1 as in the example shown in Figure 3.3.2. □

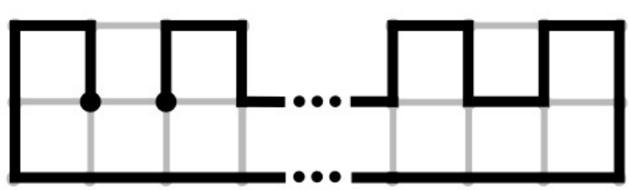

Figure 3.3.2.

We saw in Lemmas 2.2.2 and 2.2.3 that Lemma 3.3.2 does not hold if the left endpoint has odd parity.

Lemma 3.3.3. *Let $n \in \mathbb{N}$ be an even number, and let G be a $3 \times n$ grid graph. Suppose the two endpoints are adjacent and both in either the top or the bottom row. Then there is a solution.*

Proof. Without loss of generality, suppose both endpoints are in the bottom row. Then this Lemma can be proven using the same logic as in Lemma 3.3.1 as in the example shown in Figure 3.3.3. □

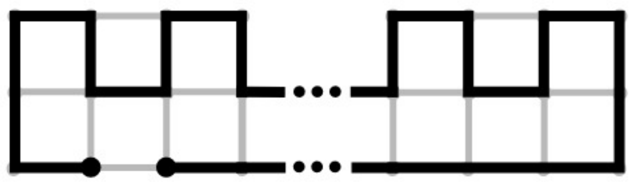

Figure 3.3.3.

Lemma 3.3.4. *Let $n \in \mathbb{N}$ be an even number, and let G be a $3 \times n$ grid graph. Suppose the endpoints are in adjacent columns and one is on the top and the other on the bottom. Then there is a solution.*

Proof. **Case 1**: Suppose the left endpoint has odd parity. Then the right endpoint has even parity. Without loss of generality, suppose the left endpoint is in the top row, thus making the right endpoint be in the bottom row. Then the configuration has a solution as follows in Figure 3.3.4. In the portion of the grid graph between the dots on the right, there may be any finite number of the segments depicted in Figure 3.2.2, and in the section on the left side there may be any finite number of an upside-down version of the same segment.

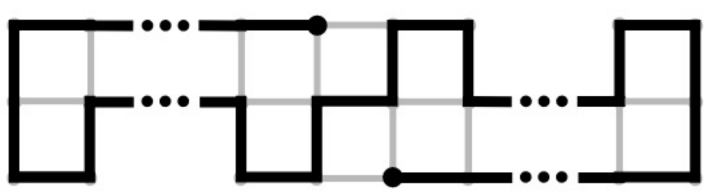

Figure 3.3.4.

Case 2: Suppose the left endpoint has even parity. Then the right point has odd parity. Without loss of generality, suppose the left endpoint is in the top row, thus making the right endpoint be in the bottom row. Then there is a solution similar to that in Case 1, but with the two sides flipped in the opposite direction by means of a column being removed on each side of the endpoints, as shown in Figure 3.3.5.

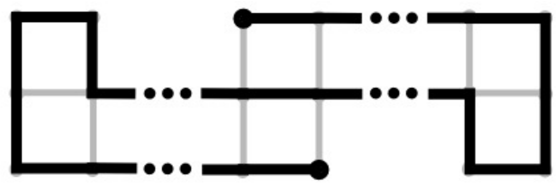

Figure 3.3.5.

This time in the portion of the grid graph between the dots on the left there may be any finite number of the segments depicted in Figure 3.2.2, and in the section on the right side there may be any finite number of an upside-down version of the same segment.

Thus this is always a solution in this case. □

Lemma 3.3.5. *Let $n \in \mathbb{N}$ be an even number and suppose $n \geq 4$. Let G be a $3 \times n$ grid graph. Suppose the two endpoints are non-co-columned and not in adjacent columns, and the left endpoint has even parity and the right endpoint has odd parity. Then there is a solution.*

Proof. The proof is by induction on n.

Base case: Suppose $n = 4$. Then there are three possible configurations of the endpoints, up to symmetry, as seen in Figures 3.3.6, 3.3.7, and 3.3.8. As shown in the figures, there is a solution in each case.

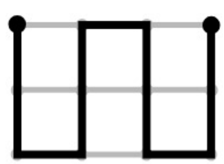

Figure 3.3.6.

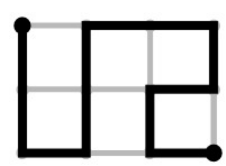

Figure 3.3.7.

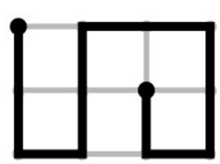

Figure 3.3.8.

Inductive step: Let $n \in \mathbb{N}$ be an even number and suppose $n \geq 4$. Assume that on a grid graph of size $3 \times n$ with two endpoints, if the endpoints satisfy the hypothesis of this conjecture, then there is a solution to the grid graph. Let G be a grid graph of size $3 \times (n + 2)$. Suppose that the endpoints satisfy the hypothesis of this conjecture.

Case 1: There are no endpoints in at least one endzone of G. Without loss of generality, assume there are no endpoints in the right endzone of G (if it were the left endzone that contained no endpoints, then if the graph is flipped horizontally the endpoints would still satisfy the hypotheses of the conjecture). Let G_1 be a grid graph of size $3 \times n$ created by removing the right endzone from G.

Removing the right endzone of G will not change the coordinates of the vertices in G since they begin from the bottom left corner. Therefore, the endpoints in G_1 have the same parity as the endpoints of G. By the inductive hypothesis, there is a solution to G_1.

Subcase 1A: Suppose the right endpoint of G is not in the right almost-endzone. Since the almost-endzone of G contains no endpoints, the portion of the path through the rightmost column of G_1 must look like in Figure 3.3.9.

Figure 3.3.9.

Then a path exists on G by extending the path on G_1 through the removed endzone of G, as in Figure 3.3.10.

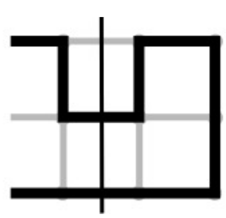

Figure 3.3.10.

Subcase 1B: Suppose the right endpoint is in the right almost-endzone. Since the almost-endzone of G contains an endpoint which has odd parity, the portion of the path through the rightmost column of G_1 must look like one of the two configurations in Figure 3.3.11, up to symmetry.

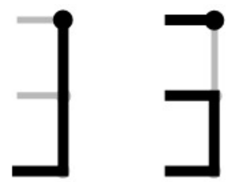

Figure 3.3.11.

Then a path exists on G by extending the path on G_1 through the removed endzone of G, as in Figure 3.3.12.

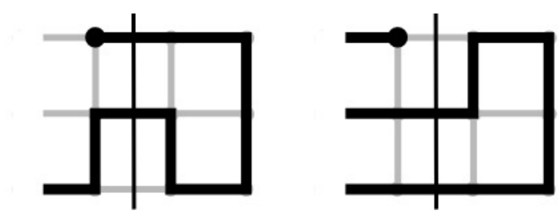

Figure 3.3.12.

Case 2: There is one endpoint in each endzone of G. Let G_1 be a grid graph of size $3 \times n$ created by removing the right endzone from G. Removing the right endzone of G will not change the coordinates of the vertex in the other endzone. Since one endpoint will be removed with the right endzone, we will create a "false" endpoint on an odd vertex in the almost-endzone adjacent to the right endzone. By the inductive hypothesis there must be a path on G_1 between the false endpoint and the remaining original endpoint. Since there is now one endpoint in the almost-endzone of G, the portion of the path through the rightmost column of G_1 must look like one of the two configurations in Figure 3.3.13.

Then a path exists on G by extending the path on G_1 through the false endpoint and into the removed endzone, ending at the original endpoint. There are three different ways the false endpoint and original endpoint can be configured, shown in Figure 3.3.14.

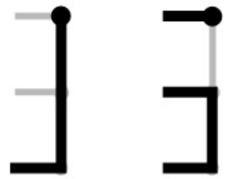

Figure 3.3.13.

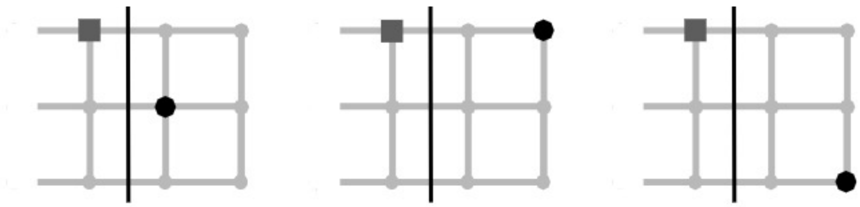

Figure 3.3.14.

Therefore the possible extensions look like one of the six configurations in Figure 3.3.15, depending on both the configuration of the initial path and the relative positioning of the false and original endpoint.

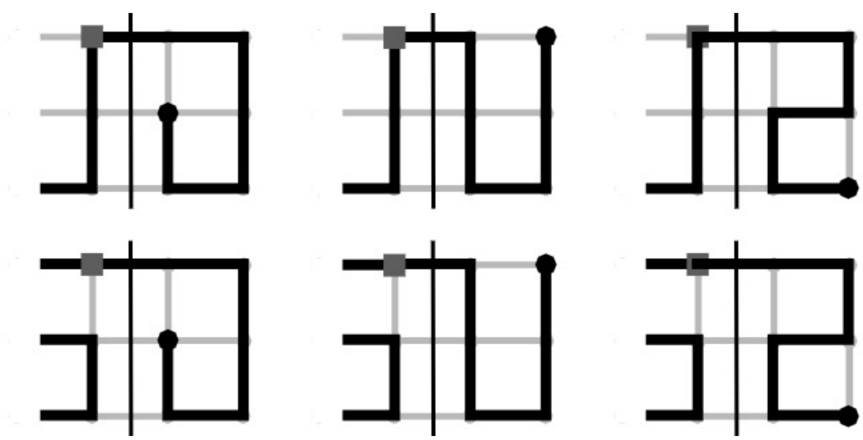

Figure 3.3.15.

By induction we prove that if the left endpoint of a grid graph G of size $3 \times n$ where $n \geq 4$ and n is even is on an even vertex and the right endpoint is on an odd vertex, then there is a solution to G. $\square$

3.4 Solution

We now have all the necessary lemmas to prove Theorem 3.1.1.

We restate the theorem as follows, in two parts. For a grid graph G of size $3 \times n$ where $n \geq 4$ and n is even there is a solution if the endpoints have opposite parity and any of the following hold.

1. The endpoints are both in the same column

2. The endpoints are in adjacent columns but not in the same column, and neither one is in the center.

3. The endpoints are not in the same column or adjacent columns, and the left endpoint has even parity (thus the right endpoint has odd parity).

There is no solution if any of the following hold.

1. The endpoints are not both in the same column, and the left endpoint is in the center row and has odd parity.

2. The two endpoints are not in the same column and not in adjacent columns, and the left endpoint is on the top or bottom and has odd parity.

For these last two cases, the same holds if the right endpoint has even parity, due to symmetry.

The cases together account for every possible pair of endpoints in a a $3 \times n$ grid graph where $n \in \mathbb{N}$ is an even number when the two endpoints have opposite parity, so if every item in the above list is proven, then so is Theorem 3.1.1.

Proof. Let G be a grid graph of size $3 \times n$ where $n \geq 4$ and n is even. Suppose the endpoints have opposite parity.

There is a solution to G if the two endpoints are in the same column, as stated in Lemma 3.3.1.

There is a solution to G if the two endpoints are in adjacent columns but not in the same column, and neither one is in the center, as stated in Lemma 3.3.3, which accounts for the case when both endpoints are on the bottom or top row; and Lemma 3.3.3, which accounts for the case when one endpoint is on the bottom row and the other is on the top row.

There is a solution to G if the two endpoints are not in the same column or adjacent columns, and the left endpoint has even parity (thus the right endpoint has odd parity), as stated in Lemma 3.3.5.

There is no solution to G if the endpoints are not both in the same column, and the left endpoint is in the center row and has odd parity, as stated by the combination of Lemma 2.2.2, which is specific to the end columns, and Lemma 2.2.3, which applies to all columns that are not the end columns.

There is no solution to G if the endpoints are not in the same column and not in adjacent columns, and the left endpoint is on the top or bottom and has odd parity, as stated by the combination of Lemma 2.2.4, which is specific to the next to end columns, and Lemma 2.2.5, which applies to all columns that are not the next to end columns.

Therefore every criteria of the theorem is proven. $\square$

4
Grid Graphs of the form $3 \times n$ where n is Odd

4.1 Lemmas

The following lemmas are required to prove the only if criteria for grid graphs such that $3 \times n$ where n is odd. The if criteria has already been proven in [2].

Lemma 4.1.1. *Let $n \in \mathbb{N}$ be an odd number, and let G be a $3 \times n$ grid graph. Suppose the endpoints have odd parity and are co-columned. Then there is no solution.*

Proof. The only possible configuration here is if the endpoints are on the top and bottom of an even column. We use proof by contradiction. Suppose there is a solution.

There is a special case when both points are in the column next to the end, as shown in Figure 4.1.1. Suppose the two endpoints are in the column next to an end column. Without loss of generality, suppose it is the left end column. The only way for the path to get to the top left and bottom left corners of G is if the path goes directly there by going left from one of the endpoints. However, if the path starts by going left from both endpoints as it must to reach the corners, it will meet itself in the center of the end column, thus not being able to reach any of the remaining vertices. Thus there is no solution in this special case. This argument applies in reverse if the endpoints are in the column next to the right end.

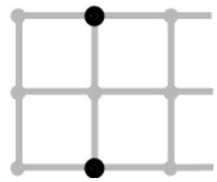

Figure 4.1.1.

Now suppose the two endpoints are in a column not next to an end column. Let the top endpoint be x, the bottom endpoint be y, and the number of the column with x and y be p.

If the path starts by going to the center from either endpoint, it will have blocked itself from filling one of the two sides of G, since when it goes in one direction or the other there is no way for it to get back to the other side of column p. Thus the path must start by going left or right from one of the endpoints. Without loss of generality we suppose it goes to the left; suppose the path starts at x. Let H be a $3 \times (p-1)$ sub-graph within G such that such that H is everything to the left of column p. Then the path will enter H from the top right vertex of H and fill in every vertex in H before going back right through the center of column p, since there is no way for the path to return to the left of column p after going through the center of column p. Thus H is a self-contained grid graph with endpoints as shown in Figure 4.1.2. However, there is no solution to H in this configuration since these endpoints have opposite parity and $(p-1)$ is an odd number. Thus there is a contradiction by Lemma 2.2.1, and there is no solution to G. $\square$

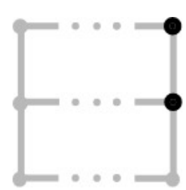

Figure 4.1.2.

Lemma 4.1.2. *Let $n \in \mathbb{N}$ be an odd number, and let G be a $3 \times n$ grid graph. Suppose the endpoints are non-co-columned and one endpoint is in the center of an odd column. Then there is no solution.*

Proof. If an endpoint in the center of an odd column is in an end column, then the proof is the same as that of Lemma 2.2.2. Otherwise, the proof is the same as that of Lemma 2.2.3. □

Lemma 4.1.3. *Let $n \in \mathbb{N}$ be an odd number, and let G be a $3 \times n$ grid graph. Suppose the endpoints are not co-columned, and an endpoint is on the top or bottom of an even column. Then there is no solution.*

Proof. First, suppose that the endpoints are not in adjacent columns. Then, if an endpoint at the top or bottom of an even column is in a column next to the end column, then the proof is the same as that of Lemma 2.2.4; otherwise, the proof is the same as that of Lemma 2.2.5.

Second, suppose the endpoints are in adjacent columns. Then, because the endpoints have the same parity by Lemma 2.2.1, the only possible case is that the endpoint not at the top or bottom of an even column is in the center of an odd column. By Lemma 4.1.2, there is no solution in this case. □

4.2 Theorem

Theorem 4.2.1. *Let $n \in \mathbb{N}$ be an odd number. Let G be a $3 \times n$ grid graph. Then there is a solution if and only if both endpoints have even parity.*

Proof. **Case 1**. Suppose both endpoints have even parity. Then there is a solution, as proven by Lemma 2.2.1.

Case 2. We use proof by contradiction. Suppose there is a solution and one of the endpoints has odd parity. As we know, the two endpoints on G must have the same parity in order for there to be a solution by Lemma 2.2.1. Thus, both endpoints have odd parity. From Lemma 4.1.1, there is no solution if the two endpoints are in the same column. From Lemma 4.1.2, there is no solution if either one of the endpoints is in the center row. From Lemma 4.1.3, there is no solution if the endpoints are in different columns and either one of the endpoints is in either the top or bottom rows. Thus, there is no solution in this case. □

5

Grid Graphs of the form $m \times n$

5.1 Preliminary Lemmas

The following lemma is very useful for solving theorems for larger grid graphs, and is used in every remaining theorem in this project.

Lemma 5.1.1. *Let $n \in \mathbb{N}$. Suppose $n \geq 3$. Let G be a $2 \times n$ grid graph. Let one endpoint be in a bottom corner of G. If the other endpoint has opposite parity, then there is a solution.*

Proof. Without loss of generality, let the endpoint in a bottom corner be in the bottom right corner. Let the point in the bottom left corner of G be a, and let the other endpoint be b. There is a method to draw out a solution for any length of G that depends upon whether b is on the top row or the bottom row.

Case 1: Suppose b is on the bottom row. The path starts from a by going upwards, and then turns 90 degrees in the only direction possible at every vertex until it reaches the vertex next to b on the bottom row, which it will because there is an even number of vertices between a and b as they have opposite parity. Then the path goes upwards, then turns left and goes all the way to the end column, turns left twice, then goes straight to b. Thus there is a solution to G in this case. An example of this solution is shown in Figure 5.1.1.

Figure 5.1.1.

Case 2: Suppose b is it the top row. The path starts from a by going upwards, and then turns 90 degrees in the only direction possible at every vertex until it reaches the vertex under to b on the bottom row, which it will because there is an even number of vertices between a and b as they have opposite parity. Then the path continues straight until it reaches reaches the left end column, turns right twice, then goes straight to the endpoint. Thus there is a solution to G in this case. An example of this solution is shown in Figure 5.1.2.

Therefore, since there is a solution in both cases, there is a solution to G. $\square$

Figure 5.1.2.

5.2 $m \times n$ Grid Graphs where m, n are Odd

Theorem 5.2.1. *Let $n, m \in \mathbb{N}$ be odd numbers. Suppose $m, n \geq 3$. Let G be a $m \times n$ grid graph. If both endpoints have even parity, then there is a solution.*

Proof. Suppose both endpoints have even parity. We prove that there is a solution by induction on m.

Base Case: Suppose $m = 3$. Then the situation is as in Theorem 4.2.1. Thus there is a solution if and only if the endpoints have even parity.

Inductive Case: Assume that on a grid graph of size $m \times n$ with two endpoints, if the endpoints satisfy the hypothesis of this theorem, then there is a solution to the grid graph. Let G be a grid graph of size $(m+2) \times n$. Suppose that the endpoints satisfy the hypothesis of this conjecture.

Case 1: Suppose there are no endpoints in at least one vertical endzone of G. Without loss of generality, assume there are no endpoints in the top vertical endzone of G.

Let G_1 be a grid graph of size $m \times n$ created by removing the top vertical endzone from G.

Removing the top endzone of G will not change the parity of the vertices in G since they begin in the bottom left corner. Therefore, the endpoints in G_1 have the same parity as the endpoints of G, so by the inductive hypothesis, there is a solution to G_1.

Every vertex in the top row of G_1 except any vertex that contains an endpoint, if there are any, must have a portion of path going to the left or right of it along the row, since when the path runs into the top row of the grid graph it must either turn along the row or end. If a break of length 1 is made in one of these path segments, then the path can be extended into a loop through the removed endzone of G. The method of doing so would be for the path extension to start to the left side of the break, go up to the next row, turn left, go all the way to the end, turn right, go up one to the top row of G, turn right, go all the way to the end, turn right, turn right again, then go all the way forward until it runs into the previous path, then turn down to meet up with the right side of the break, as shown in the example in Figure 5.2.1. Thus there is a solution to G in this case.

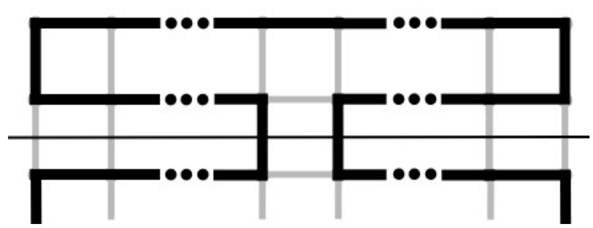

Figure 5.2.1.

Case 2: Suppose there is an endpoint in both vertical endzones of G. Let G_1 be as in Case (1), but note that there is now only one endpoint in G_1. We define a false endpoint in the top right corner of G_1. Since the top vertical endzone of G is a $2 \times n$ grid graph, by Lemma 5.1.1 there is a solution to the endzone that maps from the vertex directly above the false endpoint to the real endpoint, wherever that may be. Thus a solution to G can be found by connecting the solution to G_1 with the solution to the top vertical endzone of G by adding one edge going upward from the false endpoint to the bottom right corner of the top vertical endzone.

Therefore there is a solution to G in all cases. □

Thus we know through the above theorem under what conditions there is a solution to an $m \times n$ grid graph where $n, m \in \mathbb{N}$ are odd numbers. However, we do not know whether those are only if conditions, as that would require a separate proof. My thoughts for proving that are stated below, however I was unable to find a valid proof. Without such a proof, we cannot know for certain whether these are the only conditions in which there is a solution for such a grid graph.

Conjecture 5.2.2. *Let $n, m \in \mathbb{N}$ be odd numbers. Suppose $m, n \geq 3$. Let G be a $m \times n$ grid graph. Then there is a solution only if both endpoints have even parity.*

The following are my ideas to prove that if a path starts at a vertex with odd parity of an $n \times m$ grid graph G where m, n are odd integers, there is no way for the path to reach every vertex in G.

A path on G can reach every vertex if it starts at a vertex with even parity. In a 3×3 grid graph, there is no way to reach every vertex if the path starts at a vertex with odd parity. If a path starts at a vertex adjacent to a corner, then the only way for the path to reach the corner is for it to end in the corner. However, this would mean that the endpoints have opposite parity, which is impossible by Lemma 2.2.1. If a path starts at a point of even parity in an even column, then it is possible (would need proof) for it to reach every vertex within an $o \times n$ sub-graph consisting of the column containing the starting point and everything to the left or right of that column, without leaving that sub-graph. If it does so then by the previously referenced lemma it will end at a point with odd parity, because o is even. Then in order for the path to continue into the remainder of the grid graph, the path must go directly right or left in the empty $(m-o) \times n$

remainder of the graph, which requires the path to again start at a point with even parity and attempt to fill an odd by odd grid graph.

These ideas are very incomplete, and an idea for a proof does not immediately lend itself from them. However, the proof would most likely require induction.

5.3 $m \times n$ Grid Graphs where at least one of m, n is Even

The following conjecture states if and only if criteria for the conditions required for there to be a solution for all grid graphs where m, n are not odd and $m, n > 3$. The conditions are much less restrictive than they were for $3 \times n$ grid graphs where n is even, which is why this theorem does not include the $m = 3$ case.

Conjecture 5.3.1. *Let $m \in \mathbb{N}$ and let $n \in \mathbb{N}$. Let at least one of m, n be an even number. Suppose $m, n > 3$. Let G be a $m \times n$ graph. Then there is a solution if and only if the endpoints have opposite parity.*

I have the majority of a proof for this conjecture, however I am missing two minor cases. The portions of the proof which I have completed are as follows.

Suppose the endpoints have opposite parity. We prove that there is a solution to G. Suppose n is even.

Case 1: Suppose there is an endpoint in both vertical endzones of G. We treat the top vertical endzone as a self-contained subgraph, denoted H. Then there is a way to create a path through H that starts at the endpoint in H, ends at whichever bottom corner of H has opposite parity to the endpoint in H, and reaches every vertex in H, as stated in Lemma 5.1.1.

We will then continue to construct a solution by continuing downwards one vertex from the end of the path in H that is a bottom corner of H, as shown in Figure 5.3.1.

Let a be the vertex that is at the end of the current path and that is not in H. From here there are four subcases, $m = 4$, $m = 5$, $m > 5$ where m is odd, and lastly $m > 5$ where m is even.

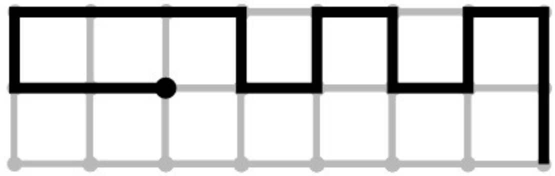

Figure 5.3.1.

Subcase 1: Suppose $m = 4$. Then the path we have constructed so far has covered every vertex except for those in the bottom two rows (other than a). These two rows can be treated as a self-contained sub-graph. Let that sub-graph be I. We observe that a has the same parity as the original endpoint that is in H, because a was reached by moving one vertex down from the bottom corner of H which has the opposite parity as the original endpoint. Thus a has the opposite parity as the endpoint in I, and hence there is a solution to I by Lemma 5.1.1. Thus there is a solution in this case.

Subcase 2: Suppose $m = 5$. Then, at this point, the path has covered every vertex except for those in the bottom three rows. These three rows can be treated as a self-contained sub-graph. Let that sub-graph be J. We observe that a has the same parity as the original endpoint that is in H, because a was reached by moving one vertex down from the bottom corner of H which has the opposite parity as the original endpoint. If a is in the top left corner, then it has even parity, and is either the left endpoint on J or co-columned with the original endpoint in J. If the a is in the top right corner, then it has odd parity, and is either the odd endpoint on H or co-columned with the original endpoint in J. Thus there is a path that solution to J either way by Theorem 3.1.1. Therefore, there is a solution to G in this subcase.

Subcase 3: Suppose $m > 5$ and m is odd. Then it is possible to draw a path from a to the vertex two rows below a while going through every vertex in those two rows. This can be accomplished by going from the a sideways to the opposite side of the grid graph, down one segment, sideways all the way back, then down one segment to reach the point the point two rows below a. This point has the same parity as a. This process can be repeated as many times as necessary until the path reaches the 3rd row from the bottom. This method is shown in Figure 5.3.2, where a is represented as a square.

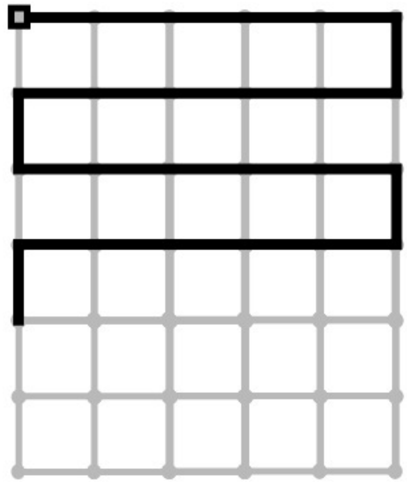

Figure 5.3.2.

At this point, the path has reached a top corner of the bottom three rows, and that corner has the same parity as a. Thus due to the same reasons as in Subcase 2 there is a solution by Theorem 3.1.1.

An example grid graph solved in this way is shown in Figure 5.3.3.

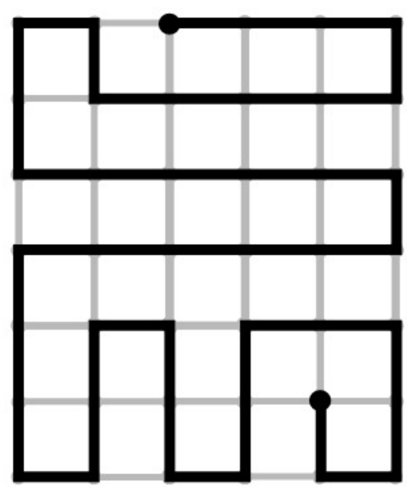

Figure 5.3.3.

Subcase 4: Suppose $m > 5$ and m is even. Then it is possible to draw a path from a to the vertex two rows below a while going through every vertex in those two rows. This can be accomplished by the same method as in Subcase 3. This process can be repeated as many times as necessary until the path reaches the 2nd row from the bottom. At this point, the path has reached a top corner of the bottom two rows, and that corner has the same parity as a. Thus due to the same reasons as in Subcase 1 there is a solution by Lemma 5.1.1.

An example grid graph solved in this way is shown in Figure 5.3.4.

Thus there is a solution if there is an endpoint in each endzone.

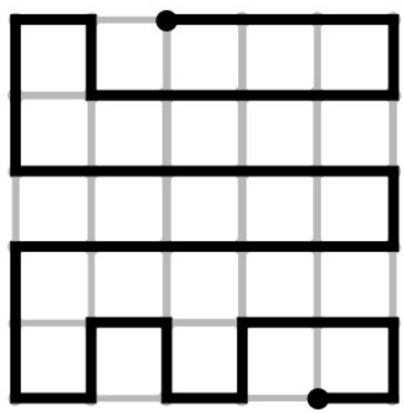

Figure 5.3.4.

Case 2: Suppose there is not an endpoint in exactly one vertical endzone of G and the other vertical endzone has exactly one endpoint. Without loss of generality, suppose that vertical endzone is the top vertical endzone. Let the bottom portion of the graph containing the two endpoints be solved as in Case 1, such that the solved portion of the graph is the minimum height such that it contains both endpoints and above it there are an even number of unsolved rows remaining. Every vertex in the top row of the solved portion of the graph except any vertex that contains an endpoint, if there are any, must have a portion of path going to the left or right of it along the row, since when the path runs into the top row of the graph it must either turn along the row or end. If a break of length 1 is made in one of these path segments, then the path can be extended into a loop through the next two rows of G. The method of doing so would be for the new portion of path to start to the left side of the break, go up to the next row, turn left, go all the way to the end, turn right, go up one row, turn right, go all the way to the end, turn right, turn right again, then go all the way forward until it runs into the previous path, then turn down to meet up with the right side of the break, as shown in the example in Figure 5.2.1. If there were only two unsolved rows in G, then this is a solution to G. Otherwise, another break can be made in the top row of the new path, and the same process of creating a two-row loop can be repeated. This process can be repeated as many times as necessary until the path reaches the top row of G. Thus there is a solution to G in this case.

Case 3: There are no endpoints in either endzone and the two endpoints are not in the same row or adjacent rows. Let the top portion of the graph containing the two endpoints be solved as in Case 2, such that the solved portion of the graph is the minimum height such that it contains both endpoints and above, and there are an even number of unsolved rows remaining at the bottom of the graph. Every vertex in the bottom row of the solved portion of the graph except any vertex that contains an endpoint, if there are any, must have a portion of path going to the left or right of it along the row, since when the path runs into the top row of the graph it must either turn along the row or end. If a break of length 1 is made in one of these path segments, then the path can be extended into a loop through the next two rows of G below the row with the break. The method of doing so would be for the new portion of path to start to the left side of the break, go down to the next row, turn left, go all the way to the end, turn right, go down one row, turn right, go all the way to the end, turn right, turn right again, then go all the way forward until it runs into the previous path, then turn up to meet up with the right side of the break. If there were only two unsolved rows in G, then this is a solution to G. Otherwise, another break can be made in the bottom row of the new path, and the same process of creating a two-row loop can be repeated. This process can be repeated as many times as necessary until the path reaches the top row of G. Thus there is a solution to G in this case.

Case 4: Suppose both endpoints are in the same vertical endzone. I have not completed a proof for this case.

Case 5: Suppose the two endpoints are in the same row or adjacent rows and not in either vertical endzone. I have not completed a proof for this case either.

If Case 4 and Case 5 were to be proven, then this would be a complete one direction of the if and only if proof.

By Lemma 2.2.1, there is no solution to a grid graph of size $m \times n$ such $m, n \in \mathbb{N}$ and at least one of m, n is an even number. Hence there is only a solution if the endpoints have opposite parity. Thus there is a solution to G only if the hypotheses hold.

5.4 $m \times n$ Grid Graphs

This conjecture is the final result of this project. This would be the final theorem for m, n grid graphs such that $m, n \in \mathbb{N}$ and $m, n > 3$; however, certain proofs are incomplete.

Conjecture 5.4.1. *Let $n, m \in \mathbb{N}$. Suppose $m, n > 3$. Let G be a $m \times n$ grid graph. Then if m, n are both odd, there is a solution if and only if both endpoints have even parity. Otherwise, there is a solution if and only if the endpoints have opposite parity.*

The portions of this conjecture which are incomplete are the only if criteria for the case when m, n are both odd; and the two remaining cases for the case when at least one of m or n is even. The next steps for if someone does another project on this topic in the future would be to determine that only if criteria, prove those two cases, and then move on to exploring criteria for two pairs of endpoints.

Bibliography

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