Spring 2021

Tverberg Type Partitions: Sub-Regular and Elliptical Polygons

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A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
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Annandale-on-Hudson, New York
May, 2021

Abstract

Tverberg's theorem states that given a set S of $T(r,d) = (r-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of S into r subsets whose convex hulls intersect. A feature of Tverberg's theorem is that $T(r,d)$ is tight, so in this senior project we investigate Tverberg-type results when $|S| < T(r,d)$. We found that in $\mathbb{R}^2$, given a set S of $T(r,2) - 2 = 3r - 4$ points, and assuming $r = r_1 r_2$, there exists a partition of S into r sets such that when grouped into r_1 collections of r_2 sets, the convex hulls of each collection overlap, and we can find the vertex set of a regular r_1-gon with one point from the intersection of each collection. We also show that given a similar construction but with $|S| = 3r - 6$, we can find the vertices of an r_1-gon in the intersections of convex hulls, with vertices on an ellipse, and other nice regularity properties.

Contents

Dedication

For my parents.

Acknowledgments

I did not come to Bard with a major in mind. Rather, I was drawn to the math department which provided such a fun and enriching atmosphere in which to learn. Therefore, I would like to express my profound gratitude to all the math faculty I worked with.

Thank you, Steven Simon, for your generous support and guidance. In addition to always taking time to help me understand, your explanations would often let me peek into what mathematics lies ahead, both at Bard and beyond. I am so happy that I was able to do my senior project with you. You are an amazing teacher.

Thank you, John Cullinan. No matter what worry I frantically brought to you, you always made it feel manageable. You are unbelievably kind, and I thoroughly enjoyed all of your classes. Your love of math is infectious, and your classes were incredibly exciting and made me want to learn more.

Thank you, Stefan Méndez-Diez. You are always so caring and helpful in our discussions. It was with you that I had my first taste of mathematical research, and it is In part because of this wonderful experience that I have decided to pursue applied math.

I would also like to thank Richard van Den Doel who, during my study abroad at UCR, introduced me to the wonders of finite abelian groups.

Thank you, Raman Ramakrishnan for being such an outstanding cello teacher these 4 years at Bard. you have taught me so much.

I am also incredibly grateful to all the friends I made at Bard, they are all precious to me. I would like to thank a few individually. Thank you, Isabela, the resident mixologist, and oatmeal maker extraordinaire, for literally (figuratively) everything. Thank you, Stephanie, for supplying me with the ramen I needed to finish sproj. Thank you, Nathanael, for reminding me how capitalization works. Thank you, Riti, for being my partner in mathematical crime. Thank you, Sadia, for those wonderful extended work sessions (and those dates!).

1
Introduction

In 1921 Radon published a theorem that is now foundational in convex geometry. It states that given a set S of $d+2$ points in $\mathbb{R}^d$, there exists a partition of S into two subsets whose convex hulls intersect. In 1966 Tverberg generalized this result to partition S by an arbitrary number of subsets (2.0.3). His theorem states that, for any $r \geq 2$, and given a set S of $T(r,d) = (r-2)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of S into r subsets such that all of their resulting convex hulls intersect (2.0.4).

An important aspect of Tverberg's theorem is that $T(r,d)$ (called Tverberg's number 2.0.6) is tight, meaning almost any set with fewer points can't be partitioned as in Tverberg's theorem (2.0.8). This leads us to wonder, with a smaller number of points than what Tverberg's theorem requires, what can we conclude? In the pursuit of some answers to the above question, we use an alternative description of Tverberg's theorem that we can more easily manipulate in order to draw conclusions for collections of points where Tverberg's theorem fails.

In Chapter 2, we show how Tverberg's theorem can be restated such that instead of picking points in a plane, we're choosing an arbitrary affine map from a simplex to $\mathbb{R}^d$ (2.0.18). For even d, this reformulation in terms of affine maps allows us to break them up into maps from a simplex to $\mathbb{R}^2 \cong \mathbb{C}$.

In Chapter 3, we discuss finite Fourier analysis on finite abelian groups, which we use to construct a basis for particular affine maps that we can identify with these smaller affine maps from a simplex to $\mathbb{C}$. This basis allows us to deconstruct any of the smaller Tverberg-type affine maps to $\mathbb{C}$ (3.3.4), and by extension any affine map from a simplex to $\mathbb{R}^{2d} \cong \mathbb{C}^d$.

In Chapter 4 we demonstrate how this Fourier deconstruction of affine maps can be used to obtain Tverberg-type theorems. We show all affine maps are completely determined by the coefficients of their Fourier decompositions, then it can be proven exactly which coefficients need to be eliminated to produce a full Tverberg partition (4.1.1). Intuitively eliminating more coefficients from a decomposition imposes a strong condition on the function and therefore requires a greater number of initial points. Thus, the question of how many points are required for a conclusion becomes a question of how many coefficients were eliminated from the Fourier decomposition of the affine map. With this in mind, we can eliminate fewer coefficients than are required for a Tverberg partition in order to conclude something about a smaller number of initial points. Eliminating one coefficient less than for a Tverberg partition leads us to a different kind of partition called a regular $r-$gon partition ([1] Theorem 1.1). Geometrically, in an $r-$gon partition, instead of finding the same point in r convex hulls (an intersection), you find r points, one from each convex hull, that form the vertex set of a regular $r-$gon (4.3.4). We then use Theorem 3.1 from paper [1] to give us the conditions under which we can make a selection of coefficients vanish (restated in 4.3.1). This theorem implies that an $r-$gon partition can occur in $\mathbb{R}^2$ when given $3r-4 = T(r,2)-2$ points (4.3.5).

In Chapter 5 we prove a main result of this senior project, concerning the sub $r-$gon partition. The same number of coefficients are eliminated as for a regular $r-$gon partition, but because of the coefficients we chose to have vanish, we can create overlap in some convex hulls, and find a point from each intersection of convex hulls to form the vertex set of a regular r_1-gon, where r_1 is a factor of r. More specifically,

Theorem 1.0.1. *(5.0.4) If $r = r_1 r_2$, almost any set of $3r-4$ points in $\mathbb{R}^2$ can be partitioned into r subsets, $A_1^1, \dots, A_{r_2}^1, \dots, A_1^{r_1}, \dots, A_{r_1}^{r_1}$, such that* *(a)* $\cap_{i=1}^{r_2} Conv(A_i^1) \neq \emptyset, \ldots, \cap_{i=1}^{r_2} Conv(A_i^{r_1}) \neq \emptyset$*, and*

(b) there exists a set of points $y_1 \in \cap_{i=1}^{r_2} Conv(A_i^1), \ldots, y_{r_1} \in \cap_{i=1}^{r_2} Conv(A_i^{r_1})$ *which form the vertex set of a regular* r_1-*gon*

This generalizes the regular $r-$gon result since it can be recovered when $r_1 = r$ and $r_2 = 1$.

In Chapter 6 we look at extensions of the $r-$gon result, as well as our sub $r-$gon result, to higher dimensions. These results are similar to the ones of two cartesian dimensions, only instead of finding just regular polygons, we find analogous partitions for products of polygons in $\mathbb{R}^{2d}$, called multiprisms (6.1.5). We also give results for sub $r-$gon partitions in $\mathbb{R}^{2d}$.

In Chapter 7 we ask, as we did initially with Tverberg's theorem, what can be said for collections of points even smaller than required for an $r-$gon partition. Our second major result comes with the introduction of elliptical $r-$gons. The points of a regular $r-$gon lie on a circle. The points of an elliptical $r-$gon lie on an ellipse, in addition to having some other nice symmetries (7.1.4). For instance, when r is even, opposing edges will have the same length and be parallel. Specifically we have the following result:

Theorem 1.0.2. *(7.2.4) Given almost any set* S *of* $3r-6$ *points in* $\mathbb{R}^2$*, we can partition* S *into* r *subsets* $A_1, \ldots, A_r$ *such that there exist points* $y_1 \in Conv(A_1), \ldots, y_r \in Conv(A_r)$ *that form the vertex set of an elliptical* $r-$*gon.*

Finally, we also have results analogous to Chapter 5 for elliptical sub $r-$gons, once again getting the points from the intersections of convex hulls.

2

Convex Geometry Background

First, we introduce the reader to the topic with contextual definitions and theorems in convex geometry. Then we'll be able look at Radon's theorem, and Tverberg's generalization thereof.

Definition 2.0.1. We call $A \subset \mathbb{R}^d$ **convex** if given any two points, $x, y \in A$, then the line segment joining them is also in the subset, i.e. $\{tx + (1-t)y | 0 \leq t \leq 1\} \subset A$.

In $\mathbb{R}^2$ convex sets can be demonstrated pictorially.

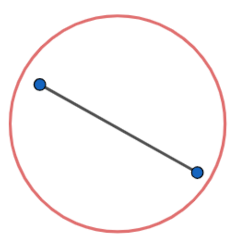

Figure 2.0.1. Convex Set Example

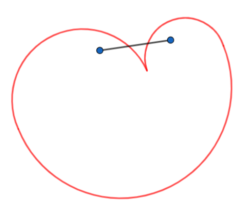

Figure 2.0.2. Convex Set Non-Example

Given any two points inside of a disk, the line segment joining them will also be contained disk. In Figure 2.0.2 we see a subset of $\mathbb{R}^2$ in which there exist two points in the subset whose line segment is not contained in the set, implying that the set is not convex.

Definition 2.0.2. Given a set, S, of points in $\mathbb{R}^d$, the **convex hull** of this set, denoted $\text{Conv}(S)$, is the intersection of all convex sets containing S.

The convex hull of a set of points is the smallest convex set containing all those points. In $\mathbb{R}^2$ this results in a line segment or a polygon whose vertices are points from S (Though not necessarily all of the points from S, e.g. The convex hull of three colinear points is a line). Examples of convex hulls for sets of 2 and 3 points are shown below.

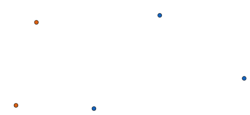

Figure 2.0.3. Two sets of points

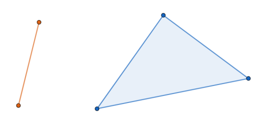

Figure 2.0.4. Two convex hulls

The convex hull of the two orange points is a line, and the convex hull of the three blue points is a triangle. Note that the three blue points in Figure 2.0.9 are affinely independent (see Definition 2.0.14 below), and if they weren't then their convex hull would be a line segment instead. With this we can understand Radon's thoerem.

Theorem 2.0.3 (Radon's Theorem)**.** *For any set, $S \subset \mathbb{R}^d$, where $|S| = d + 2$, there exist $A_1, A_2 \subset \mathbb{R}^d$, where $A_1 \cup A_2 = S$, and $A_1 \cap A_2 = \emptyset$, such that $Conv(A_1) \cap Conv(A_2) \neq \emptyset$.*

This means that given $d + 2$ points in $\mathbb{R}^d$, we can partition them into two sets whose convex hulls intersect. In $\mathbb{R}^2$ this can happen in one of two ways.

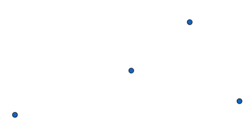

Figure 2.0.5. First scenario points

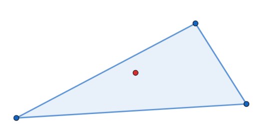

Figure 2.0.6. First scenario convex hulls

Figure 2.0.7. Second scenario points

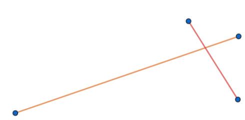

Figure 2.0.8. Second scenario convex hulls

The two cases are distinguished by the the convex hulls chosen for them to overlap. In Figure 2.0.5 we see a collection 4 points in $\mathbb{R}^2$, and in Figure 2.0.6 we see that those points have been partitioned into one set of three points and a singleton set such that the convex hulls of those sets overlap. In the second scenario, Figure 2.0.7 and Figure 2.0.8, we can see that it was instead required to partition the four points into two sets of two whose convex hulls cross each other.

Radon's result was later generalized by Tverberg into r disjoint sets.

Theorem 2.0.4 (Tverberg's Theorem)**.** *For any set, $S \subset \mathbb{R}^d$, where $|S| = (r-1)(d+1)+1$, there exist pairwise disjoint subsets $A_1, \ldots, A_r \subset \mathbb{R}^d$, where $\cup_{i=1}^{r} A_i = S$, such that $\cap_{i=1}^{r} Conv(A_i) \neq \emptyset$. This is called an $r-$fold tverberg partition.*

Informally, this means that given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, we can partition them into r sets whose convex hulls intersect.

Remark 2.0.5. The number of points required for Tverberg an Radon are the minimum. The same is true for any greater set of points, because adding more points only makes convex hulls larger, which will not hinder their overlapping. We'll discuss the implications of this more shortly. What 'minimum' means in this case will be explained further below.

The following is an example of Tverberg's theorem with $d = 2$ and $r = 3$. Note that this means $(r-1)(d+1)+1 = (2)(3)+1 = 7$ points are required.

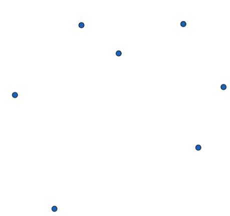

Figure 2.0.9. Set of 7 points

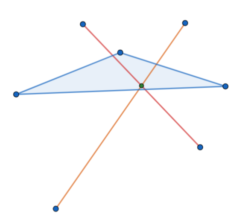

Figure 2.0.10. 3−fold Tverberg intersection

Note that we recover Radon's theorem when $r = 2$. The required number of points for a tverberg partition will be of significant interest to us in this paper. For this reason, we define this number.

Definition 2.0.6. The number of required points for a $r-$fold Tverberg partition, $T(r,d) = (r-1)(d+1)+1$, is called **Tverberg's Number**.

Definition 2.0.7. We say a statement is **almost always** true if it occurs always except on a set of measure zero.

Definition 2.0.8. If a statement is dependent on a number N, then N is **tight** if the statment is almost always false given $n < N$.

Remark 2.0.9. An important feature of $T(r,d)$ is that it is tight. Thus given almost any collection of points less than $T(r,d)$, we cannot find a full Tverberg partition.

We illustrate with an example. Tverberg's number being tight indicates that you almost always fail to get a $2-$fold Tverberg partition given less than $T(2,2) = 4$ points in $\mathbb{R}^2$. Consider the following case in which we can Tverberg partition 3 points.

Figure 2.0.11. 3 points with 2-fold Tverberg partition

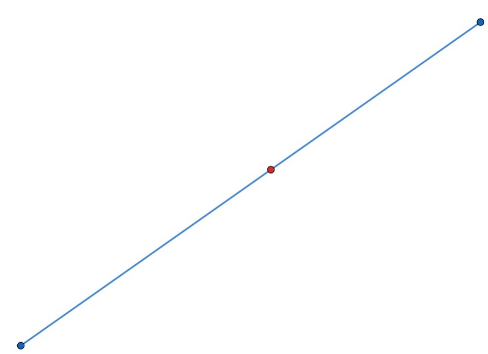

Figure 2.0.12. Tverberg partition

It can be seen pictorially, that in order to get a $2-$fold Tverberg partition with 3 points, it is necessary for the those three points to be colinear, as the points in 2.0.11 are. Since this is a very specific orientation for the points have relative to one another, it follows that for most sets of 3 points we would fail to get a $2-$fold Tverberg partition. The same is true for $T(r,d)$ in higher dimension, or with greater r.

Tverberg's theorem can be understood in other contexts as well. For that we'll need some more definitions.

Definition 2.0.10. Let V be a vector space, and $\vec{u} \in \mathbb{R}^d$. Then $A = V + \vec{u}$, is an **Affine Space**.

Affine spaces can be described as shifted vector spaces.

Definition 2.0.11. Let X be an affine space. A **convex combination** of $v_1, \cdots, v_n \in X$ is a linear combination $\sum_{i=1}^n t_i v_i$, where $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$.

This leads us to an alternate description of convex hull.

Proposition 2.0.12. *Let X be an affine space, and Y be the set containing all convex combinations of $v_1, \cdots, v_n \in X$. Then $Y =$ Conv$(\{v_1, \ldots, v_n\})$.*

Proof. Let Y be the set of all convex combinations of $v_1, \cdots, v_n \in X$. Thus $Y = \{\sum_{i=1}^n t_i v_i |$ for all sets of $t_i \geq 0$ where $\sum_{i=1}^n t_i = 1\}$. If we express the edge between v_i and v_j, as $t_i v_i + t_j v_j$ where $t_j = 1 - t_i$, we can see that this edge is contained in Y. Similarly, if we consider the convex combination of a third v_p, and the edge $t_i v_i + t_j v_j$. Assuming $t_p, t' \geq 0$, and $t_p + t' = 1$, it follows $t_p v_p + t'(t_i v_i + t_j v_j) = t_p v_p + t' t_i v_i + t' t_j v_j$ and $t_p + t' t_i + t' t_j = (1 - t') + t' t_i + t'(1 - t_i) = 1$. Therefore, the line connecting v_j and any point on the edge $t_i v_i + t_j v_j$ is contained Y. Expanding to the total of n initial points, it follows that Y contains all lines between those points, all lines between points on those lines, and so on. Since Y contains all edges between points it contains, it follows that it is convex, $Y \supset \text{Conv}(\{v_1, \ldots, v_n\})$. But also, since the convex hull must contain all edges between its points, it follows that it must also contain all convex combinations of those points and therefore $Y \subset \text{Conv}(\{v_1, \ldots, v_n\})$. Thus $Y = \text{Conv}(\{v_1, \ldots, v_n\})$ □

Definition 2.0.13. Let X and Y be affine spaces. The map $f : X \to Y$ is an **affine map** if $f(t_1 v_1 + \cdots + t_n v_n) = t_1 f(v_1) + \cdots + t_n f(v_n)$ for $v_1, \cdots, v_n \in X$, and $t_1, \cdots, t_n \in \mathbb{R}$, where $t_1 + \cdots + t_n = 1$.

Note that this implies that, with an affine map, convex combinations in X are mapped to convex combinations in Y. However, it is also important to note that the definition 2.0.13 includes $t_i \notin [0, 1]$. This means the definition of affine maps is more general than just sending convex combinations to convex combinations. These maps send points in X to points in Y, lines in X to lines in Y, etc.. Affine maps also preserve parallel lines.

Definition 2.0.14. The points $x_0, \cdots, x_n \in \mathbb{R}^k$ are **Affinely Independent** if $x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0$ are linearly independent.

Now we define the simplex, which is a triangle generalized to arbitrary dimensions.

Definition 2.0.15. A $N-$**Simplex** (symbolized by $\triangle^N$) is the Convex Hull of $N+1$ affinely independent points in $\mathbb{R}^d$ for $d \geq N$.

For example, a $0-$simplex is a point, a $1-$simplex is a line segment, a $3-$simplex is a triangle, and so on. In the case of the $3-$simplex, it is clear that it's vertices and edges are subsets of it. The following generalizes this concept.

Definition 2.0.16. A $k-$**face** of a $N-$simplex is the convex hull of $k+1$ of the N orignal affinely independent points. Thus a $k-$face is a $k-$simplex.

By 2.0.12, the $N-$Simplex made from a set of affinely independent points, $x_1, \cdots, x_{N+1} \in \mathbb{R}^N$ is the set of all convex combinations of those points:

$$\triangle^N = \{a_1x_1 + \cdots + a_{k+1}x_{N+1} | a_i \geq 0 \ \forall i, \text{ and } a_1 + \cdots + a_{N+1} = 1\} \tag{2.0.1}$$

As the the only convex combination of a point is itself, the $0-$faces of $\triangle^N$ are the vertices $x_1, \cdots, x_{N+1}$. Similarly, the $1-$faces are the line segments between any two vertices, and a $k-$face is the convex combination of k vertices.

Now we can restate Radon's theorem in terms of affine maps. If X is an affine space, and $v_1, \cdots, v_n \in X$, then for an affine map f, by definition (2.0.13) it follows $f(\sum_{i=1}^n t_i v_i) = \sum_{i=1}^n t_i f(v_i)$. Therefore f is completely determined by the images of $v_1, \cdots, v_n$. For this reason we can equate arbitrary points in $\mathbb{R}^d$ with affine maps from a simplex.

Theorem 2.0.17 (Affine Radon Theorem)**.** *Let $\triangle^{d+1}$ be a $(d+1)-$dimensional simplex. For any affine map $f : \triangle^{d+1} \to \mathbb{R}^d$, there exist two disjoint faces σ_1, σ_2 of $\triangle^{d+1}$ such that the images of the faces overlap, i.e. $f(\sigma_1) \cap f(\sigma_2) \neq 0$.*

We show two examples with affine maps f and g, where $d = 2$, below.

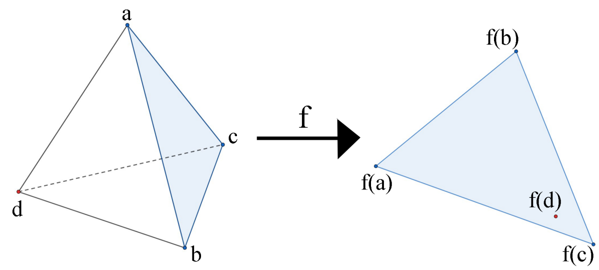

Figure 2.0.13. Affine Radon First Scenario

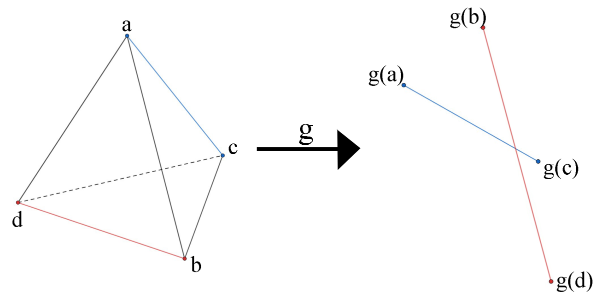

Figure 2.0.14. Affine Radon Second Scenario

Above we see two scenarios in which an affine map is mapping faces of a $3-$simplex onto the plane.

In Figure 2.0.13, given the images of the points a, b, c, and d, the selection of disjoint faces of the simplex, $\text{Conv}(\{a, b, c\})$ and $\text{Conv}(\{d\})$, allowed the images of those faces to overlap, i.e. $f(\text{Conv}(\{a, b, c\})) \cap f(\text{Conv}(\{d\})) \neq \emptyset$. In Figure 2.0.14, the affine map g mapped a, b, c, and d differently than f. This required a different selection of disjoint faces of the simplex, $\text{Conv}(\{a, c\})$ and $\text{Conv}(\{b, d\})$. With this choice, we see that $f(\text{Conv}(\{a, c\})) \cap f(\text{Conv}(\{b, d\})) \neq \emptyset$

In the original formulation of Radon's theorem, points in the plane were chosen arbitrarily, in this version, the affine map is arbitrary, and consequently maps the vertices of the simplex onto the plane arbitrarily. Thus Affine maps from an $N-$simplex to $\mathbb{R}^d$ are equivalent to $N + 1$ points in $\mathbb{R}^d$.

We noted earlier that in two dimensions there are two types of intersections that one can get with Radon's theorem (Figures 2.0.6 and 2.0.8). This is reflected in the affine version, because there are only two ways to pick two disjoint faces of a $3-$simplex.

Similarly, we can restate Tverberg's Theorem in terms of affine maps.

Theorem 2.0.18 (Affine Tverberg Theorem)**.** *Let $r \geq 2$, and $d \geq 1$. Now let $N = (r-1)(d+1)$, and $\triangle^N$ be a $N-$dimensional simplex. For any affine map $f : \triangle^N \to \mathbb{R}^d$, there exist r pairwise disjoint faces, $\sigma_1, \cdots, \sigma_r$ of $\triangle^N$ such that the images of the faces all overlap, $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Here we see $N = T(r, d) - 1$ because the $N-$simplex has $N + 1 = T(r, d)$ points, as Tverberg's original theorem requires. As $\mathrm{T}(r, d)$ is tight, so is N. In this paper we will explore conclusions that can be drawn with smaller N.

3

Fourier Analysis for Finite Abelian Groups

In this chapter we set up a finite Fourier basis to deconstruct affine maps like the ones found in the Affine Tverberg (Theorem 2.0.18). This is all in the effort to produce Tverberg-type results when $N < T(r, d) - 1$. It is important to note that the results in this chapter are not unique to this senior project, and are standard. The following is one reference for this material [2]. I include them because they were new to me, and if the reader chooses to read through them, they give a better understanding of why we are able to get the results we do in subsequent chapters. Section 3.1 and 3.2 set up for Section 3.3, which has the main result of this chapter that will be referenced frequently.

3.1 Groups and Operations

The affine maps we're hoping to deconstruct map from a simplex to $\mathbb{C}$. In chapter 4 we'll be able to associate points chosen from pairwise disjoint faces of a simplex with the elements of finite abelian groups. For this reason, since groups have regularity we can exploit, we'll be able perform finite Fourier analysis on maps from G to $\mathbb{C}$ instead. First, we'll define a set of such maps.

Definition 3.1.1. Let G be a finite abelian group. Define $L^2(G) = \{\text{All maps } f | f : G \to \mathbb{C}\}$.

The set $L^2(G)$ can be shown to be a complex vector space under function addition and scalar mutiplication (Note that a complex vector space is a vector space whose field of scalars is the complex numbers). This follows from the definitions of function addition and scalar multiplication of functions.

Remark 3.1.2. It can actually be shown that $L^2(G)$ is isomorphic to the set of all linear combinations of group elements, $\mathbb{C}[G] = \{\sum_{g\in G} \lambda_g g | \lambda_g \in \mathbb{C}\}$. Specifically, by letting $\lambda_g = f(g)$ for any $f : G \to \mathbb{C}$, $L^2(G)$ can be identified with $\mathbb{C}[G]$.

This chapter focuses on establishing a basis for $L^2(G)$ so that we can decompose its elements. For our later work, like verifying the orthonormality of the elements of our basis, we will need to define an inner product for $L^2(G)$.

Definition 3.1.3. For $f_1, f_2 \in L^2(G)$, let

$$\langle f_1, f_2\rangle = \tfrac{1}{|G|} \textstyle\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Note that $f_2(g) \in \mathbb{C}$, and $\overline{f_2(g)}$ denotes the complex conjugate of $f_2(g)$.

It can be shown that this inner product is:

(a) anti-commutative: $\langle f_1, f_2\rangle = \overline{\langle f_2, f_1\rangle}$ for all $f_1, f_2 \in L^2(G)$.

(b) Positive definite: $\langle f, f\rangle \geq 0$ for all $f \in L^2(G)$, and $\langle f, f\rangle$ if and only if $f = 0$.

(c) Linear/Conjugate Linear: For all $f_1, f_2, f_2 \in L^2(G)$, $\langle f_1, f_2 + f_3\rangle = \langle f_1, f_2\rangle + \langle f_1, f_3,\rangle$, and $\langle f_1 + f_2, f_3\rangle = \langle f_1, f_3\rangle + \langle f_2, f_3,\rangle$, and for $\lambda \in \mathbb{C}$, $\langle \lambda f_1, f_2\rangle = \lambda\langle f_1, f_2\rangle$ and $\langle f_1, \lambda f_2\rangle = \overline{\lambda}\langle f_1, f_2\rangle$

Our inner product being positive definite helps us define a norm on $L^2(G)$.

Definition 3.1.4. The L^2-**norm** is defined as $||f|| = \sqrt{\langle f, f\rangle}$

3.2 Introducing $H^1(G)$

In this section we introduce the set that will become our basis for $L^2(G)$. Interestingly this set will be made up of homomorphisms $\chi : G \to S^1$ where S^1 is the unit circle in $\mathbb{C}$. First we'll give this set a name.

Definition 3.2.1. Let G be a finite abelian group. We define $H^1(G) = \{\text{Homomorphisms } \chi : G \to S^1\}$.

This is a very general definition though. In order for this set to be useful to us, we'll need a better picture of what it's elements look like. We'll find an equivalent formulation in this section. We now introduce a set of complex numbers important for this.

Definition 3.2.2. The **m-th roots of unity** are complex numbers c satisfying $c^m = 1$. Equivalently, these are the complex numbers $\left(e^{\frac{2\pi i}{m}}\right)^k = \cos\left(\frac{2\pi k}{m}\right) + i\sin\left(\frac{2\pi k}{m}\right)$ for some integer $0 \leq k < m$.

We show an alternative view of $H^1(G)$. We first set this up for cyclic G.

Let $G \cong \mathbb{Z}_m$ be a cyclic group, and define

$$\omega_m = e^{\frac{2\pi i}{m}} \tag{3.2.1}$$

(Note that ω_m is one of the mth roots of unity).

For each $0 \leq \epsilon < m$, define $\chi_\epsilon : \mathbb{Z}_m \to \mathbb{C}^\star$ by

$$\chi_\epsilon(k) = [\omega_m^\epsilon]^k \text{ for all } k \in \mathbb{Z}_m. \tag{3.2.2}$$

Theorem 3.2.3. *Let $m \geq 1$. Then $H^1(\mathbb{Z}_m) = \{\chi_\epsilon | \epsilon \in \mathbb{Z}_m\}$.*

Proof. ($\subset$) Let ϵ be an integer such that $0 \leq \epsilon < m$. Let $a, b \in \mathbb{Z}_m$. It follows that

$$\chi_\epsilon(a+b) = [\omega_m^\epsilon]^{a+b} = \omega_m^{\epsilon a + \epsilon b} = \omega_m^{\epsilon a} \cdot \omega_m^{\epsilon b} = \chi_\epsilon(a) \cdot \chi_\epsilon(b).$$

Thus χ_ϵ is a homomorphism for any arbitrary $0 \leq \epsilon < m$. Since $\chi_\epsilon : \mathbb{Z}_m \to \mathbb{C}^\star$, by the definition of $H^1(\mathbb{Z}_m)$ it follows that $\chi_\epsilon \in H^1(\mathbb{Z}_m)$. Hence $\{\chi_\epsilon | 0 \leq \epsilon < m\} \subset H^1(\mathbb{Z}_m)$.

($\supset$) Let $f \in H^1(G)$. Thus $f : \mathbb{Z}_m \to \mathbb{C}^\star$ is a homomorphism. It follows that

$$\begin{aligned}
f(1)^m &= f(1) \cdot f(1) \cdots f(1) \\
&= f(1 + 1 + \cdots + 1) && \text{since } f \text{ is a homomorphism} \\
&= f(m) \\
&= f(0) && \text{since } f \text{ maps from } \mathbb{Z}_m \\
&= 1 && \text{as any homomorphism maps 0 to 1}
\end{aligned}$$

We can conclude, by definition, that $f(1)$ is an mth root of unity. Thus $f(1) = \left(e^{\frac{2\pi i}{m}}\right)^\epsilon = \omega_m^\epsilon$ for some $0 \leq \epsilon < m$. Now, consider for some $k \in \mathbb{Z}_m$

$$\begin{aligned}
f(k) &= f(1)^k && \text{since } f \text{ is a homomorphism} \\
&= (\omega_m)^k && \text{as shown above} \\
&= \chi_\epsilon(k).
\end{aligned}$$

It follows that $f \in \{\chi_\epsilon | 0 \leq \epsilon < m\}$. Hence $H^1(\mathbb{Z}_m) \subset \{\chi_\epsilon | 0 \leq \epsilon < m\}$.

Thus, since each is contained in the other, we conclude that $H^1(\mathbb{Z}_m) = \{\chi_\epsilon | 0 \leq \epsilon < m\}$. □

Now we can set this up for arbitrary G. Let $G \cong \oplus_{j=1}^r \mathbb{Z}_{m_j}$ be an arbitrary finite abelian group. A new function to facilitate the correspondence between G and $H^1(G)$ can be defined as follows. For $\epsilon = (\epsilon_1, \cdots, \epsilon_r) \in G$, define

$$\chi_\epsilon(k_1, \cdots, k_r) = \prod_{j=1}^{r} \chi_{\epsilon j}(k_j) \text{ for all } (k_1, \ldots, k_r) \in G. \tag{3.2.3}$$

Given this new function, the proof of Theorem 3.2.4, is similar to the proof of Theorem3.2.3.

Theorem 3.2.4. *Let* $G \cong \oplus_{j=1}^r \mathbb{Z}_{m_j}$. *Then* $H^1(G) = \{\chi_\epsilon | \epsilon \in G\}$.

The above implies a bijective correspondence between G and $H^1(G)$. It can actually be proven that $H^1(G)$ is a group under function multiplication, and that there exists an isomorphism between G to $H^1(G)$. Importantly for this senior project however, Theorem3.2.4 gives us concrete, defined functions to work with in showing $H^1(G)$ is a basis.

3.3 Verifying $H^1(G)$ as a Basis

We will now show that $H^1(G)$ forms an orthonormal basis for $L^2(G)$. This means the elements are orthogonal to each other, of unit length, and span $L^2(G)$. We will use the norm we defined earlier (3.1.4) to check if the elements are of unit length.

Proposition 3.3.1. *Let $G \cong \mathbb{Z}_m$. The elements in $\{\chi_\epsilon | \epsilon \in G\}$ are orthonormal.*

Before proving this, we'll need to verify the following lemma.

Lemma 3.3.2.

$$\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon} = \begin{cases} 0 & \text{if } \epsilon \neq 0 \\ 1 & \text{if } \epsilon = 0 \end{cases}$$

Proof. Let $\omega_m = e^{\frac{2\pi i}{m}}$, and let $0 \leq \epsilon < m$. Note that if $\epsilon = 0$, It follows that

$$\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon} = \frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{0} = \frac{1}{m}\sum_{k=0}^{m-1}1 = \frac{1}{m}(m) = 1.$$

Now suppose that $\epsilon \neq 0$. We continue with cases. Either ϵ and m are relatively prime, or they are not.

Case 1: Suppose ϵ and m are relatively prime. Then

$$\begin{aligned}
\left(\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon}\right)(1-\omega_m^\epsilon) &= \frac{1}{m}(1+\omega_m^\epsilon+\cdots+\omega_m^{(m-1)\epsilon})(1-\omega^\epsilon) \\
&= \frac{1}{m}(1-\omega_m^{m\epsilon}) \\
&= \frac{1}{m}(1-1) && \text{By the definition of the mth root of unity} \\
&= 0.
\end{aligned}$$

It follows that either $\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon} = 0$ or $1-\omega_m^\epsilon = 0$. Since ϵ and m are relatively prime, and $\epsilon \neq 0$, it follows that $\omega_m^\epsilon = e^{\frac{2\pi i\epsilon}{m}} \neq 1$. Therefore $1-\omega_m^\epsilon$ can't be zero, and $\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon}$ must be zero.

Case 2: Now suppose that ϵ and m are not relatively prime. Since they have a common factor, it follows that there exists $b < m$ such that $(b, m) = 1$, and $b\epsilon/m = c \in \mathbb{Z}$. Thus

$\omega_m^{\epsilon b} = \omega_m^{mc} = (\omega_m^c)^m = 1$. In other words the order of ω_m^ϵ is $b < m$. It follows that

$$\begin{aligned}
\left(\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon}\right)(1-\omega_m^b) &= \frac{1}{m}(1+\omega_m^\epsilon+\cdots+\omega_m^{(m-1)\epsilon})(1-\omega_m^b)\\
&= \frac{1}{m}(1+\omega_m^\epsilon+\cdots+\omega_m^{(b-1)\epsilon}+\cdots+1+\omega_m^\epsilon+\cdots+\omega_m^{(b-1)\epsilon})(1-\omega_m^b)\\
&\qquad\text{since the order of } \omega_m^\epsilon \text{ is b}\\
&= \frac{1}{b\frac{\epsilon}{c}}(1+\omega_m^\epsilon+\cdots+\omega_m^{(b-1)\epsilon})\left(\frac{\epsilon}{c}\right)(1-\omega_m^b)\\
&\qquad\text{since } b\epsilon/c = m\\
&= \frac{1}{b}(1+\omega_m^\epsilon+\cdots+\omega_m^{(b-1)\epsilon})(1-\omega_m^b)\\
&= \frac{1}{b}(1-\omega_m^{b\epsilon})\\
&= \frac{1}{b}(1-1)\\
&= 0.
\end{aligned}$$

By the same reasoning as the previous case, it follows $\frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{k\epsilon} = 0$. □

Now to the proof of 3.3.1.

Proof. (Unit) Let $\epsilon \in G$. It follows that

$$\begin{aligned}
\langle\chi_\epsilon,\chi_\epsilon\rangle &= \frac{1}{|G|}\sum_{k\in G}\chi_\epsilon(k)\overline{\chi_\epsilon(k)}\\
&= \frac{1}{|G|}\sum_{k\in G}1\\
&\qquad\text{Since } \overline{\chi_\epsilon} \text{ can be shown to be the inverse of } \chi_\epsilon\\
&= \frac{1}{|G|}|G|\\
&= 1.
\end{aligned}$$

Thus $||\chi_\epsilon|| = \sqrt{\langle\chi_\epsilon,\chi_\epsilon\rangle} = 1$ (3.1.4). This implies that all elements of $H^1(G)$ are of unit length.

(Pairwise Orthogonality) Let $\epsilon, \delta \in G = \mathbb{Z}_m$. Consider

$$\langle\chi_\epsilon,\chi_\delta\rangle = \frac{1}{|G|}\sum_{k\in G}\chi_\epsilon(k)\overline{\chi_\delta(k)} = \frac{1}{m}\sum_{k\in G}\omega_m^{\epsilon k}\overline{\omega_m^{\delta k}} = \frac{1}{m}\sum_{k\in G}\omega_m^{(\epsilon-\delta)k}.$$

By lemma 3.3.2 it follows that if $\epsilon \neq \delta$, then $\langle\chi_\epsilon,\chi_\delta\rangle = 0$. It follows that the elements of $H^1(G)$ are pairwise orthogonal. □

This orthogonality gives our basis contender another nice quality, that is the linear independence of its elements. The proof quickly follows, so we don't show it here.

Now that we know $H^1(G)$ has the nice properties of a basis, like orthonormality and linear independence, we need to show that it is indeed a basis. The final step in doing so is the demonstration of span. This is done by taking an arbitrary element of $L^2(G)$ and showing that it can be written as a linear combination of the basis functions. In this case however, we'll need to clarify what the coefficients of this linear combination will look like beforehand.

Lemma 3.3.3. *Let $f \in L^2(G)$. If $f = \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon$, then $c_\epsilon = \langle f, \chi_\epsilon \rangle$.*

Proof. Let $\delta \in G$. It follows that

$$\begin{aligned}
\langle f, \chi_\delta \rangle &= \Big\langle \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon, \chi_\delta \Big\rangle \\
&= \sum_{\epsilon \in G} c_\epsilon \Big\langle \chi_\epsilon, \chi_\delta \Big\rangle && \text{by the linearity of the inner product} \\
&= 0 + 0 + \cdots + c_\delta + \cdots + 0 && \text{since } \langle \chi_\epsilon, \chi_\delta \rangle = 0 \text{ unless } \delta = \epsilon \text{, in which case } \langle \chi_\epsilon, \chi_\delta \rangle = 1 \\
&= c_\delta
\end{aligned}$$

Thus if f can be expressed as a linear combination $\sum_{\epsilon \in G} c_\epsilon \chi_\epsilon$, then $c_\epsilon = \langle f, \chi_\epsilon \rangle$. □

With this lemma to inform what the coefficients will look like, we can prove span. We show this for arbitrary G.

Theorem 3.3.4. *Let $G \cong \oplus_{j=1}^{r} \mathbb{Z}_{m_j}$. If $f \in L^2(G)$, then f can be written as a linear combination of elements in $H^1(G)$, $f = \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon$ with $c_\epsilon = \langle f, \chi_\epsilon \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\chi_\epsilon(g)}$.*

Proof. Let $f \in L^2(G)$, Then for $h \in G$

$$\sum_{\epsilon\in G} c_\epsilon \chi_\epsilon(h) = \sum_{\epsilon\in G}\Big(\frac{1}{|G|}\sum_{g\in G} f(g)\chi_\epsilon^{-1}(g)\Big)\chi_\epsilon(h)$$

By Lemma 3.3.3

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\overline{\prod_{j=1}^{r}\chi_{\epsilon_j}(g_j)}\prod_{j=1}^{r}\chi_{\epsilon_j}(h_j)$$

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\prod_{j=1}^{r}\overline{\chi_{\epsilon_j}(g_j)}\chi_{\epsilon_j}(h_j)$$

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\prod_{j=1}^{r}\overline{\omega_{m_j}^{\epsilon_j g_j}}\omega_{m_j}^{\epsilon_j h_j}$$

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\prod_{j=1}^{r}\omega_{m_j}^{m_j-\epsilon_j g_j}\omega_{m_j}^{\epsilon_j h_j}$$

by the properties of complements of roots of unity

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\prod_{j=1}^{r}\omega_{m_j}^{m_j-\epsilon_j(g_j-h_j)}$$

$$= \frac{1}{|G|}\sum_{\epsilon\in G}\sum_{g\in G} f(g)\prod_{j=1}^{r}\overline{\omega_{m_j}^{\epsilon_j(g_j-h_j)}}$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{\epsilon\in G} f(g)\prod_{j=1}^{r}\overline{\omega_{m_j}^{\epsilon_j(g_j-h_j)}}$$

since the sums of G are finite

$$= \frac{1}{|G|}\sum_{g\in G} f(g)\sum_{\epsilon\in G}\prod_{j=1}^{r}\overline{\omega_{m_j}^{\epsilon_j(g_j-h_j)}}$$

since $f(g)$ is constant in a sum of ϵ

$$= \frac{1}{|G|}\sum_{g\in G} f(g)\sum_{\epsilon_1\in\mathbb{Z}_{m_1}}\sum_{\epsilon_2\in\mathbb{Z}_{m_2}}\cdots\sum_{\epsilon_r\in\mathbb{Z}_{m_r}}\Big(\overline{\omega_{m_1}^{\epsilon_1(g_1-h_1)}}\cdot\overline{\omega_{m_2}^{\epsilon_2(g_2-h_2)}}\cdots\overline{\omega_{m_r}^{\epsilon_r(g_r-h_r)}}\Big)$$

$$= \frac{1}{|G|}\sum_{g\in G} f(g)\sum_{\epsilon_1\in\mathbb{Z}_{m_1}}\overline{\omega_{m_1}^{\epsilon_1(g_1-h_1)}}\sum_{\epsilon_2\in\mathbb{Z}_{m_2}}\overline{\omega_{m_2}^{\epsilon_2(g_2-h_2)}}\cdots\sum_{\epsilon_r\in\mathbb{Z}_{m_r}}\overline{\omega_{m_r}^{\epsilon_r(g_r-h_r)}}$$

$$= \frac{1}{|G|}\Big(0+0+\cdots+f(h)\sum_{\epsilon_1\in\mathbb{Z}_{m_1}}\overline{\omega_{m_1}^{\epsilon_1(h_1-h_1)}}\sum_{\epsilon_2\in\mathbb{Z}_{m_2}}\overline{\omega_{m_2}^{\epsilon_2(h_2-h_2)}}\cdots\sum_{\epsilon_r\in\mathbb{Z}_{m_r}}\overline{\omega_{m_r}^{\epsilon_r(h_r-h_r)}}+\cdots+0\Big)$$

since $\sum_{\epsilon\in m}\overline{\omega_m^{\epsilon(g-h)}} = 0$ unless $g = h$

$$= \frac{1}{|G|}\Big(0+0+\cdots+f(h)\sum_{\epsilon_1\in\mathbb{Z}_{m_1}}1\sum_{\epsilon_2\in\mathbb{Z}_{m_2}}1\cdots\sum_{\epsilon_r\in\mathbb{Z}_{m_r}}1+\cdots+0\Big)$$

$$= \frac{1}{|G|}\Big(f(h)\cdot m_1\cdot m_2\cdots m_r\Big)$$

$$= \frac{1}{|G|}\Big(f(h)\cdot|G|\Big)$$

$$= f(h).$$

Since this is true for all $h \in G$, it follows that any $f \in L^2(G)$ can be expressed as a linear combination $\chi_\epsilon \in H^1(G)$. This implies that $H^1(G)$ spans $L^2(G)$. $\square$

Thus, with span, linear independence, and orthogonality, we can conclude that $H^1(G)$ is a complete orthonormal basis of $L^2(G)$. We can deconstruct any element of $L^2(G) = \{\text{all } f : G \to \mathbb{C}\}$ into a linear combination of elements in $H^1(G)$.

Remark 3.3.5. It is important to note that this decomposition is unique.

This result is crucial for every following result in this senior project.

4
Applying Fourier

4.1 Fourier Applied to Affine Tverberg

Let us look again at the affine version of Tverberg's theorem (2.0.18), and see what the implications for a Tverberg partition are if it is achieved using a decomposed affine map. We'll look at the cyclic case in two dimensions, but otherwise set things up the same way as the theorem.

Let $r \geq 2$, and $d = 2$. Now let $N = (r-1)(d+1)$, and $\triangle^N$ be a $N-$dimensional simplex. Let $G \cong \mathbb{Z}_r$, and $f : \triangle^N \to \mathbb{R}^2 \cong \mathbb{C}$ be an affine map. Consider the set $\{\sigma_g\}_{g \in G}$, of r pairwise disjoint faces of $\triangle^N$, that are parameterized by G. Let $\{x_g\}_{g \in G}$ be a collection of points from $\triangle^N$, also parameterized by G, where $x_g \in \sigma_g$ for all $g \in G$. We define

$$F : G \to \mathbb{C} \text{ by } g \mapsto f(x_g). \tag{4.1.1}$$

Then, by Theorem3.3.4, it follows

$$f(x_g) = F(g) = \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon(g) \text{ for all } g \in G. \tag{4.1.2}$$

This map is well defined because we are already indexing our points x_g from the simplex by elements of G. Then, the decomposition comes directly from our choices of domain and range (3.3.4).

With this new function and our Fourier decomposition we can effectively deconstruct the affine maps presented in the affine version of Tverberg's theorem (2.0.18). We can now learn what a Tverberg partition means in the context of such a decomposition.

Theorem 4.1.1. *Let $G = \mathbb{Z}_r$ for $r \geq 2$. Let $f : \triangle^N \to \mathbb{C}$ be an affine map. Let $\sigma_1, \ldots, \sigma_r$ be r disjoint faces of $\triangle^N$. Then there exists a set of r points $\{x_g\}_{g \in G}$ such that $x_g \in \sigma_g$ for all $g \in G$, such that $f(x_1) = \ldots = f(x_r)$ if and only if given the Fourier decomposition of F from 4.1.1, $c_\epsilon = 0$ for all $\epsilon \in G - \{0\}$.*

Proof. Let $G = \mathbb{Z}_r$.

($\Longleftarrow$) Suppose $c_\epsilon = 0$ for all $\epsilon \neq 0$. It follows that for $g \in G$

$$\begin{aligned} f(x_g) &= F(g) \\ &= \sum_{\epsilon \in \mathbb{Z}_r} c_\epsilon \chi_\epsilon(g) \\ &= c_0 \chi_0(g) \\ &= c_0 (\omega_m^0)^g \\ &= c_0. \end{aligned}$$

Note that c_0 is just a constant coefficient, and therefore all the points $\{x_g\}_{g \in G}$ map to the same point. Since $f(x_1) = \ldots = f(x_r)$, and $x_g \in \sigma_g$, it follows that $f(x_1) \in (f(\sigma_1) \cap \ldots \cap f(\sigma_r)) \neq \emptyset$. This implies a full Tverberg partition.

($\implies$) Now, suppose $\{x_g\}_{g \in G}$ gives a Tverberg partition. Then there must exist an element in $f(\sigma_1) \cap \ldots \cap f(\sigma_r)$. Or, equivalently, $f(x_g) = c$ for all $g \in G$ and some constant c. Therefore for $g \in G$

$$\begin{aligned}
c &= F(g) \\
&= \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon(g) \\
&= c_0 \chi_0(g) + \sum_{\epsilon \in G - \{0\}} c_\epsilon \chi_\epsilon(g) \\
&= c_0 + \sum_{\epsilon \in G - \{0\}} c_\epsilon \chi_\epsilon(g) && \text{since } \chi_0(g) = 1
\end{aligned}$$

Since $c_0 = c$, and $c_\epsilon = 0$ for all $\epsilon \neq 0$ is a viable decomposition, by the uniqueness of these Fourier decompositions, it follows that it is the only one.

□

A more general case with an arbitrary even dimension can be proven by splitting up affine maps $f : \triangle^N \to \mathbb{C}^d$ into $f_i : \triangle^N \to \mathbb{C}$. This technique will be explored further later in the paper, when we construct higher dimensional structures.

This demonstrates how the Fourier coefficients determine the characteristics of the function they are used to deconstruct. We will see more of this shortly.

4.2 Regular r-gon partitions

An interesting consequence of the Fourier basis we've chosen to deconstruct our maps is the ease with which it can describe regular polygons. Before we start to describe how this is, let us show what a regular polygon looks like as it will be represented by our basis.

Consider the set of the rth roots of unity (3.2.1)

$$\left\{\omega_r^k\right\}_{k=0}^{r-1} = \left\{e^{\frac{2\pi i k}{r}}\right\}_{k=0}^{r-1} = \left\{\cos\left(\frac{2\pi k}{r}\right) + i \sin\left(\frac{2\pi k}{r}\right)\right\}_{k=0}^{r-1}.$$

This cosine and sine will give the x and y values of points on the unit circle (in $\mathbb{C}$), determined by the input angle. Notice that k varies between 0 and $r-1$, implying $0 \leq \frac{2\pi k}{r} < 2\pi$ for every k, and that each of these angles will be unique in the set. Moreover, the ascending integer values of k, ensure that every point is a $\frac{2\pi}{r}$ rotation of the previous point. Since these points are on the unit circle, it follows that their distance from the origin doesn't change with k. The regularity of their angle, and placement on the unit circle imply that $\{\omega_r^k\}_{k=0}^{r-1}$ forms the vertex set of a regular $r-$gon.

However, we want an arbitrary $r-$gon. If we multiply every vertex by $e^{i\theta}$ for some $\theta \in \mathbb{R}$, then we get

$$\left\{e^{i\theta}\omega_r^k\right\}_{k=0}^{r-1} = \left\{e^{i\theta}e^{\frac{2\pi ik}{r}}\right\}_{k=0}^{r-1} = \left\{e^{\left(\frac{2\pi k}{r}+\theta\right)i}\right\}_{k=0}^{r-1}.$$

This adds θ to the angle of every point, preserving their relative position, but rotating the imagined polygon by θ. To scale the shape to an arbitrary size, we can simply multiply by a nonzero scalar $s \in \mathbb{R} - \{0\}$, to get $\{se^{i\theta}\omega_r^k\}_{k=0}^{r-1}$. Finally, we can displace all the vertices, and therefore the location of the $r-$gon, by adding an arbitrary complex number $c \in C$ to the vertices, $\{c + se^{i\theta}\omega_r^k\}_{k=0}^{r-1}$. These transformations allow us to express an $r-$gon of arbitrary size, orientation, and position. Note that adding an extra power $j \in \mathbb{N}$ to the mth root of unity only changes which vertices the individual values of k map to, as long as j and r are relatively prime $((r, j) = 1)$. We record this observation as a remark

Remark 4.2.1. Let $r \geq 3$. The vertex set of an arbitrary regular polygon with r sides, some rotation, some location, and some size, in the complex plane is $\{c + s\omega_r^{kj}\}_{k=0}^{r}$ with $c \in \mathbb{C}$, $s \in \mathbb{C} - \{0\}$, and $j \in \mathbb{N}$ such that $(r, j) = 1$.

Now we show when such a regular $r-$gon shows up in the context of the affine maps from the previous section. It is important to note that this is a result in the cyclic case, where $G = \mathbb{Z}_r$, and with dimension $d = 2$.

Proposition 4.2.2. [1] *Let $G = \mathbb{Z}_r$ for $r \geq 3$ and $j \in \mathbb{N}$ such that $(j, r) = 1$. Let $f : \triangle^N \to \mathbb{C}$ be an affine map. There exists a set of r points $\{x_g\}_{g\in G}$ such that $x_g \in \sigma_g$ for all $g \in G$, where σ_g are disjoint faces, such that $\{f(x_g)\}_{g\in G}$ is the vertex set of a regular r-gon if and only if in the decomposition of F from 4.1.1 (1) $c_\epsilon = 0$ for all $\epsilon \neq 0, j$, and (2) $c_j \neq 0$.*

Proof. ($\Longleftarrow$) Suppose $c_\epsilon = 0$ for all $\epsilon \neq 0, j$ and $c_j \neq 0$ where $j \in \mathbb{N}$ such that $(j, r) = 1$. it follows that

$$\begin{aligned}
f(x_h) &= F(h) \\
&= \sum_{\epsilon=0}^{r-1} c_\epsilon \chi_\epsilon(h) && \text{because F has a Fourier decomposition} \\
&= c_0 + c_j \chi_j(h) && \text{since } c_\epsilon = 0 \text{ for all } \epsilon \neq 0, j \\
&= c_0 + c_j \omega_r^{hj} && \text{by definition of } \chi_\epsilon.
\end{aligned}$$

Thus $\{f(x_g)\}_{g \in G} = \{c_0 + c_j \omega_r^{gj}\}_{g \in G}$ defines the vertex set of an arbitrary, regular r-gon, by 4.2.1.

($\Longrightarrow$) Now, suppose that $\{f(x_g)\}_{g \in G}$ is the vertex set of a regular r-gon. It follows by 4.2.1 that $f(x_g) = c + z\omega_r^{jg}$ for $r \geq 3$, $c \in \mathbb{C}$, $z \in \mathbb{C} - \{0\}$, and $j \in \mathbb{N}$ such that $(r, j) = 1$, and $g \in G$.

$$\begin{aligned}
F(g) &= f(x_g) && \text{by definition of F} \\
&= c + z\omega_r^{jg} \\
&= c + z\chi_j(g) + \sum_{\epsilon \in G - \{o, j\}} 0 \cdot \chi_\epsilon(g)
\end{aligned}$$

Since F must have a Fourier decomposition, $F(g) = \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon(g)$, and since that decomposition is unique, it follows that $c_0 = c \in \mathbb{C}$, $c_j = z \in \mathbb{C} - \{0\}$, and $c_\epsilon = 0$ for all $\epsilon \neq 0, j$. □

4.3 Coefficient elimination

The results of the previous two sections are very closely tied to which Fourier coefficients we commit to being zero. The question then becomes, when are those coefficients zero? Under what circumstances will we be able to find a regular $r-$gon? The following Theorem4.3.1 from [1] helps us answer this question.

Up until now, in our use of the Tverberg type setup we have considered affine maps $f : \triangle^N \to \mathbb{C}$. However, we will later consider questions in higher dimensions too. Tverberg's theorem is shown in arbitrary dimension after all. For our purposes we will consider even dimension, because we can use a direct equivalency to products of $\mathbb{C}$. With affine maps to $\mathbb{C}^d$ it is important to notice that we can rewrite them in terms of smaller maps to $\mathbb{C}$ that we're more familiar with. Thus given $f : \triangle^N \to \mathbb{C}^d$, it follows $f = (f_1, \ldots, f_d)$ where $f_i : \triangle^N \to \mathbb{C}$. Then, as in 4.1.1 we can define

$$F_i : G \to \mathbb{C} \text{ such that } g \mapsto f_i(x_g). \text{ Then each } F_i \text{ has a Fourier decomposition } F_i = \sum_{\epsilon \in G} c_{i,\epsilon}\chi_\epsilon. \tag{4.3.1}$$

Now we can introduce a helpful theorem.

Theorem 4.3.1. [1] *Let $G = \oplus_{j=1}^d \mathbb{Z}_{m_j}$ for $d \geq 1$. Choose sets $S_i \subset G - \{0\}$ for $1 \leq i \leq d$. Let $m = |G|$ and $s = \sum_{j=1}^d |S_j|$. Let $N = 2s + m - 1$ and $f : \triangle^N \to \mathbb{C}^d$ be an affine map. Then there exists a set of points $\{x_g\}_{g \in G}$, where $x_g \in \sigma_g$ for all $g \in G$ and the σ_g are disjoint, such that if we have the decomposed maps F_i, as shown in 4.3.1, the following is true:*

(a) for all $1 \leq i \leq d$, $c_{i,\epsilon} = 0$ for all $\epsilon \in S_i$

(b) for almost every f, $c_{i,\epsilon} \neq 0$ for all $\epsilon \notin S_i$ an all $1 \leq i \leq d$.

(c) If $N < 2|S| + m - 1$, then (a) fails for almost every f (N is tight).

This theorem allows us to find an N for which we will almost always get our desired decomposition of F.

An important aspect of 4.3.1 is that it's statement is true for 'almost every' affine map. This helps us define a generic affine map.

Definition 4.3.2. [1] Let $G = \oplus_{j=1}^r \mathbb{Z}_{m_j}$ with $r \geq 1$ and $f : \triangle^N \to \mathbb{C}^d$ be an affine map. Let $m = |G|$. Then f is **Fourier generic** if given any collection of subsets $\{S_i\}_{i=1}^r$, where $S_i \subset G - \{0\}$ and where $s = \sum_{j=1}^d |S_d|$ with $N < 2s + m - 1$, there do not exist disjoint faces, $\sigma_1, \ldots, \sigma_r$, and a set of points $\{x_g\}_{g \in G}$ where $x_g \in \sigma_g$, such that when we consider the decomposition of F_i from 4.3.1, we have $c_{i,\epsilon} = 0$ for all $\epsilon \in S_i$ and all $1 \leq i \leq r$.

A fourier generic map is the name for a typical affine map in the setting of Theorem4.3.1. This means that in Theorem4.3.1, given a Fourier generic map from an $N-$simplex whose N is too small, you cannot eliminate all the coefficients in S. Importantly, this also means that with an N that is big enough, a Fourier generic map will not eliminate too many coefficients. This means that generically we only eliminate the coefficients we intend to. As we will see later when we partially recover Tverberg's theorem, this property will be what recovers the tightness of Tverberg's number. It will also enforce new tight numbers of points for our own results. This sense of a generic function will be useful later as we generate further results using 4.3.1.

As an example of the utility of this new theorem, we can see that we recover part of Tverberg's theorem if we choose $G = \mathbb{Z}_r$ with $r \geq 2$, $S = G - \{0\}$ and $d = 1$. In this case

$$N = 2|S| + r - 1 = 2(|G| - 1) + r - 1 = 3r - 3.$$

Note that $N = (r-1)(2+1)$ which is the two dimensional case of the affine version of Tverberg's Theorem2.0.18. Let $f : \triangle^N \to \mathbb{C}$ be an affine map. Thus if we take $F : G \to \mathbb{C}$ (4.1.1) with the Fourier decomposition $F = \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon$. By 4.3.1 it follows that there exists a set of points $\{x_g\}_{g \in G}$ from r pairwise disjoint faces of $\triangle^N$ such that $c_\epsilon = 0$ for all $\epsilon \in S$ by . By 4.1.1 it follows that we have a full Tverberg partition.

Remark 4.3.3. Note that Theorem 4.3.1 implies $N = 3r - 3 = T(r, d) - 1$ is tight, meaning that with $n < N$ a Fourier generic map will not admit a full Tverberg partition.

Now that we've recovered something we already knew, let's investigate when the regular $r-$gons we described in section 4.2 show up. We claim

Theorem 4.3.4. [1] *Let $G = \mathbb{Z}_r$ for $r \geq 3$, and $N = 3r - 5$. Now let $\triangle^N$ be an $N-$dimensional simplex. Let $f : \triangle^N \to \mathbb{R}^2$ be a Fourier generic affine map. There exist r points $\{x_g\}_{g \in G}$ where $x_g \in \sigma_g$ for all $g \in G$ and the σ_g are disjoint, such that $\{f(x_g)\}_{g \in G}$ forms the vertex set of a regular $r-$gon.*

Proof. Let $r \geq 3$, $G = \mathbb{Z}_r$ and $S = G - \{0, j\}$ for some $j \in \mathbb{N}$ such that $(r, j) = 1$. Let $N = 2|S| + r - 1 = 2(r - 2) + r - 1 = 3r - 5$, and $f : \triangle^N \to \mathbb{C}$ be a Fourier generic affine map.

Consider r pairwise disjoint faces of $\triangle^N$, $\sigma_1, \ldots, \sigma_r$. By 4.3.1, there almost always exist a set of points $\{x_g\}_{g \in G}$, where $x_i \in \sigma_i$, so that in the Fourier decomposition of the map F from 4.1.1, we know $c_\epsilon = 0$ for all $\epsilon \in G - \{0, j\}$. Because f is fourier generic, it follows that $c_j \neq 0$. It follows by 4.2.2 that in this case $\{f(x_g)\}_{g \in G}$ is the vertex set of a regular $r-$gon. □

In much the same way that we can restate Tverberg's theorem in a setting of affine maps, we can restate 4.3.4 in a context of convex hulls. Note that since the domain of f in 4.3.4 is an $N-$dimensional simplex which implies that from the geometric point of view $N+1 = 3r-5+1 = 3r-4$ points are needed.

Theorem 4.3.5. [1] *Almost any set of $3r-4$ points in $\mathbb{R}^2$ can be partitioned into r disjoint sets, $A_1, \ldots, A_r$, such that there exist r points, $x_1 \in Conv(A_1), \ldots, x_r \in Conv(A_r)$, which form the the vertex set of a regular $r-$gon.*

We call an application of 4.3.5 a regular $r-$gon partition. Notice that at a dimension $d = 2$, Tverberg's number $T(r, d) = 3r - 2$, which is greater than the number of points $(3r - 4)$ required for an $r-$gon partition (4.3.5). Since $T(r, d)$ is tight, this result (4.3.5) also sheds interesting light on what can be said about collections of points smaller than $T(r, d)$. It's also important to note that by 4.3.1, the requirement of $3r - 4$ points for a regular $r-$gon partition is tight as well.

5

Regular Sub $r-$gons

It was shown in [?leinersimon], that given $3r-4$ points in $\mathbb{R}^2$ partitioned into r subsets so that one can find the vertex set of a regular $r-$gon with one vertex from each of their convex hulls. But now we ask what other types of partitions there are given the same $3r-4$ points.

Definition 5.0.1. Let $r=r_1r_2$, and S be a set of points in $\mathbb{R}^2$. A **sub $r-$gon partition** of S is a partition of S into r sets, grouped into r_1 collections of r_2 sets, $A_1^1,\dots,A_{r_2}^1,\dots,A_1^{r_1},\dots,A_{r_2}^{r_1}$, such that

a) $\cap_{i=1}^{r_2}\mathrm{Conv}(A_i^1)\neq\emptyset,\dots,\cap_{i=1}^{r_2}\mathrm{Conv}(A_i^{r_1})\neq\emptyset$, and

b) there exist points $y_1\in\cap_{i=1}^{r_2}\mathrm{Conv}(A_i^1),\dots,y_{r_1}\in\cap_{i=1}^{r_2}\mathrm{Conv}(A_i^{r_1})$ that form the vertex set of a regular r_1-gon.

First we show what a sub $r-$gon partition is equivalent to in the affine formulation, with our Fourier decomposition.

Proposition 5.0.2. *Let $G=\mathbb{Z}_r$ for $r\geq 3$, where $r=r_1r_2$, and $N\in\mathbb{N}$. Let $f:\triangle^N\to\mathbb{C}$ be an affine map. Consider r disjoint faces of $\triangle^N$ grouped into r_1 collections of r_2 faces, $C_1=\{\sigma_0,\sigma_{r_1},\cdots,\sigma_{(r_2-1)r_1}\},\cdots,C_{r_1}=\{\sigma_{r_1-1},\cdots,\sigma_{(r_1-1)+(r_2-1)r_1}\}$. Let $\{x_g\}_{g\in G}$ be a set of points from $\triangle^N$ such that $x_g\in\sigma_g$ for all $g\in G$. For some $j\in\mathbb{N}$ such that $(r_1,j)=1$, we can say that* *given the Fourier decomposition of the map F from 4.1.1, (1) $c_\epsilon = 0$ for all $\epsilon \neq 0, jr_2$, and (2) $c_{jr_2} \neq 0$ if and only if*

a) $\cap_{\sigma \in C_1} f(\sigma) \neq \emptyset, \cdots, \cap_{\sigma \in C_{r_1}} f(\sigma) \neq \emptyset$, and

b) $f(x_0) \in \cap_{\sigma \in C_1} f(\sigma), \cdots, f(x_{r_1-1}) \in \cap_{\sigma \in C_{r_1}} f(\sigma)$ are the vertices of a regular r_1-gon.

Proof. ($\implies$) Let $j \in \mathbb{N}$ such that $(j, r_1) = 1$. Suppose $c_\epsilon = 0$ for all $\epsilon \neq 0, jr_2$ and that $c_{jr_2} \neq 0$. It follows that for $g \in G$

$$\begin{aligned}
f(x_g) &= F(g) \\
&= \sum_{\epsilon \in G} c_\epsilon \chi_\epsilon(g) && \text{since } F: G \to \mathbb{C}\text{, we use Thm 3.3.4} \\
&= c_0 + c_{jr_2}\chi_{jr_2}(g) && \text{since all other } c_\epsilon\text{'s are zero by assumption} \\
&= c_0 + c_{jr_2}\omega_r^{jr_2 g} \\
&= c_0 + c_{jr_2} e^{\frac{2\pi i g j r_2}{r_1 r_2}} \\
&= c_0 + c_{jr_2}\omega_{r_1}^{gj}
\end{aligned}$$

Since g will run through all $r = r_1 r_2$ elements of $G = \mathbb{Z}_r$, and there are only r_1 distinct powers of ω_{r_1}, it follows that in many cases multiple points from the simplex will map to the same point in $\mathbb{C}$. We investigate where and how these overlaps happen. Since $f(x_g) = c_0 + c_{jr_2}\omega_{r_1}^{gj}$ we can see

$$\begin{aligned}
f(x_{g+nr_1}) &= c_0 + c_{jr_2}\omega_{r_1}^{(g+nr_1)j} \\
&= c_0 + c_{jr_2}\omega_{r_1}^{gj}\omega_{r_1}^{nr_1 j} \\
&= c_0 + c_{jr_2}\omega_{r_1}^{gj} \\
&= f(x_g)
\end{aligned}$$

Notice that because of the root of unity, only $0 \leq n \leq r_2 - 1$ give unique results. Therefore, since n can be one of r_2 values, every point in $\{f(x_g)\}_{g=0}^{r_1-1}$ is getting mapped onto by r_2 points from the simplex, and the indices of these points differ by multiples of r_1,

$$f(x_g) = f(x_{g+r_1}) = \cdots = f(x_{g+(r_2-1)r_1}).$$

It follows that

$$f(\sigma_g) \cap f(\sigma_{g+r_1}) \cap \ldots \cap f(\sigma_{g+(r_2-1)r_1}) \neq \emptyset$$

This means that the images of the faces in the collection C_{g+1} overlap. It follows that $\cap_{\sigma \in C_1} f(\sigma) \neq \emptyset, \cdots, \cap_{\sigma \in C_{r_1}} f(\sigma) \neq \emptyset$.

Now, if we consider the points $f(x_0) \in \cap_{\sigma \in C_1} f(\sigma), \ldots, f(x_{r_1-1}) \in \cap_{\sigma \in C_{r_1}} f(\sigma)$, they will all be distinct. Moreover, since we showed $f(x_g) = c_0 + c_{jr_2}\omega_{r_1}^{gj}$, they will form the vertex set of a regular r_1-gon.

($\Longleftarrow$) Suppose $\cap_{\sigma \in C_1} f(\sigma) \neq \emptyset, \ldots, \cap_{\sigma \in C_{r_1}} f(\sigma) \neq \emptyset$,

and $f(x_0) \in \cap_{\sigma \in C_1} f(\sigma), \ldots, f(x_{r_1-1}) \in \cap_{\sigma \in C_{r_1}} f(\sigma)$ are the vertices of a regular r_1-gon.

It follows that $\{f(x_g)\}_{g=0}^{r_1-1} = \{c + z\omega_{r_1}^{jg}\}_{g=0}^{r_1-1}$ with $(j, r_1) = 1$ and $z \neq 0$, by 4.2.1. Therefore

$$\begin{aligned} F(g) &= f(x_g) \\ &= c + z(\omega_{r_1}^{j})^g \\ &= c + z\big(e^{\frac{2\pi i j r_2}{r_1 r_2}}\big)^g \\ &= c + z(\omega_r^{jr_2})^g \\ &= c + z\chi_{jr_2}(g) \\ &= \sum_{\epsilon=0}^{r-1} c_\epsilon \chi_\epsilon(g) \qquad \text{where } c_0 = c,\ c_{jr_2} = z, \text{ and } c_\epsilon = 0 \text{ for } \epsilon \neq 0, jr_2 \end{aligned}$$

This shows that $F(g)$ can be expressed in our usual Fourier decomposition as $\sum_{\epsilon \in G} c_\epsilon \chi_\epsilon(g)$ with $c_\epsilon = 0$ for all $\epsilon \neq 0, jr_2$. By the uniqueness of the Fourier decomposition, it follows that this one, with $c_\epsilon = 0$ unless $\epsilon = 0, jr_2$ is the only decomposition. □

Now we can use Theorem 4.3.1 to show under what circumstances we can find sub $r-$gon partitions.

Theorem 5.0.3. *Consider $G = \mathbb{Z}_r$ where $r = r_1 r_2$. Let $N = 3r - 5$ and let $f : \triangle^N \to \mathbb{C}$ be a Fourier generic affine map. Then there exist r disjoint faces in $\triangle^N$, grouped into r_1 collections of r_2 faces, $C_1 = \{\sigma_0, \sigma_{r_1}, \cdots, \sigma_{(r_2-1)r_1}\}, \cdots, C_{r_1} = \{\sigma_{r_1-1}, \cdots, \sigma_{(r_1-1)+(r_2-1)r_1}\}$ such that:*

a) $\cap_{\sigma\in C_1} f(\sigma) \neq \emptyset, \cdots, \cap_{\sigma\in C_{r_1}} f(\sigma) \neq \emptyset$*, and*

b) there exist points $y_1 \in \cap_{\sigma\in C_1} f(\sigma), \cdots, y_{r_1} \in \cap_{\sigma\in C_{r_1}} f(\sigma)$ *that are the vertices of a regular* r_1*-gon.*

c) The dimension of the simplex, N *is tight.*

Proof. Let $G = \mathbb{Z}_r$, where $r = r_1 r_2$, and $S = G - \{0, jr_2\}$. Let $N = 2|S| + r - 1 = 2(r-2) + r - 1 = 3r - 5$, and $f : \triangle^N \to \mathbb{C}$ be a Fourier generic affine map. By Theorem 4.3.1 there exist r pairwise disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g\in G}$, and a set of points, $\{x_g\}_{g\in G}$, where $x_g \in \sigma_g$, such that for the Fourier decomposition of the map F from 4.1.1, we have $c_\epsilon = 0$ for $\epsilon \in S$. Since f is fourier generic, $c_\epsilon = 0$ unless $\epsilon = 0, jr_2$. By Theorem 5.0.2 it follows that $\cap_{\sigma\in C_1} f(\sigma) \neq \emptyset, \cdots, \cap_{\sigma\in C_{r_1}} f(\sigma) \neq \emptyset$ and $f(x_0) \in \cap_{\sigma\in C_1} f(\sigma), \cdots, f(x_{r_1-1}) \in \cap_{\sigma\in C_{r_1}} f(\sigma)$ are the vertices of a regular r_1-gon.

To prove (c) note that if the simplex had dimension $n < N = 2|S| + r - 1$, then by 4.3.1 the above would fail since f is Fourier generic. Thus N is tight. □

Note that N being tight holds for all our results, since it is a direct consequence of Theorem 4.3.1. Since the proof will always be the same, we'll omit stating it explicity in theorems in later on.

Again, just as we can state Tverberg's theorem in terms of both affine maps, and convex hulls, we can restate this theorem in terms of convex hulls too. Since the domain of the affine map f in 5.0.3 is a simplex with dimension $N = 3r - 5$, it follows that we'll need $N + 1 = 3r - 4$ points in the plane to create that simplex and find an r_1-gon in the intersection of convex hulls.

Theorem 5.0.4. *Let* $r \geq 3$ *and* $r_1, r_2 \in \mathbb{Z}$ *such that* $r_1 r_2 = r$*. Almost any set of* $3r - 4$ *points in* $\mathbb{R}^2$ *can be partitioned into* r *sets,* $A_1^1, \ldots, A_{r_1}^1, \ldots, A_1^{r_1}, \ldots, A_{r_2}^{r_1}$*, such that there exist* r_1 *points,* $y_1 \in \cap_{i=1}^{r_2} Conv(A_i^1), \ldots, y_{r_1} \in \cap_{i=1}^{r_2} Conv(A_i^{r_1})$*, that form the vertex set of a regular* r_1-*gon. Moreover, almost any collection of fewer points can't be a sub* $r-$*gon partition.*

This result gives us a more general understanding of what's going on than the $r-$gon partition we proved in the previous section (4.3.5). Indeed, if either r_1 or r_2 equal 1, then we recover

Theorem 4.3.5. What is also interesting is that there is no change to the number of points required. You need $3r-4$ points no matter the choice of r_1 and r_2. This means that given $3r-4$ points in the plane, you can almost always form a regular polygon with vertices whose number is a factor of r.

What follows is an example of the versatility sub $r-$gon partitions provide. This example will be for $r = 12$, and therefore $T(12, 2) - 2 = 32$ points in $\mathbb{R}^2$.

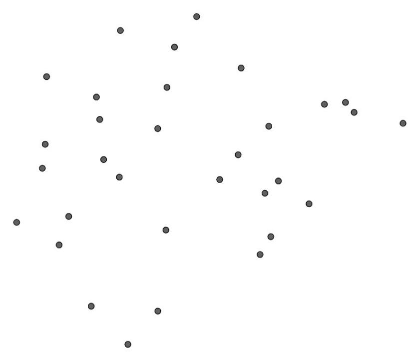

Figure 5.0.1. An arbitrary set of 32 points

Now, since $12 = 4 \cdot 3$, we should be able to find a square in the intersection of convex hulls.

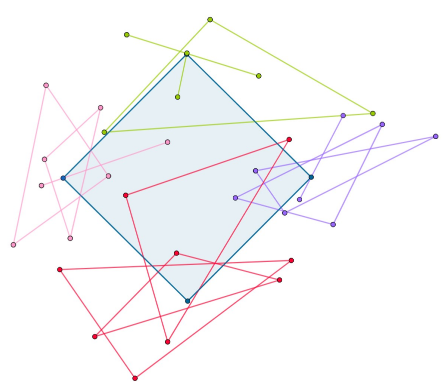

Figure 5.0.2. Z_{12} Sub square partition

Here we see the overlapping collections of convex hulls distinguished by their colors, and the vertices of the square being taken from the overlap of the convex hulls of each collection. Of course, since $12 = 3 \cdot 4$ we should be able to find a regular triangle in the intersection of convex hulls too.

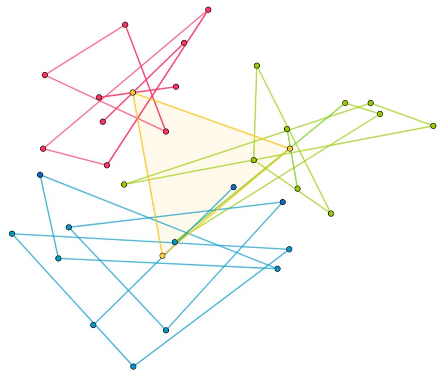

Figure 5.0.3. Z_{12} Sub triangle partition

It is interesting to notice how even though this result is with a smaller number of points than Tverberg's original theorem, we find the vertices of this r_1-gon from what could almost be described as 'mini' Tverberg partitions.

6
Multi Dimensional Applications

So far we've discussed Tverberg type results in the two dimensional plane. Tverberg's theorem (2.0.4) is, however, stated for the more general $\mathbb{R}^d$. This leads us to ask how the two dimensional polygon results of the previous chapter may be extended to multiple dimensions. Our transition to higher dimensions will also involve using more general finite abelian groups, rather than just cyclic ones as before.

6.1 Multiprism

Ironically, the quickest application of what we know is also the hardest to visualize. Let's begin by defining what what type of object we'll be looking for:

Definition 6.1.1. A **Polytope** is the convex hull of some finite set of points in $\mathbb{R}^d$.

This is a general object in d dimensions. We'll be focused on a particular kind of polytope, one that is the product of shapes in lower dimensions. To express this, we introduce some new notation. Let P_r denote a regular $r - gon$ for $r \geq 3$, and let P_2 denote a line segment.

Definition 6.1.2. Let $r_i \geq 3$ for all i. A **Multiprism** in $\mathbb{R}^{2d+k}$ is $P = P_{r_1} \times \ldots \times P_{r_d} \times P_2^k$ for some $k \geq 0$.

Multiprisms are polytopes which are the cartesian product of two dimensional polygons and line segments.

Note that the cartesian product adds the dimension of the sets it's applied to. This is why, if $P = P_{r_1} \times \ldots \times P_{r_d}$ for $r_i \geq 3$, then P has dimension $2d$. Similarly, if we were to also include a product of k lines segments to get $P = P_{r_1} \times \ldots \times P_{r_d} \times P_2^k$ for some $k \geq 0$, this multiprism P would have dimension $2d + k$, as shown in the definition.

Like polygons in $\mathbb{R}^2$, multiprisms are made of smaller components.

Definition 6.1.3. A $k-$**face** of a $d-$dimensional polytope is a $k-$dimensional polytope subset of the $d-$polytope.

So the $0-$faces are the vertices, the $1-$faces are the edges, etc.

Definition 6.1.4. A **facet** of a $d-$dimensional polytope is a $(d-1)-$face.

The facets of a cube, for example, are the square $2-$faces on its exterior.

Remark 6.1.5. We're considering multiprisms, which are special polytopes. It's important to note that if we look at $P = P_{r_1} \times \ldots \times P_{r_d} \times P_2^k$, then the $k-$faces are constructed using the faces of the component P_{r_i} and P_2 (the entire polygons, edges, or vertices). So a $k-$face of P is the product of $0-$faces, $1-$faces, and $2-$faces of P_{r_i} and the edges P_2.

It follows that the vertices of a multiprism are all the possible cartesian products of vertices of the component P_{r_i} and P_2. as in Figure 6.1.1

As an example of a multiprism, we could consider a $P_3 \times P_2$.

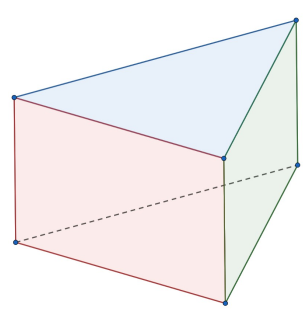

Figure 6.1.1. Triangular prism

As we can see, this is a three dimensional regular triangular right prism. The facets on the side are rectangles, while the facets on the top on bottom are equilateral triangles.

Given a $4-$dimensional multiprism $P = P_4 \times P_3$, we won't be able to present an image of it, but we can describe it's faces.

($3-$faces) It's facets should be $3-$dimensional prisms, so they'll be products of 1 and 2 dimensional faces from P_4 and P_3. It follows that the two facets of P are a right prism with a square base, and a right prism with a triangular base.

($2-$faces) It's $2-$faces will be products of $1-$dimensional faces from P_4 and P_3, or the $2-$face of one times the vertices of the other. The products of edges from P_4 and P_3 give us $4 \cdot 3 = 12$ $2-$faces. Then P_4 times the vertices of P_3 will give us $3\ \ 2-$faces, while P_3 times the vertices of P_4 will give us $4\ \ 2-$faces. Thus in total P has $12 + 3 + 4 = 19\ \ 2-$faces.

($1-$faces) It's $1-$faces will be products of $1-$faces in one, and $0-$faces in the other. Taking vertices from P_4 and edges from P_3, we get $4 \cdot 3 = 12$ edges. Then taking vertices from P_3 and edges from P_4, we get $3 \cdot 4 = 12$ edges. Thus P has $12 + 12 = 24$ edges.

($0-$faces) Finally, the $0-$faces will be products of the $0-$faces from both. Thus P has $4 \cdot 3 = 12$ vertices.

6.2 Finding Multiprisms

Previously we used cyclic groups of order r in the construction of our Fourier basis and the subsequent regular $r-$gons and sub $r-$gons in $\mathbb{R}^2 \cong \mathbb{C}$. In multiple dimensions we'll use a more general group structure, $G = \oplus_{j=1}^{d} \mathbb{Z}_{m_j}$. Because our previous work used maps onto the complex plane, this initial result will be deconstructing maps to $\mathbb{C}^d \cong \mathbb{R}^{2d}$. Notice that this means we are only considering multiprisms occuring in even dimension.

Theorem 6.2.1. *Let $G = \oplus_{j=1}^{d}\mathbb{Z}_{m_j}$ for $d \geq 1$. Let $m = |G|$ and $N = 2(m-2)^d + m - 1$. Let $m_i = m_i^1 m_i^2$ for all $1 \leq i \leq d$. Let $f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. Then there exist m disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g\in G}$, such that*

(a) $\bigcap_{b\in\oplus_{j=1}^{d}\mathbb{Z}_{m_j^2}} f\left(\sigma_{p+(b_1m_1^1,\ldots,b_dm_d^1)}\right) \neq \emptyset$ for all $p \in \oplus_{j=1}^{d}\mathbb{Z}_{m_j^1}$, and

(b) There exists a set of points $\{y_p\}_{p\in\oplus_{j=1}^{d}\mathbb{Z}_{m_j^1}}$, where $y_p \in \bigcap_{b\in\oplus_{j=1}^{d}\mathbb{Z}_{m_j^2}} f\left(\sigma_{p+(b_1m_1^1,\ldots,b_dm_d^1)}\right)$ for all $p \in \oplus_{j=1}^{d}\mathbb{Z}_{m_j^1}$, and $\{y_p\}_{p\in\oplus_{j=1}^{d}\mathbb{Z}_{m_j^1}}$ is the vertex set of $P = P_{m_1^1} \times \ldots \times P_{m_d^1}$ where each regular m_i^1-gon is parallel to a $\mathbb{C}-$plane generated by a standard basis vector.

In much the same way as the points of a sub $r-$gon (4.3.5) are taken from the intersection of convex hulls in $\mathbb{C}$, the points in this vertex set of a multiprism in $\mathbb{C}^d$ will be taken from the intersection of convex hulls in $\mathbb{C}^d$, only these convex hulls will be polytopes.

Let us run through an example for clarity. Let $G = \mathbb{Z}_6 \oplus \mathbb{Z}_4$. Then, $|G| = 6 \cdot 4 = 24$, and let $N = 2((24)-2)^2 + (24) - 1 = 991$. then we pick any generic affine map from $\triangle^N$ to $\mathbb{C}^2 \cong \mathbb{R}^4$. Given such a setup, we can find 24 disjoint faces of $\triangle^N$, whose images overlap in 6 collections of $2 \cdot 2 = 4$ each, and we can find vertices of a $P_3 \times P_2$ multiprism with one vertex from each overlap. But, we could also choose a different set of disjoint faces so that the images of the faces overlap in 8 collections of $3 \cdot 1 = 3$, and in the overlaps of their images we find the vertices of a $P_2 \times P_4$ multiprism.

Now we proceed to the proof of Theorem 6.2.1.

Proof. For $1 \leq k \leq d$ let $j^{(k)} = (0, \ldots, 0, j_k, 0, \ldots, 0)$ where $(m_k^1, j_k) = 1$. We choose a collection of sets, $S_i = G - \{0, m_i^2 j^{(i)}\}$. Let $s = \sum_{j=1}^{d} |S_i| = (|G| - 2)^d = (m-2)^d$, and $N = 2s + m - 1$. By theorem 4.3.1 it follows that there exist m disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g \in G}$, and a set of points $\{x_g\}_{g \in G}$ where $x_g \in \sigma_g$, such that if $p \in G$, then

$$
\begin{aligned}
& f(x_p) \\
&= (f_1(x_p), f_2(x_p), \ldots, f_d(x_p)) \\
&= (F_1(p), F_2(p), \ldots, F_d(p)) \\
&\qquad \text{from 4.3.1} \\
&= \left(\sum_{\epsilon \in G} c_{1,\epsilon} \chi_\epsilon(p), \sum_{\epsilon \in G} c_{2,\epsilon} \chi_\epsilon(p), \ldots, \sum_{\epsilon \in G} c_{d,\epsilon} \chi_\epsilon(p) \right) \\
&= \left(c_{1,0}\chi_0(p) + c_{1,m_1^2 j^{(1)}} \chi_{m_1^2 j^{(1)}}(p), c_{2,0}\chi_0(p) + c_{2,m_2^2 j^{(2)}} \chi_{m_2^2 j^{(2)}}(p), \ldots, c_{d,0}\chi_0(p) + c_{d,m_d^2 j^{(d)}} \chi_{m_d^2 j^{(d)}}(p) \right) \\
&\qquad \text{by 4.3.1 since } c_{k,\epsilon} = 0 \text{ for all } \epsilon \in S_k \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \chi_{m_1^2 j^{(1)}}(p), c_{2,0} + c_{2,m_2^2 j^{(2)}} \chi_{m_2^2 j^{(2)}}(p), \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \chi_{m_d^2 j^{(d)}}(p) \right) \\
&\qquad \text{since } \chi_{(0,\ldots,0)}(p) = \prod_{j=1}^{d} \chi_0(p) = \prod_{j=1}^{d} 1 = 1 \\
&= \left(c_{1,0} + c_{1,m_1^2(j_1,\ldots,0)} \chi_{m_1^2(j_1,\ldots,0)}(p), c_{2,0} + c_{2,m_2^2(0,j_2,\ldots,0)} \chi_{m_2^2(0,j_2,\ldots,0)}(p), \ldots, c_{d,0} + c_{d,m_d^2(0,\ldots,j_d)} \chi_{m_d^2(0,\ldots,j_d)}(p) \right) \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \prod_{t=1}^{d} \chi_{m_1^2 j_t^{(1)}}(p_t), c_{2,0} + c_{2,m_2^2 j^{(2)}} \prod_{t=1}^{d} \chi_{m_2^2 j_t^{(2)}}(p_t), \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \prod_{t=1}^{d} \chi_{m_d^2 j_t^{(d)}}(p_t) \right) \\
&\qquad \text{by 3.2.3 where } j^{(1)} = \left(j_1^{(1)}, \ldots, j_d^{(1)} \right), \ p = (p_1, \ldots, p_d) \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \chi_{m_1^2 j_1^{(1)}}(p_1), c_{2,0} + c_{2,m_2^2 j^{(2)}} \chi_{m_2^2 j_2^{(2)}}(p_2), \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \chi_{m_d^2 j_d^{(d)}}(p_d) \right) \\
&\qquad \text{because } j_t^{(k)} = 0 \text{ unless } t = k \text{ by construction, and } \chi_0(p_t) = 1 \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \chi_{m_1^2 j_1}(p_1), c_{2,0} + c_{2,m_2^2 j^{(2)}} \chi_{m_2^2 j_2}(p_2), \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \chi_{m_d^2 j_d}(p_d) \right) \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \omega_{m_1^1 m_1^2}^{m_1^2 j_1 p_1}, c_{2,0} + c_{2,m_2^2 j^{(2)}} \omega_{m_2^1 m_2^2}^{m_2^2 j_2 p_2}, \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \omega_{m_d^1 m_d^2}^{m_d^2 j_d p_d} \right) \\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}} \omega_{m_1^1}^{j_1 p_1}, c_{2,0} + c_{2,m_2^2 j^{(2)}} \omega_{m_2^1}^{j_2 p_2}, \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}} \omega_{m_d^1}^{j_d p_d} \right)
\end{aligned}
$$

Now we have a description of what our points map to, but we still to show what points overlap in the image. Let $b = (b_1, \ldots, b_d)$ and $t = (b_1 m_1^1, \ldots, b_d m_d^1)$. Consider

$$\begin{aligned}
f(x_{p+t}) &= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}}\omega_{m_1^1}^{j_1(p_1+b_1m_1^1)}, \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}}\omega_{m_d^1}^{j_d(p_d+b_dm_d^1)}\right)\\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}}\omega_{m_1^1}^{j_1p_1}\omega_{m_1^1}^{j_1b_1m_1^1}, \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}}\omega_{m_d^1}^{j_dp_d}\omega_{m_d^1}^{j_db_dm_d^1}\right)\\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}}\omega_{m_1^1}^{j_1p_1}(1), \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}}\omega_{m_d^1}^{j_dp_d}(1)\right)\\
&= \left(c_{1,0} + c_{1,m_1^2 j^{(1)}}\omega_{m_1^1}^{j_1p_1}, \ldots, c_{d,0} + c_{d,m_d^2 j^{(d)}}\omega_{m_d^1}^{j_dp_d}\right)\\
&= f(x_p)
\end{aligned}$$

This described the overlapping images. Since $m_i = m_i^1 m_i^2$ for all i, it follows that the ith slot in the tuple has m_i^2 possible multiples of m_i^1. Therefore, the ith slot contributes m_i^2 overlapping points. Since $x_g \in \sigma_g$, it follows

$$\bigcap_{b \in \oplus_{j=1}^d \mathbb{Z}_{m_j^2}} f\left(\sigma_{p+(b_1m_1^1,\ldots,b_dm_d^1)}\right) \neq \emptyset \text{ for all } p \in G.$$

We can also conclude that

$$\{f(x_g)\}_{g \in \oplus_{j=1}^d \mathbb{Z}_{m_j^1}} \subset \{f(x_g)\}_{g \in G}$$

are a set of non overlapping points.

From 4.2.1 we know that for $e \neq 0, c \in \mathbb{C}$, the set $\{c + e\omega_{m_i^1}^{ap}\}_{p \in \mathbb{Z}_{m_i^1}}$ where $(a,p) = 1$, is the vertex set of a regular m_i^1-gon. Consider

$$\{f(x_g)\}_{g \in \oplus_{j=1}^d \mathbb{Z}_{m_j^1}} = \left\{\left(c_{1,0} + c_{1,j^{(1)}}\omega_{m_1}^{j_1g_1}, c_{2,0} + c_{2,j^{(2)}}\omega_{m_2}^{j_2g_2}, \ldots, c_{d,0} + c_{d,j^{(d)}}\omega_{m_d}^{j_dg_d}\right)\right\}_{g \in \oplus_{j=1}^d \mathbb{Z}_{m_j^1}}$$

Since f is Fourier generic, we know that $c_{1,j^{(1)}}, \ldots, c_{d,j^{(d)}} \neq 0$, and thus we can see that each place in the tuple will form the vertex set of its own m_i^1-gon. Therefore the set of points as a whole will form a multiprism in $\mathbb{C}^d$. Because each m_i-gon is only represented in the ith tuple spot, it follows that every m_i-gon will be parallel to the $\mathbb{C}-$plane generated by the standard basis vector of the ith slot. $\square$

6.3 Sub $r-$gons in $\mathbb{C}^d$

In extending our work from the two dimensional plane to $\mathbb{C}^d$ we may also ask if a more direct conversion is possible. Under what conditions can we find a sub $r-$gon in $\mathbb{C}^d$. There turn out to be a couple answers to this question.

6.3.1 Arbitrary sub $r-$gon

Because we're looking for a sub $r-$gon partition, despite being in $\mathbb{C}^d$, we'll still use a cyclic group. Notice however, that the dimension of the simplex needed (and consequently the number points needed in a geometric setting) will remain the same as with the multiprism. This is because we're still eliminating all but two coefficients from every spot in the tuple.

Theorem 6.3.1. *Let $G = \mathbb{Z}_m$ with $m \geq 3$. Let $m = n_1 n_2$, and $N = 2(m-2)^d + m - 1$. Let $f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. Then there exists m disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g\in G}$ such that*

(a) $\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset$ *for all* $0 \leq g < n_1$, *and*

(b) There exist a set of points $\{y_g\}$ where $y_g \in \bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1})$ for all $0 \leq g < n_1$, and $\{y_g\}$ is the vertex set of a regular n_1-gon.

Proof. Let $j \in \mathbb{Z}$ such that $(n_1, j) = 1$. We choose a collection of sets, $S_i = G - \{0, jn_2\}$. Let $s = \sum_{j=1}^{d} |S_i| = (m-2)^d$, and $N = 2s + m - 1$. Let $f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. By theorem 4.3.1 it follows that there exist m disjoint faces of $\triangle^N$ and a set of points $\{x_g\}_{g\in G}$ from distinct faces such that if $p \in G$, then

$$\begin{aligned}
f(x_p) &= (f_1(x_p), f_2(x_p), \ldots, f_d(x_p)) \\
&= (F_1(p), F_2(p), \ldots, F_d(p)) \\
&\quad \text{from 4.3.1} \\
&= \left(\sum_{\epsilon\in G} c_{1,\epsilon}\chi_\epsilon(p), \sum_{\epsilon\in G} c_{2,\epsilon}\chi_\epsilon(p), \ldots, \sum_{\epsilon\in G} c_{d,\epsilon}\chi_\epsilon(p)\right) \\
&= (c_{1,0}\chi_0(p) + c_{1,n_2j}\chi_{n_2j}(p), c_{2,0}\chi_0(p) + c_{2,n_2j}\chi_{n_2j}(p), \ldots, c_{d,0}\chi_0(p) + c_{d,n_2j}\chi_{n_2j}(p)) \\
&\quad \text{by 4.3.1 since } c_{k,\epsilon} = 0 \text{ for all } \epsilon \in S_k \\
&= (c_{1,0} + c_{1,n_2j}\chi_{n_2j}(p), c_{2,0} + c_{2,n_2j}\chi_{n_2j}(p), \ldots, c_{d,0} + c_{d,n_2j}\chi_{n_2j}(p)) \\
&\quad \text{since } \chi_0(p) = \omega_m^0{=}1 \\
&= \left(c_{1,0} + c_{1,n_2j}\omega_m^{n_2jp}, c_{2,0} + c_{2,n_2j}\omega_m^{n_2jp}, \ldots, c_{d,0} + c_{d,n_2j}\omega_m^{n_2jp}\right) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \left(c_{1,n_2j}\omega_m^{n_2jp}, c_{2,n_2j}\omega_m^{n_2jp}, \ldots, c_{d,n_2j}\omega_m^{n_2jp}\right) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \omega_{n_1n_2}^{n_2jp}\left(c_{1,n_2j}, c_{2,n_2j}, \ldots, c_{d,n_2j}\right) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \omega_{n_1}^{jp}\left(c_{1,n_2j}, c_{2,n_2j}, \ldots, c_{d,n_2j}\right)
\end{aligned}$$

Let $\vec{c}_0 = (c_{1,0}, c_{2,0}, \ldots, c_{d,0})$ and $\vec{c}_1 = (c_{1,n_2j}, c_{2,n_2j}, \ldots, c_{d,n_2j})$. Then the set $\{f(x_g)\}_{g\in G} = \{\vec{c}_0 + \omega_{n_1}^{jg}\vec{c}_1\}_{g\in G}$. We're looking for a sub $r-$gon, so we know there's going to be some overlap. But what points overlap? For $p \in G$ and $k \in \mathbb{Z}$, consider

$$\begin{aligned}
f(x_{p+kn_1}) &= \vec{c}_0 + \omega_{n_1}^{j(p+kn_1)}\vec{c}_1 \\
&= \vec{c}_0 + \omega_{n_1}^{jp}\omega_{n_1}^{kn_1}\vec{c}_1 \\
&= \vec{c}_0 + \omega_{n_1}^{jp}(1)\vec{c}_1 \\
&= f(x_p).
\end{aligned}$$

Of course, since the points $\{x_g\}_{g\in G}$ are parametrized by $G = \mathbb{Z}_m$, and $m = n_1n_2$, it follows that $0 \le k < n_2$. Therefore With the same argument as

$$f(x_p) = f(x_{p+n_1}) = \ldots = f(x_{p+(n_2-1)n_1}).$$

Since each point comes from a distinct disjoint face, it follows that

$$\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset \text{ for all } 0 \leq g < n_1.$$

Thus we can take the following set of distinct points.

$$f(x_g) \in \bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \text{ for all } 0 \leq g < n_1.$$

Since $\omega_{n_1}^{jg} = e^{\frac{2\pi ijg}{n_1}}$, it follows that $\vec{c}_0 + \omega_{n_1}^{jg}\vec{c}_1$ is a rotation of $\vec{c}_1$ by $\frac{2\pi jg}{n_1}$ radians and translation by $\vec{c}_0$. It follows that $\{f(x_g)\}_{g\in G} = \{\vec{c}_0 + \omega_{n_1}^{jg}\vec{c}_1\}_{g\in G}$ will form a regular n_1-gon in $\mathbb{C}^d$, parallel to $\langle\vec{c}_1\rangle = \{\lambda\vec{c}_1 | \lambda \in \mathbb{C}\}$. □

6.3.2 Parallel Sub $r-$gon

Theorem 6.3.1 gives a sub $r-$gon partition on an arbitrary plane. This leads us to wonder if we could find a sub $r-$gon partition on a plane of our choosing. For this result we will require an intermediary lemma. The lemma will be similar to Theorem 6.3.1, only we'll be eliminating more coefficients in order to have a stronger condition on the orientation of the sub $r-$gon in $\mathbb{C}^d$. Of course this also results in the dimension of the simplex, N, being larger. In the geometric setting this means that we'll have to have more points initially in $\mathbb{C}^d$ in order to find a sub $r-$gon in the intersection of convex hulls.

Lemma 6.3.2. *Let $G = \mathbb{Z}_m$ with $m \geq 3$. Let $n_1, n_2 \in \mathbb{Z}$ such that $m = n_1 n_2$, and $N = 2(m-2)(m-1)^{d-1} + m - 1$. Let $f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. Then there exist m pairwise disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g\in G}$ such that*

(a) $\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset$ for all $0 \leq g < n_1$, and

(b) There exist a set of points $\{y_g\}$ where $y_g \in \bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1})$ for all $0 \leq g < n_1$, and $\{y_g\}$ is the vertex set of a regular n_1-gon parallel to the complex plane generated by the standard basis vector $(1, 0, \ldots, 0) \in \mathbb{C}^d$.

Proof. Let $j \in \mathbb{Z}$ such that $(n_1, j) = 1$. We choose a collection of sets where $S_1 = G - \{0, n_2 j\}$, and for $i > 1$, $S_i = G - \{0\}$. Let $s = \sum_{j=1}^{d} |S_i| = (m-2)(m-1)^{d-1}$, and $N = 2s + m - 1$. Let

$f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. By theorem 4.3.1 it follows that there exist m disjoint faces of $\triangle^N$ and a set of points $\{x_g\}_{g\in G}$ from distinct faces such that if $p \in G$, then

$$\begin{aligned}
f(x_p) &= (f_1(x_p), f_2(x_p), \ldots, f_d(x_p)) \\
&= (F_1(p), F_2(p), \ldots, F_d(p)) \\
&\quad \text{from 4.3.1} \\
&= \left(\sum_{\epsilon\in G} c_{1,\epsilon}\chi_\epsilon(p), \sum_{\epsilon\in G} c_{2,\epsilon}\chi_\epsilon(p), \ldots, \sum_{\epsilon\in G} c_{d,\epsilon}\chi_\epsilon(p)\right) \\
&= (c_{1,0}\chi_0(p) + c_{1,n_2j}\chi_{n_2j}(p), c_{2,0}\chi_0(p), \ldots, c_{d,0}\chi_0(p)) \\
&\quad \text{by 4.3.1 since } c_{k,\epsilon} = 0 \text{ for all } \epsilon \in S_k \\
&= (c_{1,0} + c_{1,n_2j}\chi_{n_2j}(p), c_{2,0}, \ldots, c_{d,0}) \\
&\quad \text{since } \chi_0(p) = \omega_m^0 {=} 1 \\
&= \left(c_{1,0} + c_{1,n_2j}\omega_m^{n_2jp}, c_{2,0}, \ldots, c_{d,0}\right) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \left(c_{1,n_2j}\omega_m^{n_2jp}, 0, \ldots, 0\right) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \omega_{n_1n_2}^{n_2jp}\,(c_{1,n_2j}, 0, \ldots, 0) \\
&= (c_{1,0}, c_{2,0}, \ldots, c_{d,0}) + \omega_{n_1}^{jp}\,(c_{1,n_2j}, 0, \ldots, 0)
\end{aligned}$$

If we define $\vec{c}_0 = (c_{1,0}, c_{2,0}, \ldots, c_{d,0})$ and $\vec{c}_1 = (c_{1,n_2j}, 0, \ldots, 0)$, then $\{f(x_g)\}_{g\in G} = \{\vec{c}_0 + \omega_{n_1}^{j_1g_1}\vec{c}_1\}_{g\in G}$. With the same argument as in the proof of 6.3.1, we can conclude

$$f(x_p) = f(x_{p+n_1}) = \ldots = f(x_{p+(n_2-1)n_1}).$$

And since each point comes from a distinct disjoint face, it follows that

$$\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset \text{ for all } 0 \le g < n_1.$$

Thus we can take the following set of distinct points.

$$f(x_g) \in \bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \text{ for all } 0 \le g < n_1.$$

Just as in the proof of 6.3.1 it follows that $\{f(x_g)\}_{g=0}^{n_1-1} = \{\vec{c}_0 + \omega_{n_1}^{j_1 g_1}\vec{c}_1\}_{g=0}^{n_1-1}$ is the vertex set of an n_1-gon. In this case however $\vec{c}_1 = (c_{1,n_2 j}, 0, \ldots, 0) = c_{1,n_2 j}\,(1, 0, \ldots, 0)$. This indicates $\vec{c}_1$ lies on the $\mathbb{C}-$plane $\langle(1, 0, \ldots, 0)\rangle = \{\lambda(1, 0, \ldots, 0)|\lambda \in \mathbb{C}\}$. What makes the above n_1-gon interesting is that since it's made by multiplying the vector $\vec{c}_1$ by a complex number, it too lies on $\langle(1, 0, \ldots, 0)\rangle$. Thus, when translated by $\vec{c}_0$, we can conclude that $\{f(x_g)\}_{g=0}^{n_1-1} = \{\vec{c}_0 + \omega_{n_1}^{j_1 g_1}\vec{c}_1\}_{g=0}^{n_1-1}$ is parallel to $\langle(1, 0, \ldots, 0)\rangle$. $\square$

Since $(1, 0, \ldots, 0)$ is a standard basis vector of $\mathbb{C}^d$, we can ask if we can find a sub $r-$gon that is parallel to any given complex plane. In order to prove this we'll need an understanding of what it means to be perpendicular. For this reason we introduce

Definition 6.3.3. Let $\vec{u} = (u_1, \ldots, u_d)$ and $\vec{v} = (v_1, \ldots, v_d)$ be two vectors in $\mathbb{C}^d$. The **Hermition Inner Product** is defined as $\langle\vec{u}, \vec{v}\rangle_{\mathbb{C}} = u_1\overline{v_1} + \ldots + u_d\overline{v_d}$.

Note that on the real axes this is the same inner product as in $\mathbb{R}^d$. Additionally, just as in $\mathbb{R}^d$, in $\mathbb{C}^d$ two vectors are defined to be orthogonal when their Hermition Inner product is zero. With this new inner product we can establish a couple new terms.

Definition 6.3.4. Let $\vec{v} \in \mathbb{C}^d$, and $\langle\vec{v}\rangle = \{\lambda\vec{v}|\lambda \in \mathbb{C}\}$ be the linear subspace generated by $\vec{v}$. Then the **Orthogonal Complement** of $\langle\vec{v}\rangle$ is $\langle\vec{v}\rangle^{\perp} = \{\vec{w} \in \mathbb{C}^d|\langle\vec{w}, \vec{v}\rangle_{\mathbb{C}} = 0\}$.

Definition 6.3.5. A **Unitary Map** is a complex linear map $L : \mathbb{C}^d \to \mathbb{C}^d$ such that $\langle L(\vec{u}), L(\vec{v})\rangle = \langle\vec{u}, \vec{v}\rangle$ for all $\vec{u}, \vec{v} \in \mathbb{C}^d$.

Theorem 6.3.6. *Let $G = \mathbb{Z}_m$ with $m \geq 3$. Let $n_1, n_2 \in \mathbb{Z}$ such that $m = n_1 n_2$, and $N = 2(m-2)(m-1)^{d-1}+m-1$. Choose a linear $\mathbb{C}-$plane generated by a nonzero $\vec{v} \in \mathbb{C}^d$, $\langle\vec{v}\rangle = \{\lambda\vec{v}|\lambda \in \mathbb{C}\}$. Let $f : \triangle^N \to \mathbb{C}^d$ be a Fourier generic affine map. Then there exist m pairwise disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g\in G}$, such that*

(a) $\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset$ *for all* $0 \leq g < n_1$, *and*

(b) There exist a set of points $\{y_g\}$ *where* $y_g \in \bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1})$ *for all* $0 \leq g < n_1$, *and* $\{y_g\}$ *is the vertex set of a regular n_1-gon parallel to the complex plane* $\langle\vec{v}\rangle$.

Proof. Let $\langle\vec{v}\rangle = \{\lambda\vec{v}|\lambda \in \mathbb{C}\}$ be some arbitrary complex plane generated by a nonzero vector $\vec{v} = (v_1, \dots, v_d) \in \mathbb{C}^d$.

Let $\Phi : \mathbb{C}^d \to \mathbb{C}^d$ be a unitary map that sends $(1, 0, \dots, 0)$ to $\vec{v}$, and the remaining standard basis vectors of $\mathbb{C}^d$ to the mutually orthogonal vectors in the orthogonal complement of $\langle\vec{v}\rangle$.

Since Φ is linear and f is affine, which is an offset linear map , it follows that their composition $(\Phi^{-1} \circ f) : \triangle^N \to \mathbb{C}^d$, is affine.

Our previous work in 6.3.2 has shown that for $p \in G$

$$(\Phi^{-1} \circ f)(x_p) = (c_0^1 + c_j^1\omega_{n_1}^{jp}, c_0^2, \dots, c_0^d)$$

Note that this is an n_1-gon in a complex plane parallel to $\langle(1, 0, \dots, 0)\rangle = \{\lambda(1, 0, \dots, 0)|\lambda \in \mathbb{C}\}$.

It follows that

$$\begin{aligned}
f(x_p) &= \Phi((c_0^1 + c_j^1\omega_{n_1}^{jp}, c_0^2, \dots, c_0^d)) \\
&= \Phi((c_0^1, c_0^2, \dots, c_0^d) + \omega_{n_1}^{jp}c_j^1(1, 0, \dots, 0)) \\
&= \Phi((c_0^1, c_0^2, \dots, c_0^d)) + \Phi(\omega_{n_1}^{jp}c_j^1(1, 0, \dots, 0)) \\
&= \Phi((c_0^1, c_0^2, \dots, c_0^d)) + \omega_{n_1}^{jp}c_j^1\Phi((1, 0, \dots, 0)) && \text{since } \Phi \text{ is unitary} \\
&= \Phi((c_0^1, c_0^2, \dots, c_0^d)) + \omega_{n_1}^{jp}c_j^1\vec{v} && \text{by construction of } \Phi
\end{aligned}$$

Since f is Fourier generic, it follows that it cannot have a full Tverberg partition. Thus if $c_j^1 = 0$, then f would always map to the constant $\Phi((c_0^1, c_0^2, \dots, c_0^d))$, and therefore admit a full Tverberg partition. Since this would be contradictory, we conclude that $c_j^1 \neq 0$.

The same argument used in the proof of 6.3.1 allows us to conclude that

$$\bigcap_{j=0}^{n_2-1} f(\sigma_{g+jn_1}) \neq \emptyset \text{ for all } 0 \leq g < n_1$$

Similarly, the set $\{f(x_g)\}_{g=0}^{n_1-1} = \{\Phi((c_0^1, c_0^2, \dots, c_0^d)) + \omega_r^{jg}c_j^1\vec{v}\}_{g=0}^{n_1-1}$ will then give us distinct values which form the vertex set of a regular n_1-gon parallel to $\langle\vec{v}\rangle$.

□

7
Elliptical Polygons

Until now our work has focused on results related to $r-$gon and sub $r-$gon partitions. These polygonal partitions came from asking what can be said about collections of points smaller than the Tverberg number (2.0.6). In the two dimensional case, Tverberg's theorem (2.0.4) states that we can find an $r-$fold Tverberg partition given $T(r, 2) = 3r - 2$ points in $\mathbb{R}^2$, and that this fails for almost any collection of fewer points. Our main sub $r-$gon result (4.3.5) then claimed that given a generic set of $3r-4$ points in $\mathbb{R}^2$, where $r = r_1 r_2$, we can partition the set into r subsets, and find the vertex set of a regular r_1-gon where each vertex is taken from the intersection of r_2 convex hulls. Again this fails for almost any collection of fewer points (4.3.1). This reduction in the required number of points is directly correlated with the number of coefficients of our Fourier decompositions we had vanish via theorem 4.3.1. Eliminating more coefficients reduces the number of terms in the sum and makes it easier to parse out where our affine map is sending points, but this also increases the dimension of the required simplex, and therefore the number of required initial points in the plane. One might now ask what can be said about an even smaller collection of points than required for a regular sub $r-$gon partition. We'll now investigate this question.

7.1 Elliptical $r-$gons

In this chapter, instead of finding a regular polygon whose points lie on a circle, we'll be finding special polygons with points on an ellipse.

Definition 7.1.1. An **Ellipse** in $\mathbb{C}$ with foci $F_1, F_2 \in \mathbb{C}$, and a major axis of length $s \in \mathbb{R}_{>0}$ is described by the set $E(F_1, F_2) = \{z| \ \ |z - F_1| + |z - F_2| = s\}$.

An example ellipse is shown below.

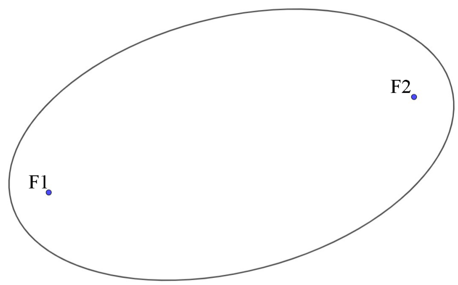

Figure 7.1.1. Ellipse

Notice that if $F_1 = F_2$, then The ellipse equation becomes $|z - F_1| = s/2$, which means the set describes a circle of radius $s/2$, centered at F_1. Also note that if the distance between F_1 and F_2 equals the length of the major axis, then $E(F_1, F_2)$ is a line segment. This degenerate case of an ellipse will have implications for results later in this chapter.

Now we need to know what a set of points on an ellipse is in terms of the elements of our Fourier basis, roots of unity.

Proposition 7.1.2. *Let $r > 2$ be an integer, let $c_0 \in \mathbb{C}$, and let $j \in \mathbb{Z}_r$ such that $(r, j) = 1$. Let $c_1 = r_1 e^{i\theta_1}$, and $c_2 = r_2 e^{i\theta_2}$ be non zero complex numbers. Let $F_1 = 2\sqrt{r_1 r_2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$, $F_2 = -2\sqrt{r_1 r_2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$. Then the set of points $\{c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg}\}_{g\in\mathbb{Z}_r}$ lie on the ellipse $E(F_1 + c_0, F_2 + c_0) = \{z| \ \ |(z - c_0) - F_1| + |(z - c_0) - F_2| = 2(r_1 + r_2)\}$.*

Remark 7.1.3. Consider the case where $|c_1| = r_1 = r_2 = |c_2|$. Then Since $F_1 = -F_2$, the distance between F_1 and F_2 is $2|F_1| = 2\left|2\sqrt{r_1 r_2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right| = 4r_1$. Also note that the major axis is of length $2(r_1 + r_2) = 4r_1$. Thus this is the degenerate case of an ellipse in which the set is a line segment. Moreover, if $|c_1| \neq |c_2|$, then you get a genuine ellipse.

Proof. We'll show that the points described above satify the equation of our given ellipse, and are therefore situated on it.

Let $p \in \mathbb{Z}_r$ and $z_p = c_0 + c_1\omega_r^{jp} + c_2\omega_r^{-jp}$. It follows that

$$
\begin{aligned}
&|(z_p - c_0) - F_1| + |(z_p - c_0) - F_2| \\
&= |(c_0 + c_1\omega_r^{jp} + c_2\omega_r^{-jp} - c_0) - F_1| + |(c_0 + c_1\omega_r^{jp} + c_2\omega_r^{-jp} - c_0) - F_2| \\
&= |c_1\omega_r^{jp} + c_2\omega_r^{-jp} - F_1| + |c_1\omega_r^{jp} + c_2\omega_r^{-jp} - F_2| \\
&= \left|c_1\omega_r^{jp} + c_2\omega_r^{-jp} - 2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right| + \left|c_1\omega_r^{jp} + c_2\omega_r^{-jp} - \left(-2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right)\right| \\
&= \left|r_1e^{i\theta_1}\omega_r^{jp} + r_2e^{i\theta_2}\omega_r^{-jp} - 2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right| + \left|r_1e^{i\theta_1}\omega_r^{jp} + r_2e^{i\theta_2}\omega_r^{-jp} + 2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right| \\
&= \left|r_1e^{i\theta_1}\omega_r^{jp} + r_2e^{i\theta_2}\omega_r^{-jp} - 2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\omega_r^{\frac{jp}{2}-\frac{jp}{2}}\right| + \left|r_1e^{i\theta_1}\omega_r^{jp} + r_2e^{i\theta_2}\omega_r^{-jp} + 2\sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\omega_r^{\frac{jp}{2}-\frac{jp}{2}}\right| \\
&= \left|\left(\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}} - \sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right)^2\right| + \left|\left(\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}} + \sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right)^2\right| \\
&= \left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}} - \sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right|^2 + \left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}} + \sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right|^2 \\
&= \left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}\right|^2 - \sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}\overline{\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}} - \overline{\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}}\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}} + \left|\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right|^2 \\
&\quad + \left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}\right|^2 + \sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}\overline{\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}} + \overline{\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}}\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}} + \left|\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right|^2 \\
&\qquad \text{since } |z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2 \\
&= 2\left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}\omega_r^{\frac{jp}{2}}\right|^2 + 2\left|\sqrt{r_2}e^{\frac{i\theta_2}{2}}\omega_r^{\frac{-jp}{2}}\right|^2 \\
&= 2\left|\sqrt{r_1}e^{\frac{i\theta_1}{2}}e^{i\frac{jp/2}{r}}\right|^2 + 2\left|\sqrt{r_2}e^{\frac{i\theta_2}{2}}e^{i\frac{-jp/2}{r}}\right|^2 \\
&= 2\left|\sqrt{r_1}e^{i\left(\frac{\theta_1}{2}+\frac{jp/2}{r}\right)}\right|^2 + 2\left|\sqrt{r_2}e^{i\left(\frac{\theta_2}{2}-\frac{jp/2}{r}\right)}\right|^2 \\
&= 2(\sqrt{r_1})^2 + 2(\sqrt{r_2})^2 \\
&= 2(r_1 + r_2)
\end{aligned}
$$

Since this satifies the equation of our chosen ellipse, it follows that this point lies on the ellipse, $z_p \in E(F_1+c_0, F_2+c_0)$. Since this is true for any $p \in \mathbb{Z}_r$, it follows that $\{c_0+c_1\omega_r^{jg}+c_2\omega_r^{-jg}\}_{g\in\mathbb{Z}_r} = \{z_g\}_{g\in\mathbb{Z}_r} \subset E(F_1+c_0, F_2+c_0)$. □

Now we know that the points $\{c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg}\}_{g\in\mathbb{Z}_r}$ lie on an ellipse. It follows that they form the vertex set of an $r-$gon, except in the degenerate case where the ellipse set is a line segment. In the non-degenerate case, we define these special $r-$gons.

Definition 7.1.4. Let $c_0 \in \mathbb{C}$, and let $c_1, c_2 \in \mathbb{C} - \{0\}$ such that $|c_1| \neq |c_2|$. Let $r \geq 2$ and $j \in \mathbb{Z}_r$ such that $(j, r) = 1$. If $p(g) = c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg}$, define an **Elliptical** $r-$**gon** to be the convex hull of the set of points $\{p(g)\}_{g\in\mathbb{Z}_r}$.

Because they are described using roots of unity, Elliptical $r-$gons have some symmetry properties in addition to being on an ellipse.

Proposition 7.1.5. *Consider any p from 7.1.4. For any t, the midpoint between $p(g+t)$ and $p(g-t)$ is on the line segment between $c_0 + (p(g) - c_0)$ and $c_0 - (p(g) - c_0)$.*

Proof. Let $g, t \in \mathbb{Z}_r$. Consider

$$\begin{aligned}
p(g+t) + p(g-t) &= c_0 + c_1\omega_r^{j(g+t)} + c_2\omega_r^{-j(g+t)} + c_0 + c_1\omega_r^{j(g-t)} + c_2\omega_r^{-j(g-t)} \\
&= 2c_0 + c_1\omega_r^{jg+jt} + c_2\omega_r^{-jg-jt} + c_1\omega_r^{jg-jt} + c_2\omega_r^{-jg+jt} \\
&= 2c_0 + c_1\omega_r^{jg}\omega_r^{jt} + c_2\omega_r^{-jg}\omega_r^{-jt} + c_1\omega_r^{jg}\omega_r^{-jt} + c_2\omega_r^{-jg}\omega_r^{jt} \\
&= 2c_0 + c_1\omega_r^{jg}\omega_r^{jt} + c_2\omega_r^{-jg}\omega_r^{-jt} + c_1\omega_r^{jg}\omega_r^{-jt} + c_2\omega_r^{-jg}\omega_r^{jt} \\
&= 2c_0 + (c_1\omega_r^{jg} + c_2\omega_r^{-jg})(\omega_r^{jt} + \omega_r^{-jt}) \\
&= 2c_0 + (c_1\omega_r^{jg} + c_2\omega_r^{-jg})\left(e^{i\frac{jt2\pi}{r}} + e^{i\frac{-jt2\pi}{r}}\right) \\
&= 2c_0 + (c_1\omega_r^{jg} + c_2\omega_r^{-jg})\left(2\mathrm{Re}\left(e^{i\frac{jt2\pi}{r}}\right)\right) \\
&= 2c_0 + (p(g) - c_0)2\cos\left(\frac{jt2\pi}{r}\right).
\end{aligned}$$

From this it follows that the midpoint satisfies

$$\frac{p(g+t)+p(g-t)}{2} = c_0 + (p(g)-c_0)\cos\left(\frac{jt2\pi}{r}\right). \tag{7.1.1}$$

As $-1 \leq \cos\left(\frac{jt2\pi}{r}\right) \leq 1$, we can conclude that 7.1.1 implies that the midpoint of $p(g+t)$ and $p(g-t)$ is somewhere along the line between $c_0 + (p(g)-c_0)$ and $c_0 - (p(g)-c_0)$. □

An example of Proposition 7.1.5 is shown with an elliptical 5−gon in Figure 7.1.3 below.

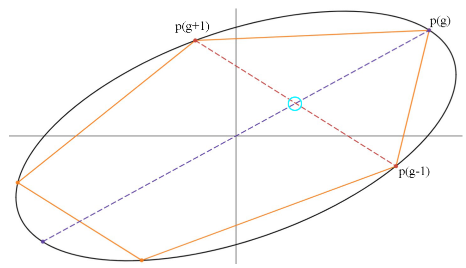

Figure 7.1.2. An example of an elliptical 5−gon

In this case we see that $c_0 = 0$, which means the ellipse is centered at the origin. The ellipse is in black, and the orange lines outline the corresponding elliptical polygon with points on that ellipse. This showcases a point $p(g)$, and how the midpoint between $p(g+1)$ and $p(g-1)$ lies on the line extending between $p(g)$ and $-p(g)$. A nonzero c_0 would simply shift this entire diagram, by c_0, away from the origin.

We also show this example for a different g, to demonstrate 7.1.5 working for multiple points.

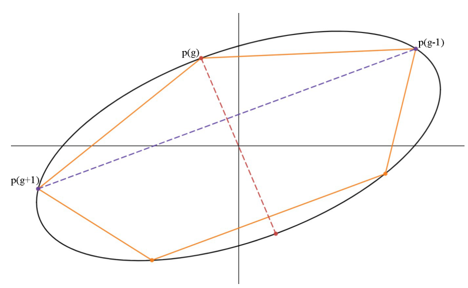

Figure 7.1.3. Alternate example of an elliptical 5−gon

When r is even, there are other properties we can find among Elliptical r−gons.

Proposition 7.1.6. *Consider a p from 7.1.4 with $r = 2m$. Then the midpoint of $p(g)$ and $p(g+m)$ is c_0 for all $g \in \mathbb{Z}_r$.*

Proof. Note that since $(j, r) = 1$ and r is even, we can conclude that j is odd. It follows that for $g \in \mathbb{Z}_r$

$$\begin{aligned}
&p(g) + p(g+m) \\
&= c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg} + c_0 + c_1\omega_r^{j(g+m)} + c_2\omega_r^{-j(g+m)} \\
&= 2c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg} + c_1\omega_r^{jg}\omega_r^{jm} + c_2\omega_r^{-jg}\omega_r^{-jm} \\
&= 2c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg} + c_1\omega_r^{jg}\omega_{2m}^{jm} + c_2\omega_r^{-jg}\omega_{2m}^{-jm} \\
&= 2c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg} + c_1\omega_r^{jg}(-1)^j + c_2\omega_r^{-jg}(-1)^{-j} \\
&= 2c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg} + c_1\omega_r^{jg}\,(-1) + c_2\omega_r^{-jg}\,(-1) \\
&\qquad \text{because } j \text{ is odd} \\
&= 2c_0.
\end{aligned}$$

It follows that

$$\frac{p(g) + p(g+m)}{2} = c_0. \tag{7.1.2}$$

This indicates that every vertex of the elliptical $r-$gon has an opposing vertex with which it shares the midpoint c_0. □

An example of this property can be observed in Figure 7.1.4.

The final symmetry we'll show is related. Again, if p is constructed as in 7.1.4 but with even r.

Proposition 7.1.7. *Consider some p from 7.1.4 with $r = 2m$. Then for all $g, k \in \mathbb{Z}_r$ the edge between $p(g)$ and $p(k)$ is parallel and has the same length as the edge between $p(g+m)$ and $p(k+m)$.*

Proof. For $g, k \in \mathbb{Z}_r$,

$$\begin{aligned}
p(g) - p(g+m) &= 2c_0 && \text{by 7.1.2} \\
&= p(k) - p(k+m).
\end{aligned}$$

From the above it follows that

$$p(g) - p(k) = p(g+m) - p(k+m). \tag{7.1.3}$$

The difference between two points in the complex plane yields what can be thought of as the vector between them. Therefore 7.1.3 indicates that opposing edges are equal and parallel. □

In particular, we are interested in $k = g + 1$. Then Proposition 7.1.7 tells us that the edge between $p(g)$ and $p(g+1)$ is parallel and has equal length to the edge between $p(g+m)$ and $p(g+m+1)$. This indicates that opposing edges on an elliptical $r-$gon of even r, have equal length and are parallel.

We can see an example of this phenomenon (along with Proposition 7.1.6) in the depiction of an elliptical 8−gon below.

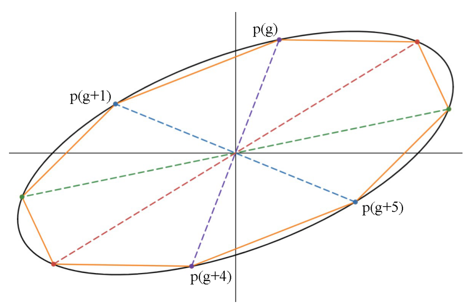

Figure 7.1.4. An example of an elliptical 8−gon with symmetries

7.2 Finding Elliptical $r-$gons

An important note to make about the points $p(g) = c_0 + c_1\omega_r^{jg} + c_2\omega_r^{-jg}$ is that the exponent of the second root of unity is the negative of the first. Since $j \in \mathbb{Z}_r$, this means $-j$ is equivalent to an integer $0 \le d < r$ where $j + d = r = 0 \in \mathbb{Z}_r$. Now we can see under what circumstances we can find these special polygons.

Theorem 7.2.1. *Let $G = \mathbb{Z}_r$ with $r \ge 3$, and $N = 2(r-3)+r-1 = T(r,2)-5$. Let $f : \triangle^N \to \mathbb{C}$ be a Fourier generic affine map. Then there exists a set of points $\{x_g\}_{g\in G}$ from distinct disjoint faces of $\triangle^N$ such that $\{f(x_g)\}_{g\in G}$ is the vertex set of an Elliptical $r-$gon or are all colinear.*

Proof. Let $j \in \mathbb{Z}_r$ such that $(r, j) = 1$, and $S = G - \{0, j, -j\}$. Let $N = 2|S| + |G| - 1 = 2(r-3) + r - 1 = 3r - 7$, and $f : \triangle^N \to \mathbb{C}$ be a Fourier generic map. By theorem 4.3.1 it follows that there exist r disjoint faces of $\triangle^N$ and a set of points $\{x_g\}_{g \in G}$ from distinct faces such that if $p \in G$, then

$$\begin{aligned}
f(x_p) &= F(p) && \text{from equation 4.1.1} \\
&= \sum_{\epsilon \in \mathbb{Z}_r} c_\epsilon \chi_\epsilon(p) && \text{by 3.3.4} \\
&= c_0 + c_j \chi_j(p) + c_{-j}\chi_{-j}(p) && \text{by 4.3.1} \\
&= c_0 + c_j \omega_r^{jp} + c_{-j}\omega_r^{-jp}
\end{aligned}$$

We know $c_j, c_{-j} \neq 0$ because f is Fourier generic. Since we do not know if $|c_j| = |c_{-j}|$, it follows that $\{f(x_g)\}_{g \in G}$ is either the vertex set of an Elliptical $r-$gon, by definition 7.1.4, or it's a set of colinear points (7.1.3). □

Intuitively it makes sense that as we eliminate fewer coefficients from our Fourier decomposition, and consequently require fewer initial points, we get a weaker result. However, it is fascinating to see that given only $T(r, 2) - 5$ points, we can still find nice $r-$gons with interesting symmetries. This is 4 fewer points than for a Tverberg partition, and 2 fewer than for a regular $r-$gon partition.

As with Theorem 5.0.3 we can again get a more general sub $r-$gon result of this type.

Theorem 7.2.2. *Let $G = \mathbb{Z}_r$ with $r \geq 2$. Let $r_1, r_2 \in \mathbb{Z}$ such that $r = r_1 r_2$, and $N = 2(r-3) + r - 1 = T(r, 2) - 5$. Let $f : \triangle^N \to \mathbb{C}$ be a Fourier generic affine map. Then there exist r pairwise disjoint faces of $\triangle^N$ that we can group into r_1 collections of r_2 faces, $C_1 = \{\sigma_0, \sigma_{r_1}, \cdots, \sigma_{(r_2-1)r_1}\}, \cdots, C_{r_1} = \{\sigma_{r_1-1}, \cdots, \sigma_{(r_1-1)+(r_2-1)r_1}\}$ such that:*

a) $\cap_{\sigma \in C_1} f(\sigma) \neq \emptyset, \cdots, \cap_{\sigma \in C_{r_1}} f(\sigma) \neq \emptyset$, and

b) there exist points $y_1 \in \cap_{\sigma \in C_1} f(\sigma), \cdots, y_{r_1} \in \cap_{\sigma \in C_{r_1}} f(\sigma)$ that form the vertex set of an Elliptical r_1-gon or are colinear.

Proof. Let $j \in \mathbb{Z}_{r_1}$ such that $(r_1, j) = 1$, and $S = G - \{0, jr_1, -jr_1\}$. Let $N = 2|S| + |G| - 1 = 2(r-3) + r - 1 = 3r - 7$, and $f : \triangle^N \to \mathbb{C}$ be a Fourier generic map. By theorem 4.3.1 it follows that there exist r disjoint faces of $\triangle^N$, $\{\sigma_g\}_{g \in G}$, and a set of points $\{x_g\}_{g \in G}$ where $x_g \in \sigma_g$, such that if $p \in G$, then

$$
\begin{aligned}
f(x_p) &= F(p) && \text{from equation 4.1.1} \\
&= \sum_{\epsilon \in \mathbb{Z}_r} c_\epsilon \chi_\epsilon(p) && \text{by 3.3.4} \\
&= c_0 + c_j \chi_j(p) + c_{-jr_2} \chi_{-jr_2}(p) && \text{by 4.3.1} \\
&= c_0 + c_j \omega_r^{jr_2 p} + c_{-j} \omega_r^{-jr_2 p} \\
&= c_0 + c_j \omega_{r_1 r_2}^{jr_2 p} + c_{-j} \omega_{r_1 r_2}^{-jr_2 p} \\
&= c_0 + c_j \omega_{r_1}^{jp} + c_{-j} \omega_{r_1}^{-jp}
\end{aligned}
$$

Because f maps to r_1th roots of unity and there are r points in $\{x_g\}_{g \in G}$, we're going to get some overlap in the image. Thus we ask where the overlap occurs. Consider for $p \in G$ and $n \in \mathbb{Z}$

$$
\begin{aligned}
f(x_{p+nr_1}) &= c_0 + c_j \omega_{r_1}^{j(p+nr_1)} + c_{-j} \omega_{r_1}^{-j(p+nr_1)} \\
&= c_0 + c_j \omega_{r_1}^{jp} \omega_{r_1}^{nr_1} + c_{-j} \omega_{r_1}^{-jp} \omega_{r_1}^{nr_1} \\
&= c_0 + c_j \omega_{r_1}^{jp} + c_{-j} \omega_{r_1}^{-jp} \\
&= f(x_p).
\end{aligned}
$$

Thus $f(x_p) = f(x_{p+r_1}) = \ldots = f(x_{p+(r_2-1)r_1})$. It follows that $\cap_{n=0}^{r_2-1} \sigma_{p+nr_1} \neq \emptyset$ for all $p \in G$. We can also conclude that the images of the first r_1 points, $\{x_g\}_{g=0}^{r_1-1}$ do not overlap. We know $c_j, c_{-j} \neq 0$ since f is Fourier generic. Since we don't know if $|c_j| = |c_{-j}|$, it follows that $\{f(x_g)\}_{g=0}^{r_1-1}$ is the vertex set of an Elliptical r_1−gon, by definition 7.1.4, or is a set of colinear points (7.1.3). □

Remark 7.2.3. Both Theorem 7.2.1 and Theorem 7.2.2 include a case in which the images of the points from the simplex are colinear. However, we believe it can be shown that for almost any $3r-6$ points, an ellitical sub $r-$gon partition (and elliptical $r-$gon partition) exists where the points are not colinear and instead form the vertex of an elliptical polygon. We don't prove this, because it involves the proof of Theorem 4.3.1 ([1]) which we didn't go into in this senior project. Intuitively, it's because colinearity is equivalent to the condition $|c_j| = |c_{-j}|$. This imposes an extra condition on the Fourier coefficients, thereby requiring an additional dimension of the $N-$simplex if f is generic.

Note that $N = 3r-7$ is tight for elliptical $r-$gons or sub $r-$gons, as it was for regular sub $r-$gons. We didn't include this in their theorems, but the proof is the same as in Theorem 5.0.3.

As we've done with the original sub $r-$gon, we can state theorems 7.2.1 and 7.2.2 in terms of convex hulls.

Theorem 7.2.4. *Let $r \geq 3$. Almost any set of $3r-6$ points in $\mathbb{R}^2$ can be partitioned into r disjoint sets $A_1, \dots, A_r$ such that there exist $x_1 \in Conv(A_1), \dots, x_r \in Conv(A_r)$ that form the vertex set of an elliptical $r-$gon, or are colinear.*

And similarly

Theorem 7.2.5. *Let $r \geq 3$ and $r = r_1 r_2$. Almost any set of $3r-6$ points in $\mathbb{R}^2$ can be partitioned into r_1 collections of r_2 disjoint sets, $A^1_1, \dots, A^1_{r_2}, \dots, A^{r_1}_r, \dots, A^{r_1}_{r_2}$ such that there exist $x_1 \in \cap_{i=1}^{r_2} Conv(A^1_i), \dots, x_{r_1} \in \cap_{i=1}^{r_2} Conv(A^{r_1}_i)$ that form the vertex set of an elliptical r_1-gon, or are colinear.*

We can illustrate these theorems with examples.

Theorem 7.2.4 states that with almost any collection $3(4) - 6 = 12 - 5 = 6$ points in $\mathbb{R}^2$, we can find either the vertex set of an elliptical $4-gon$ (a parallelogram), or a set of colinear points. Though as we noted in 7.2.3, we believe the former is almost always true. An elliptical $4-$gon can happen in one of two ways.

Figure 7.2.1. Case 1: Six arbitrary points

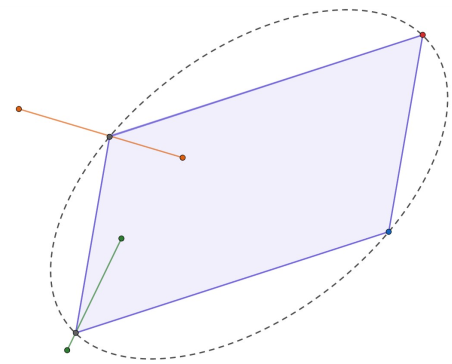

Figure 7.2.2. Case 1: Elliptical 4−gon

We see that taking the points in Figure 7.2.1, we can group the points into four collections, two collections of 2 points and two collections of 1 point. We can then find the vertex set of the Elliptical 4−gon in Figure 7.2.2 with points from the convex hulls of those collections.

In the second case, shown in Figures 7.2.3 and 7.2.4, the points are instead grouped into three collections of 1 point, and one collection of 3 points.

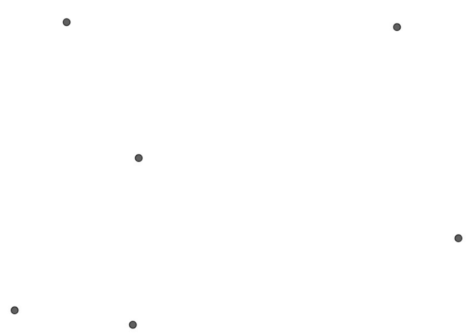

Figure 7.2.3. Case 2: Six arbitrary points

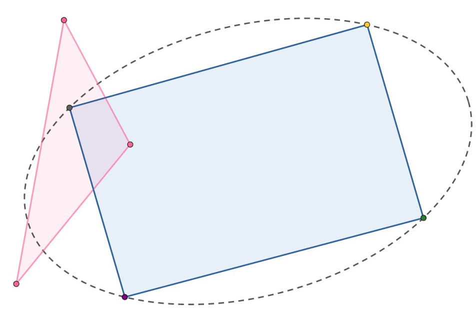

Figure 7.2.4. Case 2: Elliptical 4−gon

Similarly, Theorem 7.2.5 states that with almost any collection of $3(10) - 6 = 30 - 6 = 24$ points in $\mathbb{R}^2$, we can find an elliptical 5−gon in the intersection of convex hulls.

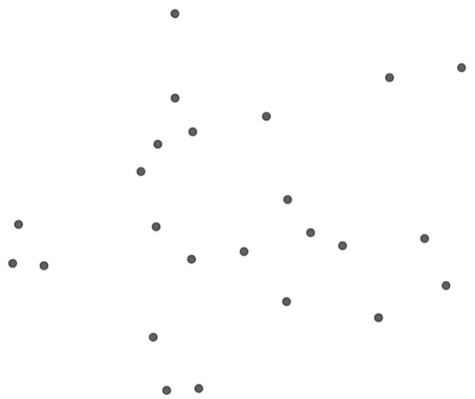

Figure 7.2.5. Twenty four arbitrary points

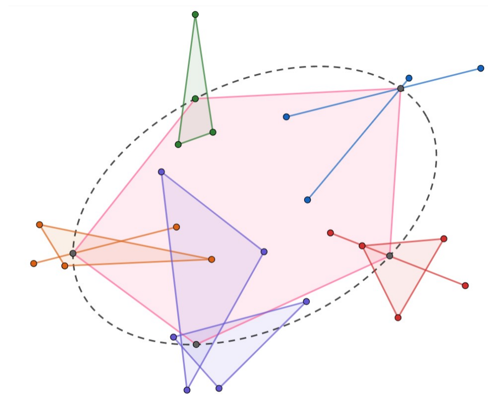

Figure 7.2.6. Elliptical 5−gon

Here, we see that with the points in Figure 7.2.5 we can group them into 10 collections of points such that the convex hulls of collections intersect in pairs, and we can find the vertex set of an Elliptical 5−gon in the intersections of those convex hulls. The end result is depicted in Figure 7.2.6.

Bibliography

[1] Leah Leiner, and Steven Simon. *Regular Polygonal Partitions of a Tverberg Type*. arXiv:1908.10810, 2021.

[2] Benjamin Steinberg. *Fourier Analysis on Finite Groups*. ISBN:978-1-4614-0776-8 , Springer 2011.