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Orthogonal Projections of Lattice Stick Knots

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Abstract

A lattice stick knot is a closed curve in $\mathbb{R}^3$ composed of finitely many line segments (*sticks*) that lie parallel to the three coordinate axes in $\mathbb{R}^3$, such that the line segments meet at points in the 3-dimensional integer lattice. The lattice stick number of a knot is the minimal number of sticks required to realize that knot as a lattice stick knot. A right angle lattice projection is a projection of a knot in $\mathbb{R}^3$ onto the plane such that the edges of the projection lie parallel to the two coordinate axes in the plane, and the edges meet at points in the 2-dimensional integer lattice. This project examines when right angle lattice projections are projections of lattice stick knots, with the aim to get an upper bound on lattice stick number.

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Acknowledgments

I would like to thank my advisor, Ethan Bloch, for his guidance and for always pushing me to think more critically and try new approaches throughout this rewarding process. I thank you for supporting me reassuring me when I needed it most.

1
Introduction

1.1 Background

A knot is a closed curve in $\mathbb{R}^3$ that does not intersect itself. An everyday example of a knot can be found in an extension cord. Imagine you are holding an extension cord with one end in each hand. Twist the two ends around each other to create a knot in the cord. Then plug the two ends of the cord together. This forms a closed loop that does not intersect itself.

In this paper we will be looking at knots composed of straight line segments, called *sticks*. These knots can be made by replacing curves with a finite number of line segments. When a knot is made entirely out of sticks it is referred to as a **stick knot**. The point where two sticks meet are referred to as a **vertex**.

A further introduction into the principles of knot theory can be found in [1]. From this text we use the notion of nontrivial knot, unknot, and crossing number, which we will denote here as $C(K)$.

Two nontrivial projections of knots that have 3 and 4 crossings, are pictured in Figure 1.1.1, they are called 3_1 and 4_1 respectively.

Once we have some knot K, one question to ask is, how could this knot be represented using sticks? Additionally, what is the smallest number of sticks we would need to construct this knot? Figure 1.1.2 shows 3_1 and 4_1 from Figure 1.1.1 realized as stick knots.

Another type of stick knot is the lattice stick knot.

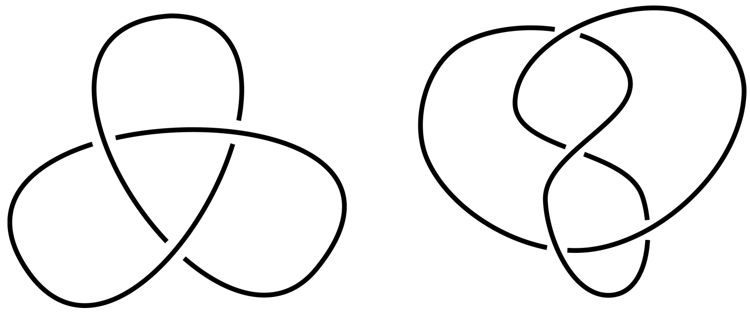

Figure 1.1.1.

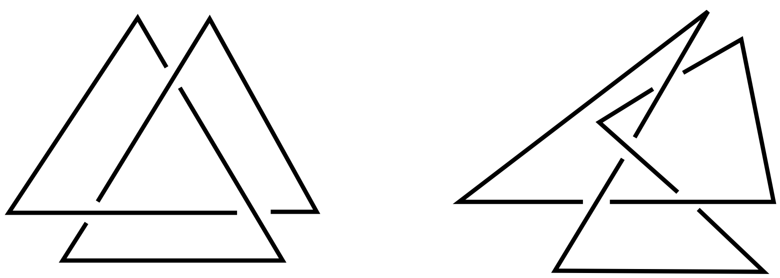

Figure 1.1.2.

Definition 1.1.1. A **lattice stick knot**, abbreviated, LSK, is a stick knot whose vertices are in $\mathbb{Z}^3$ and whose sticks lie parallel to one of the three coordinate axes. $\triangle$

Figure 1.1.3 shows 3_1 and 4_1 realized as LSKs. Sticks in a LSK are referred to based on which of the three coordinate axes they lie parallel to. For example, a stick in an LSK which lies parallel to the x-axis is referred to as an x-stick. Similarly a stick that lies parallel to the y-axis or z-axis would be called a y-stick or a z-stick, respectively.

Some application of LSKs to chemistry can be seen in [4] and [2].

Stick knots and LSKs each have their own invariant that is used to refer to the minimal number of sticks needed to construct each type of knot.

Definition 1.1.2. Let K be a knot. The **minimal stick number** for some knot K, $S(K)$, is the minimal number of sticks needed to realize K as a stick knot. $\triangle$

Definition 1.1.3. Let K be a knot. The **lattice stick number**, $S_L(K)$, is the minimal number of sticks needed to realize K as an LSK. $\triangle$

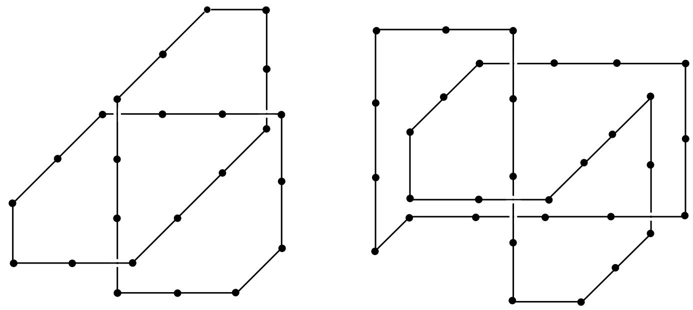

Figure 1.1.3.

Definition 1.1.4. Let K be a knot. The **lattice edge number**, $S_E(K)$, is the minimal number of unit length sticks needed to realize K as an LSK. △

The inequality below describes the relationship between the three invariants defined above. This follows directly from the added levels of restriction placed on the sticks during construction

$$C(K) \leq S(K) \leq S_L(K) \leq S_E(K) \tag{1.1.1}$$

1.2 Previous Results

Of the three invariants mentioned above, crossing number and stick number are known for many more knots than the other invariants. A brief history in the tabulation of knots with up to 13 crossings can be found in Chapter 2 of [1]. Stick number has been calculated for knots with up to 10 crossings.

In [10] Janse van Rensburg and Promislow proved that $S_L(K) \geq 12$ for any nontrivial knot K. Continuing the study of LSKs, in [6] Huh and Oh proved that the lattice stick numbers of 3_1 and 4_1 are 12 and 14, respectively, and that $S_L(K) \geq 15$ for any other nontrivial knot K. In [3] Diao proved that $S_E(K) \geq 24$ for any nontrivial knot. Additionally, in [11] Scharein et al. showed that the lattice edge number of $3_1, 4_1$ and 5_1 is 24, 30 and 34, respectively.

In addition to results pertaining to specific knots, the relationship between these different knot invariants has also been studied. In [9] Negami proved the following result about upper and lower bounds on the stick number of a knot K in terms of the crossing number $C(K)$:

$$\frac{5+\sqrt{25+8(C(K)-2)}}{2} \le S(K) \le 2C(K). \tag{1.2.1}$$

Later, in [7] Huh and Oh improved Negami's upper bound to $S(K) \le \frac{3}{2}(C(K)+1)$, for a non-alternating prime knot K. Furthermore, in [5], Hong, No, and Oh proved further improved the upper bound on the lattice stick number in terms of the crossing number. They showed that for any nontrival knot that was not the trefoil knot, 3_1, that $S_L(K) \le 3C(K)+2$.

1.3 Focus of This Paper

Rather than focusing on trying to improve or branch off of the inequalities described in the previous section, in this paper we will focus on a certain kind of projection of knots and examine when there is a lattice stick knot that projects onto it. This paper lays the groundwork for talking about these projections and introduces methods and notation that are used to study them.

2
Right Angle Lattice Projections

2.1 Introduction

This chapter focuses on projections of knots in $\mathbb{R}^2$ where the vertices are in $\mathbb{Z}^2$ and the sticks lie parallel to either the x- or y-axis. This type of projection is found by taking a the representation of some knot K as a stick knot and either rotating existing sticks or adding new sticks until all of the sticks in the projection lie parallel to the x- or y-axis.

Figure 2.1.1 shows the process of manipulation mentioned above for the trefoil knot, 3_1. It begins, in the top left, with a representation of 3_1 as a stick knot and ends in the lower right, with a projection where all of the sticks lie parallel to either the x- and y- axis.

Definition 2.1.1. A **right angle lattice projection (RALP)** is a projection of a knot onto the xy-plane in which all of the vertices of the projection are in $\mathbf{Z}^2$ and the edges of the projection lie parallel to the coordinate axes. △

Our hope is that RALPs are projections of LSKs and that they can be used to get an upper bound on the lattice stick number for a give knot K.

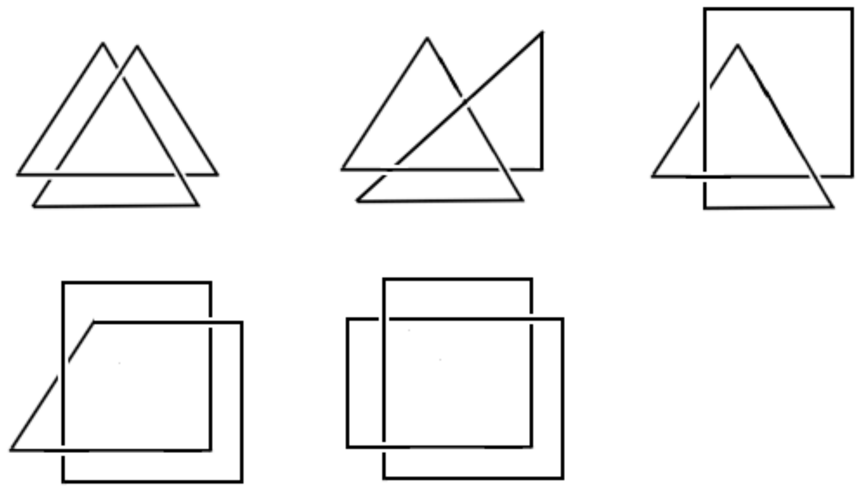

Figure 2.1.1.

2.2 Forbidden Subsets

One way to determine whether a RALP comes from the projection of an LSK is to look for subsets of RALPs that could not come from the projection of an LSK.

Definition 2.2.1. Let R be a RALP and let T be a subset of R. If there isn't a subset of an LSK that projects onto T then T is **LSK forbidden**. $\triangle$

Figure 2.2.1 shows a subset of a RALP, called f_1, that is LSK forbidden. Before moving onto the proof that f_1 is LSK forbidden, we introduce the following definition.

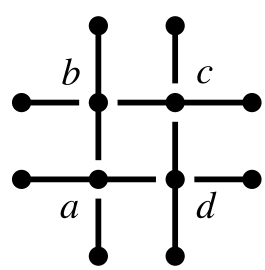

Figure 2.2.1.

Definition 2.2.2. Let R be a RALP and let p be a crossing in a R. Define p^+ to be the point on the stick that crosses over the other stick at p. Define p^- to be the point on the stick that crosses under p^+.

Let q be a point in R. If there is a z-stick in the LSK that projects onto R, that projects onto q then, define q_t to be the maximum z-value on that z-stick. Define q_s to be the minimum z-value on that z-stick.

If there is not a z-stick in the LSK that projects onto R, that projects onto q then, define q_z to be the z-value at q. $\triangle$

If there is a z-stick in the LSK that projects onto R that projects onto a crossing r in a RALP, the maximum and minimum values on the z-stick that projects onto r would be r_t^+ and r_s^+; and r_t^- and r_s^-, respectively.

If there is not z-stick in the LSK that projects onto a RALP that projects onto a point r in a RALP, it can be thought of an having a z-stick of height 0 at that point. This allows us to apply Definition 2.2.2 to see that $r_z = r_t = r_s$.

With these definitions in hand, we move on to the following Lemma in which the crossings in f_1 will be referred to using the same labels that are used in Figure 2.2.1.

Lemma 2.2.3. *f_1 is LSK forbidden*

Proof. Let R be a RALP that contains the subset f_1 shown in Figure 2.2.1. Assume K is an LSK whose projection onto the xy-plane is R.

At each crossing in R there are four possibilities for what could be projecting onto the points at the crossing from K. WLOG we will examine these four cases from the perspective of the crossing at d.

Case 1 Neither d_z^+ nor d_z^- have z-sticks from K projecting onto them. In this case, $d_z^+ > d_z^-$

Case 2 Both d_z^+ and d_z^- have z-sticks from K projecting onto them. In this case, $d_s^+ > d_t^-$

Case 3 There is a z-stick from K projecting onto d_z^+, but not onto d_z^-. In this case, $d_s^+ > d_z^-$

Case 4 There is a z-stick from K projecting onto d_z^-, but not onto d_z^+. In this case, $d_z^+ > d_t^-$

In all four of these cases $d_s^+ > d_t^-$. The analogous statements hold for the other three crossings in R.

Next, we look at the interactions between two adjacent vertices in f_1. Take for example the vertices a^+ and d^- from Figure 2.2.1. Figure 2.2.2 shows the nine generic subsets of K that could project onto a^+ and d^- and the stick that connects them, as viewed from the side.

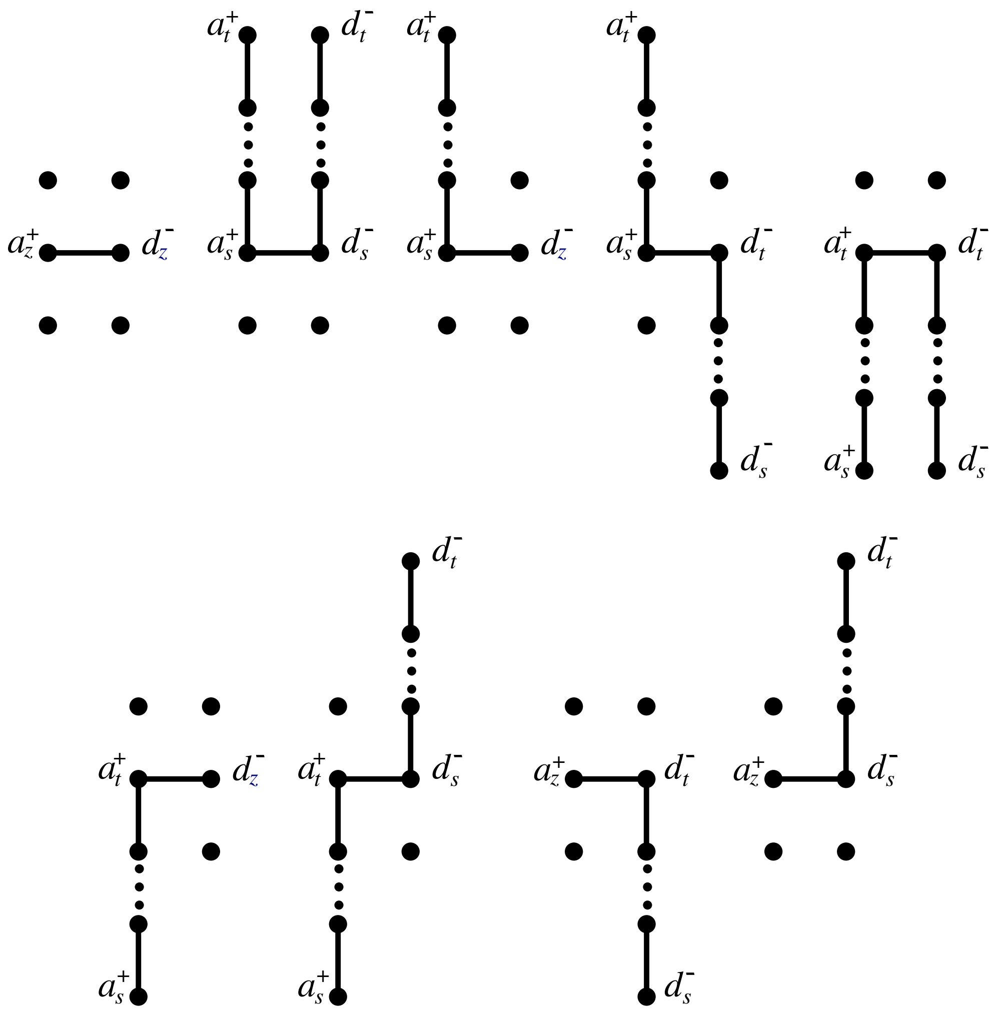

Figure 2.2.2.

For each case shown in Figure 2.2.2 the inequality $d_t^- \geq a_s^+$ holds. An analogous relationship holds for the other three pairs of adjacent crossings in f_1.

When combined, these two observations leads to the following chain of inequalities that describe the relationship of the four crossings in f_1 in relation to each other as one moves around the four crossings f_1 in a clockwise fashion beginning at d:

$d_s^+ > d_t^- \geq a_s^+ > a_t^- \geq b_s^+ > b_t^- \geq c_s^+ > c_t^- \geq d_s^+$.

From this we come to the conclusion that $d_s^+ > d_s^+$, which is a contradiction. Therefore, there is no LSK K that projects onto R. □

Based on Lemma 2.2.3 as well as other examples that were computed throughout this process, we have come to the following conjecture.

Conjecture 2.2.4. *Let R be a RALP. If R does not contain f_1 then R is the projection of an LSK.*

Now we introduce methods that lay the groundwork for examining RALPs and when they are projections of LSKs.

2.3 Cutting RALPs

Definition 2.3.1. Let R be a RALP and let p be a crossing in R. When the stick containing p^+ lies parallel to the y-axis, it is a **type i crossing**. When the stick containing p^+ lies parallel to the x-axis, it is called a **type ii crossing**. A collection of crossings that are either all type i or all type ii crossings, are said to have the same **crossing type**. △

Figure 2.3.1 shows the two types of crossings described in Definition 2.3.1.

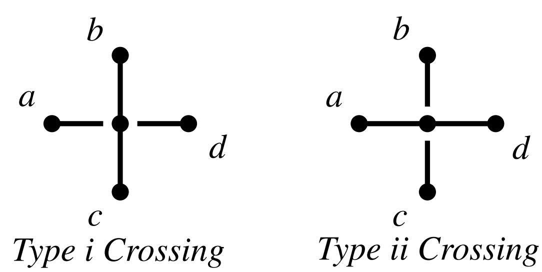

Figure 2.3.1.

Definition 2.3.2. Let R be a RALP and p be a crossing in R. Performing a **cut** at p is the process of changing p from the projection of a crossing to the projection of a point where two corners of RALPs meet. There are two ways a cut can be performed.

(a) When the crossing is split into two corners by dividing along the line $y = -x$ it is called a **type -1** cut.

(b) When the crossing is split into two corners by dividing along the line $y = x$ it is called a **type 1** cut.

△

The left-most image in Figure 2.3.2 shows a RALP, K, that has a single crossing p. To the right of K in the figure are the results of cutting p in the two ways described in Definition 2.3.2. The two images on the right of this figure show the standard way that we will draw RALPs after a crossing has been cut. The crossing is still drawn at p to enable us to tell which type of crossing was at p before the cut. The lines that bend around p show the orientation of the corners of the RALP after p has been cut.

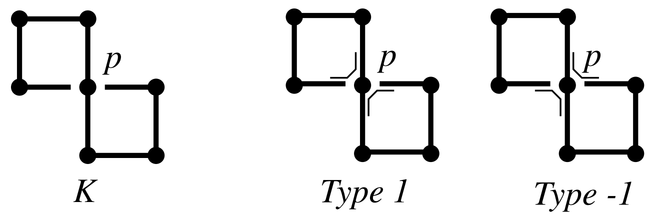

Figure 2.3.2.

This small example highlights the fact that there are two possible outcomes when cutting at a crossing. In Figure 2.3.2, a type 1 cut results in two separate RALPs, and p becomes a point where two corners of different RALPs touch. In Figure 2.3.2, a type -1 cut does not result in two separate the RALP, and p becomes a point where two corners of the same RALP touch. To describe the effects of cutting a crossing in a RALP, we introduce the following definition.

Definition 2.3.3. Let R be a RALP and let p be a crossing in R. Suppose cutting p does not separate the RALP. The resulting object is called a **RALP with overlap**. When p becomes the point where two corners of the same RALP with overlap meet, p is called a **self-overlap**. △

In the example shown in Figure 2.3.2 a type -1 cut turns p into a self-overlap and a type 1 cut does not.

The following definition provides a way to talk about the area surrounding a crossing in a RALP.

Definition 2.3.4. Let R be a RALP and let p be a crossing in R. The **neighborhood** of p contains the vertices in $\mathbb{R}^2$ adjacent to p and the sticks connecting those vertices to p. △

Figure 2.3.3 shows the neighborhood of a crossing p, it contains the vertices a, b, c and d and the two sticks that connect a and d to b to c, respectively.

To more precisely describe the vertices in the neighborhood of a crossing, we introduce the following definitions.

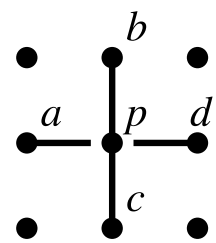

Figure 2.3.3.

Definition 2.3.5. Let R and a RALP and let p be a vertex in R. The two pairs of vertices in the neighborhood of p that are on opposite sides of p are the **opposing pairs** of p. The other four pairs of vertices in the neighborhood of p are the **non-opposing pairs** of p. △

In Figure 2.3.3 the two opposing pairs are: c and b, and a and d; and the four non-opposing pairs are: a and b, b and d, d and c, and c and a.

Definition 2.3.6. Let R be a RALP and p be a crossing in R. Suppose R is cut at p. The two non-opposing pairs pairs of p that are connected by x- and y- sticks are called **strands**. △

The following lemma show how to cut any RALP so that it becomes a collection of RALPs that meet only at the points where the crossings used to be. The method described in the proof is based on the standard method used for constructing Seifert circles. For more on this topic see Section 4.3 in [1].

Lemma 2.3.7. *Let R be a RALP. There is a finite number of cuts that separate R into RALPs that only meet in places where cuts occurred.*

Proof. Assign an orientation on R. At each crossing in R there will be two sticks oriented towards the crossing and two sticks oriented away from the crossing. Crossings will be cut by connecting the stick oriented towards the crossing to the adjacent stick oriented away from the crossing with a diagonal line. Figure 2.3.4 shows an example of how this line is drawn.

Cutting all crossings in the way described above creates a collection of non-intersecting polygons. In order to make these polygons RALPs the diagonal lines have to be replaced with sticks that lie parallel to the coordinate axes to form the two corners that meet at the points where there used to be crossings.

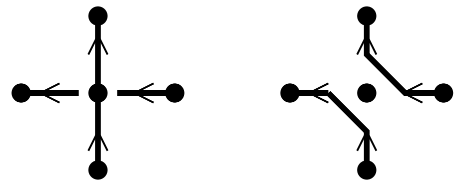

Figure 2.3.4.

This is done by removing the diagonal lines and extending the x- and y-sticks that were connected to the diagonal stick so that the non-opposing pair meets at the point that used to be a crossing.

The only thing that remains to be shown is that after the corners are added back on there are no RALPs with overlap.

Suppose that when the corners are added to R, R becomes a RALP with overlap. Let q be the self-overlap in R. WLOG an orientation can be assigned to one of the two strands in the neighborhood of q. There are two ways an orientation can be assigned on the other strand in the neighborhood of q, these are shown in Figure 2.3.5, where the left strand is the one that has its orientation assigned initially.

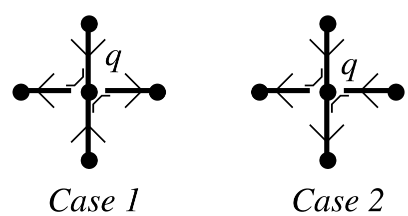

Figure 2.3.5.

The orientation on the sticks in Case 1 from Figure 2.3.5 could not have come from the projection of a crossing because the two y-sticks are oriented towards each other.

This issue does not arise in Case 2. Thus, what remains to be shown is how the two corners can be connected to form a RALP with overlap, where q is a self-overlap. As a result of the orientation on the sticks, there are two ways to connect the corners, either two non-opposing pairs of q can be connected or two opposing pairs of q can be connected.

Connecting two non-opposing pairs of q results in two distinct RALPs, which contradicts the assumption that r is a self-overlap. The general case of the two corners being connected in this way is shown in

Case 2a of Figure 2.3.6. The curves in the picture represent the multitude of possible configurations of x- and y-sticks that could be connecting the non-opposing pairs of q.

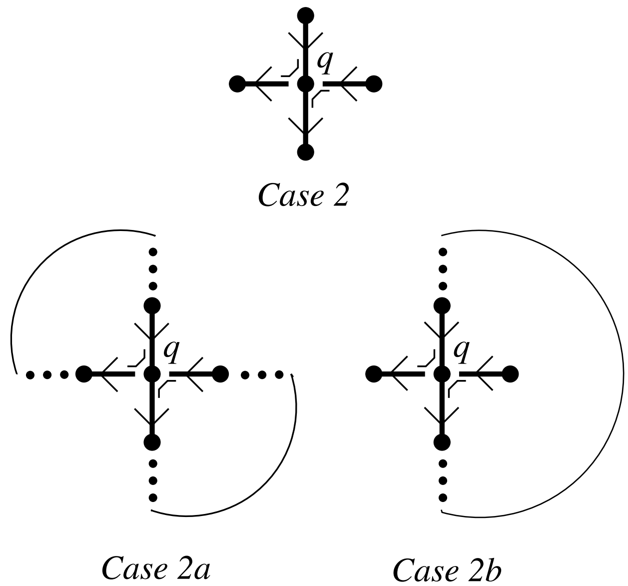

Figure 2.3.6.

Another possibility is to connect the opposing pairs of q. After one opposing pair of q has been connected it is not possible to connect the second opposing pair, because one vertex is inside the section marked off by the x- and y-sticks that were added in the previous step, and the vertex it needs to be connected to is outside of this section defined by the x-and y-sticks that were added in the previous step. This is shown in Case 2b of Figure 2.3.6. Thus q is not a self-overlap.

Therefore, adding the corners back on does not result in any self-overlaps. □

Figure 2.3.7 shows the process described in Lemma 2.3.7 for a specific representation of 3_1 as a RALP. The left-most image in the figure is 3_1 realized as a RALP with an orientation assigned on the sticks. The middle image shows the collection of non-intersecting polygons that result from cutting all crossings. The right-most image shows the RALPs after the crossings have been added back on.

Definition 2.3.8. Let R be a RALP. Suppose all crossings in R are cut using the method described in Lemma 2.3.7. The resulting RALPs that meet only at the corners where there used to be crossings are called **RALP-circles**. △

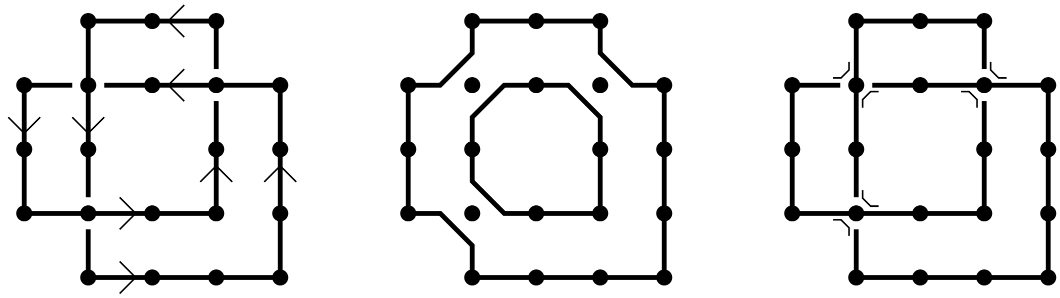

Figure 2.3.7.

After all crossing in a RALP have been cut, there cannot be any crossings in the RALP-circles. This implies that RALP-circles are projections of themselves realized as LSKs. This observation combined with Lemma 2.3.7 results in the following corollary.

Corollary 2.3.9. *Let R be a RALP that comes from the projection of a stick knot representation of a nontrivial knot. Using a finite number of cuts R can be separated into a collection of RALP-circles all of which are projections of LSKs.*

2.4 Sewing RALPs

By Lemma 2.3.7 we know it is possible to take any RALP and perform a finite number of cuts that separate the original RALP into a collection of RALP-circles that only meet at points were there used to be crossings. Additionally, we know that these RALP-circles are projections of LSK, by Corollary 2.3.9.

This section introduces a method that, contrary to cutting, combines LSKs that project onto RALPs to form a different LSK that projects onto a RALP. This method will informally be referred to as *sewing*. The following small example outlines the concept of sewing.

Let T be the RALP shown in the top left image in Figure 2.4.1. Let q be the only crossing in T. After q is cut, T becomes two RALPs R and S that each have no crossings and meet only at q, as seen in the top right image in Figure 2.4.1. Since R and S have no crossing they are projections of themselves realized as LSKs. R and S are shown in the left-most image of the second row of the figure. The middle image in the second row of the figure shows two possible LSKs, R' and S', that could project onto R and S.

Finally in the farthest right image in the second row, we see how the two LSKs can be combined to form an LSK that projects onto T.

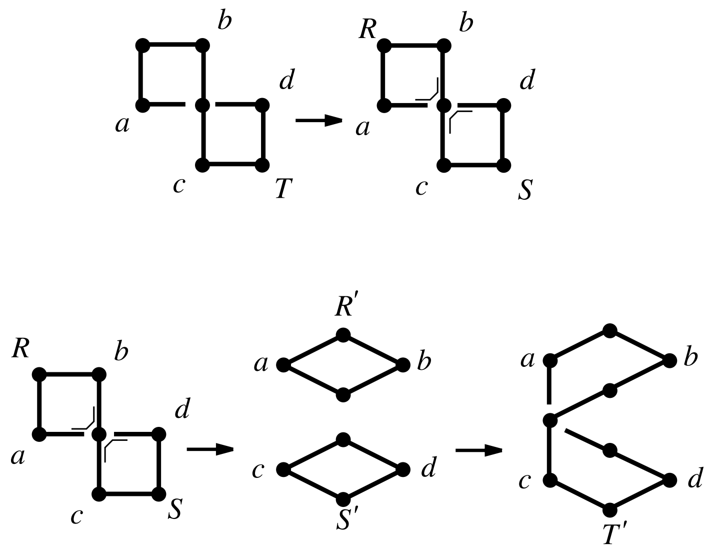

Figure 2.4.1.

Before moving on to show more instances where two LSKs that project onto RALPs can be sewn together, we introduce the following definition.

Definition 2.4.1. Let R be a RALP and let p be a crossing in R. If there is a non-opposing pair of p such that both vertices have the opposite crossing type to that of p, then p is called a **problem crossing**. The crossings in the non-opposing pair of p that have the opposite crossing type of p are the **rivals** of p. $\triangle$

Notice that in the drawing of f_1 shown in Figure 2.2.1 all four of the crossings are problem crossings. The following theorem shows some of the instances when it is possible to sew two LSKs together.

Theorem 2.4.2. *Let R be a RALP and let p be a crossing in R. Suppose R is cut at p. Suppose there are LSKs K_1 and K_2, or an LSK K_1, that project onto the two RALPs, or onto the RALP with overlap, respectively, that are formed when p is cut.*

(a) Suppose there are at most two crossings on R one unit away from p, and p is not a problem crossing. Then there is some LSK K that projects onto R.

(b) If R satisfies the conditions outlined in (a) then, K will require at most three more z-sticks than there were in K_1 and K_2 combined, or in K_1. The additional z-sticks in K are added either at p or at vertices adjacent to p.

Proof. Part (a) It can be verified that there are 98 ways there could be at most two crossings adjacent to p, taking into account the different locations of crossings adjacent to p and the different combinations of crossing types at crossings adjacent to p and at p. In sixteen of these instances p was a problem crossing before it was cut. Therefore, there are 82 cases to consider. These 82 cases can be obtained by rotating and reflecting in $\mathbb{R}^3$ the twelve generic cases shown in Figure 2.4.2. Therefore, only these twelve generic cases will be considered when forming K. The circles in Cases 4 and 5 signify that there could be a crossing,

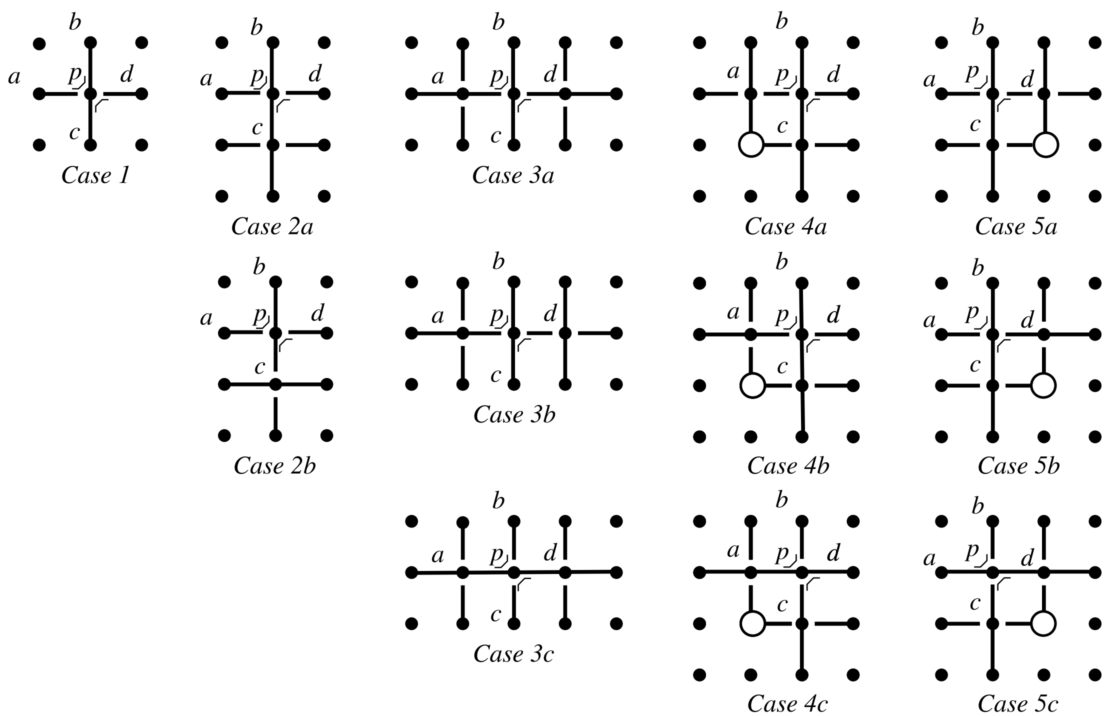

Figure 2.4.2.

a single corner, or another point where two corners of RALPs meet at that point. It does not matter what is at those points, because only the vertices adjacent to p will be affected when K is formed.

We will consider the case where there are LSKs K_1 and K_2 that project onto two RALPs that have corners that meet at p; the other case is similar, so we omit the details. Let S_1 and S_2 be the strands at p in the two RALPs, let T_1 and T_2 be the subsets of K_1 and K_2 that project onto S_1 and S_2, respectively.

There are nine cases to consider for T_1 and T_2, taking into account the presence of z-sticks at the points in T_1 and T_2 that project onto p, as seen in Figure 2.4.3.

In Figure 2.4.3 T_1 is the subset containing a and b, and T_2 is the subset containing c and d. Figure 2.4.3 shows T_2 with a higher z-value than T_1, the analogous figure where T_1 has a higher z-value than T_2 is shown in Figure 2.4.4.

In Figure 2.4.3 and Figure 2.4.4 Cases *(a), (b), (e), (f), (g),* and *(h)* are the six generic cases which can be rotated and reflected in $\mathbb{R}^3$ to find all other cases. Therefore, only these six generic cases will be considered when forming K.

It is possible to find ways to form K in these six cases that work in all twelve generic cases shown in Figure 2.4.2.

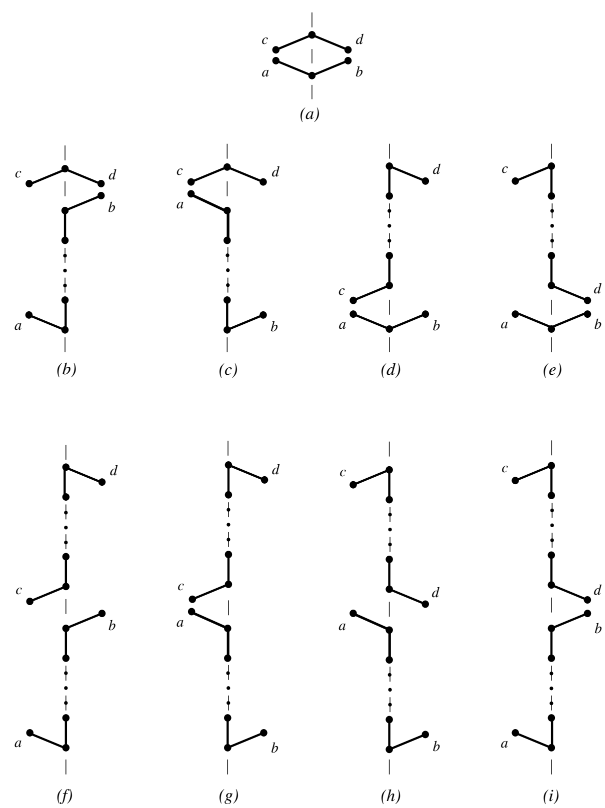

Figure 2.4.3.

There are no z-sticks shown at vertices a, b, c or d in Figure 2.4.3 and Figure 2.4.4 because they would not have an effect on how K is formed.

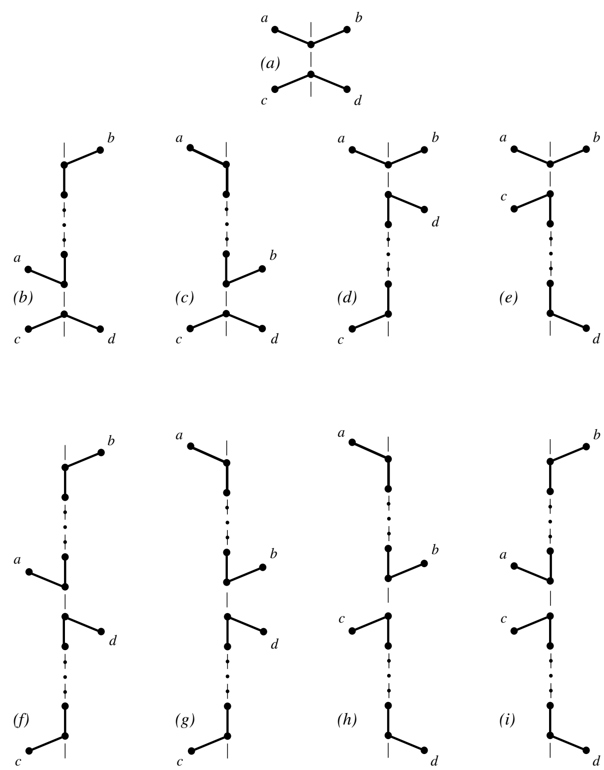

Figure 2.4.4.

Figure 2.4.5 show how to form K by combining T_1 and T_2 to form a type i crossing when T_1 has a higher z-value than T_2 at p.

Figure 2.4.6 show how to form K by combining T_1 and T_2 to form a type i crossing when T_2 has a higher z-value than T_1 at p.

Figure 2.4.7 shows how to form K by combining T_1 and T_2 to form a type ii crossing when T_1 has a higher z-value than T_2 at p.

Figure 2.4.8 shows how to form K by combining T_1 and T_2 to form a type ii crossing when T_2 has a higher z-value than T_1 at p.

Part (b) The last thing that remains to be shown is that at most three z-sticks were added when K was formed. This follows directly from Part (a). Notice that in Figure 2.4.5, Figure 2.4.6, Figure 2.4.7, and Figure 2.4.8 there are either two or three z-sticks added in each solution. Furthermore, they are added at one of the four points adjacent to p or at the point that projects onto p. □

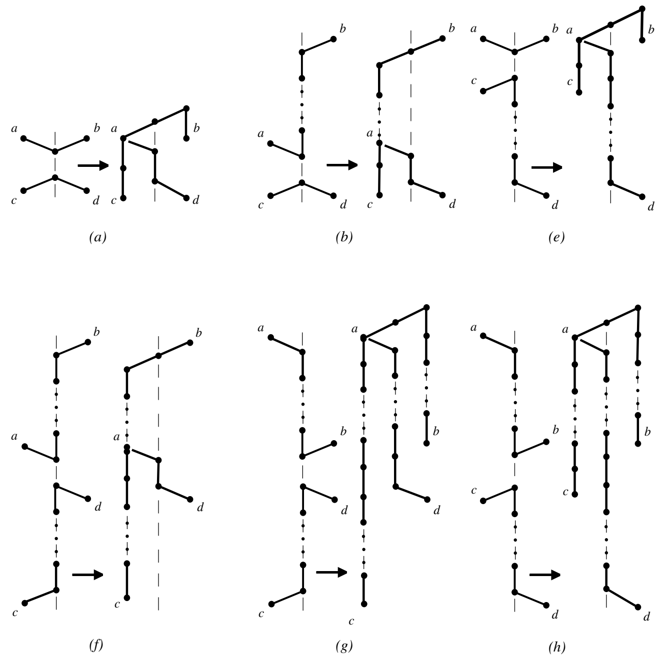

Figure 2.4.5.

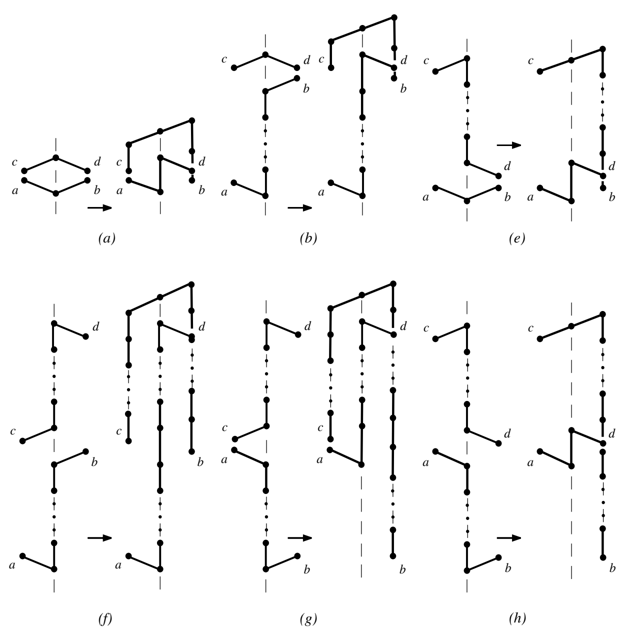

Figure 2.4.6.

Combining Theorem 2.4.2 and Lemma 2.3.7 yields the following corollary,

Corollary 2.4.3. *A RALP with no adjacent crossings is a projection of an LSK.*

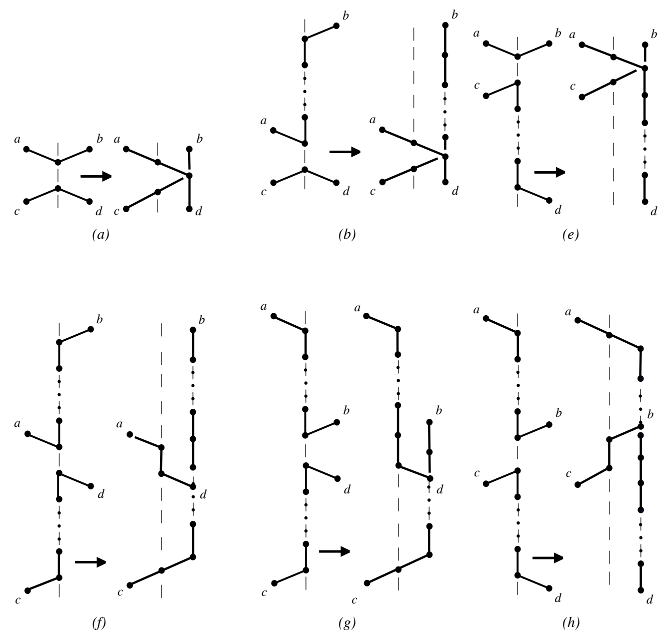

Figure 2.4.7.

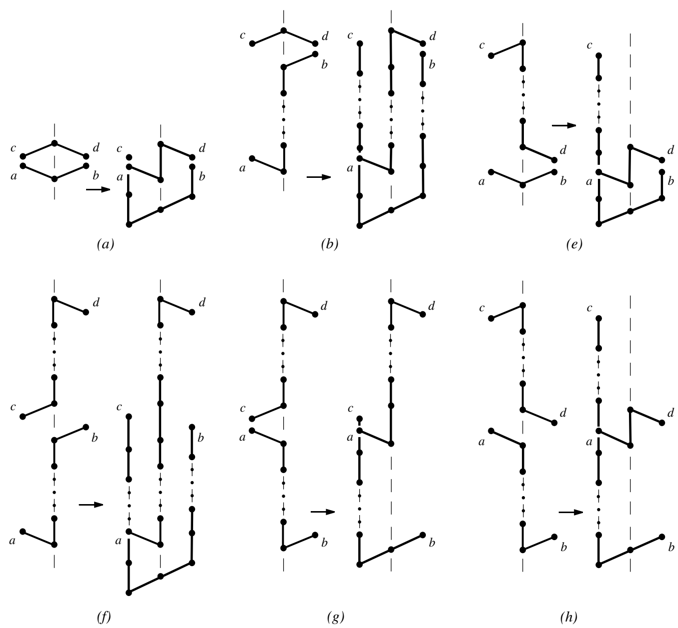

Figure 2.4.8.

Proof. Let R be a RALP with no adjacent crossings. By Lemma 2.3.7 it is possible cut all crossings in R, which results in RALP-circles that only meet at the points where cuts occurred and are projections of LSKs.

Since there are no adjacent crossings, all the points where two corners of RALP-circles meet satisfy the conditions of Theorem 2.4.2. Thus, using the methods presented in Theorem 2.4.2 the LSKs that project onto the RALP-circles can be combined to form a LSK that projects onto the original RALP R. Let K be the LSK that projects onto R.

Let q be a vertex in R, and let p and r be crossings in R. It remains to be shown that when q is adjacent to p and r that p and r can be connected without causing a conflict between the z-sticks that might've been added to K that project onto q when p and r were constructed. Figure 2.4.9 shows one way that q could be adjacent to p and r.

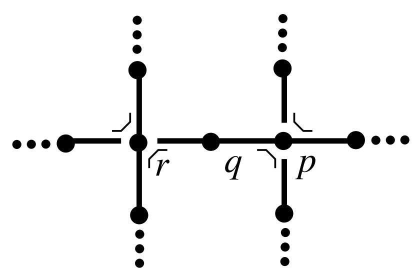

Figure 2.4.9.

Depending on the way that the crossings were constructed at p and r, there are three possible configurations of z-sticks in K that could project onto q. These possibilities are shown, as viewed from the side, in Figure 2.4.10. Figure 2.4.10 shows these configurations when they are adjacent to r, the same possibilities arise when they are adjacent to p. The points that project onto q are labelled using the notation defined in Definition 2.2.2.

Let q_p and q_r be the points that project onto q that are connected to the crossings p and r, respectively. When the crossings at p and r are constructed, either $q_p = q_r$ or $q_p > q_r$ or $q_p < q_r$.

When $q_p = q_r$ we see that, since q_p and q_r have the same z-value they are the same point. Thus, p and q are already connected, so all that we have to do is remove any z-sticks that project onto q_p or q_r from the LSKs that were formed when p and q were constructed.

When $q_p \neq q_r$ we see that, p and q can be connected by first removing any z-sticks in K that project onto q_p or q_r and then add a new z-stick to the LSK that was formed when p and q were constructed that connects q_p and q_r. □

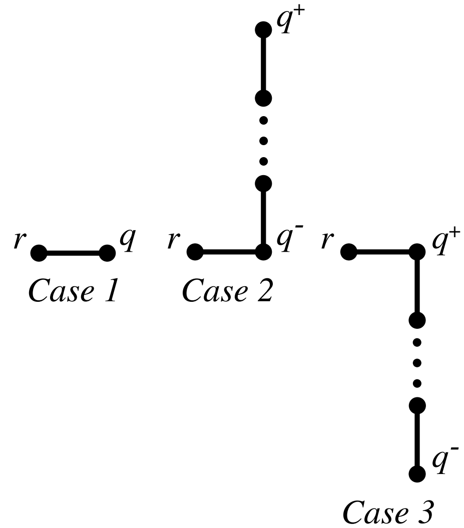

Figure 2.4.10.

Corollary 2.4.3 opens the door to a large number of situations in which RALPs are projections of LSK. Leaving the only thing left to show is how to deal with RALPs that have adjacent crossings. This will be left for the following chapter.

3
Problem Graphs

3.1 Introduction

We saw in the previous chapter, in Corollary 2.4.3, that when there are no adjacent crossings in a RALP it is the projection of an LSK. This allows us to identify a large number of RALPs that are projections of LSKs. However, this still leaves a large category of RALPs untouched. This chapter focuses on RALPs that have groups of adjacent crossings. In order to explore how adjacent crossings influence each other, we introduce a way to encode the relationship between adjacent crossings. By looking at how adjacent crossings influence each other we are able to make conclusions about how to sew the RALP-circles that result from cutting RALPs that have groups of adjacent crossings.

3.2 Construction

We will begin our examination of RALPs with multiple adjacent crossings by looking at the RALPs shown in Figure 3.2.1. These RALPs each have four crossings in a 1×1 square. When these RALPs are cut using the method introduced in Lemma 2.3.7, they look like the RALP-circles shown in Figure 3.2.2.

Using the methods described in Theorem 2.4.2 we will sew the crossings back into the RALP-circles shown in Figure 3.2.2 to get the original RALPs shown in Figure 3.2.1. The crossings will be sewn back

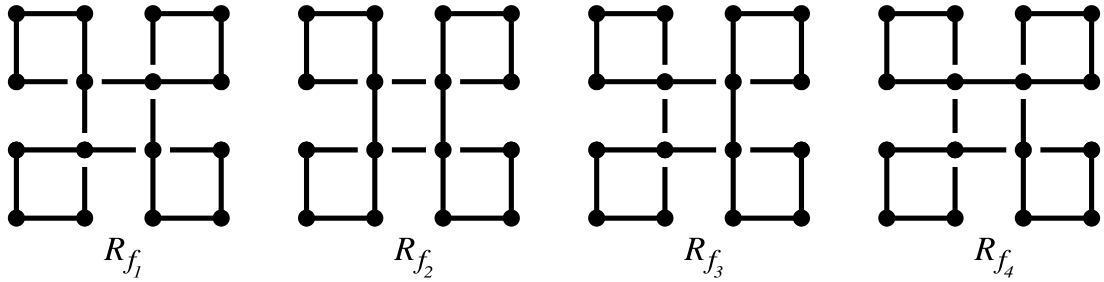

Figure 3.2.1.

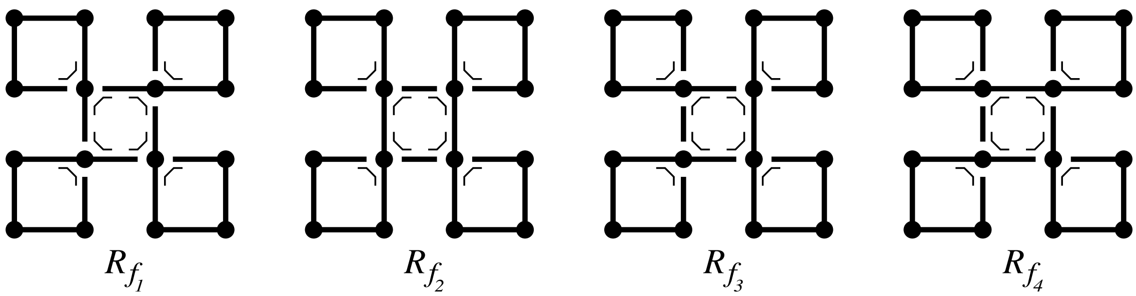

Figure 3.2.2.

into each collection of RALP-circles in the same order. We will begin at the crossing in the lower right and move to adjacent crossings in a counterclockwise fashion.

In the first step, we are able to sew the corners together in all four instances because there are no crossings adjacent to the point where the crossing is being formed. We see in the next two steps that it is possible to sew the crossings at the top right and top left points in all four cases because there is only one crossing adjacent to the point being sewn.

In the final step, we have to sew the crossing in the lower right. The RALP-circles at this step are shown in Figure 3.2.3. They are labelled as R'_{f_i}.

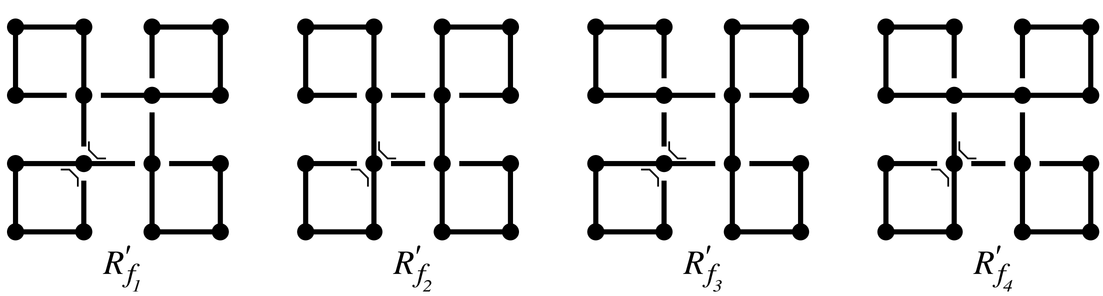

Figure 3.2.3.

At this step there are two crossing adjacent to the point being sewn in all four cases. However, the only crossing that cannot be sewn is the one in R'_{f_1}, because the original crossing at that point is a problem crossing.

It is interesting to note that the last crossing that we tried to sew was not the only problem crossing present in the original RALPs. There were four problem crossings in R_{f_1} and one problem crossing in R_{f_4}. The other problem crossings were able to be sewn without issue because they were sewn before their rivals.

This small example suggests that there might be a systematic way to sew problem crossings in RALPs that are projections of LSKs so that the conditions of Theorem 2.4.2 are satisfied each time sewing occurs. Before moving forward we introduce a way to encode the relationships between problem crossings in the following definition.

Definition 3.2.1. Let R be a RALP. The **problem graph** of R, denoted G_R, is the directed graph whose set of vertices contains the problem crossings of R and their rivals, and whose set of edges contains the directed edges that go from a problem crossing to its rival. If two adjacent problem crossings are rivals of each other, there will be two directed edges between them. When this occurs the edges are called **antiparallel edges**. △

Vertices that correspond to problem crossings in G_R are marked with circles, vertices corresponding to rivals are colored black, and other points in the RALP are colored gray.

Figure 3.2.4 shows the problem graphs for R_{f_1} and R_{f_4} alongside the original RALPs. This figure shows two different types of problem graphs. The problem graph for R_{f_4} can come from a RALP that is a projection of an LSK. Whereas the problem graph for R_{f_1} cannot come from a RALP that is a projection of an LSK.

After seeing these two basic examples of problem graphs in Figure 3.2.4 the question of what other configurations of problem graphs could come from RALPs that do not contain f_1 comes up. The next sections begins with a basic analysis of some of the properties of problem graphs. Then it examines which subsets of problem graphs cannot come from RALPs without f_1 and which can.

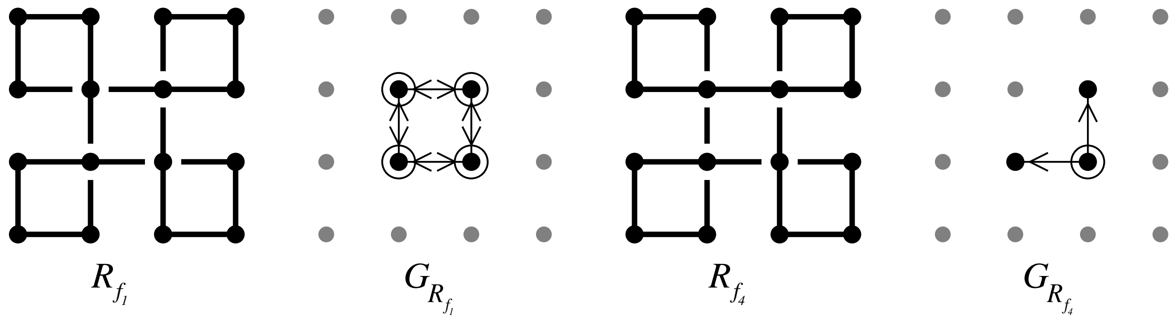

Figure 3.2.4.

3.3 Structure of Problem Graphs

The lemmas in this section analyze the structure of problem graphs. In the following lemmas and discussion, when reference is made to the crossing type of a vertex in a problem graph, it is actually referring to the type of crossing at the problem crossing in the original RALP that corresponds to that vertex. Similarly, when the rival of a vertex in the problem graph is mentioned, it is actually referring to the rival of the problem crossing in the original RALP that corresponds to that vertex.

As a direct result of Definition 2.4.1 and Definition 3.2.1 we get the follow lemma.

Lemma 3.3.1. *Let R be a RALP and let $\mathbf{G}_R$ be its problem graph. Let p and q be adjacent vertices in $\mathbf{G}_R$.*

(a) There cannot be a single directed edge in $\mathbf{G}_R$ between p and q.

(b) If there are antiparallel edges connecting in $\mathbf{G}_R$ p and q, then p and q are not the same crossing type.

(c) If there are no directed edges in $\mathbf{G}_R$ between p and q, then p and q are the same crossing type.

Proof. Part (a) Assume there is one directed edge between p and q. WLOG, assume the directed edge goes from p to q. Since q is the rival of p and p is not the rival of q, it follows from Definition 2.4.1 that p and q are not the same crossing type.

Since q is also a problem crossing and p is not one of its rivals, the rivals of q must fall on at least two of the other three vertices that are one unit away from q in its neighborhood such that they form a non-opposing pair of q. Figure 3.3.1 shows p and q, and the three places where the rivals of q can be located are shown in grey. By Definition 2.4.1, the rivals of q will be the opposite type of crossing, which means they will be the same type of crossing as p. Regardless of where the other two rivals of q are placed, at least one will form a non-opposing pair of q with p. This means that p will be a rival of q, which implies that there would be a directed edge between q and p, which contradicts our assumption that there was only one directed edge between p and q.

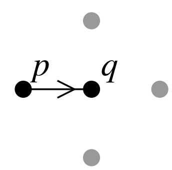

Figure 3.3.1.

Part (b) Assume there are antiparallel edges between p and q. By Definition 3.2.1, this means that p and q are rivals of each other. By Definition 2.4.1 a problem crossing's rival is the opposite type of crossing. Therefore, p and q cannot be the same crossing type.

Part (c) Assume there is no directed edge between p and q. By Definition 3.2.1, this means that p and q are not rivals of each other and Definition 2.4.1 states that rivals are opposite crossing types. Therefore, since p and q are not rivals, this means that p and q must be the same crossing type. □

The following lemma introduces some subsets of problem graphs that cannot come from RALPs that contain f_1.

Lemma 3.3.2. *Let R be a RALP without f_1 and let $\mathbf{G}_R$ be its problem graph. The configurations shown in Figure 3.3.2 cannot exist in $\mathbf{G}_R$.*

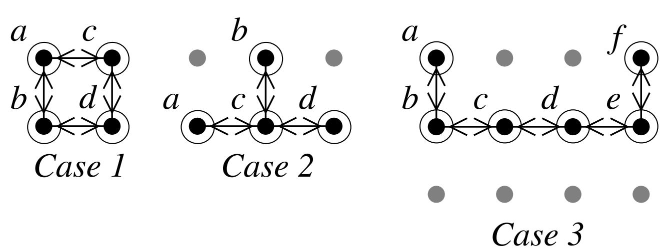

Figure 3.3.2.

Proof. Let R be a RALP. Suppose R does not contain f_1.

Case 1 Suppose that there are four vertices in G_R that are connected by antiparallel edges as shown in Case 1 of Figure 3.3.2. By Lemma 3.3.1 a and d will be of one crossing type and b and c will be of the other crossing type. Which as we saw in Figure 2.2.1 is the configuration of crossings in f_1. This means that this subset of a problem graph could not come from $\boldsymbol{R}$.

Case 2 Suppose there are four vertices in G_R that are connected by antiparallel edges as shown in Case 2 of Figure 3.3.2. By Lemma 3.3.1 we know that the crossing type of a, b, and d is not the same as the crossing type of c. Since rivals of a problem crossing form a non-opposing pair with that point, we know that the second rival of b has to be added at one of the two grey points adjacent to it in the figure. By Definition 2.4.1 the rival of b has crossing type opposite that of b. Which means the rival of b would also be the opposite crossing type of a and d. Therefore no matter which grey point the rival of b is added at, it would also be a rival of a or d, and in return a or d would be one of its rivals. Which means that the rival of b would also be a problem crossing. This would result in four problem crossings in a 1×1 square, each of which would be connected to adjacent vertices by antiparallel edges. Which as we already saw in Case 1 cannot come from $\boldsymbol{R}$.

Case 3 Suppose that there are six vertices in G_R that are connected by antiparallel edges as shown in Case 3 of Figure 3.3.2. By Lemma 3.3.1 a, c, and e will be the opposite crossing type of b, d, and f. The directed edges pointing from c and d to their respective rivals cannot point in the same direction, because each rival would be the rival of the other which would result in four problem crossings in a 1×1 square, each of which would be connected to adjacent vertices by antiparallel edges. Which as we already saw in Case 1 cannot come from $\boldsymbol{R}$.

This means one of the directed edges will have to point in the positive y-direction. Suppose that this occurs at d. Adding the rival of d at this point would again result in four problem crossings in a 1×1 square, each of which would be connected to adjacent vertices by antiparallel edges. Which as we already saw in Case 1 cannot come from $\boldsymbol{R}$. □

Additionally, we see that any problem graph containing one of the configurations shown in Lemma 3.3.2 can't come from a RALP that does not contain f_1. Figure 3.3.3 shows two configurations of problem graphs that each contain Case 2 from Figure 3.3.2. For the same reason that Case 2 cannot come from a RALP that does not contain f_1, these two configurations cannot as well.

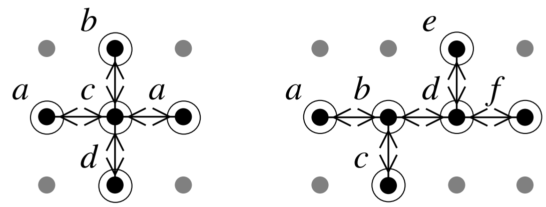

Figure 3.3.3.

Lemma 3.3.1 allows us to gather information about the crossing type of adjacent problem crossings in a RALP. This is done by looking at whether or not there is a pair of antiparallel edges between adjacent vertices in the problem graph. We know that if there are antiparallel edges between adjacent vertices in a problem graph, the problem crossings that correspond to those vertices are not the same crossing type; and that if there are not antiparallel edges between two adjacent vertices in a problem graph, the problem crossings that correspond to those vertices are the same crossing type.

The following lemma shows a configuration of problem graphs that cannot come from any RALP with or without f_1. It uses the information gathered about crossing type of adjacent problem crossings from Lemma 3.3.1 to show that these subsets cannot exist in a problem graph.

This lemma concerns a configuration of problem graphs which does not have any antiparallel edges between adjacent vertices. While there are other instances where this configuration of vertices could exist with antiparallel edges between them, this lemma rules out only the cases where there are no antiparallel edges between vertices.

Lemma 3.3.3. *Let R be a RALP and let G_R be its problem graph. The configuration shown in Figure 3.3.4 cannot exist in G_R.*

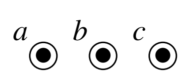

Figure 3.3.4.

Proof. Suppose there are three vertices in G_R arranged in a line with no antiparallel edges between adjacent vertices, as shown in Figure 3.3.4. By Lemma 3.3.1 a, b, and c will be the same crossing type. This configuration of vertices cannot occur in a problem graph because there isn't a non-opposing pair of b where its rivals could be located. This contradicts the assumption that b was a problem crossing. □

It follows from Lemma 3.3.3 that any subset of a problem graph containing the configuration shown in Figure 3.3.4 could not come from a problem graph for the same reason. Some examples of problem graphs that contain this configuration are shown in Figure 3.3.5

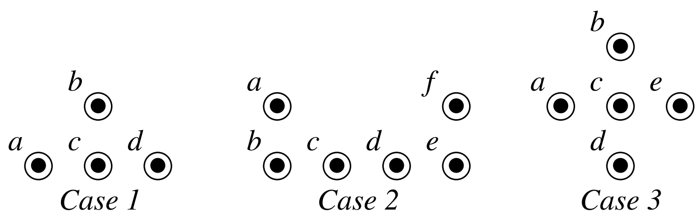

Figure 3.3.5.

The configurations of problem graphs shown in Figure 3.3.6 are some of those that could come from the problem graph for a RALP that does not contain f_1.

In Cases 2 and 3 of Figure 3.3.6 the rivals could be located in different places. For either a or b in Case 2, there could be a third rival at the grey point shown in the figure, or the rivals for the vertex could be the other non-opposing pair in the neighborhood of that vertex. In Case 3, the rivals of problem vertices could be at the grey points opposite the current location of rivals shown in the figure. Additionally, in Case 3 there could be rivals of a and d on the grey points that are one unit away from each in the x-direction.

In Cases 1, 4, 5, and 6 of Figure 3.3.6 the location of the rivals are fixed. In Case 1, this is because there's no where else the rivals could be placed. In Cases 4, 5, and 6 this is because adding another rival at any of the vertices would either form f_1 or change the configuration of the problem graph.

We saw in the small example at the beginning of Section 3.2 that when the problem crossing was sewn before its rivals it wasn't treated as a problem crossing. With this in mind we can order the problem crossings that correspond to the vertices in the configurations of problem graphs shown in Figure 3.3.6, so that when the crossings are sewn the conditions of Theorem 2.4.2 are met.

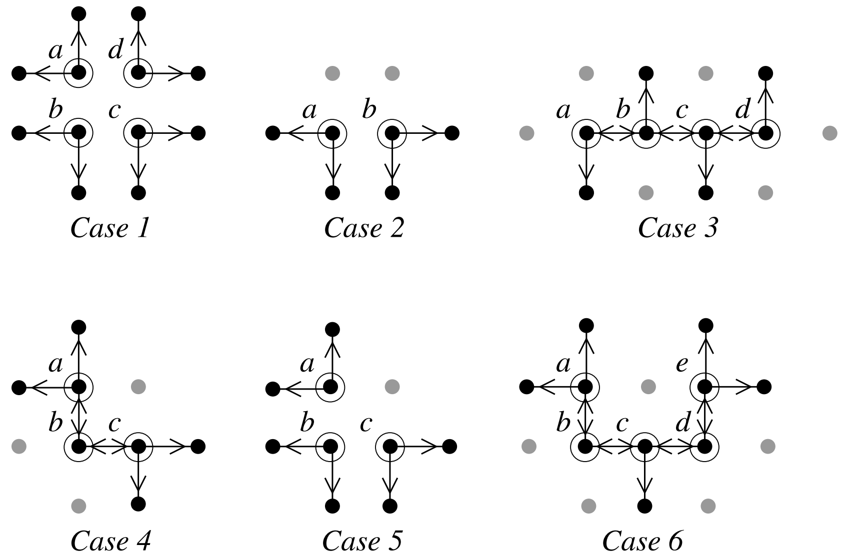

Figure 3.3.6.

Since there are no antiparallel edges between the vertices in Cases 1, 2, and 5 of Figure 3.3.6, none of the problem crossings corresponding to those vertices are rivals of each other. This means that sewing one of the crossings would not affect any of the other crossings.

However, when there are antiparallel edges between vertices in the problem graph, as in Cases 3, 4, and 6, the order matters when sewing crossings. It remains to be shown how to choose this order. We hypothesize that the problem crossings whose rivals are also problem crossings should be sewn first, then problem crossings whose rivals are not other problem crossings. Finally, leaving the other rivals of problem crossings to be sewn last.

4

Future Work

While we haven't seen a RALP that disproves Conjecture 2.2.4, it still remains to be shown that if a RALP does not contain f_1 it is a projection of an LSK. If the conjecture is true, we hope that the method of cutting described in Lemma 2.3.7 and the method of sewing described in Theorem 2.4.2 can be used to get an upper bound on the number of sticks required to realize a knot as an LSK.

We also believe that problem graphs may hold promise. The small example at the beginning of Section 3.2 showed us that when a problem crossing was sewn before its rivals it wasn't treated as a problem crossing when it was sewn. Then in Section 3.3 we saw some configurations of problem graphs that cannot come from RALPs that do not contain f_1 as well as some configuration that can come from RALPs that could contain f_1. At the end of the same section we saw how certain configurations of problem graphs can suggest an order the crossings should be sewn in. We hope that through a further analysis of problem graphs, we can find a systematic way to order crossings in RALPs that are the projections of LSKs so that when crossings are sewn back between RALP-circles the conditions of Theorem 2.4.2 are satisfied.

Theorem 2.4.2 deals only with cases where there are at most two crossings adjacent to the point being sewn. This suggest that another area to explore would be whether or not it is possible to sew two corners of RALPs together if there are three or four crossings adjacent to the point being sewn.

The next section goes through an example that shows how the crossings in a RALP with multiple problem crossings can be ordered so that the conditions of Theorem 2.4.2 are satisfied each time two corners are sewn.

4.1 Examples

The following example takes a RALP that has more than one problem crossing and shows how the problem graph can be used to order the problem crossings so the conditions of Theorem 2.4.2 are satisfied when the crossings are sewn back between RALP-circles

Figure 4.1.1 shows R, a RALP that does not contain f_1, and G_R to the right of it in the same image. The collection of RALP-circles that results from cutting all crossings in R using the method in Lemma 2.3.7 are shown in image (1) of Figure 4.1.2.

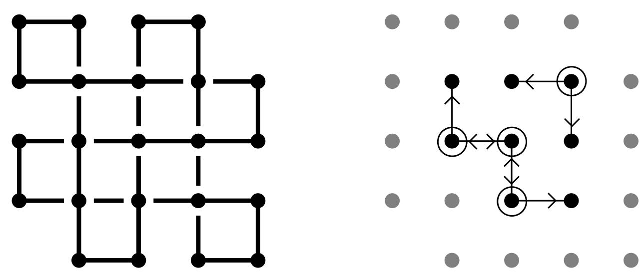

Figure 4.1.1.

Based on the small example that began Section 3.2 we will want to sew problem crossings before their rivals. We see from looking at G_R, that there is one problem crossing whose two rivals are also problem crossings. This crossing is marked with an * in image (2) of the Figure 4.1.2, and it will be sewn first. After sewing this crossing the other three problem crossings will be sewn next, these crossings are marked with an * in Image (3) of the figure.

The corners that will be sewn next have to be picked a little more precisely, because even though none of the other crossings are problem crossings they have to be sewn in such a way that we are not left trying to sew any corners that are adjacent to three or four crossings. After examining the figure, we see that if we sew the corners marked with an * in image (4) of the figure the three crossings left to be sewn will all be adjacent to two crossings.

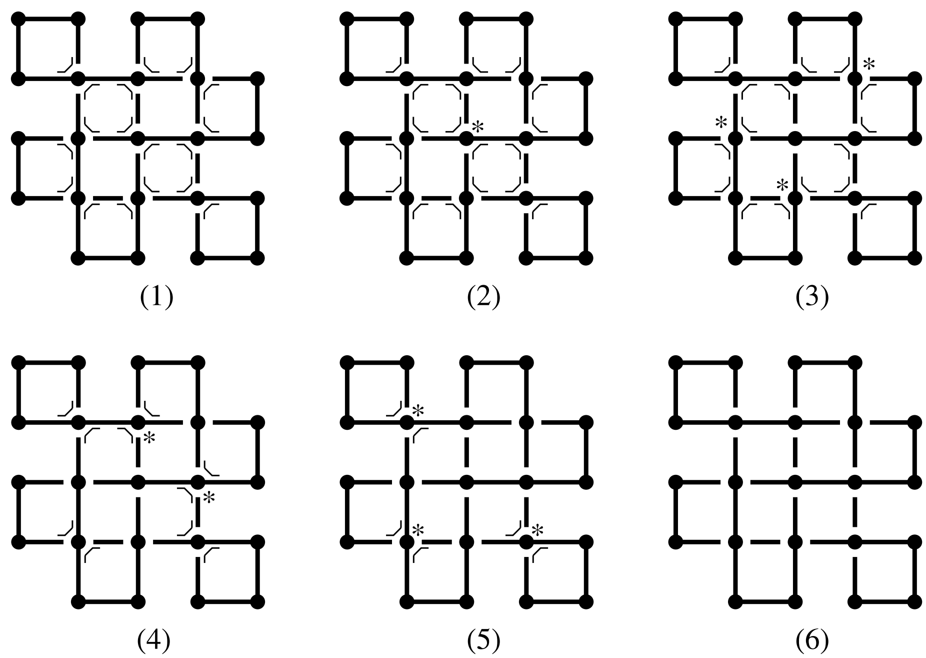

Figure 4.1.2.

The remaining three crossings to be sewn are marked with an * in image (5). All three of these corners satisfy the conditions outlined in the statement of Theorem 2.4.2. At this point we have sewn all the crossings. The original RALP R is shown in image (6) of the figure.

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