Exploring Sequences of Tournament Graphs with Draws

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Abstract

Tournaments occur all over the world and they are used to decide championships in various competitions. For this project, I will be exploring tournaments in the round robin style in which every team plays one another. This is based on Sadiki Lewis' senior project, *Exploring Tournament Graphs and Their Win Sequences.* I will be expanding his project by including the possibility of a draw between two teams, in addition to a win or a loss. Teams and games will be modeled by complete graphs where each vertex represents a team and each directed edge between two vertices represents the outcome of a game between the two corresponding teams. Since I am including draws, I will be exploring values placed on wins, draws, and losses and how they affect the score sequences derived from those tournaments graphs.

Contents

Dedication

Dedicated to my Mom and family. Thanks for all of your love and constant support.

Acknowledgments

Thank you to my senior project adviser, Professor Lauren Rose. Your guidance and support were super helpful during this process and you kept me on task. We've explored things that we both weren't familiar with and it was an amazing experience. Much appreciation and many thanks to the Bard Math Department. You've made me challenge and approach problems in ways that I never thought I could. To my professors, Amir Barghi, Maria Belk, Ethan Bloch, John Cullinan, and Stefan Mendez-Diaz, thank you all for your support throughout my time at Bard and for pushing me to be a better mathematician. Thank you to Steven Simon for making abstract algebra and real analysis so enjoyable and for always offering your help.

To my friends, Philippine and Molly, thanks for all of your love and patience with me. Thanks for helping me grow into the person I am now. Love you weirdos.

Thank you to my volleyball and lacrosse teammates and coaches. Your care, love, and support were greatly appreciated in times of stress and difficultly. Thank you all for standing by me throughout my years and especially during my last days as a collegiate athlete. I'll miss playing with you all and I hope for the best in the seasons to come.

To my Possemates , the Posse foundation and BEOP, thank you for being there for me from before the beginning of my collegiate career to the end. I don't know what I would do and where I'd be without y'all.

1
Introduction

Throughout this project, I will be exploring the score sequences of graphs that model Round Robin tournaments. This is an extension of Sadiki Lewis' Senior Project on *Exploring Tournament Graphs and Their Win Sequences* [1]. In his senior project, Sadiki explored win sequences of tournament graphs and the relationships between a graph's win sequence and lose sequence. In his future work section, Sadiki introduced the idea of adding draws between two teams as an option. Two questions that arose are as follows:

1. "How do score sequences of Win/Lose/Draw Tournaments differ from the Win/Lose Tournaments?"

2. "Is it possible to find a formula to count all possible n-player tournaments for Win/Lose/Draw?"

To begin, I will first define a graph's "score sequence" and then focus on finding possible graphs up to 4 vertices, by finding formulas to guarantee that all possible score sequences are found.

Tournaments can be viewed as a series of games or competitions between a number of people, groups, or teams. Tournaments arise in many sports and games, such as by chess, golf, soccer, and volleyball. There are single elimination tournaments, double elimination tournaments, round robin tournaments, and multistage tournaments, to name a few. Typically for elimination tournaments, games are planned ahead of time. In these types, teams are not guaranteed to play every other team. In the round robin style, every team plays every single other team in the tournament exactly once.Multistage tournaments can be a combination of round robin and elimination or a mixture of something else; the possibilities are endless!

Example 1.0.1. Let us look at a real life example of a tournament. Figure 1.0.1 shows the semi-finals and final results of the 2018's Division I Men's Basketball post-season, also known as the Final Four in March Madness [3].

Figure 1.0.1.

In Figure 1.0.1, the winners are in bold. From this figure, we can see that the two semi-final games (the first and last pairing), lead result to the middle pairing, which is the championship. In the semi-finals, Michican beats Loyola Chicago and Villanova beats Kansas. These two winners face off in the championship game in which Villanova beats Michigan and is deemed the national champion. Figure 1.0.2 shows the tournament modeled as a graph.

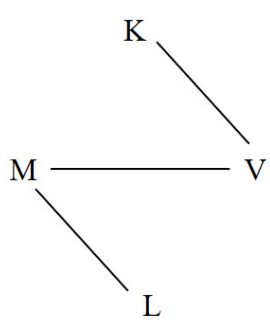


Figure 1.0.2. Visual Representation of the Tournament in Figure 1.0.1

For the graph in Figure 1.0.2, V represents Villanova, M represents Michigan, K represents Kansas, and L represents Loyola Chicago. Looking at the graph, we see that each vertex represents a team and each edge connecting two vertices represents a game between the two. We will explore more in depth how to determine the winner of a tournament later in this project but only for Round Robin tournaments.

◊

The tournament in Figure 1.0.2 is a single elimination style tournament. In this case, not every team plays each other (Kansas and Loyola Chicago are eliminated in one round and do not play each other). In this project, I will be focusing primarily on Round Robin Tournaments, where all competitors play against everyone and no one is eliminated.

Example 1.0.2. Let us look at a Round Robin Tournament to better understand how they work. Figure 1.0.3 shows a list of schools that we will use as competitors in a tournament. From the table in Figure 1.0.4, we are able to see that each team plays each other in different rounds.

Teams
A) Bard College
B) Clarkson University
C) Ithaca College
D) Vassar College

Figure 1.0.3.

Round 1	Round 2	Round 3
Bard vs Ithaca	Bard vs Clarkson	Bard vs Vassar
Clarkson vs Vassar	Ithaca vs Vassar	Clarkson vs Ithaca

Figure 1.0.4. Tournament for the teams in Figure 1.0.3

All of the prior figures show a tournament of 4 college teams. A visual representation of this tournament can be seen in Figure 1.0.6.

Round 1	Round 1 Winner	Round 2	Round 2 Winner	Round 3	Round 3 Winner
Bard vs Ithaca	Ithaca	Bard vs Clarkson	Bard	Bard vs Vassar	Bard
Clarkson vs Vassar	Vassar	Ithaca vs Vassar	Ithaca	Clarkson vs Ithaca	Clarkson

Figure 1.0.5. Results from the Tournament in Figure 1.0.4

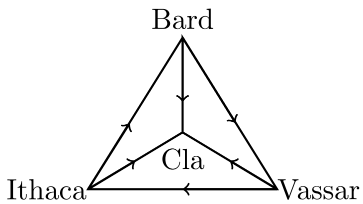


Figure 1.0.6. Visual Representation of the Tournament in Figure 1.0.5

The vertices or points of the graph in Figure 1.0.6 represent each team and the line connecting two of the vertices represents a game between those two teams. Since this is a round robin tournament, every team plays each other so there will always be a line between two vertices which is also known as a complete graph.

To notate what team won in a game, arrows are placed on the line representing the game. Note that in the graph in Figure 1.0.6, there is an arrow pointing away from Bard and towards Vassar. This means that for that game, Bard has won and Vassar has lost. We will be exploring patterns and restrictions resulting from tournaments like these but with the possibility of a draw between two teams. ◊

In Chapter 2, we introduce basic graph theory needed to understand this project. This information will mainly consist of definitions and examples. Chapter 3 will narrow down information towards tournaments and give a sense of how tournaments are modeled. In this chapter, we review the properties of win and lose sequences that had been previously found. In Chapter 4, we introduce the concept of draws in a tournament and a new type of sequence called a score sequence. In Chapter 5, we present our findings about score sequences for 2-1-0 scoring and their tournament graphs with draws. In Chapter 6, we will introduce a new type of scoring, 3-1-0 scoring, and show the difference between 2-1-0 scoring and 3-1-0 scoring. We will also present findings from the 3-1-0 scoring in this chapter. We include a chapter on future work that provides questions for future students who wish to explore other paths within this project.

2 Preliminaries

In this chapter we introduce the basic graph theory needed to understand this project. These definitions are based on Robin J Wilson's fourth edition of *Introduction to Graph Theory* [4].

2.1 Graph Theory Definitions

Definition 2.1.1. A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. △

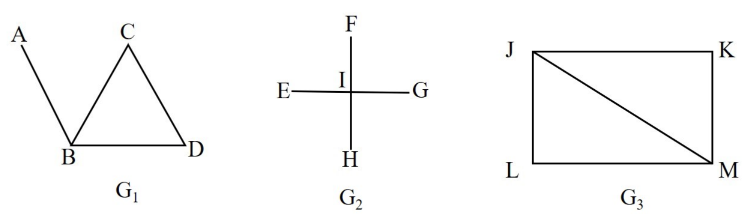


Figure 2.1.1.

Figure 2.1.1 shows visual representations of some simple graphs. We see that $V(G_1) = \{A, B, C, D\}$ and $E(G_1) = \{AB, BC, CD, BD\}$, $V(G_2) = \{E, F, G, H, I\}$ and $E(G_2) = \{EI, FI, GI, HI\}$ and then $V(G_3) = \{J, K, L, M\}$ and $E(G_3) = \{JK, KM, ML, JL, JM\}$.

Definition 2.1.2. We may remove the restriction that an edge joins two distinct vertices, and allow loops - edges joining a vertex to itself. The resulting object, in which loops and multiple edges are allowed, is called a **general graph** - or, simply, a **graph**. Thus every simple graph is a graph, but not every graph is a simple graph. △

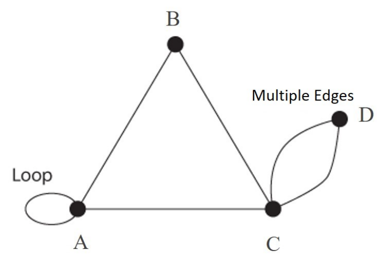


Figure 2.1.2.

Figure 2.1.2 gives an example of a general graph. This graph is not simple since this graph contains multiple edges between two vertices, C and D, and a loop, A to A, where an edge starts and ends with the same vertex.

Definition 2.1.3. The **degree** of a vertex v, written $d(v)$, in a graph G is the number of edges incident to v. △

Example 2.1.4. In the graph in Figure 2.1.2, $d(A) = 4$, $d(B) = 2$, $d(C) = 4$, and $d(D) = 2$. ◊

Definition 2.1.5. Two graphs G_1 and G_2 are **isomorphic** if there is a one-to-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2. △

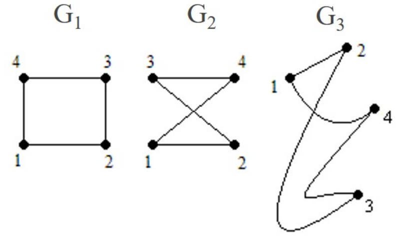


Figure 2.1.3.

In Figure 2.1.3, we show an example of three isomorphic graphs. G_1, G_2, and G_3 each have edges 1-2, 2-3, 3-4, and 4-1. Hence, each of these graphs are isomorphic to one other and can be considered the same.

Definition 2.1.6. A simple graph in which each pair of distinct vertices are adjacent and connected by one distinct edge is called a **complete graph**. △

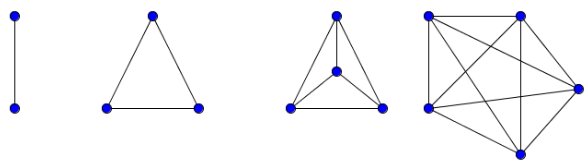


Figure 2.1.4.

Figure 5.0.1 shows a visual representation of complete graphs with 2, 3, 4, and 5 vertices, respectively.

Definition 2.1.7. A **directed graph**, or digraph, D consists of a non-empty finite set V(D) of elements called **vertices**, and a finite family E(D) of ordered pairs of elements of V(D) called **edges** where the edges are directed from one vertex to another. △

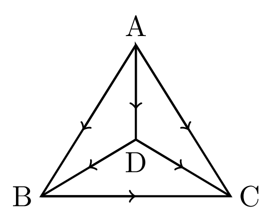


Figure 2.1.5.

Figure 2.1.5 shows an example of a directed complete graph. Not only does each pair of vertices have a distinct edge, but each edge is "directed" i.e. pointing from one vertex to another. The direction of the arrows will be used to determine scoring in subsequent chapters.

Definition 2.1.8. Two digraphs G_1 and G_2 are **isomorphic digraphs** if their underlying graphs are isomorphic and the direction of the corresponding edges of G_1 and G_2 are same. △

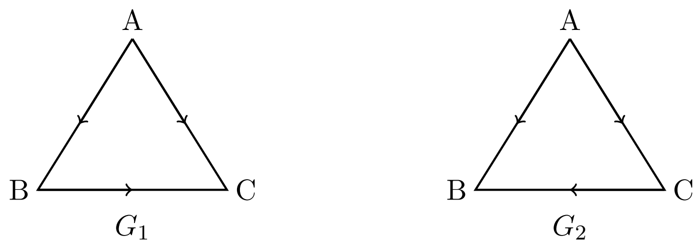


Figure 2.1.6.

In Figure 2.1.6, we give an example of two isomorphic digraphs, G_1 and G_2. The only difference between G_1 and G_2 is the direction of the edge between vertices B and C. If we were to take out the labels for the vertices as seen in Figure 2.1.7, it is easier to see how the two digraphs are actually the same. Looking at G_2 in Figure 2.1.7, we can see that it is just a vertical reflection of G_1.

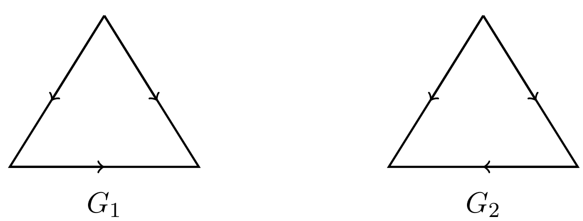


Figure 2.1.7.

Definition 2.1.9. The **out-degree** of a vertex v of digraph D is the number of edges of the form vw, and is denoted by $d^+(v)$. Similarly, the **in-degree** of v is the number of edges of digraph D of the form wv, and is denoted by $d^-(v)$. △

In the digraph in Figure 2.1.5, vertex D has two edges that have an arrow pointing away from the vertex, so then vertex D has out-degree of 2 denoted $d^+(D) = 2$. Vertex D also has an edge with an arrow pointing towards vertex D, so then vertex D has in-degree of 1 denoted $d^-(D) = 1$. For vertex A, $d^+(A) = 3$ and $d^-(A) = 0$, vertex B has $d^+(B) = 1$ and $d^-(B) = 2$, and vertex C has $d^+(C) = 0$ and $d^-(C) = 3$.

Note: The degree of vertex v can be written as $d(v) = d^+(v) + d(v)$.

Definition 2.1.10. A vertex v is called a **source** if all edges connected to that vertex are directed away from v. In other words, if $d^+(v) = d(v)$, the number of edges connected to it, then v is a source. △

Definition 2.1.11. A vertex v is a **sink** if all edges connected are directed towards v. In other words, if $d^-(v) = d(v)$, then v is a sink. △

Example 2.1.12. Figure 2.1.5 shows an example of a digraph with both a source and a sink. Vertex A is the source since $d^+(A) = d(A) = 3$ and vertex C is the sink since $d^-(v) = d(v) = 3$. ♢

It is important to note that the presence of a sink does not necessarily mean the presence of a source and vice versa. This will be proven later.

2.2 Applying Preliminary Terms

Let us apply the preliminary terminology explained in the previous section to the following graph in order to have a better grasp of what each term means.

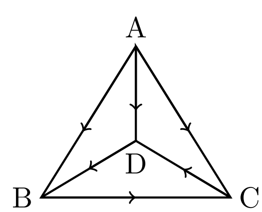


Figure 2.2.1.

Example 2.2.1. The out-degree of each vertex of the complete digraph in Figure 2.2.1 are as follows: $v^+(A) = 3, v^+(B) = 1, v^+(C) = 1, v^+(D) = 1$. Since vertex A has the maximum out-degree value, it is a source. As noted earlier, the presence of a source does not mean there is a sink as can be seen here.

The digraph in Figure 2.2.2 is isomorphic to the digraph in Figure 2.2.1. The vertices C and D essentially just trade positions between the two figures. In the graph in Figure 2.2.1, there are arrows going from B to C, from C to D and from D to B. For the graph in Figure 2.2.2, it

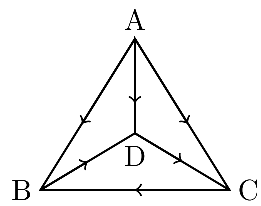


Figure 2.2.2.

is the opposite. There are arrows from B to D, from D to C, and from C to B. By relabeling what we call vertices C and D, we can get the same graph as Figure 2.2.1. If vertex D was now called C and vertex C was now called D, we would get that arrows are going from B to C, from C to D, and from D to B which is the same as in Figure 2.2.1. $\Diamond$

3

Modeling Tournaments with Graphs

Various sports and events use tournaments as a means of competition and scoring. Many tournaments can be visually represented by the use of graphs. For this project, we will focus on round robin tournaments where every team plays every other team exactly once.

Definition 3.0.1. A **tournament** is a digraph in which any two vertices are joined by exactly one edge. △

What we had originally called a complete digraph earlier, we now call a tournament graph. For a tournament graph, a vertex represents a team and the edges represent the games between two teams. In a tournament with n teams, each team plays $n - 1$ games, one with each other team.

Definition 3.0.2. The out-degree of a vertex v represents the number of **wins** for that team and the in-degree of a vertex v represents the number of **losses** for that specific team.

△

A vertex's out-degree is represented by $d^+ = \#$ of wins, and a vertex's in-degree is represented by $d^- = \#$ of losses.

Since d^+ represents the number of wins, a source of a tournament graph would be the tournament's winner, since that team has beaten every other team. Similarly, since d^- represents the number of losses, a sink of a tournament graph is the overall loser since that team has lost to every other team. Later we will prove that there can be at most one source and at most one sink in a tournament graph.

Example 3.0.3. In Figure 3.0.1, we have a list of teams in a tournament. Figure 3.0.2 shows a schedule of the games. Figure 3.0.3 shows the visual representation tournament graph T.

Teams
A) Bard College
B) Sage College
C) Sarah Lawrence College
D) SUNY New Paltz

Figure 3.0.1.

Round 1	Round 2	Round 3
Bard vs Sarah Lawrence	Bard vs Sage	Bard vs New Paltz
Sage vs New Paltz	Sarah Lawrence vs New Paltz	Sage vs Sarah Lawrence

Figure 3.0.2.

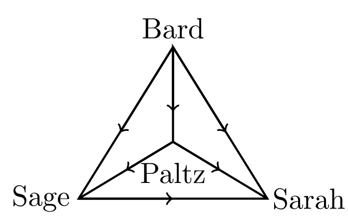


Figure 3.0.3. Tournament Graph T

We see that Bard has out-degree 3, Sage has out-degree 1, Sarah Lawrence has out-degree 0, and New Paltz has out-degree 2. Translating this into tournament terminology, Bard has 3 wins, Sage has 1 win, Sarah Lawrence has 0 wins, and New Paltz has 2 wins. $\Diamond$

3.1 Win and Lose Sequences

Now that the notion of wins and losses is established, we will go over win and lose sequences, as explored by Sadiki Lewis in his senior project, *Exploring Tournament Graphs and Their Win*

Sequences [1]. He discovered various patterns and made conjectures that formed the basis of this project. The following are different propositions, lemmas, and theorems from Lewis' project.

Definition 3.1.1. A **win sequence** $S^+ = (s_1^+, s_2^+, ..., s_n^+)$ represent the numbers of wins of every team in a tournament graph T_G written in non-increasing order $s_1^+ \geq s_2^+ \geq ... \geq s_n^+$. For a vertex v_i the number of wins $s_i^+ = d^+(v_i)$. △

Example 3.1.2. From Figure 3.0.3, Bard has 3 wins, Sage has 1 win, Sarah Lawrence has 0 wins, and New Paltz has 2 wins. Putting these wins in non-increasing order we get that the win sequence for the tournament is $S^+ = (3, 2, 1, 0)$. ♢

Definition 3.1.3. A **lose sequence** $S^- = (s_1^-, s_2^-, ..., s_n^-)$ represents the numbers of losses of every team in a tournament T_G written in non-decreasing order $s_1^- \leq s_2^- \leq ... \leq s_n^-$. For a vertex v_i the number of losses $s_i^- = d^-(v_i)$. △

Example 3.1.4. In Figure 3.0.3, Bard has 0 losses, Sage has 2 losses, Sarah Lawrence has 3 losses, and New Paltz has 1 loss. Putting these losses in non-decreasing order we get that the lose sequence for this tournament is $S^- = (0, 1, 2, 3)$. ♢

3.2 Win and Lose Sequences Results

The following results can be found in *Exploring Tournament Graphs and Their Win Sequences* [1]. After each result, we interpret the formula in less technical terms.

Lemma 3.2.1. *Let T_G be a tournament graph on n vertices, if S^+ is a win sequence, then* $\sum s_i^+ = \binom{n}{2} = \frac{n(n-1)}{2}$.

Proof. $\sum s_i^+$ is the sum of all the wins of a tournament graph. There is an arrow associated with each edge and we know that each edge contributes an in and out degree. So the sum of all the wins which is represented by the out-degree equals the number of edges. The number of edges in any complete graph is $\binom{n}{2}$, so $\sum s_i^+ = \binom{n}{2} = \frac{n(n-1)}{2}$. □

Lemma 3.2.1 states that the sum of all the wins equals the total number of edges in a tournament graph. This makes sense since every edge in a tournament graph designates exactly one winner and loser. So the number of wins is the number of edges, $\frac{n(n-1)}{2}$.

Lemma 3.2.2. *Let T_G be a tournament graph. If S^- is a lose sequence, then $\sum s_i^- = \binom{n}{2}$.*

Proof. Similar to Lemma 3.2.1, $\sum s_i^-$ is the sum of all losses in a tournament graph. We know every arrow on an edge contributes an in and out degree for a vertex. Thus, the sum of all losses, represented by in-degrees, is $\binom{n}{2}$, the number of edges. □

Lemma 3.2.2 states that the sum of all the losses equals the total number of edges in a tournament graph. This makes sense since every edge in a tournament graph designates exactly one winner and loser.

Lemma 3.2.3. *Let T_G be a tournament graph, S^+ be the win sequence of T_G, and S^- be the lose sequence of T_G. Then $\sum s_i^+ = \sum s_i^-$.*

Proof. From Lemma 3.2.1 and 3.2.2, we get that $\sum s_i^+ = \binom{n}{2}$ and $\sum s_i^- = \binom{n}{2}$, respectively. Thus $\sum s_i^+ = \sum s_i^-$. □

It makes sense that the number of wins equals the number of losses since for every win there is a loss and vice versa.

Lemma 3.2.4. *Let T_G be a tournament graph, S^+ be the win sequence of T_G, and S^- be the lose sequence of T_G. Then $s_i^+ + s_i^- = n - 1 = d(v)$.*

Proof. By definition, $d(v) = d^+(v) + d^-(v)$, so $d^+(v) + d^-(v) = n - 1$ since the number of edges coming out of a vertex in a complete graph is $n - 1$. Since $s_i^+ = d^+(v_i)$ and $s_i^- = d^-(v_i)$ it follows that $s_i^+ + s_i^- = n - 1$. □

Lemma 3.2.4 says that the sum of a vertex's in and out degree equals $n - 1$, which is the degree of any vertex in a complete graph with n-vertices.

Theorem 3.2.5. *If T_G is a tournament graph on n vertices, then T_G has a 1 source if and only if $s_1^+ = n - 1$ and T_G has a 1 sink if and only if $s_n^+ = 0$.*

Proof. 1. Any tournament graph T_G has at most one source.

Let $v, w \in V$ where v and w are vertices of V and v is a source. Since v is a source, v has an out-degree towards all other vertices. Then w already has an in-degree of at least one. Hence w cannot be a source and T_G has at most 1 source.

2. If $s_i^+ = n - 1$, T_G has a source.

Suppose $s_1^+ = n - 1$, then v_1 has won every game in the tournament. Thus $v_1^+ = n - 1$ and is a source and the only source.

3. Any tournament graph T_G has at most one sink.

Let $v, w \in V$ where v and w are vertices of V and v is a sink. All edges connected to v provide an in-degree. Then vertex w has at least one win and is hence not a sink.

4. If $s_n^+ = 0$, then T_G has a sink.

Suppose $s_n^+ = 0$. Then vertex v_n has lost every game in the tournament. So $v_n^- = n - 1$ and is a sink. Hence, every tournament graph T_G has at most one sink. □

Note: The presence of a sink in a tournament graph does not necessarily mean that there exists a sink in that same tournament graph and vice versa. In addition, tournament graphs do not necessarily have to have a sink or source at all, as can be seen in Figure 3.2.1.

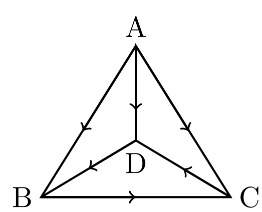


Figure 3.2.1.

The following theorem by Lewis creates restrictions on win sequences. Later on in this project, we will have a similar type of theorem but for creating restrictions on score sequences which will be explained further in the next chapter.

Theorem 3.2.6. *Let $S^+ = (s_1^+, ..., s_n^+,)$ be a win sequence for a tournament graph T_G, where $s_1^+ \geq ... \geq s_n^+$ then:*

1. $s_1^+ \leq n - 1$.
2. $s_2^+ \leq n - 2$.
3. $s_1, ..., s_{n-1}$.
4. $s_n \leq [\frac{n-1}{2}]$.
5. $\sum s_i = \binom{n}{2}$.

These restrictions on the scores of a win sequence are used to produce tournament graphs and were observed by examining previousl explored tournament graphs.

4

Tournament Graphs with Draws

Now we will add the possibility of draws in tournament graph and explore the resulting graphs. Tournaments such as chess, various stages of FIFA, and Swiss system tournaments use win, draw, and lose as their possible tournament results. Note that the addition of draws in to a tournament graph T_G no longer makes the graph a tournament graph in the mathematical sense since tournaments are complete digraphs. With the addition of draws, the entire graph is no longer a complete digraph since there is no "win" direction. Although these graphs are not technically tournament graphs, we will call them tournament graphs for this project.

Example 4.0.1. Figure 4.0.1 shows the result from one of the stages of the World Cup in 2014 for group A [2]. In each row there are two numbers in bold, the top number is the score of the team on the left of it and the bottom number is the score of the team on the right of it.

Figure 4.0.2 shows us how many wins, draws, and losses each team had in Figure 4.0.1. In that tournament's results, Brazil won twice and had one tie, Mexico won twice and had one tie, Cameroon lost all three games and Croatia won once and lost twice. In Figure 4.0.3 we see a visual representation of the tournament shown in Figure 4.0.1.

$\Diamond$

Fixtures / Results			
12/06/2014	Brazil	3 - 1	Croatia
13/06/2014	Mexico	1 - 0	Cameroon
17/06/2014	Brazil	0 - 0	Mexico
18/06/2014	Cameroon	0 - 4	Croatia
23/06/2014	Croatia	1 - 3	Mexico
23/06/2014	Cameroon	1 - 4	Brazil

Figure 4.0.1. FIFA World Cup 2014 Group A

Group A	W	D	L
Brazil	2	1	0
Mexico	2	1	0
Croatia	1	0	2
Cameroon	0	0	3

Figure 4.0.2. Results from the tournament in Figure 4.0.1.

Note: The segment mark for a draw between two teams is denoted by two short parallel lines perpendicular to and on the edge representing that game. We see that Brazil has two out arrows towards Croatia and Cameroon that represents its wins against them. For the game between Brazil and Mexico, we have the two short parallel lines that represent the draw between those two teams.

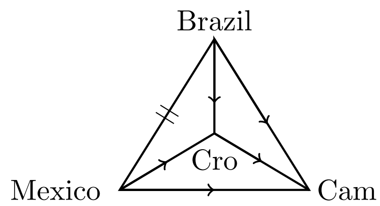


Figure 4.0.3. Visual representation of the tournament in FIgure 4.0.2.

For this tournament graph, we could make a win and lose sequence but that would give some misleading information since there is also a tie and neither of those sequences provide information about the tournament in that aspect. To account for wins, losses, and draws, we will look at the graph's score sequence but first, we need to determine how scoring should work in a tournament graph.

4.1 Scoring in Tournaments

There are many different ways to determine scores. One has to choose how many points to give for a win, loss, or draw. For the next couple of chapters, we will use a scoring system where the winning team gets 2 points, the losing team gets 0, and if there is a draw, both teams receive 1 point.

Example 4.1.1. If we look back to the World Cup representation in Figure 4.0.3, we can now assign each team a score. Since Brazil won twice and had a draw, it receives 5 points, two for each win and one for the draw. Mexico also won twice and had a draw so Mexico receives a score of 5. Cameroon lost all three games so Cameroon receives a score of 0. Finally, Croatia won once and lost twice so Croatia has a score of 2. ◊

Looking at a tournament generally, we note that each edge provides a score of 2. That is, each directed edge results in 2 points for the winner and 0 for the loser. If the edge represents a draw, both teams get 1 point.

4.1.1 Score Sequences

Definition 4.1.2. The **score sequence** of a tournament graph T_G can be denoted by $(s_1, s_2, ..., s_n)$ where $s_1, s_2, ...s_n$ are in non-increasing order i.e. $s_1 \geq s_2 \geq ... \geq s_n$. Each number in the score sequence represents the score of a team. △

Example 4.1.3. In the graph in Figure 4.0.3, Brazil scores 5, Mexico scores 5, Cameroon scores 0, and Croatia scores 2. So the score sequence for this tournament is $(5, 5, 2, 0)$.

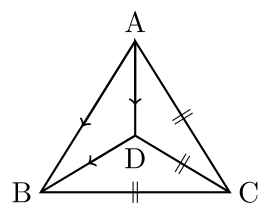


Figure 4.1.1.

	Wins	Draws	Losses	Points
A	2	1	0	5
B	0	1	2	1
C	0	3	0	3
D	1	1	1	3

Figure 4.1.2.

The graph in Figure 4.1.1 can be represented by the table in Figure 4.1.2. Team A has 2 wins and a draw so team A has a score of 5. Team B has a draw and 2 losses so team B has a score of 1. Team C tied all of their games so C has a score of 3. Team D has 1 win, 1 loss, and 1 draw so team D has a score of 3. Putting this all together we get that the graph in Figure 4.1.1 has score sequence $(5, 3, 3, 1)$.

◊

5

2-1-0 Scoring Findings

In this chapter, we investigate the properties of score sequences. We found that multiple tournament graphs can have the same score sequence even though they are not isomorphic. We came up with a list of restrictions on sequences with 2-1-0 scoring. From these restrictions we were able to list all possible score sequences. These results are in the following theorem.

Theorem 5.0.1. *Let $(s_1, s_2, ..., s_n)$ be a score sequence for a tournament graph T_G using 2-1-0 scoring where $s_1 \geq s_2 \geq ... \geq s_n$ then:*

1. $\sum_1^n s_i = n(n-1)$.
2. *Max* $s_1 \leq 2(n-1)$.
3. *If* $s_1 = 2(n-1)$, *then* $s_2 \leq 2(n-2)$.
4. $s_1 \geq (n-1)$.
5. *If* $s_n = 0$, *then* $s_{n-1} \geq 2$.

The following are proofs of each restriction/guideline.

1. $\sum(s_1 + ...s_n) = n(n-1)$.

Proof. Since a total of two points is alloted for each game (2 points for one team and 0 to another for a win/lose and 1 point for each team for a draw), each edge allots 2 points. The number of edges for any complete graph is $\binom{n}{2}$ or $\frac{n(n-1)}{2}$. Thus then the sum of a score sequence should be $2 * \#$ of edges $= 2 * (\frac{n(n-1)}{2}) = n(n-1)$. □

Example 5.0.2. In a game of two teams, there is only on game occurring and the options are for both teams to tie or for one team to win or one team to lose i.e. $s = (1, 1)$ or $(2, 0)$. Then the sum of the score sequence would be 2.

For a tournament of 3 teams, the score sequence total will always be $3 * 2 = 6$ and for a 4 team tournament, the total of the score sequence will always be $4 * 3 = 12$. Finally, a 5 team tournament would have a score sequence total of $5 * 4 = 20$. Each of these tournaments can be seen in Figure 5.0.1. ◊

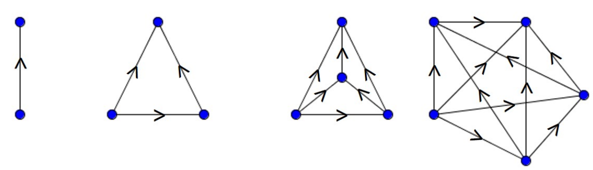


Figure 5.0.1. This figure shows complete Round Robin Tournaments for 2, 3, 4, and 5 teams, respectively.

2. $s_1 \leq 2(n-1)$.

Proof. The best a team can do is win every game. Since there are n teams and $n-1$ games, if a team wins all $n-1$ games, that team will receive a maximum score of $2(n-1)$. □

3. $s_2 \leq 2(n-2)$ when $s_1 = 2(n-1)$.

Proof. In the case that s_1 is it's maximum score of $= 2(n-1)$, the second highest score s_2 already has a loss. The maximum score that s_2 can now receive is by winning the remaining

$(n-2)$ games since it already has lost to s_1. So then the max s_2 can be when s_1 is the maximum is $2(n-2)$. □

4. $s_1 \geq n-1$.

Proof. Let $s_1 < n-1$. So then $s_2 < n-1, s_3 < n-1, ..., s_n < n-1$ since $s_1 \geq s_2 \geq ... \geq s_n$. Summing all the score up, we would get $s_1 + s_2 + ... + s_n < n(n-1)$ which is a contradiction because the sum of the score sequences is always equal $n(n-1)$ as proven in part 1 of this theorem. Hence, $s_1 \geq n-1$. □

5. $s_{n-1} \geq 2$ when $s_n = 0$.

Proof. Let $s_n = 0$. That means that the team who has score 0 has lost to every other team. Since every other team has at least one win, the second lowest team who has the score s_{n-1} cannot have a score of 0 or 1 and must be at least 2 since it has at least one win. □

Note: All score sequences and their respective graphs created from these restrictions were acquired up to tournaments of 4 teams which can be seen in the Appendix. For a tournament of 2 teams, we achieve 2 unique tournament graphs. For a tournament of 3 teams, we achieved 7 unique tournament graphs. Finally, for a tournament of 4 teams, we achieved 41 unique tournament graphs.

When looking at score sequences that had only one graph associated with it, it was seen that at least 2 of the scores in those score sequences could only be achieved in one way. For example, the score of 6 for a 4 team tournament can only be obtained by winning all 3 games, the score of 0 can only be achieved by losing all games, and the score of 5 can only be achieved by winning two games and tying the other in a 4 team tournament. Score sequences with multiple graphs contained more team scores such as 4 and 3 where scores could be obtained multiple ways.

6

2-1-0 scoring vs 3-1-0 scoring

6.1 Purpose of the 3-1-0 scoring

The 3-1-0 scoring system puts an emphasis on games won by allotting 3 points for a won game instead of 2. If the vertices were to be ranked by their score sequences using the 2-1-0 scoring and two vertices had the same score sequence, the scoring can be changed to 3-1-0 in order put an emphasis on winning. In the event that the score sequences remain the same for two ties teams after there was an emphasis placed on wins, other methods of ranking can be implemented.

Earlier in Chapter 4, an example involving the World Cup in 2014 was used. For this tournament, we used the 2-1-0 scoring to create that tournaments score sequence. Now that 3-1-0 scoring has been explained, we can now use this same example to make a score sequence for this same tournament. This 3-1-0 scoring is actually the scoring that FIFA uses during this stage; they allot 3 for winning a game, 1 for tying, and 0 for losing. Figure 6.1.1 shows what each team's score is with the 3-1-0 scoring compared to the 2-1-0 scoring.

3-1-0 scoring	W	D	L	Pts
Brazil	2	1	0	7
Mexico	2	1	0	7
Croatia	1	0	2	3
Cameroon	0	0	3	0

2-1-0 scoring	W	D	L	Pts
Brazil	2	1	0	5
Mexico	2	1	0	5
Croatia	1	0	2	2
Cameroon	0	0	3	0

Figure 6.1.1.

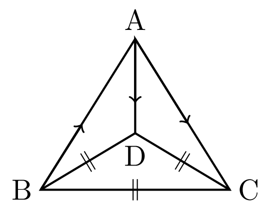


Figure 6.1.2. Visual Representation of a Tournament Graph with 4 Teams

2-1-0 scoring	W	D	L	Pts
A	2	0	1	4
B	1	2	0	4
C	0	2	1	2
D	0	2	1	2

3-1-0 scoring	W	D	L	Pts
A	2	0	1	6
B	1	2	0	5
C	0	2	1	2
D	0	2	1	2

Figure 6.1.3.

Example 6.1.1. Figure 6.1.2 and Figure 6.1.3 gives us the same tournament shown in three different ways. We have a visual representation of the tournament graph as seen in Figure 6.1.2 and we also have two tables that allow us to read how each team did and the points they received as seen in Figure 6.1.3. Looking at the 2-1-0 scoring table, we see that teams A and B are tied for first and teams C and D are tied for third. If we then use the 3-1-0 scoring, we now have that the tie for first is now broken but the tie for third remains. Since team A had more actual wins than team B, team A places first for the 3-1-0 scoring. $\Diamond$

6.2 Can two graphs have the same score sequence for 3-1-0 scoring?

When dealing with the 3-1-0 scoring, the allotment of points is not symmetrical like the 2-1-0 scoring. That is, when looking at the 2-1-0 scoring, there is always 2 points given for a game whether that be a point for each team because there was a tie or 2 points for one team and 0 for the other for a win lose situation. With the 3-1-0 scoring, there is an emphasis on winning. Due to this type of scoring, two graphs are unable to even have the possibility of having the same score sequence if they do not have the same number of draws. This is because for this specific scoring, the sum of the score sequences for a graph goes down by one from the maximum total of points for every draw added to the tournament graph.

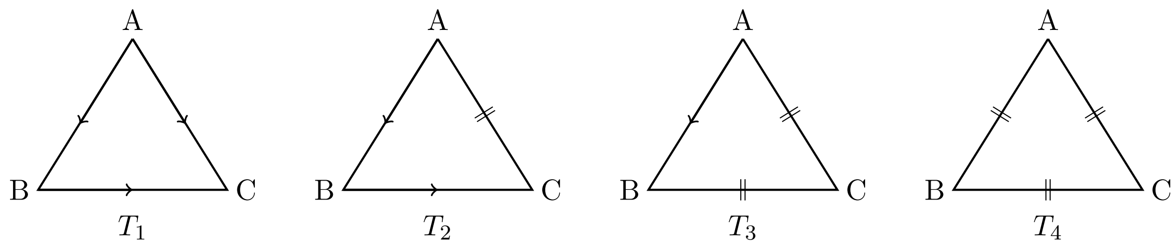


Figure 6.2.1.

Sub-Figures	Score Sequence	# of Draws	Sum
T_1	(6, 3, 0)	0	9
T_2	(4, 3, 1)	1	8
T_3	(4, 2, 1)	2	7
T_4	(2, 2, 2)	3	6

Figure 6.2.2. Score Sequences acquired using 3-1-0 scoring

Example 6.2.1. Figure 6.2.1 shows 4 tournament graphs with different amounts of draws: T_1 has no draws, T_2 has one draw, T_3 has two draws, and T_4 has all draws. Figure 6.2.2 shows a table of each tournament graph's respective score sequence using 3-1-0 scoring, the number of draws, and the score sequence sum. As it can be clearly seen in Figure 6.2.2, as the number of draws increases in a tournament graph, the sum of the score sequence decreases by one for every draw added. ◊

Since there is an emphasis on wins by giving a higher point value to wins, the allotment of points is not symmetrical. When an edge results in a draw, two points are alloted from that edge. If the game results in a win and loss, three points are alloted, 3 to the winner and 0 to the loser. So every draw that occurs in a tournament reduces the sum of a score sequence's maximum by 1. So if a graph had a different number of draws than another graph, there is no way that those two graphs could have the same score sequence using 3-1-0 scoring because the sum of those scores would already be different. two graphs with the same number of draws are the only ones that have the possibility of having the same tournament graph.

6.3 3-1-0 Scoring Findings

When we translated all the score sequences from 2-1-0 scoring to 3-1-0 scoring and their respective tournament graphs for teams of 2, 3, and 4, we found that all the converted score sequences were unique except for one pair of graphs with two ties from the 2-1-0 score sequence $(5, 3, 3, 1)$. when converted to the 3-1-0 scoring, both of these graphs had the score sequence of $(7, 4, 4, 1)$ as can be seen in Figure 6.3.1. These are the only two graphs for a tournament up to 4 teams that have both the same 2-1-0 score sequence and 3-1-0 score sequence.

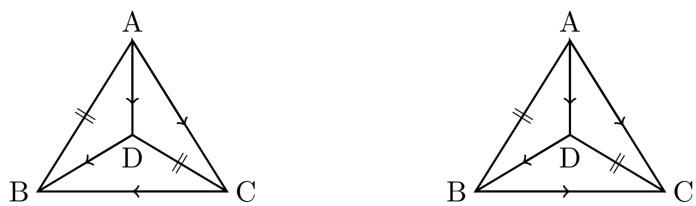


Figure 6.3.1. The score sequence for both of these figures in 2-1-0 scoring are $(5, 3, 3, 1)$ when translated to 3-1-0 scoring they remain the same. The 3-1-0 score sequence for these two graphs is $(7, 4, 4, 1)$.

The difference between the two graphs in Figure 6.3.1 is the direction of the arrow between teams B and C. For most of the other graphs, such a change just provides an isomorphic graph or a completely different score sequence, but for this one, the sequences stay the same but the graphs are considered different and not isomorphic.

6.3.1 3-1-0 Score Sequences

When looking at the score sequences for tournament graphs using the 3-1-0 scoring, it was seen that a lot of the restrictions placed on 2-1-0 scoring also held true with minor adjustments.

Theorem 6.3.1. *Let $(s_1, s_2, ..., s_n)$ be a score sequence for a tournament graph T_G where $s_1 \geq s_2 \geq ... \geq s_n$ then:*

1. $\sum_1^n s_i = 3 * \frac{n(n-1)}{2} - \#$ *of draws.*
2. $n(n-1) \leq s_1 \leq 3(n-1)$.
3. *If* $s_1 = 3(n-1)$, *then* $s_2 \leq 3(n-2)$.
4. $s_1 \geq n - 1$.
5. *If* $s_n = 0$, *then* $s_{n-1} \geq 3$.

Below are proofs of each restriction/guideline.

1. $\sum_1^n s_i = 3 * \frac{n(n-1)}{2} - \#$ of draws.

Proof. If there are no draws, the max sum of a score sequence is $3*$ the # of edges $= 3 * \frac{n(n-1)}{2}$, since each edge contributes 3 points to the winner and 0 to the loser. With each additional draw added onto the tournament graph, the sum decreases by one since the edge is no longer contributing 3 points to the tournament but instead it is contributing 2 points, one for each of the teams associated with that game. So the sum of a score sequence equals $3 * \frac{n(n-1)}{2}$ and for every draw the sum decreases by 1. Hence $\sum_1^n s_i = 3 * \frac{n(n-1)}{2} - \#$ of draws. □

2. $n(n-1) \leq \sum s_i \leq \frac{3n(n-1)}{2}$.

Proof. From part 1 of this theorem, it was proven that the max sum of a score sequence using 3-1-0 scoring is $\frac{3n(n-1)}{2}$. With every additional draw, the sum decreases. The min sum of a score sequence would occur when all games result in ties. If every game is a tie, then there are $\frac{n(n-1)}{2}$ ties since that is how many edges there are. Then the max sum of the score sequence minus the number of edges is $\frac{3n(n-1)}{2} - \frac{n(n-1)}{2} = \frac{2n(n-1)}{2} = n(n-1)$ is the min sum of a score sequence using 3-1-0 scoring. □

3. $s_1 \leq 3(n-1)$.

Proof. The best a team can do is win every game. Since there are $n-1$ games and 3 points awarded for every win, if a team wins all $n-1$ games, that team will receive a max score of $3(n-1)$. □

4. If $s_1 = 3(n-1)$, then $s_2 \leq 3(n-2)$.

Proof. If $s_1 = 3(n-1)$, then this team wins every game so every other team in the tournament has a loss. The best score s_2 can be is if a team wins all of the remaining $n-2$ games. So then the max s_2 can be in this case is $3(n-2)$. □

5. $s_1 \geq n-1$.

Proof. Let $s_1 < n-1$. Then $s_2 < n-1, s_3 < n-1, ..., s_n < n-1$. Taking the sum of s_1 to s_n, we get that $s_1 + s_2 + ... + s_n < n(n-1)$. But since the minimum score sequence sum is equal to $n(n-1)$ by part 2 of this theorem, we get a contradiction. Therefore, $s_1 \geq n-1$. □

6. If $s_n = 0$, then $s_{n-1} \geq 3$.

Proof. Let $s_n = 0$. This means that the team that is associated with that score has lost to every other team. Since every other team has at least one win, the second lowest scoring team cannot score lower than 3 since it already has a win. □

Note: Part 2 of this theorem was created for the 3-1-0 scoring to account for the variability of the sum of a score sequence. The equivalent theorem for 2-1-0 does not contain a restriction like this.

7
Future Explorations

7.0.1 Questions

There are various questions that had been thought of but were not able to be explored. Some of these my result in one sentence simple answers to a plethora of future projects. Many of these questions build off of what was discovered in this project with the addition of draws and score sequences. Possible future questions are:

1. Are there examples of graphs in which the team that placed first in a tournament changes due to the conversion of a 2-1-0 scoring to another type of scoring.

We have seen examples where the placement of two teams ties for 1st can be altered. Throughout the graphs up to 4 teams, we did not find a sequence in which the actual 1st place team was changed completely due to different scoring. To explore this option further, more teams can be added and the sequences can be analyzed or a different possible scoring option could be put into place.

2. Do different allotments leave findings the same or change them drastically i.e. using a scoring of $2, 1, -\frac{1}{2}$ where losing causes a penalty?

The scoring of $2-1-0$ and $1, 0, -1$ result in the same findings since these two scoring systems are affine transformations of one another. Researching other types of scoring could be interesting to observe and see if the same theorems hold true.

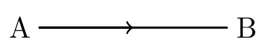
3. Does a unique score sequence for 2-1-0 necessarily convert to a unique 3-1-0 score sequence or vice versa?

We found one unique sequence that stayed the same when converted so it can only be assumed that there would be more. If we take our two unique sequences and make it into a 5 team tournament where the added team just ties with every other team, we receive the same type of anomaly.

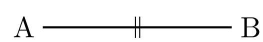
Appendix A

Tournament Graphs with Draws

A.1 Tournaments with Two Teams and Their Score Sequences

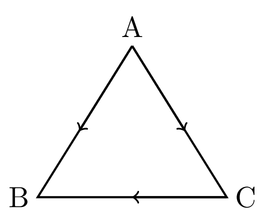


$(2, 0)$

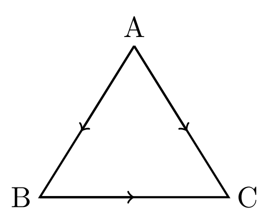


$(1, 1)$

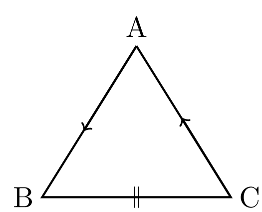
A.2 Tournaments with Three Teams and Their Score Sequences



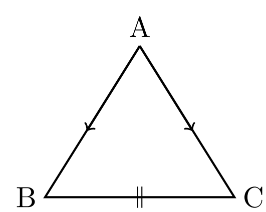
$(2, 2, 2)$



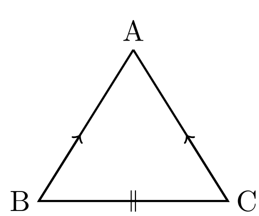
$(4, 2, 0)$



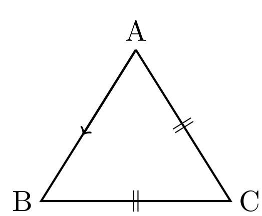
$(4, 1, 1)$



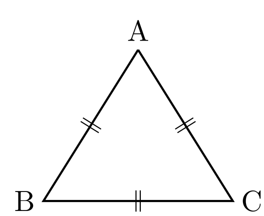
$(3, 2, 1)$



$(3, 3, 0)$



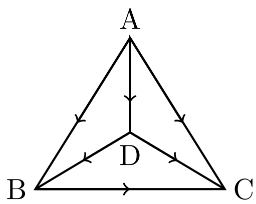
$(3, 2, 1)$



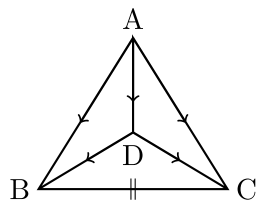
$(2, 2, 2)$

A.3 Tournaments with Four Teams and Their Score Sequences

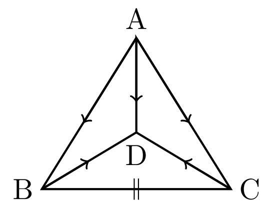
The following tournament graphs are place in Lexicographic order of decreasing s_1 value.



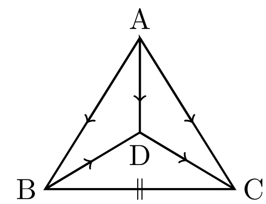
$(6, 4, 2, 0)$



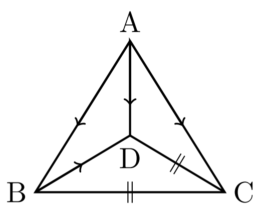
$(6, 4, 1, 1)$



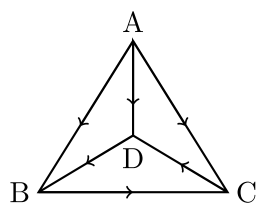
$(6, 3, 3, 0)$



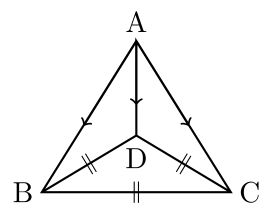
$(6, 3, 2, 1)$



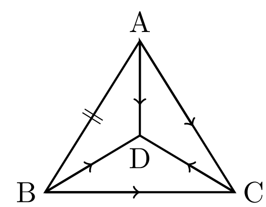
$(6, 3, 2, 1)$



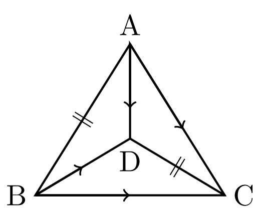
$(6, 2, 2, 2)$



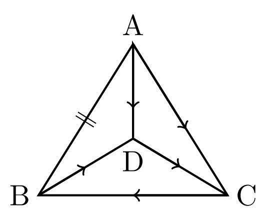
$(6, 2, 2, 2)$



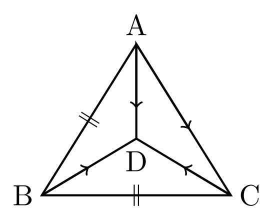
$(5, 5, 2, 0)$



$(5, 5, 1, 1)$



$(5, 4, 3, 0)$



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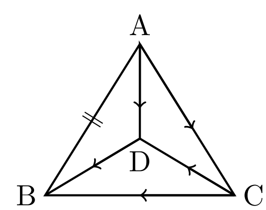

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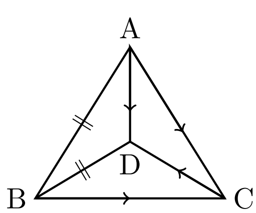

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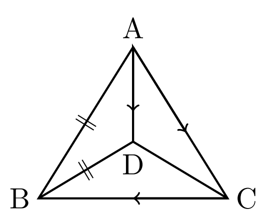

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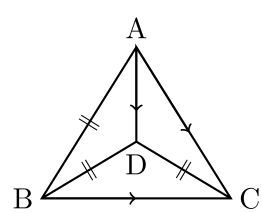

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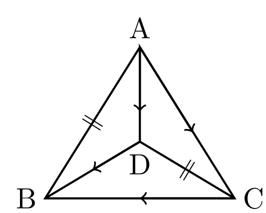

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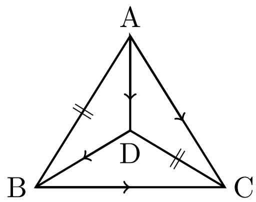

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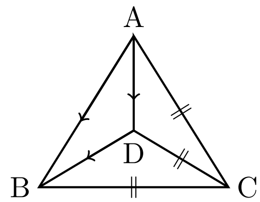

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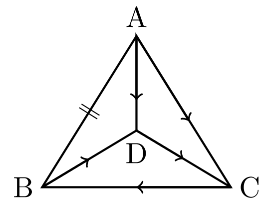

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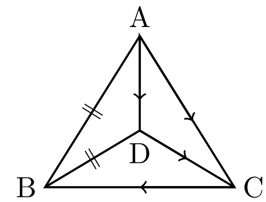

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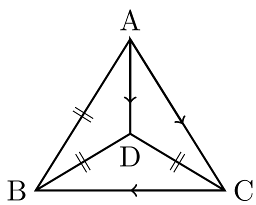

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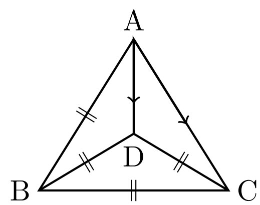

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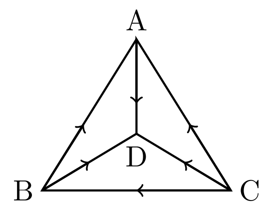

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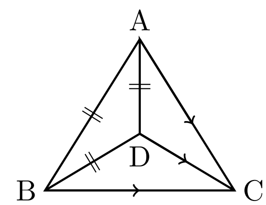

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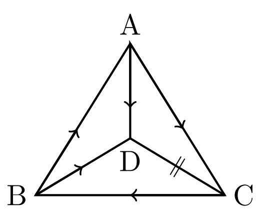

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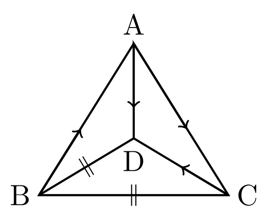

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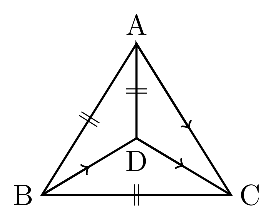

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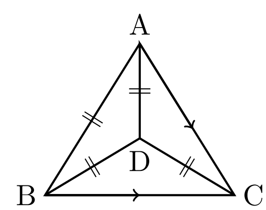

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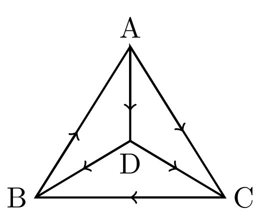

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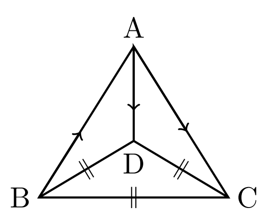

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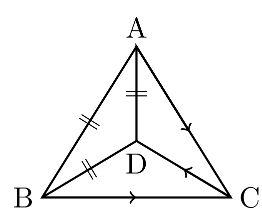

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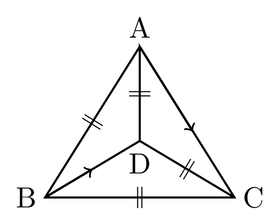

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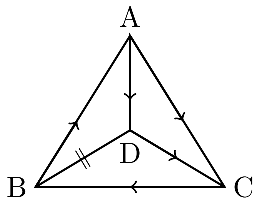

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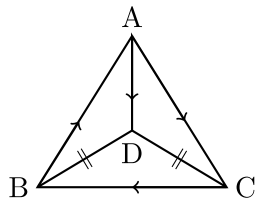

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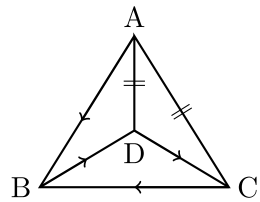

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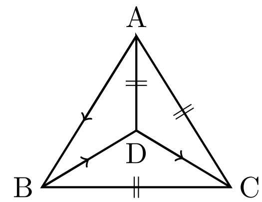

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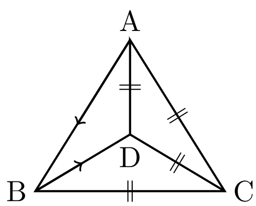

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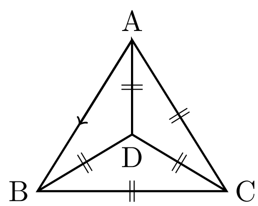

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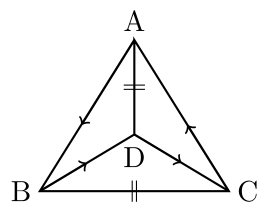

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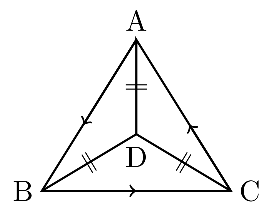

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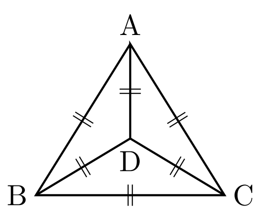

$(3, 3, 3, 3)$

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