Win Big(k), Lose Big(k): A further exploration of the Win Big and Lose Big voting systems

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by
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Abstract

This project focuses on the mathematical study of ranked voting systems, where voters rank their candidates in order of preference. There are many different voting systems used to determine a winner from a given set of ballots, for example Plurality, Borda Count, Coombs, Instant Runoff, and Anti-plurality. Looking further into a previous senior project where the author created a new voting system that selects an average alternative, we analyze the long term results of those systems. Taking the Win Big(k), and Lose Big(k) methods we compare them to various pre-existing ranked voting methods.

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1 Introduction

Throughout this project, I will be exploring the mathematical study of ranked voting systems. Looking further into a previous senior project by Erin Stuckenbruck, where she created a new voting system which selects an average alternative, I will be analyzing the long term results of those systems. Taking the Win Big(k), and Lose Big(k) systems I will compare them to various pre-existing ranked voting methods.

There are a wide variety of voting systems each with their own advantages and disadvantages leading to specific outcomes. Each voting system satisfies or violates properties referred to as fairness criteria. I will mainly focus on ranked voting systems. A ranked voting system is a system where a voter ranks alternatives given in a specific order depending on their preferences (most preferred to least preferred or least preferred to most preferred).

I will be further examining Erin Stuckenbruck's senior project, where she developed the systems Win Big and Lose Big. Those systems place more importance on a voter's likes and dislikes, giving a more preferred candidate a greater reward for being liked, and vice versa giving a candidate that is least preferred a greater penalty therefore selecting a winner that is least disliked. Further exploring her voting systems, we have Win Big(k) and Lose Big(k). They expand upon Pascal's triangle and increase the amount of points in the system. They also demonstrate the importance of both Plurality and Anti-plurality winners since there is such an importance on the first place candidate in Plurality and last place candidate in Anti-plurality because of the points allotted to each. Win Big(k) and Lose Big(k) indicate the ultimate winner after a specific k, and we will be proving why that occurs.

2
Preliminaries

Voting is a method for a group to make a collective decision on an opinion or idea, where each person can express their opinion on their preferred choice. Voting is used all over the world. In this chapter, we define several new systems that we will be using to compare to **Win Big(k)** and **Lose Big(k)**. In this chapter we will go over definitions of these systems, along with examples with how they work.

2.1 Basic Voting Systems

In this section, we will be defining some basic voting systems as well as the terminology used when talking about voting systems. We often use voting to reach a peaceful decision, and voting theory or social choice theory applies mathematical principles to studying these types of social choice procedures.

Definition 2.1.1. A **Preference Ballot** is a ballot B in which the voter ranks the choices in order of preference, $A = \{a_1, a_2, ..., a_n\}$, where $n \in \mathbb{N}$. [1]

Definition 2.1.2. A **profile**, P, is a set of ballots.

Example 2.1.3. From Figure 2.1.1, $A = \{W, CT, B, P\}$ is the set of alternatives, and the profile $P = \{B1, B2, B3, B4, B5, B6, B7\}$.

Weights	B1	B2	B3	B4	B5	B6	B7
W1	W	CT	P	CT	B	B	CT
W2	CT	B	CT	P	W	P	W
W3	B	P	W	B	CT	W	B
W4	P	W	B	W	P	CT	P

Figure 2.1.1: Ranked Meal Choices

Definition 2.1.4. A **Ranked Voting System** is a voting system in which voters rank their candidates or choices in a sequence of 1st, 2nd, 3rd, etc. [2]

Example 2.1.5. A committee of 7 people are voting for their favorite DTR meal. The choices are a wrap (W), pizza (P), burger (B), chicken tenders (CT). They have ranked the meals from their most preferred down to the least preferred. Their ranked meal choices are in Figure 2.1.1:

Definition 2.1.6. A ranked voting systems **Societal Preference Order** is the ranking of candidates on preferential ballots.

Example 2.1.7. Using Figure 2.1.3, the societal preference order is shown in Figure 2.1.2.

Definition 2.1.8. A **Partially Ordered Set (Poset)** formalizes and generalizes the intuitive concept of a set's components being ordered, sequenced, or arranged. A poset is made up of a set and a binary relation stating that one of the elements in the set precedes the other in the ordering for particular pairs of elements. [11]

Definition 2.1.9. A **Positional Voting System** is a ranked voting system in which the candidates receive points based on their rank position on the ballot and the one with the most points overall wins. [3]

The weight vector being used in this Positional Voting System is $\overrightarrow{w} = (4, 3, 2, 1)$.

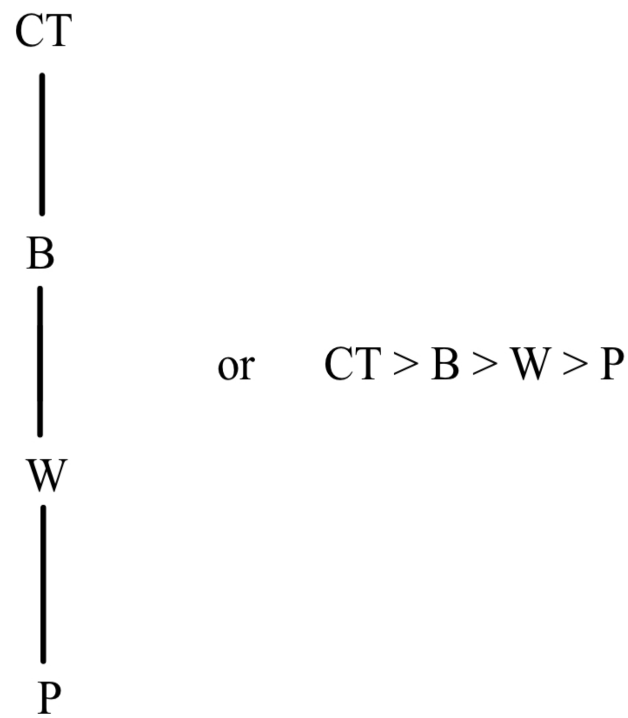

Figure 2.1.2: Societal Preference Order of Figure 2.1.3

Example 2.1.10. In Figure 2.1.3 are the rankings of the DTR meals with first place receiving 4 points, second place receiving 3 points, third place receiving 2 points, and fourth place receiving 1 point.

The scores of the weights for each candidate depending on their position on the ballot in Figure 2.1.1 where first place receives w_1 points, second place receives w_2 points, third place receives w_3 points, and last place receives w_4 points:

- $s(W) = w_1 + 2w_2 + 2w_3 + 2w_4$

  Here wrap receives 1 first place weight, 2 second place weights, 2 third place weights, and 2 last place weights.

- $s(P) = w_1 + 2w_2 + w_3 + 3w_4$

  Here pizza receives 1 first place weight, 2 second place weights, 1 third place weight, and 3 last place weights.

Weights	B1	B2	B3	B4	B5	B6	B7	Pts.
W1	W	CT	P	CT	B	B	CT	(4pts)
W2	CT	B	CT	P	W	P	W	(3pts)
W3	B	P	W	B	CT	W	B	(2pts)
W4	P	W	B	W	P	CT	P	(1pt)

Figure 2.1.3: Scores and Rankings of DTR meals

- $s(B) = 2w_1 + w_2 + 3w_3 + w_4$

  Here burger receives 2 first place weights, 1 second place weight, 3 third place weights, and 1 last place weight.

- $s(CT) = 3w_1 + 2w_2 + w_3 + w_4$

  Here chicken tenders receives 3 first place weights, 2 second place weights, 1 third place weight, and 1 last place weight.

Next plugging in the points corresponding to the weights we achieve the following:

- $s(W) = 4 + 1 + 2 + 1 + 3 + 2 + 3 = 16$
- $s(P) = 1 + 2 + 4 + 3 + 1 + 3 + 1 = 15$
- $s(B) = 2 + 3 + 1 + 2 + 4 + 4 + 2 = 18$
- $s(CT) = 3 + 4 + 3 + 4 + 2 + 1 + 4 = 21$

Therefore wrap receives 16 points, chicken tenders receives 21 points, burger receives 18, and pizza receives 15. Chicken tenders receive the most points and hence is the winner in this positional voting system.

In the **Plurality** voting system, the choice with the most first-choice votes is declared the winner. Hence only the first place row is weighted. [1]

W	B1	B2	B3	B4	B5	B6	B7	Pts.
W1	W	CT	P	CT	B	B	CT	(1pt)
W2	CT	B	CT	P	W	P	W	(0pts)
W3	B	P	W	B	CT	W	B	(0pts)
W4	P	W	B	W	P	CT	P	(0pts)

Figure 2.1.4: Plurality

Definition 2.1.11. The **Plurality** voting system, uses the weight vector $\overrightarrow{w}_P = (1, 0, 0, 0, 0, ..., 0)$.

Example 2.1.12. A committee of 7 people are voting for their favorite DTR meal. The choices are a wrap (W), pizza (P), burger (B), chicken tenders (CT), and they have ranked their meal preferences in order in Figure 2.1.4:

Using the scores of the weights for each candidate and plugging in the following $\overrightarrow{w}_P = (1, 0, 0, 0, 0, 0, 0)$, we get:

- $s(W) = 1 + 0 + 0 + 0 + 0 + 0 + 0 = 1$
- $s(P) = 0 + 0 + 1 + 0 + 0 + 0 + 0 = 1$
- $s(B) = 0 + 0 + 0 + 0 + 1 + 1 + 0 = 2$
- $s(CT) = 0 + 1 + 0 + 1 + 0 + 0 + 1 = 3$

The Plurality winner is chicken tenders with 3 first place votes as shown in Figure 2.1.5.

Anti-plurality is a system in which the candidate with the fewest last place votes wins. Therefore the last place votes are the only votes that are weighted.[4]

Definition 2.1.13. The **Anti-plurality** voting system uses the weight vector $\overrightarrow{w}_{AP} = (0, 0, 0, 0, 0, ..., 1)$.

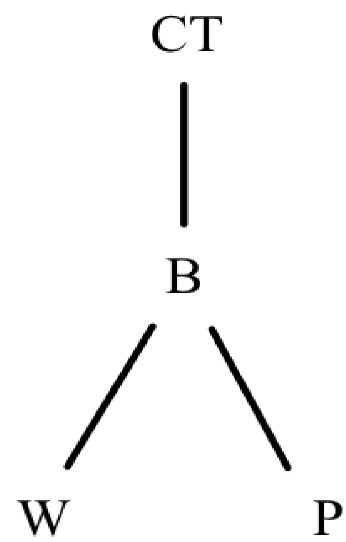

Figure 2.1.5: Plurality Winner is Chicken Tenders

Example 2.1.14. The committee is now using Anti-plurality to determine the best DTR meal. Pizza has 3 last place votes, wrap has 2 last place votes, and both burger and chicken tenders have 1 last place vote as shown in Figure 2.1.6.

Once again plugging in the following $\overrightarrow{w}_{AP} = (0, 0, 0, 0, 0, 0, -1)$, into the scores of the weights we achieve:

- $s(W) = 0 - 1 + 0 - 1 + 0 + 0 + 0 = -2$
- $s(P) = -1 + 0 + 0 + 0 + -1 + 0 + -1 = -3$

W	B1	B2	B3	B4	B5	B6	B7	Pts.
W1	W	CT	P	CT	B	B	CT	(0pts)
W2	CT	B	CT	P	W	P	W	(0pts)
W3	B	P	W	B	CT	W	B	(0pts)
W4	P	W	B	W	P	CT	P	(-1pt)

Figure 2.1.6: Anti-Plurality

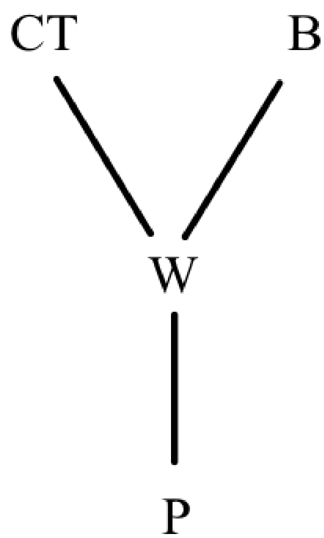

Figure 2.1.7: Anti Plurality Example

- $s(B) = 0 + 0 + -1 + 0 + 0 + 0 + 0 = -1$

- $s(CT) = 0 + 0 + 0 + 0 + 0 + -1 + 0 = -1$

Therefore the committee least prefers DTR pizza since it received the most last place rankings. Hence the chicken tenders and burgers are preferred the most out of the 4 meals since they both receive the least amount of last place rankings. The ranking is shown in Figure 2.1.7.

Borda count is a family of positional voting rules which gives each candidate, for each ballot, a number of points corresponding to the number of candidates ranked lower. In the original variant, the lowest-ranked candidate gets 0 points, the next-lowest gets 1 point, etc., and the highest-ranked candidate gets $n - 1$ points, where n is the number of candidates. Once all votes have been counted, the candidate with the most points is the winner. [5]

Definition 2.1.15. The **Borda count** system gives each alternative $(n - i)$ points for each ballot that ranks the alternatives in the i^{th} row, i.e. $\vec{w}_{BC} = ((n-1), (n-2), (n-3), ..., 1, 0)$. [5]

Example 2.1.16. The DTR meal committee decides to use the Borda Count voting system to see the general consensus on the favorite meals. The system works with first place receiving 3

W	B1	B2	B3	B4	B5	B6	B7	Pts.
W1	W	CT	P	CT	B	B	CT	(3pts)
W2	CT	B	CT	P	W	P	W	(2pts)
W3	B	P	W	B	CT	W	B	(1pt)
W4	P	W	B	W	P	CT	P	(0pts)

Figure 2.1.8: Borda Count

points, second place receiving 2 points, third place receiving 1 point, and fourth place receiving none. See Figure 2.1.8

Since there are 4 choices $n = 4$.

- $W1 =$ 1st place points $= n - 1 = 4 - 1 = 3$
- $W2 =$ 2nd place points $= n - 2 = 4 - 2 = 2$
- $W3 =$ 3rd place points $= n - 3 = 4 - 3 = 1$
- $W4 =$ 4th place points $= n - 4 = 4 - 4 = 0$

Here are the scores for each candidate:

- $s(W) = w_1 + 2w_2 + 2w_3 + 2w_4$
- $s(P) = w_1 + 2w_2 + w_3 + 3w_4$
- $s(B) = 2w_1 + w_2 + 3w_3 + w_4$
- $s(CT) = 3w_1 + 2w_2 + w_3 + w_4$

Plugging in the Borda Count voting system weights we have the following results:

- $s(W) = 3 + 0 + 1 + 0 + 2 + 1 + 2 = 9$
- $s(P) = 0 + 1 + 3 + 2 + 0 + 2 + 0 = 8$

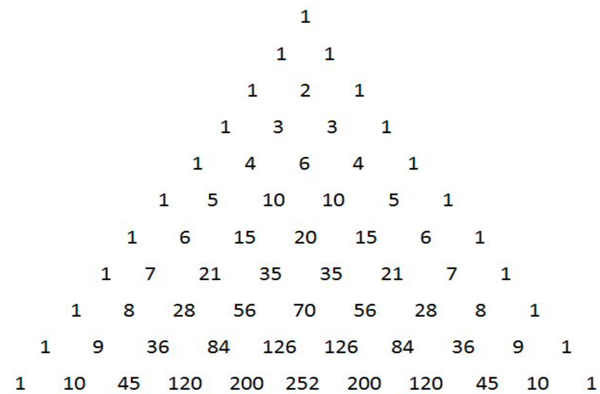

Figure 2.2.1: Pascal's Triangle

- $s(B) = 1 + 2 + 0 + 1 + 3 + 3 + 1 = 11$
- $s(CT) = 2 + 3 + 2 + 3 + 1 + 0 + 3 = 14$

Therefore wrap receives 9 points, chicken tenders receives 14 points, burger receives 11, and pizza receives 8. Thus chicken tenders is the winner using the Borda Count voting system.

2.2 Pascal's Triangle

Named after the French mathematician Blaise Pascal, Pascal's triangle shown in Figure 2.2.1 is a number pattern in the shape of a triangle. The pattern is that each number in the triangle is the sum of the two numbers above it. *Triangle Numbers* will be used to define the Win Big, Win Big(k), Lose Big, and Lose Big(k) voting systems.

The diagonals of Pascal's triangle contain the figurate numbers (polygonal number) of simplices (a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions). Only 1's are found on the diagonals that go along the left and right borders. The natural numbers are ordered on the diagonals close to the edge diagonals. The triangular numbers are ordered in the following pair of diagonals as you move inwards. The tetrahedral numbers are in order on the following pair of diagonals, and the pentatope numbers are on the next pair. Because of the triangle's symmetry, the k^{th} d-dimensional number is the same as the d^{th} k-dimensional number, as shown in Figure 2.2.2 [12]

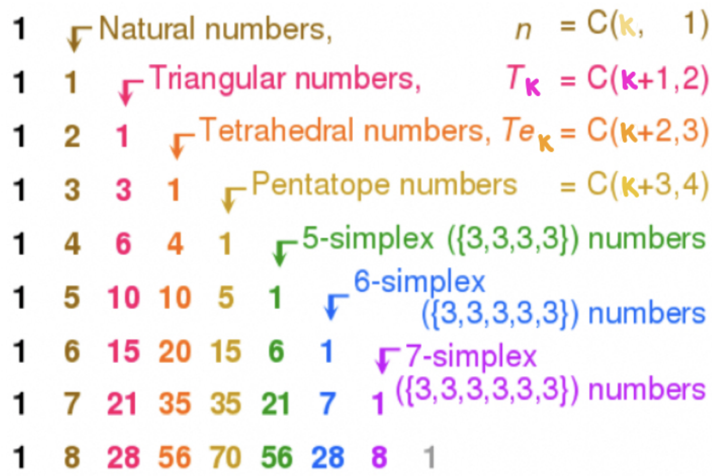

Figure 2.2.2: The Diagonal's of Pascal's Triangle shown vertically

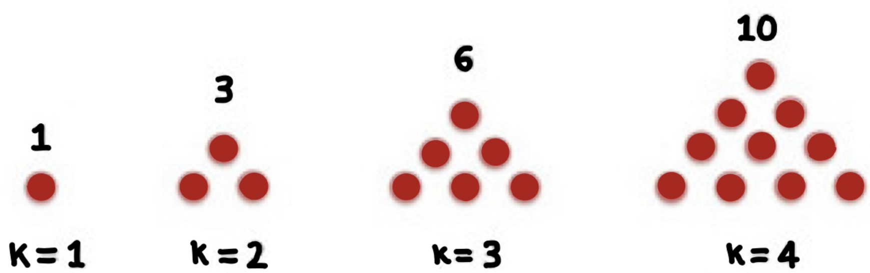

Figure 2.3.1: Triangle Number Example

2.3 Triangle Numbers

Triangle numbers are a pattern of numbers that form equilateral triangles. Each subsequent number in the sequence adds a new row of dots to the triangle. It's important to note that k equals the term in the sequence so, $k = 1$ is the first term and so forth... [9]

The first triangular number is 1, the second is 3, the third is 6, the fourth 10, the fifth 15, and so on as shown in figure 2.3.1. You can see that each triangle comes from the one before by adding a row of dots on the bottom which has one more dot than the previous bottom row. This means that the kth triangle number T_k is equal to $T_k = 1 + 2 + 3 + ... + k$ [10]

Definition 2.3.1. The k^{th} **triangle number** T_k is defined by $T_k = \sum_{n=0}^{k} n = 1+2+3+...+k = \frac{k(k+1)}{2}$.

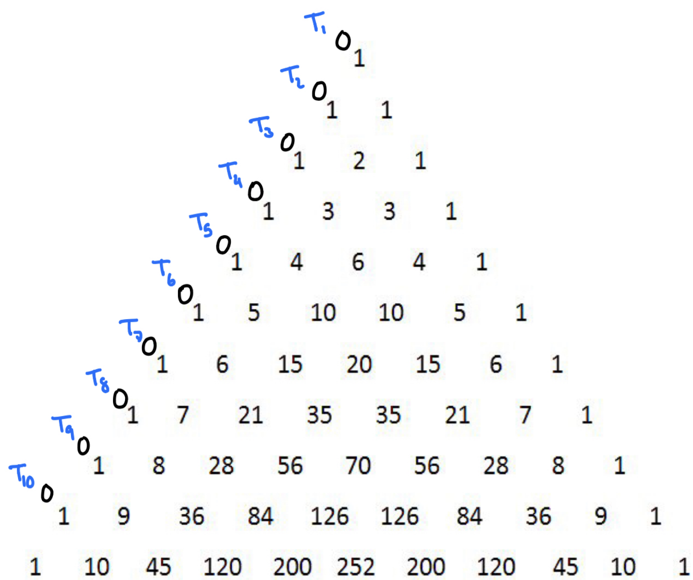

Figure 2.3.2: Pascal's Triangle with T numbers

3
Win Big(k)

In this chapter we will define the original **Win Big** voting system and then lead into the **Win Big(k)** voting system.

3.1 Win Big

Similar to Borda Count, the **Win Big** method assigns points to an alternative based on their position in a preferential ballot. The difference is that the **Win Big** method gives an alternative a greater reward for being more preferred by a voter than the Borda Count method, selecting a winner that is generally liked. Recall that C is the number of candidates, m is the number of ballots, and T_k is the triangle number. [7]

Definition 3.1.1. Let $n = \mid C \mid$. The **Win Big** method is a voting system with weight vector $\overrightarrow{w}_{WB} = (T_{n-1}, ..., T_1, T_0)$ where T_i is the i^{th} triangle number.

Example 3.1.2. With 4 candidates the weight vector for Win Big is $\overrightarrow{w}_{WB} = (T_3, T_2, T_1, T_0) = (6, 3, 1, 0)$. Let $P = \{B_1, B_2, B_3\}$ be a set of ballots and $A = \{a, b, c, d\}$ be candidates. Consider the ranked ballots in Figure 3.1.1:

For any weight vector $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$, we determine the scores for each candidate:

- $s(a) = w_1 + w_2 + w_4$
- $s(b) = 2w_2 + w_4$

Weights	B1	B2	B3
W1	a	d	c
W2	b	a	b
W3	c	c	d
W4	d	b	a

Figure 3.1.1: Ranked Ballots with 4 Candidates

B1	B2	B3	Weights
a	d	c	6pts
b	a	b	3pts
c	c	d	1pt
d	b	a	0pts

Figure 3.1.2: Ranked Ballots with 4 Candidates with weights

- $s(c) = w_1 + 2w_3$
- $s(d) = w_1 + w_3 + w_4$

In the Win Big system, the weight vector is $(6, 3, 1, 0)$ as shown in Figure 3.1.2. To form the points for each position, the points for positions below are added up and form the next amount of points for the allotted place:

Therefore the overall scores in 3.1.2 are:

- $s(a) = 6 + 3 + 0 = 9$
- $s(b) = 3 + 0 + 3 = 6$

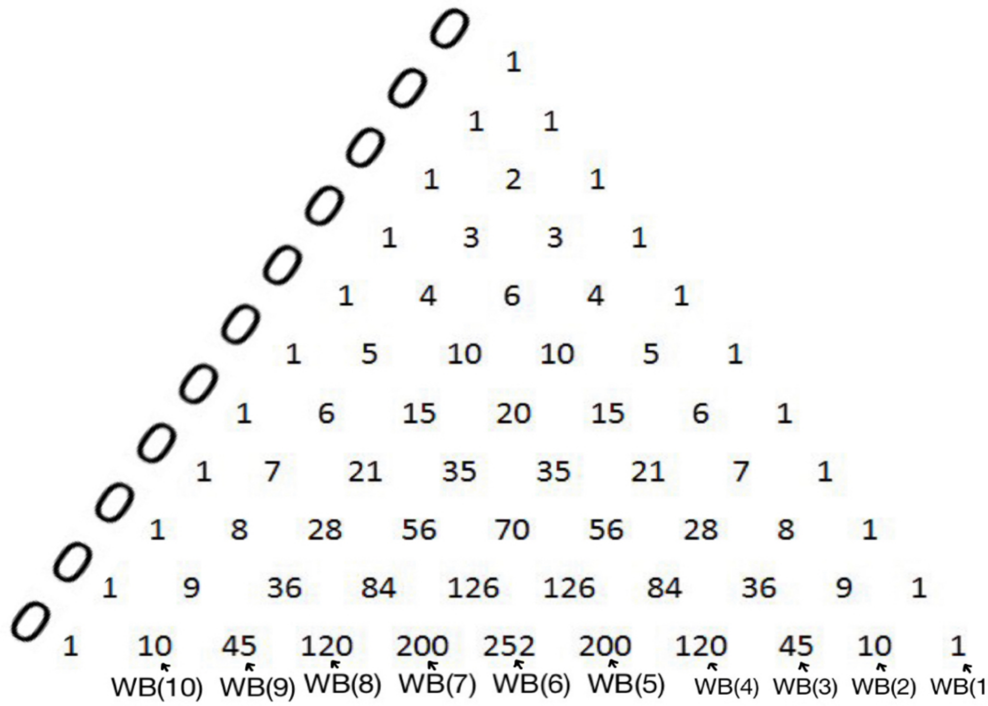

Figure 3.1.3: Pascal's Triangle Augmented with 0's

- $s(c) = 1 + 1 + 6 = 8$
- $s(d) = 0 + 6 + 1 = 7$

The results are $s(a) > s(c) > s(d) > s(b)$. Therefore $f(P) = \{a\}$, i.e. a is the winner.

3.2 Win Big(k)

We define the weight vectors of **Win Big(k)** to be the k^{th} diagonal of Pascal's Triangle augmented with a diagonal of 0's as shown in figure 3.1.3. The diagonal goes from the bottom to the top, i.e. $WB(1) = (1, 1, ..., 1, 0)$. As you move onto higher diagonals, the candidate will receive significantly more points for being the most preferred, setting them even more ahead.

Example 3.2.1.

1. $WB(1) = (1, ..., 1, 0)$, is the weight vector of the Anti-plurality voting system.
2. $WB(2) = (n-1, ..., 2, 1, 0)$, is the weight vector for Borda Count.

B1	B2	B3	B4	B5	B6	B7	Weights
d	b	a	a	c	b	b	w1
a	a	d	d	d	c	d	w2
c	d	b	c	a	a	c	w3
b	c	c	b	b	d	a	w4

Figure 3.3.1: Win Big(k) ballots

3. $WB(3) = (T_k, ..., T_2, T_1, 0)$ which is the original Win Big voting system.

4. $WB(4)$ with 6 candidates has the weight vector $\overrightarrow{w}_{WB(4)} = (35, 20, 10, 4, 1, 0)$

3.3 Win Big(k) example

Similar to **Win Big**, as $k \to \infty$ **Win Big(k)** is tending towards Plurality with a greater reward for being preferred the most. Both systems heavily weight the first place candidate similar to the Plurality voting system.

Example 3.3.1. Let $P = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ and $A = \{a, b, c, d\}$. Consider the ranked ballots in Figure 3.3.1:

Here are the scores for each candidate:

- $s(a) = 2w_1 + 2w_2 + 2w_3 + w_4$
- $s(b) = 3w_1 + w_3 + 3w_4$
- $s(c) = w_1 + w_2 + 3w_3 + 2w_4$
- $s(d) = w_1 + 4w_2 + w_3 + w_4$

Using the Plurality voting system with the weights $\overrightarrow{w}_P = (1, 0, 0, 0, 0, 0, 0)$ we get:

- $s(a) = 0 + 0 + 1 + 1 + 0 + 0 + 0 = 2$

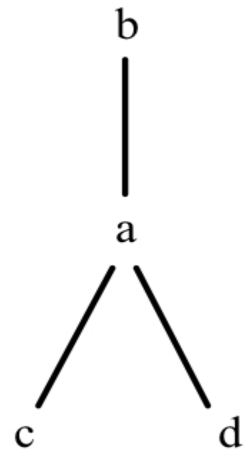

Figure 3.3.2: Societal Preference Plurality Winner

- $s(b) = 0 + 1 + 0 + 0 + 0 + 1 + 1 = 3$
- $s(c) = 0 + 0 + 0 + 0 + 1 + 0 + 0 = 1$
- $s(d) = 1 + 0 + 0 + 0 + 0 + 0 + 0 = 1$

Therefore the Plurality winner is b since $S(b) > S(a) > S(c), S(d)$. The societal preference poset using plurality is shown in Figure 3.3.2.

Using the same set of ballots and now using the Win Big system with the weights $w = (6, 3, 1, 0)$ we get the following scores:

- $S(a) = 3 + 3 + 6 + 6 + 1 + 1 + 0 = 20$
- $S(b) = 0 + 6 + 1 + 0 + 0 + 6 + 6 = 19$
- $S(c) = 1 + 0 + 0 + 1 + 6 + 3 + 1 = 12$
- $S(d) = 6 + 1 + 3 + 3 + 3 + 0 + 3 = 19$

Looking at the scores using Win Big we notice that a is the winner despite b being the Plurality winner. The social preference using Win Big is shown in Figure 3.3.3.

Now using the Win Big(k) method, if we change the weights to , $(T_k, k, 1, 0)$ with $k = 4$, we get the following scores from Figure 3.3.4 and weight vector $\overrightarrow{w} = (10, 4, 1, 0)$:

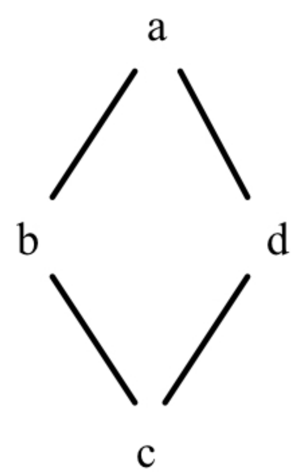

Figure 3.3.3: Societal Preferences Using Win Big

B1	B2	B3	B4	B5	B6	B7	Weights
d	b	a	a	c	b	b	10pts
a	a	d	d	d	c	d	4pts
c	d	b	c	a	a	c	1pt
b	c	c	b	b	d	a	0pts

Figure 3.3.4: Win Big(k) Weights

Figure 3.3.5: Societal Preference of Win Big(4)

- $s(a) = 4 + 4 + 10 + 10 + 1 + 1 + 0 = 30$
- $s(b) = 0 + 10 + 1 + 0 + 0 + 10 + 10 = 31$
- $s(c) = 1 + 0 + 0 + 1 + 10 + 4 + 1 = 17$
- $s(d) = 10 + 1 + 4 + 4 + 4 + 0 + 4 = 27$

The societal preference of Win Big(4) is shown in Figure 3.3.5

Using the new weights, we notice b is both the Plurality winner and the $WB(4)$ winner of this voting system. Later we will prove that b will also win $WB(k)$ for all $k \geq 4$.

Definition 3.3.2.

- For 3 candidates $\overrightarrow{w}_{WB(k)} = (k, 1, 0)$.
- For 4 candidates $\overrightarrow{w}_{WB(k)} = (T_k, k, 1, 0)$.
- For 5 candidates $\overrightarrow{w}_{WB(k)} = (T_k + \sum_{i=1}^{k=1} Ti, T_k, k, 1, 0)$.
- For n candidates use the k^{th} diagonal $P_d(k) = \frac{1}{d!} \prod_{n=0}^{d-1}(k + n) = \frac{k^{(d)}}{d!}\binom{k+d-1}{d}$, where $k^{(d)}$ is the rising factorial.

3.4 Win Big(k) Theorem

In this section, we prove the Win Big(k) theorem, which says that after a certain k a candidate will win every time afterwards, if they are the Plurality winner.

The weight vector of Win Big(k) with 3 candidates A, B, C is $(k, 1, 0)$, which means k is the weight of the first place ballots, 1 is the weight of the second place ballots, and 0 is the weight of the third place ballots. Let a_1, b_1, c_1 be the number of first place scores, a_2, b_2, c_2 be the number of second place scores, and a_3, b_3, c_3 be the number of third place scores for A, B, and C, respectively. If A wins Plurality, then at the 1st level, A wins more ballots than B and C. This means $a_1 > b_1 >$ and $a_1 > c_1$.

Notice that $a_1 + b_1 + c_1 = m$, since m equals the number of ballots. Similarly we notice that the $a_2 + b_2 + c_2 = m$ and $a_3 + b_3 + c_3 = m$. The 1st place scores for (A, B, C) are (ka_1, kb_1, kc_1). The second place scores are (a_2, b_2, c_2). The third place scores are $(0, 0, 0)$.

Therefore the total scores for A, B, C are:

$$s(A) = ka_1 + a_2$$

$$s(B) = kb_1 + b_2$$

$$s(C) = kc_1 + c_2$$

Theorem 3.4.1. *For $n = 3$, let T be a set of m ballots for A, B, C. Assume A wins plurality. Then A wins $WB(k)$ for all $k \geq K$, for some $K \geq 3$.*

Lemma 3.4.2.

1. $b_1 < a_1$

2. $kb_1 < ka_1$

3. $b_1 + 1 \leq a_1$

4. $b_1 + b_2 \leq m$

5. $b_2 \leq m$

Proof.

(1) Since a wins Plurality, then at the 1st level, a wins more ballots than b and c. This means $b_1 < a_1$ and $c_1 < a_1$.

(2) From (1) $b_1 < a_1$ and $c_1 < a_1$. Multiplying both sides by $k \geq 1$, to get $kb_1 < ka_1$ and $kc_1 < ka_1$.

(3) Since a_1 and b_1 are integers and $b_1 < a_1$, then $b_1 + 1 \leq a_1$.

(4) Since $b_1 + b_2 + b_3 = m$ which is the number of ballots since b_1 is the number of first place ballots where b is in first place , b_2 is the number of second place ballots, and b_3 is the number of 3rd place ballots, therefore by adding all the ballots together you get m equals the number of ballots. Hence $b_1 + b_2 \leq b_1 + b_2 + b_3 = m$. Therefore $b_1 + b_2 \leq m$.

(5) Since $b_2 < b_1 + b_2 \leq b_1 + b_2 + b_3 = m$, therefore $b_2 \leq m$.

□

Theorem 3.4.3. *Let $k = m$. Then $S(B) \leq S(A)$.*

Proof.

Let $k = m$, we will show $S(B) \leq S(A)$.

$S(B) = mb_1 + b_2$ (by definition)

$\leq mb_1 + m$ (by 5)

$= m(b_1 + 1)$ (by factoring)

$\leq m(a_1)$ (by 3)

$\leq ma_1 + a_2$ (since $0 \leq a_2$)

$= S(A)$

Therefore $S(B) \leq S(A)$. □

Theorem 3.4.4. *Let $k = m + 1$, then $S(B) < S(A)$.*

Proof.

Let $k = m + 1$. Plug in $k = m + 1$ for $S(A), S(B)$, and $S(C)$. Therefore the scores are,

$s(A) = ka_1 + a_2 = a_1m + a_1 + a_2$

$s(B) = kb_1 + b_2 = b_1m + b_1 + b_2$

$s(C) = kc)1 + c_2 = c_1m + c_1 + c_2$

Now follow $s(B)$. From above we have

$$s(B) = kb_1 + b_2 = b_1m + b_1 + b_2.$$

Then by (4) in Lemma 3.1.2. and factoring,

$$\leq b_1m + m$$

$$= m(b_1 + 1).$$

Next by (3) in Lemma 3.1.2.,

$$\leq ma_1.$$

Since $a_1 \geq 1$,

$$< ma_1 + a_1.$$

Then by adding a_2,

$$\leq a_1m + a_1 + a_2$$

$$= s(A).$$

Therefore this proves that $S(B) < S(A)$. Similarly, $S(C) < S(A)$, so A wins $WB(m + 1)$. □

Theorem 3.4.5. *Let A, B, C be candidates and P a profile of ballots. Assume A wins Plurality. Then $WB(k)$ for all $k \geq m + 1$, i.e. $S(A) > S(B)$ and $S(A) > S(C)$.*

Proof.

We must show A wins $WB(m+j)$ for any $j \geq 1$. By plugging in $m+j$ for k in $S(A), S(B)$, and $S(C)$, the scores are:

$$s(A) = ka_1 + a_2 \quad = \quad a_1 m + a_1 j + a_2$$

$$s(B) = kb_1 + b_2 \quad = \quad b_1 m + b_1 j + b_2$$

$$s(C) = kc_1 + c_2 \quad = \quad c_1 m + c_1 j + c_2$$

Now following $s(B)$ from above,

$$s(B) = kb_1 + b_2 = b_1 m + b_1 j + b_2.$$

$$= mb_1 + b_1 + b_1(j-1) + b_2$$

Then by (4) in Lemma 3.1.2. and expanding,

$$\leq b_1(m+j-1) + m$$

$$< mb_1 + m + b_1(j-1).$$

Next by factoring and (3) in Lemma 3.1.2.,

$$= m(b_1+1) + b_1(j-1)$$

$$\leq ma_1 + b_1(j-1).$$

Then by (1) and since $j-1 \leq j$,

$$< ma_1 + a_1(j-1)$$

$$< ma_1 + a_1 j$$

Next adding a_2,

$$\leq ma_1 + a_1 j + a_2$$

$$= s(A).$$

This proves that $s(B) < s(A)$. Similarly, $s(C) < s(A)$, therefore A wins $WB(m+j)$. $\square$

4
Lose Big(k)

In this chapter we will define the original **Lose Big** voting system, and then lead into the **Lose Big(k)** voting system which expands upon Pascal's triangle.

4.1 Lose Big

The **Lose Big** voting system also uses the triangle numbers when assigning points to alternatives. Similar to how Anti-plurality is the reversal of Plurality, **Lose Big** is the negative reversal of Win Big. The idea is that the least preferred an alternative is, the more they are penalized, selecting a winner that is generally not disliked.

Instead of counting the number of positions the alternative is above every other alternative, we count the number of positions the alternative is below each alternative, assigning a negative value. This means the first place alternative receives 0 points, the second place alternative receives -1 points, for being 1 row below the first place alternative, the third place alternative receives -3 points, -1 for being 1 row below the second place alternative and -2 for being 2 rows below the first place alternative, and so on. The alternative with the highest score wins.

Like in the Win Big method, notice that the weights are the triangle numbers, merely negated. [7]

Weights	B1	B2	B3
W1	a	d	c
W2	b	a	b
W3	c	c	d
W4	d	b	a

Figure 4.1.1: Ballots from 3.1.2

Definition 4.1.1. Let $n = \mid C \mid$. The **Lose Big** method is a voting system with weighted vector $\overrightarrow{w}_{LB} = -(0, T_1, ..., T_{n-1})$, where T_i is the i^{th} triangle number. [7]

Example 4.1.2. With 4 candidates the weight vector for Lose Big is $\overrightarrow{w}_{LB} = -(0, T_1, T_2, T_3) = (0, -1, -3, -6)$. Let $P = \{B_1, B_2, B_3\}$ and $A = \{a, b, c, d\}$. Consider the ranked ballots in Figure 4.1.1.

Here are the scores of the weights of each candidate as shown in the Win Big chapter:

- $s(a) = w_1 + w_2 + w_4$
- $s(b) = 2w_2 + w_4$
- $s(c) = w_1 + 2w_3$
- $s(d) = w_1 + w_3 + w_4$

Here are the scores of B1 from Figure 4.1.2. To form the points for each position, the points for positions below are added up and form the next amount of points for the allotted place.

- $LB_1(a) = 0$
- $LB_2(b) = 0 - 1 = -1$
- $LB_3(c) = 0 - 1 - 2 = -3$

B1	B2	B3	Weights
a	d	c	0pts
b	a	b	-1pt
c	c	d	-3pts
d	b	a	-6pts

Figure 4.1.2: Lose Big Weights

- $LB_4(d) = 0 - 1 - 2 - 3 = -6$

Therefore the final scores for each candidate in 4.1.2 are:

- $s(a) = 0 - 1 - 6 = -7$
- $s(b) = -1 - 6 - 1 = -8$
- $s(c) = -3 - 3 - 0 = -6$
- $s(d) = -6 - 0 - 3 = -9$

This shows $s(c) > s(a) > s(b) > s(d)$. Therefore $f(P) = \{c\}$.

Referring back to Example 3.1.2., notice that Win Big chose a as the winner since a was first in one of the ballots and second in the other, therefore receiving a greater reward. In the example, c was chosen as the Lose Big winner instead of a. This is because a was put last in B_3, therefore being penalized more, while c was never placed below third place, since c is not truly disliked by any of the voters.

4.2 Lose Big(k)

We define the weight vectors of **Lose Big(k)** to be the k^{th} diagonal of negative Pascal's Triangle augmented with a diagonal of 0's as shown in figure 4.2.1, starting from the upper left.

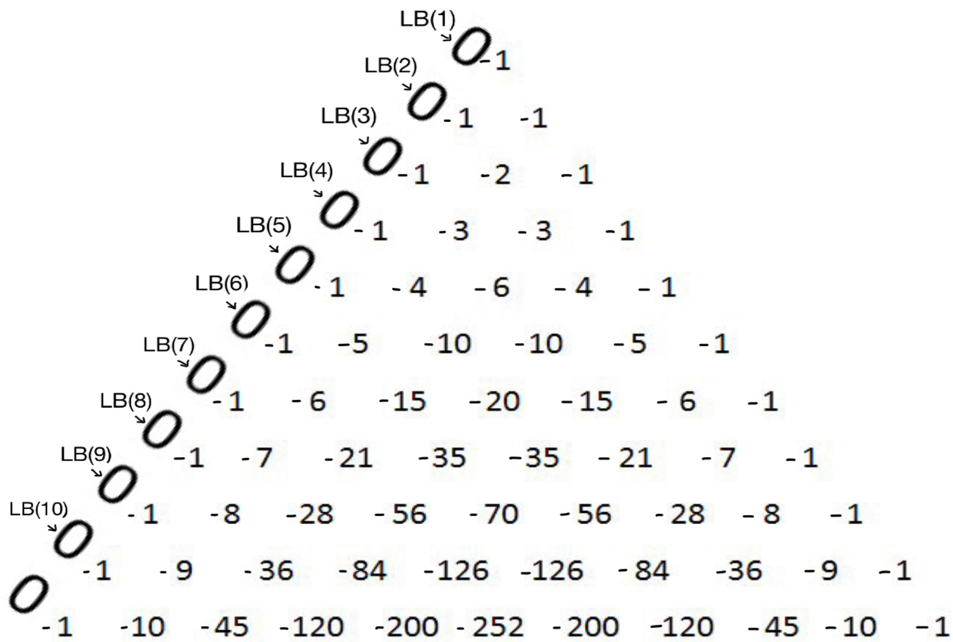

Figure 4.2.1: Negative Augmented Pascal's Triangle

Definition 4.2.1.

- For 3 candidates $\overrightarrow{w}_{LB(k)} = -(0, 1, k)$.
- For 4 candidates $\overrightarrow{w}_{LB(k)} = -(0, 1, k, T_k)$.
- For 5 candidates $\overrightarrow{w}_{LB(k)} = -(0, 1, k, T_k, T_k + \sum_{i=1}^{k=1} Ti)$.
- For n candidates use the k^{th} diagonal $P_d(k) = -(\frac{1}{d!}\prod_{n=0}^{d-1}(k+n) = \frac{k^{(d)}}{d!}\binom{k+d-1}{d}))$, where $k^{(d)}$ is the rising factorial.

Similar to Win Big(k), we can also observe that this formula moves us onto the next row in Pascal's negative augmented triangle. As you move onto higher rows, the candidate will receive significantly more points against for being the least preferred and setting them even more behind.

Notice $LB(1) = (0, -1, -1, ..., -1)$ in the first diagonal and $(0, ..., 0, 1)$ Anti-plurality give the same results and are the weight vectors of Plurality's voting system, $LB(2) =$

B1	B2	B3	B4	B5	B6	B7	Weights
d	b	a	a	c	b	b	w1
a	a	d	d	d	c	d	w2
c	d	b	c	a	a	c	w3
b	c	c	b	b	d	a	w4

Figure 4.3.1: Lose Big(k) Ballots

$(0, -1, -2, ..., -(n-1))$, is the weight vector for negative Borda Count, and $LB(3) = -(0, T_1, T_2, ..., T_{n-1})$ is the original Lose Big weight vector.

Example 4.2.2. $LB(4)$ with 6 candidates has the weight vector $\overrightarrow{W}_{LB(4)} = (0, -1, -4, -10, -20, -35)$ as shown in Figure 4.2.1.

4.3 Lose Big(k) example

Similar to **Lose Big**, as $k \to \infty$ **Lose Big(k)** is tending towards Anti-plurality with a greater consequence for being the least preferred.

Example 4.3.1. Let $P = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ and $A = \{a, b, c, d\}$. Consider the ranked ballots in Figure 4.3.1:

Here are the scores of the weights for each candidate:

- $s(a) = 2w_1 + 2w_2 + 2w_3 + w_4$
- $s(b) = 3w_1 + w_3 + 3w_4$
- $s(c) = w_1 + w_2 + 3w_3 + 2w_4$
- $s(d) = w_1 + 4w_2 + w_3 + w_4$

Using the Anti-plurality voting system with the weights $\overrightarrow{w}_{AP} = (0, 0, 0, 0, 0, 0, -1)$, we get the following scores:

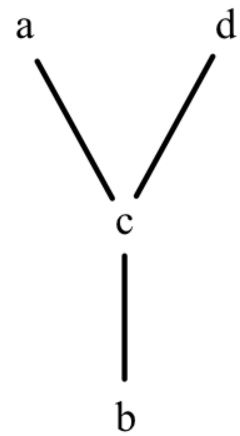

Figure 4.3.2: Anti-plurality Societal Preference

- $s(a) = 0 + 0 + 0 + 0 + 0 + 0 - 1 = -1$
- $s(b) = -1 + 0 + 0 - 1 - 1 + 0 + 0 = -3$
- $s(c) = 0 - 1 - 1 + 0 + 0 + 0 + 0 = -2$
- $s(d) = 0 + 0 + 0 + 0 + 0 - 1 + 0 = -1$

Therefore the Anti-plurality winner as shown in Figure 4.3.2 is a and d since $s(a), s(d) > s(c) > s(b)$.

Using the same set of ballots and now using the Lose Big system with the weights $\overrightarrow{w}_{LB} = (0, -1, -3, -6)$ we get the following scores:

- $s(a) = -1 - 1 - 0 - 0 - 3 - 3 - 6 = -14$
- $s(b) = -6 - 0 - 3 - 6 - 6 - 0 - 0 = -21$
- $s(c) = -3 - 6 - 6 - 3 - 0 - 1 - 3 = -22$
- $s(d) = 0 - 3 - 1 - 1 - 1 - 6 - 1 = -13$

Looking at the scores using Lose Big we notice that d is the winner despite a being the Anti-plurality winner. Therefore the societal preference using the Lose Big voting system is shown in Figure 4.3.3.

Figure 4.3.3: Societal Preference using Lose Big

B1	B2	B3	B4	B5	B6	B7	Weights
d	b	a	a	c	b	b	0pts
a	a	d	d	d	c	d	-1pt
c	d	b	c	a	a	c	-4pts
b	c	c	b	b	d	a	-10pts

Figure 4.3.4: Lose Big(k) with Weights

Now using the Lose Big(k) voting system, if we change the weights to ,$\overrightarrow{w}_{LB} = (0, -1, -k, -T_k)$ for $k = 4$, so the weights are $\overrightarrow{w}_{LB(4)} = (0, -1, -4, -10)$, we get the following scores from Figure 4.3.4:

- $s(a) = -1 - 1 - 0 - 0 - 4 - 4 - 10 = -20$
- $s(b) = -10 - 0 - 4 - 10 - 10 - 0 - 0 = -34$
- $s(c) = -4 - 10 - 10 - 4 - 0 - 1 - 4 = -33$
- $s(d) = 0 - 4 - 1 - 1 - 1 - 10 - 1 = -18$

The societal preference of Lose Big(k) is shown in Figure 4.3.5.

Figure 4.3.5: Societal Preference of Lose Big(k)

Observing the scores above using the new weights, d, the Anti-plurality winner is also the winner of this voting system and will continue to be the winner of this voting system.

4.4 Lose Big(k) Theorem

In this section, we prove the Lose Big(k) theorem which says that after a certain k a candidate will win every time after that k, since they are the Anti-plurality winner.

The weight vector of Lose Big(k) with 3 candidates, A, B, C is $(0, -1, -k)$, which means 0 is the weight of the first place ballots, -1 is the weight of the second place ballots, and $-k$ is the weight of the third place ballots. Let a_3, b_3, c_3 be the number of last place scores for A, B, and C, respectively. Let a_2, b_2, c_2 be the number of second place scores for A, B, and C, and let a_1, b_1, c_1 be the number of first place scores for A, B, and C. If A wins Anti-plurality, then at the last level, A wins fewer ballots than B and C. This means $a_3 < b_3$ and $a_3 < c_3$.

Notice that $a_1 + b_1 + c_1 = m$, since m equals the number of ballots. Similarly we notice that the $a_2 + b_2 + c_2 = m$ and $a_3 + b_3 + c_3 = m$. The 1st place scores for (A, B, C) are $(0, 0, 0)$. The second place scores are $(-a_2, -b_2, -c_2)$. The third place scores are $(-ka_3, -kb_3, -kc_3)$.

Therefore the total scores for A, B, C are:

$$S(A) = -ka_3 - a_2$$

$$S(B) = -kb_3 - b_2$$

$$S(C) = -kc_3 - c_2$$

Theorem 4.4.1. *For $n = 3$, let T be a set of m ballots for A, B, C. Assume A wins Anti Plurality. Then A wins $LB(k)$ for all $k \geq K$, for some $K \geq 3$.*

Lemma 4.4.2. *Let A be the Anti-plurality winner and let B be another candidate.*

1. $b_3 > a_3$
2. $kb_3 > ka_3$
3. $b_3 \geq a_3 + 1$
4. $b_3 + b_2 \leq m$
5. $b_2 \leq m$

Proof.

(1) A wins Anti-plurality, so at the last level A wins fewer ballots than B and C. This means $b_3 > a_3$, and $c_3 > a_3$.

(2) Multiplying both sides by $k \geq 1$, we get $kb_3 > ka_3$.

(3) Since a_1 and b_1 are integers and $b_3 > a_3$, then $b_3 \geq a_3 + 1$.

(4) $b_1 + b_2 + b_3 = m$, the number of ballots, since b_1 is the number of first place ballots where b_1 is in first place , b_2 is the number of second place ballots, and b_3 is the number of 3rd place ballots, therefore by adding all the ballots together you get $m =$ number of ballots. Then since $b_1 \geq 0$, $b_3 + b_2 \leq b_3 + b_2 + b_1 = m$. Therefore $b_3 + b_2 \leq m$.

(5) $b_2 \leq b_1 + b_2$ (since $b_1 \geq 0$) $\leq b_1 + b_2 + b_3$ (since $b_3 \geq 0$) $= m$. Therefore $b_2 \leq m$. □

Theorem 4.4.3. *Let $k = m$. Then $S(A) \geq S(B)$.*

Proof.

Let $k = m$. We will show $-S(B) \geq -S(A)$

Then $-S(B) = -(-mb_3 - b_2)$ (by definition)

$= mb_3 + b_2$

$\geq m(a_3 + 1) + b_2$ by (3)

$\geq ma_3 + m + b_2$ (by factoring)

$\geq ma_3 + a_2 + b_2$ (since $m \geq a_2$)

$\geq ma_3 + a_2 + 0$ (since $b_2 \geq 0$)

$= -S(A)$.

Hence $-S(B) \geq -S(A)$, which by multiplying by -1 implies $S(A) \geq S(B)$. $\square$

Theorem 4.4.4. *Let $k = m + 1$, then $S(A) > S(B)$.*

Proof.

Let $k = m + 1$. Plug in $k = m + 1$ for $S(A), S(B)$, and $S(C)$. Therefore the scores are,

$-S(A) = -ka_3 - a_2 = -ma_3 - a_3 - a_2$

$-S(B) = -kb_3 - b_2 = -mb_3 - b_3 - b_2$

$-S(C) = -kc_3 - c_2 = -mc_3 - c_3 - c_2$

Now follow $-S(A)$. From above we have

$$-S(A) = -ka_3 - a_2 = ma_3 + a_3 + a_2.$$

Then by (4) in Lemma 4.4.2. and factoring,

$$\leq ma_3 + m$$

$$= m(a_3 + 1).$$

Next by (3) in Lemma 4.4.2.,

$$\leq mb_3.$$

Since $b_3 \geq 1$,

$$< mb_3 + b_3.$$

Then by adding b_2,

$$\leq mb_3 + b_3 + b_2$$
$$= -S(B).$$

Therefore this proves that $-S(B) > -S(A)$. Similarly, $-S(C) > -S(A)$, so A wins $LB(m+1)$. □

Theorem 4.4.5. *Let A, B, C be candidates and P a profile of ballots. Assume A wins Anti-plurality. Then $LB(k)$ for all $k \geq m+1$, i.e. $-S(A) \leq -S(B)$ and $-S(A) \leq -S(C)$.*

Proof.

Let $k = m + j$ where $j \geq 1$. Plug in $k = m + j$ for $S(A), S(B)$,and $S(C)$. Therefore the scores are,

$$s(A) = -ka_3 - a_2 = -a_3m - a_3j - a_2$$

$$s(B) = -kb_3 - b_2 = -b_3m - b_3j - b_2$$

$$s(C) = -kc_3 - c_2 = -c_3m - c_3j - c_2$$

Now follow $s(A)$

From above we have,

$$-s(A) = -ka_3 - a_2 = ma_3 + a_3j + a_2$$
$$= ma_3 + a_3 + a_3(j-1) + a_2$$

Then by (4) in Lemma 4.4.2. and expanding,

$$\leq a_3(m + j - 1) + m$$
$$< ma_3 + m + a_3(j-1).$$

Next by factoring and (3) in Lemma 4.4.2.,

$$= m(a_3 + 1) + a_3(j - 1)$$

$$\leq mb_3 + a_3(j - 1).$$

Then by (1) in Lemma 4.4.2. and since $j - 1 \leq j$,

$$< mb_3 + b_3(j - 1)$$

$$< mb_3 + b_3 j$$

Next by adding b_2,

$$\leq mb_3 + b_3 j + b_2$$

$$= -s(B).$$

This proves that $-S(A) \leq -S(B)$, and similarly $-S(A) \leq -S(C)$. Therefore A wins $LB(m + j)$. $\square$

5
Future Work

There are a few things that I would analyze and like to complete if I had more time with this project. I would love to see how Win Big(k) and Lose Big(k) compare to Win Big and Lose Big in fairness criteria and examine the changes between them on what each system satisfies and violates. I would also love to see how Win Big(k) and Lose Big(k) work with ties and the results that come from them. Also looking at using Pascal's Triangle not diagonally but vertically or horizontally and observing whether or not there are other mathematical connections we can make.

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