1987

Harrodian Instability Notes

Anwar Shaikh PhD

$$\dot{e} = -hAe - h(1+i)d + h(1+i)\gamma \rightarrow A\bar{e} = (1+i)(\gamma - \bar{d})$$

$$\dot{d} = ae - cd - bed - \gamma$$

$$\dot{d} = e(a - bd) - cd - \gamma$$

$$\dot{d} = ae - (c + be)d - \gamma$$

$$\dot{d} = (ae - \gamma) - (c + be)d$$

let $ae - \gamma = z$, $\dot{z} = a\dot{e}$

$$\dot{d} = z - (c + \tfrac{b}{a}z + \tfrac{b}{a}\gamma)d$$

$$\boxed{\dot{d} = z - \tfrac{b}{a}zd - (c + \tfrac{b}{a}\gamma)d}$$

$$\dot{z} = a\dot{e} = -hAae - h(1+i)ad + h(1+i)a\gamma$$

$$= -hAz - hA\gamma - h(1+i)ad + h(1+i)a\gamma$$

$$= -h[Az] - h(1+i)ad + h[(1+i)a\gamma - A\gamma]$$

$$= -hAz - h(1+i)ad + h\gamma$$

since $[(1+i)a - A] = (1+i)(1+\mu m) - i + (1+i)\mu m = 1$

Suppose $y \equiv (1+i)ad - \gamma \rightarrow \dot{z} = -haz - hy$

$$\dot{y} = (1+i)a\dot{d} = (1+i)az - \tfrac{b}{a}z(1+i)ad - (c + \tfrac{b}{a}\gamma)(1+i)ad$$

$$\dot{y} = (1+i)az - \tfrac{b}{a}zy - (c + \tfrac{b}{a}\gamma)y - \tfrac{b}{a}z\gamma - (c + \tfrac{b}{a}\gamma)\gamma$$

I If we abstract from desired money reserves (& from govt), then

1. $e = a_c - s'$; $s' = s - a_f$ 2. $\dot{D}_B = E$ 3. $\dot{p}^+ = m I_c$

Hence if normal debt level $= D_B^N = d_B^N \cdot p^+$, $\dot{D}_B^N = d_B^N \dot{p}^+ = d_B^N m I_c$

so $\boxed{\dot{D}_B - \dot{D}_B^N = E - d_B^N m I_c}$ → if $\dot{D}_B = \dot{D}_B^N$, $e = m a_c$. But from 1

$e = d_B^N m a_c = d_B^N m e + d_B^N m s'$ → $\bar{e} = 0$ → if $d_B^N = 0$

→ if $I_c = 0$

Last

$$a_i = \boxed{\dot{e} = -h(Ae + Bd_B - C)}$$

$A = (i + c(1+i)\mu m)$

$B = c(1+i)$

$C = c(1+i)(\beta g - \mu m s') = c(1+i)\gamma$

$$\dot{e} = 0 \text{ if } \boxed{\begin{aligned} Bd_B &= C - Ae \\ Ae &= C - Bd_B \end{aligned}}$$

$$\boxed{\dot{d}_B = ae - bed_B - cd_B - \gamma}$$

$a = c(1+\mu m)$

$b = m - i(1+\mu m)$

$c = i\gamma + ms'$

$\dot{d}_B = 0,\quad d_B(c + be) = \cancel{ae} - \gamma$

$$\dot{d}_B = a\frac{C}{A} - a\frac{B}{A}d_B - b\left(\frac{C}{A} - \frac{B}{A}d_B\right)d_B - cd_B - \gamma = 0$$

$$\frac{a\,C/A}{a\,B/A} - d_B - \frac{b}{a}\left(\frac{C/A}{B/A} - d_B\right)d_B - \frac{c}{a\,B/A}d_B - \gamma = 0$$

$$(C/B - d_B) - b/a\,(C/B - d_B)d_B - \frac{Ac}{aB}d_B - \frac{A\gamma}{aB} = 0$$

$\frac{C}{B} = \frac{c(1+i)\gamma}{c(1+i)} = \gamma$

let $C/B - d_B = z \rightarrow d_B = C/B - z$

$$z - b/a\,z(C/B - z) - \frac{Ac}{aB}(C/B - z) - \frac{\gamma A}{aB} = 0$$

$$z - \frac{b}{a}\frac{C}{B}z + \frac{b}{a}z^2 - \frac{Ac}{aB}\frac{C}{B} + \frac{Ac}{aB}z - \frac{\gamma A}{aB} = 0$$

$\cancel{\left(1 - \frac{b}{a}\frac{C}{B}\right)z + b}$

$X \neq 0$

$$\dot{e} = -h\left[Ae + (1+i) - d_{i} - (1+i)\gamma\right] \rightarrow A\bar{e} = (1+i)[\gamma - \bar{d}]$$

$$(\bar{d} - \gamma) = -\frac{A}{1+i}e$$

$$\dot{d} = 1\ ae - bed - cd - \gamma \rightarrow \bar{d}(c+be) = a\bar{e} - \gamma$$

$$\bar{d} = \frac{ae - \gamma}{c+be}$$

$$(\bar{d} - \gamma) = \frac{ae - \gamma}{c+be} - \gamma$$

$$-\left(\frac{A}{1+i}\right)e = \frac{ae-\gamma}{c+be} - \gamma$$

$$= \frac{a}{b}\frac{(be-\gamma)}{(be+c)} = a/b\,\frac{(be+c-(\gamma+c))}{(be+c)}$$

$$\gamma - \left(\frac{A}{1+i}\right)e = \frac{ae-\gamma}{c+be}$$

Note that $A = i + (1+i)\mu m = i(1+\mu m) + \mu m$

$= (1+i)(1+\mu m) - 1$

$= (1+i)a - 1$

$\rightarrow (1+i)a = 1+A, \quad i(1+\mu m) = A - \mu m$

$$\gamma(1+i) - Ae = \frac{(1+i)ae - \gamma(1+i)}{c+be}$$

$b = m - i(1+\mu m) = m - A + \mu m$

$b = m(1+\mu) - A = m' - A$

$$(\gamma(1+i) - Ae)(c + (m'-A)e) = (1+A)e - \gamma(1+i)$$

$$\gamma(1+i)c - Ace + (m'-A)\gamma(1+i)e - \underbrace{A(m'-A)e^2} = (1+A)e - \gamma(1+i)$$

$$A(m'-A)e^2 + [(1+A) - (m'-A)\gamma(1+i) + Ac]e - \gamma(1+i)[1+c] = 0$$

$$m' - A = m + \mu m - i(1+\mu m) - \mu m = m - i - i\mu m = m(1-i\mu) - i$$

$$A[m(1-i\mu) - i]e^2 + [(1+i)(1+\mu m) - (m - i(1+\mu m))(1+i)\gamma + (i(1+\mu m) + \mu m)(i\gamma + ms')]e$$

$$- \gamma(1+i)(1+c) =$$

$$[\ \ ]e' + \{(1+i)(1+\mu m) - m(1+i)\gamma + i(1+i)(1+\mu m)\gamma + i^2(1+\mu m)\gamma + i\mu m\gamma +$$

$$i(1+\mu m)\gamma + i^2(1+\mu m)\gamma \quad ms'[i(1+\mu m) + \mu m]\}e$$

$$-\gamma(1+i)(1+c) =$$

$$A' = \frac{A}{1+i}$$

$$\frac{ae - \gamma}{be + c} = \gamma - A'e$$

$$ae - \gamma = (\gamma - A'e)(be+c) = \frac{A'}{b}\left(\gamma\frac{b}{A'} - be\right)(be+c)$$

$$ae - \gamma = \frac{A'}{b}\left[\left(\gamma\frac{b}{A'} + c\right) - (be+c)\right](be+c)$$

$$\frac{a}{b}(be+c) - (\gamma+c)\Big] =$$

$$\frac{a}{b}(e' - (\gamma+c)) = \frac{A'}{b}\left[\left(\gamma\frac{b}{A'} + c\right) - e'\right](e') \rightarrow \frac{a}{A'}(e' - \gamma') = \left[\left(\gamma\frac{b}{A'} + c\right) - e'\right]e'$$

let $e' = be + c$

$$e'^2 + \left[-\gamma\frac{b}{A'} - c + \frac{a}{A'}\right]e' - \frac{a}{A'}\gamma' = 0$$

$\gamma \equiv \beta g - \mu m s'$

$$\dot{e} = h\left(e - (1+i)\frac{D_B + \dot{D}_B}{p}\right) = -h(ie + (1+i)z)$$

$$z \equiv e - \frac{D_B + \dot{D}_B}{p}$$

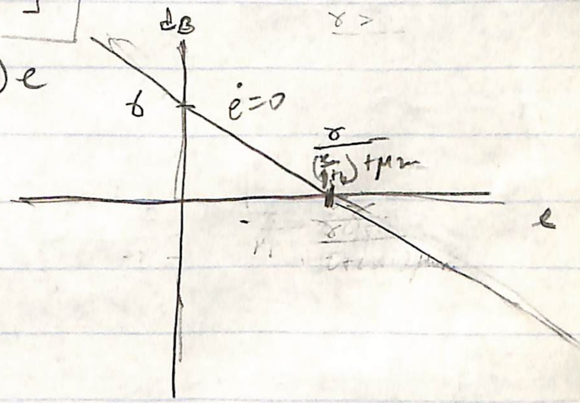
1. $D_B/p = (1+\mu m)e - \gamma$

$$\dot{e} = h\left(e - (1+i)d_B - (1+i)\frac{\dot{D}_B}{p}\right) = h\left[e - (1+i)d_B - (1+i)(1+\mu m)e + (1+i)\gamma\right]$$

$$\boxed{\dot{e} = h\left[\overbrace{(1 - (1+i)(1+\mu m))}^{-(1+i)\mu m - i}e - (1+i)d_B + (1+i)\gamma\right]}$$

$\dot{e} = 0 \rightarrow (1+i)\bar{d}_B = (1+i)\gamma - (i + (1+i)\mu m)e$

$\bar{d}_B = \gamma - \left(\frac{i}{1+i} + \mu m\right)e$



2. $\dot{d}_B = \frac{\dot{D}_B}{p} - d_B \dot{P}/p = \frac{\dot{D}_B}{\gamma} - d_B m i a c + i d_B \frac{\dot{D}_B}{p}$

$= \frac{\dot{D}_B}{p}(1 + i d_B) - d_B m e - d_B m g'$

$= [(1+\mu m)e - \gamma](1 + i d_B) - d_B m e - d_B m s'$

$= (1+\mu m)e - \gamma + (1+\mu m)i e d_B - i\gamma d_B - d_B m e - d_B m s'$

$\dot{d}_B = (1+\mu m)e - [m - (1+\mu m)i]e d_B - (i\gamma + m s')d_B - \gamma$ ✓

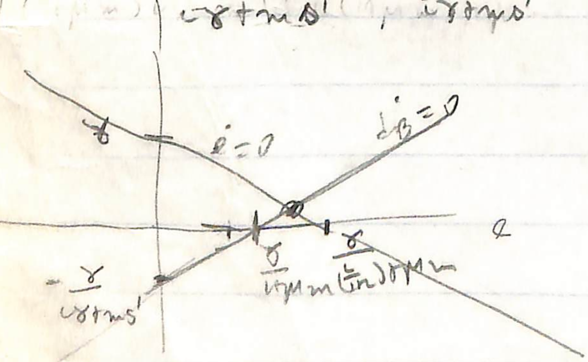
$\dot{d}_B = 0 \rightarrow \bar{e}\,[(1+\mu m) - (m - i(1+\mu m))\bar{d}_B] = (i\gamma + m s')\bar{d}_B + \gamma$

or $\bar{d}_B[(m - (1+\mu m)i)e + i\gamma + m s'] = (1+\mu m)\bar{e} - \gamma$

$$\bar{d}_B = \frac{(1+\mu m)e - \gamma}{(m - (1+\mu m)i)e + (i\gamma + m s')}$$

if $m = (1+\mu m)i$ and $\gamma > 0$
$\rightarrow m - i = i\mu m$

$$\bar{d}_B = \frac{(1+\mu m)\bar{e}}{i\gamma + m s'} - \frac{\gamma}{i\gamma + m s'}$$



if $m - i > i\mu m$, combining $\dot{e} = 0$ & $\dot{d}_B = 0$

$$\bar{d}_B = \frac{(1+\mu m)e - \gamma}{(m - (1+\mu m)i)e + (i\gamma + m s')} = \gamma - \left(\frac{i}{1+i} + \mu m\right)e$$

$(1+i)(1+\mu m)e - \gamma(1+i) = [(i + (1+i)\mu m)e - \gamma(1+i)] \cdot [m - i(1+\mu m)]e + (i\gamma + m s')$

$(1+i)(1+\mu m)e - \gamma(1+i) = (i + (1+i)\mu m)(m - i(1+\mu m))e^2 - \gamma(1+i)(i\gamma + m s')$

$$A_1 = \frac{(1+i)\beta g + i\delta}{1+i} \qquad B_1 = \frac{(1+i)\mu m + v}{1+i}$$

$$\dot{a}_c = 0 \rightarrow d_B = A_1 - B_1 a_c$$

$$\dot{d}_B = 0 \rightarrow A_2 a_c - B_2 d_B + C_2 a_c d_B - D = 0$$

$$A_2 a_c - B_2 A_1 + B_1 B_2 a_c + C_2 a_c (A_1 - B_1 a_c) - D = 0$$

$$A_2 a_c + B_1 B_2 a_c + C_2 A_1 a_c - B_2 A_1 - D + C_2 B_1 a_c^2 = 0$$

$$C_2 B_1 a_c^2 + (A_2 + B_1 B_2 + C_2 A_1) a_c - (B_2 A_1 + D) = 0$$

$$\bar{a}_c = \frac{-(A_2 + B_1 B_2 + C_2 A_1) \pm \sqrt{(A_2 + B_1 B_2 + C_2 A_1)^2 - 4(C_2 B_1)[-(B_2 A_1 + D)]}}{2 C_2 B_1}$$

$$= \frac{-(A_2 + B_1 B_2 + C_2 A_1) \pm \sqrt{(A_2 + B_1 B_2 + C_2 A_1)^2 + 4(C_2 B_1)(B_2 A_1 + D)}}{2 C_2 B_1}$$

$(x+y)^2 \geq ?$

$$x^2 - bx - c = 0$$

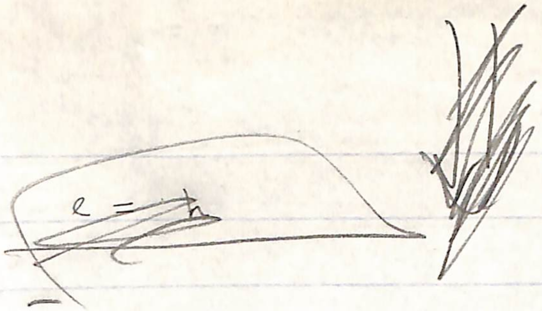
$$x^2 + bx - c = 0$$

$$(x + \alpha)(x + \beta) = 0$$

$$\alpha + \beta = b, \quad \alpha\beta = -c \qquad \alpha\beta = -\frac{a}{A'}(\gamma + c)$$

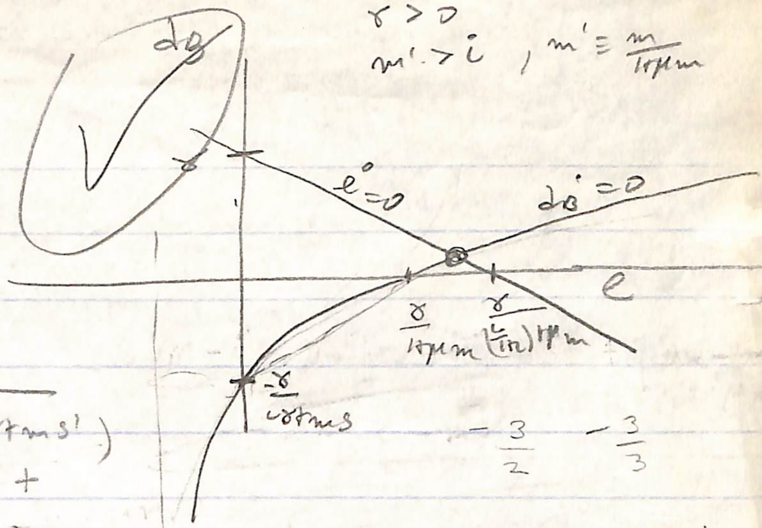
$$\alpha + \beta = \frac{a}{A'} - c - \gamma\frac{b}{A'}$$

$$\alpha\beta + \beta^2 = \beta(\quad) =$$



$\gamma > 0$

$m' > i \;,\; m' \equiv \frac{m}{1+\mu m}$



$$\bar d_B = \frac{(1+\mu m)e - \gamma}{\underset{+}{[m - i(1+\mu m)]e} + \underset{+}{(i\gamma + ms')}}$$

at $e = \frac{\gamma}{1+\mu m}$, $m - i(1+\mu m)e + i\gamma + ms' = [m - i(1+\mu m)]\frac{\gamma}{1+\mu m} + i\gamma + ms'$

$= \left(\frac{m}{1+\mu m}\right)\gamma - i\gamma + i\gamma + ms' = m'\gamma + ms' > 0$ if $s' > 0$

$= \underset{+}{(m'-i)\gamma} + (i\gamma + ms')$

at $e = 0$, $m - i(1+\mu m)e + (i\gamma + ms') = i\gamma + ms' > 0$ if $\gamma > 0, s' > 0$

So for $0 \le e \le \frac{\gamma}{1+\mu m}$ $\; m - i(1+\mu m)e + i\gamma + ms > 0$ and if $m' \ge i$, then this term increases as e increases

$\dot d_B = 0$ & $\dot e = 0$ $\rightarrow$ $\dfrac{(1+\mu m)\bar e - \gamma}{(m-i')\bar e + \gamma'} = \gamma - (\frac{i}{1+i} + \mu m)\bar e$

$$\boxed{\begin{array}{l}\gamma' = i\gamma + ms' \\ i' = i(1+\mu m)\end{array}}$$

$(1+\mu m)e - \gamma = (m-i')\gamma e + \gamma(\gamma') - (m-i')(\frac{i}{1+i}+\mu m)e^2 - \gamma(\frac{i}{1+i}+\mu m)e$

$(m-i')(\frac{i}{1+i}+\mu m)e^2 + \{(1+\mu m) - [(m-i') + (\frac{i}{1+i}+\mu m)]\gamma\}e - \gamma(1+\gamma') = 0$

$ABe^2 + [C - (A+B)\gamma]e - \gamma(1+\gamma') = 0$

$$\bar e = \frac{-[C-(A+B)\gamma] \pm \sqrt{(C-(A+B)\gamma)^2 + 4AB\gamma(1+\gamma')}}{2AB}$$

$$= \frac{[(A+B)\gamma - C] \pm \sqrt{C^2 - 2C(A+B)\gamma + (A+B)^2\gamma^2 + 4AB\gamma(1+\gamma')}}{2AB}$$

$e = a_c - \delta'$

$g_{p_t} = \dot{p}^t / p_t = m a_c \qquad \dot{p}/p = m a_c - i D_B / p$

$a_c = h(e$

$\dot{D_B}/p = a_c + a_f - \delta + \mu m a_c -$

$z = \frac{D_B + \dot{D_B}}{p} =$

$$\dot{D_B}/p = \frac{I_c + I_f - S + \mu R^* + BNG}{p}$$

$$= a_c + a_f - \delta + \mu m a_c + (1-\beta)\,\delta$$

$$= (1+\mu m) a_c + [\underbrace{a_f + (1-\beta) g - \delta}_{\gamma}]$$

$$\dot{d_B} = \dot{D_B}/p - d_B\, \dot{p}/p = (1+i d_B)\frac{\dot{D_B}}{p} - d_B m a_c$$

$$= (1+i d_B)(1+\mu m) a_c + \gamma(1+i d_B) - d_B m a_c$$

$$\dot{d_B} = (1+\mu m) a_c + [i(1+\mu m) - m] a_c d_B + \gamma i d_B + \gamma$$

$$\dot{e} = -h\left[\underbrace{i(1+\mu m)+\mu m}\, e + (1+i)\, d_B + \gamma(1+i)\right] = -h\left[Ae + (1+i)d_B - \gamma(1+i)\right]$$

Let $x = Ae - B$, $y = (1+i)d_B - \gamma(1+i) + B \rightarrow \dot{e} = -h[A(x+B) + (1+i)d$

$\dot{x} = \dot{e} = -h($

$$e = -h\left[ie + (1+i)z\right], \quad z \equiv e - \left(\frac{D_B + \dot{D}_B}{P}\right), \quad \frac{\dot{D}_B}{P} = (1+\mu m)e - \gamma$$

$$= e - d_B - z$$

$$\dot{z} = \dot{e} - \dot{d}_B - (1+\mu m)\dot{e} = -\mu m \dot{e} - \dot{d}_B$$

$$\dot{d}_B = \left(\frac{D_B}{P}\right)^{\cdot} = \frac{\dot{D}_B}{P} - d_B \dot{P}/P = \frac{\dot{D}_B}{P} - d_B\left(ma_c - i\frac{\dot{D}_B}{P}\right) = (1+id_B)\frac{\dot{D}_B}{P} - d_B me - d_B m s'$$

$$\dot{d}_B =$$

$$e = a_c - s'$$

$$\dot{a}_c = +h\left[e - (1+i)\left(\frac{D_B + \dot{D}_B}{P}\right)\right], \quad \boxed{\frac{\dot{D}_B}{P}} = \frac{E(1+\mu)R^{\alpha} - M G}{P} = e + \mu m a_c - \beta g$$

$$= \boxed{a_c(1+\mu m) - (\beta g + s')}$$

$$\dot{a}_c = h\left[e - (1+i)d_B - (1+i)e - (1+i)\mu m a_c + (1+i)\beta g\right]$$

$$\dot{a}_c = -h\left[+ie + (1+i)d_B + (1+i)\mu m a_c - (1+i)\beta g\right]$$

$$\dot{a}_c = -h\left[ia_c - is' + (1+i)d_B + (1+i)\mu m a_c - (1+i)\beta g\right]$$

$$\dot{a}_c = -h\left[(i(1+\mu m) + \mu m)a_c + (1+i)d_B - ((1+i)\beta g + is')\right]$$

$$\boxed{\dot{a}_c = 0} \rightarrow \boxed{(1+i)d_B = \left[(1+i)\beta g + is'\right] - (i(1+\mu m) + \mu m)a_c}$$

$$\dot{d}_B = \frac{\dot{D}_B}{P} - d_B \dot{P}/P = \frac{\dot{D}_B}{P}(1 + id_B) - m a_c d_B \cancel{= a_c(1+\mu m)(1+id_B)}$$

$$= a_c(1+\mu m)(1+id_B) - (\beta g + s')(1+id_B) - m a_c d_B$$

$$= a_c(1+\mu m) + i(1+\mu m)a_c d_B - (\beta g + s') - i(\beta g + s')d_B - m a_c d_B$$

$$\dot{d}_B = (1+\mu m)a_c - i(\beta g + s')d_B + \underbrace{(i(1+\mu m) - m)}_{i(1+\mu)+m(\,)}a_c d_B - (\beta g + s')$$

$$\dot{d}_B = 0 \,\&\, \dot{e} = 0 \rightarrow (1+\mu m)a_c - \frac{i}{1+i}(\beta g + s')\left[(1+i)\beta g + is'\right] + \left(\frac{i}{1+i}\right)(\beta g + s')(i(1+\mu m) + \mu m)a_c$$

$-$

$$\gamma \equiv \beta g - \mu m s'$$

$$\boxed{z} = d_B - \gamma = \frac{D_B + \dot{D}_B}{p} - (1+\mu m)e$$

$$\dot{e} = a i = h\left[e - (1+i)\left(\frac{D_B + \dot{D}_B}{p}\right)\right]$$

$$\frac{D_B + \dot{D}_B}{p} = \frac{D_B + E + (\dot{M}R^\alpha - \dot{M}_G)}{p} = d_B + e + \mu m a c - \beta g = d_B + e + \mu m e + \mu m s' - \beta g$$

$$\frac{D_B + \dot{D}_B}{p} = e(1+\mu m) + d_B - (\underbrace{\beta g - \mu m s'}_{\gamma}) = (1+\mu m)e + \underbrace{d_B - \gamma}_{z}$$

let $\boxed{d_B - \gamma = z}$ = excess of actual over "equl" ("normal") debt ← But no normal debt here

$$\boxed{z = d_B - \gamma = \frac{D_B + \dot{D}_B}{p} - (1+\mu m)e} \rightarrow \boxed{\frac{D_B + \dot{D}_B}{p}} = z + (1+\mu m)e$$

$$\boxed{\dot{e}} = h\left[e - (1+i)z - \underset{(1+i+\mu m+\mu m i)}{(1+i)(1+\mu m)}\, e\right] = \boxed{-h\left[((i(1+\mu m)+\mu m)e + (1+i)z\right]}$$

(same form as in p ERR7)

$$\dot{z} = \dot{d}_B = \frac{\dot{D}_B}{p} - d_B\, \dot{p}/p = \frac{\dot{D}_B}{p} - d_B\left(mac - i\frac{\dot{D}_B}{p}\right) = \frac{\dot{D}_B}{p}(1+iz+i\gamma) - (z+\gamma)\, mac$$

$$\dot{z} = \left[(1+\mu m)e - \gamma\right](1+iz+i\gamma) - (z+\gamma)me - (z+\gamma)ms'$$

$$\dot{z} = (1+\mu m)(1+i\gamma)e + (1+\mu m)iez - \gamma(1+i\gamma) - \gamma i z - mez - \gamma me - ms'z - \gamma ms'$$

$$z' = \left[(1+\mu m)(1+i\gamma) + \gamma m\right]e - (\gamma i + ms')z + \left[(1+\mu m)i - m\right]ez - \gamma(1+i\gamma + ms')$$

Stability: Short-Run Model

9/9/86

$$\boxed{\begin{array}{l} \varepsilon = a - 1 \qquad \dot{p}/p = ma \\ B' = rB + \dot{\varepsilon} \\ \dot{a} = k\left[\varepsilon - \frac{(1+i)B}{p}\right] \end{array}}$$

1. Basic System

$$\dot{\varepsilon} = -KC\varepsilon + K(1+C)Z$$

$$\dot{z} = -C\varepsilon - (M-c)z - M\varepsilon z$$

$$J = \begin{bmatrix} -Kc & K(1+c) \\ -c - Mz & -(M-c) - M\varepsilon \end{bmatrix}$$

2. Two Critical points

$$\dot{\varepsilon} = 0 \longrightarrow C\bar{\varepsilon} = (1+c)\bar{z}$$

$$\dot{z} = 0 \longrightarrow +C\bar{\varepsilon} + (M-c)\frac{C}{1+C}\bar{\varepsilon} + M\bar{\varepsilon}\left(\frac{C}{1+C}\right)\bar{\varepsilon} = 0$$

$$\bar{\varepsilon}\left(1 + \frac{M-c}{1+c} + \frac{M\varepsilon}{1+c}\right) = 0$$

$$\bar{\varepsilon}(1+\cancel{c} + M - \cancel{c} + M\varepsilon) = 0$$

$$\bar{\varepsilon}\left[M\varepsilon + (1+M)\right] = 0 \longrightarrow \varepsilon_1 = 0 \;;\; \varepsilon_2 = -\left(\frac{1+M}{M}\right)$$

→ Two critical points

(I) $\boxed{\varepsilon_1 = 0,\ z_1 = 0} \longrightarrow J(\bar{\varepsilon}_1, \bar{z}_1) = A_1 = \begin{bmatrix} -Kc & K(1+c) \\ -c & -(M-c) \end{bmatrix}$

(II) $\boxed{\varepsilon_2 = -\left(\frac{1+M}{M}\right),\ z_2 = -\left(\frac{1+M}{M}\right)\left(\frac{c}{1+c}\right)}$

(Note) that $\varepsilon = a - 1 \rightarrow \varepsilon_2 = -\left(\frac{1+M}{M}\right) = \bar{a}_2 - 1$
$\rightarrow \bar{a}_2 = 1 - \left(\frac{1+M}{M}\right) = \frac{M-1-M}{M} = -\frac{1}{M}$
$\rightarrow \frac{\dot{K}c}{p} = -\frac{Kc}{p} \rightarrow \boxed{\frac{\dot{K}c}{Kc} = -1}$

$$\therefore -c - Mz = -c + \frac{(1+M)}{1+c}c = c\left[\frac{1+M}{1+c} - 1\right] = \frac{c}{1+c}\left[M - c\right]$$

$$-(M-c) - M\varepsilon = -M + c + (1+M) = 1 + c$$

$$\therefore J(\bar{\varepsilon}_2, \bar{z}_2) = \begin{bmatrix} -Kc & K(1+c) \\ \frac{c}{1+c}(M-c) & 1+c \end{bmatrix} \rightarrow TR = -Kc + 1 + c = 1 + c(1-K) \gtrless 0$$

$$Det = -Kc(1+c) - Kc$$

9/9/86

3. Stability

(1) <u>First Critical point</u> : $\bar{E}_1 = 0$, $\bar{Z}_1 = 0$

$$A_1 = \begin{bmatrix} -KC & K(1+C) \\ -C & -(M-C) \end{bmatrix}$$

Thus $M > C$ is a <u>sufficient</u> condition, but <u>not</u> necessary. <u>Generally, need</u> (i) $M > C - KC$ for $TR < 0$ (ii) $M, C > 0$ for $\det A > 0$

$TRA_1 = -KC - (M-C) < 0$, since $M > C$, and $K, C, M > 0$

$\det A_1 = KC(M-C) + KC(1+C) = KC[1+M] > 0$

→ Thus $(0,0)$ is a <u>SINK</u>

→ Stability of $(0,0)$ is in<u>dependent</u> of size of K, M, C, as long as $K, C, M > 0$ and $\boxed{M > C}$ (<u>sufficient</u>)

→ Convergence is: monotonic for $0 < k < k_1$

oscillatory for $k_1 < k \leq k_2$

$\boxed{\Delta} = (TRA)^2 - 4 \det A = [-KC - (M-C)]^2 - 4KC(M+1) = \boxed{[KC + M - C]^2 - 4KC(M+1)}$

$\Delta = (KC)^2 + 2KC(M-C) + (M-C)^2 - 4KC(M+1) = (KC)^2 + 2KC[M - C - 2M - 2] + (M-C)^2$

$\Delta = (KC)^2 - 2KC(M+2+C) + (M+C+2)^2 - (M+C+2)^2 + (M-C)^2$

$\Delta = [KC - (M+2+C)]^2 - [(M+C)^2 + 2(M+C)(2) + 4] + (M-C)^2$

$= [-M - 2 - C(1-K)]^2 - 4(M+C) - 4 \cancel{-M^2} - 2MC \cancel{-C^2} \cancel{+M^2} - 2MC \cancel{+C^2}$

$= [M + 2 + C(1-K)]^2 - 4M - 4C - 4 - 4MC = [\text{—}] - 4M(1+C) - 4(1+C)$

$\Delta = [M + 2 + C(1-K)]^2 - 4(1+C)(1+M)$

FOR $k = 0$, $\Delta = (M-C)^2 > 0$ (see original form). Then, as $k \uparrow$, $\Delta \downarrow$ until $\Delta = 0$ at $k = k_1^*$

As $k \uparrow$, above k_1^* ($k > k_1^*$), $\Delta \downarrow$ until some k^*. Then $\Delta \uparrow$ with k until $\Delta = 0$ at $k = k_2^*$

DOMAIN OF OSCILLATORY CONVERGENCE

1. $\Delta \equiv \left[kc + (M-c)\right]^2 - 4kc(M+1) =$

when $k = 0$, $\Delta_0 = (M-c)^2 > 0$ but Δ_0 is a small number if $M, c << 1$

2. $$\frac{\partial \Delta}{\partial k} = 2\left[kc + (M-c)\right]c - 4c(M+1) = 2c\left[kc + M - c - 2M - 2\right]$$

$$= 2c\left[kc - (M+c+2)\right] =$$

$\therefore \frac{\partial \Delta}{\partial k} \gtrless 0$ as $k \gtrless \frac{(M+c+2)}{c}$

$\therefore \frac{\partial \Delta}{\partial k} = 0 \Longleftrightarrow$ MINIMUM Δ at $k^* \equiv \frac{M+c+2}{c} = 1 + \frac{M+2}{c} > 1$

Note that since $M > c$, so $k^* > 2$. For $c < 1$, $\frac{M+2}{c} > 2$, so that $k^* > 4$.

3. We can also express $\boxed{\Delta = \left[M + 2 + c(1-k)\right]^2 - 4(1+c)(1+M)}$

so that

$$\Delta = 0 \Leftrightarrow k = \Big[\underbrace{1 + \frac{M+2}{c}}\Big] \pm \frac{2}{c}\sqrt{(1+c)(1+M)}$$

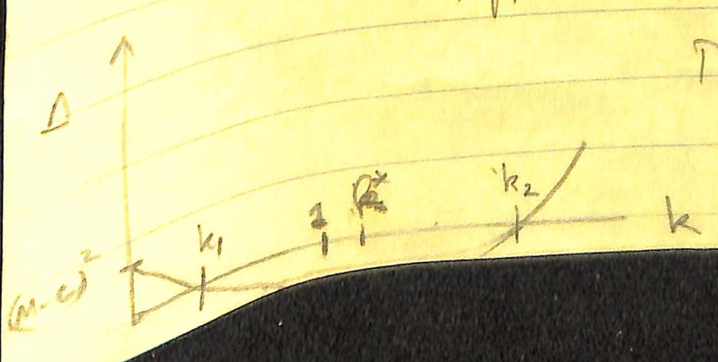
$$k = k^* \pm \frac{2}{c}\sqrt{(1+c)(1+M)}$$

where $k^* \equiv 1 + \frac{M+2}{c} > 1$

$\therefore \rightarrow k_1 = k^* - \frac{2}{c}\sqrt{(1+c)(1+M)} < k^*$; $k_2 = k^* + \frac{2}{c}\sqrt{(1+c)(1+M)}$

4. For $k = 0$, $\Delta = (M-c)^2 \geq 0$. Since $\frac{\partial \Delta}{\partial k} < 0$ for $k < k^*$, and $k^* > \cancel{0}$, as k rises above zero $\Delta \downarrow$ until $k = k^* > 1$. At $k = k_1$, $\Delta = 0$, where $k_1 < k^*$ from above. But $\frac{\partial \Delta}{\partial k} < 0$ even here, since $k_1 < k^*$, so $\Delta \downarrow$ further as $k \uparrow$ until $k = k^*$. After that, $\Delta \uparrow$ until $k = k_2$

Thus for any M, c, the domain of oscillation is all k between k_1 and k_2.



($\rightarrow$ since $k_2 > 4$, the crucial parameter is k_1.
$\rightarrow$ for all $M, c > 0$, $M > c$, calculate k_1.

2B

9/9/86

$$\Delta = \left[M+2+c(1-k)\right]^2 - 4c(1+c)(1+M)$$

$$\therefore \quad \Delta = 0 \longrightarrow M+2+c(1-k^*) = \pm 2\sqrt{c(1+c)(1+M)}$$

$$c(1-k^*) = \frac{-(M+2) \pm 2\sqrt{c(1+c)(1+M)}}{-c}$$

Two values at which $\Delta=0$, of which lower value is $k_1^* = 1 + \frac{M+2 - 2\sqrt{c(1+c)(1+M)}}{c}$ $\longleftarrow$ $\rightarrow$

$$k^* = 1 + \frac{M+2 \pm 2\sqrt{(1+c)c(1+M)}}{c}$$

$1 + 2 \cdot 2 - 2\sqrt{(1-1)(1+2)}$ / -1

~~But $\sqrt{(1+c)c(1+M)} > \sqrt{(1+c)(1+c)} = 1+c$, since $c < M$~~

$$\Delta = 0 \rightarrow k_2^* \equiv 1 + \frac{M+2 + 2\sqrt{c(1+c)(1+M)}}{c} > 1 + \frac{M+2+2(1+c)}{c} = 1 + \frac{M+2c+4}{c}$$

$$= 3 + \frac{M+4}{c} = 45 \quad \text{for } M=.2,\ c=.1$$

$$k_2^* = 1 + \frac{M+2+2\sqrt{c(1+c)(1+M)}}{c} < 1 + \frac{M+2+2(1+M)}{c} = 1 + \frac{3M+4}{c}$$

$$= 47 \text{ for } M=.2,\ c=.1$$

$$\boxed{\underset{(45)}{3 + \frac{M+4}{c}} < k_2^* < \underset{(47)}{1 + \frac{3M+4}{c}}}$$

Note that lower limit is $\frac{(c+2c)+(M+4)}{c}$ and upper is $\frac{(c+2M)+M+4}{c}$

so that if $M, c \ll 1$, $k_2^* \cong \frac{c+M+4+(2c+2M)/2}{c} = \frac{c+M+4+M+c}{c} = \frac{2c+2M+4}{c}$

$$\boxed{k_2^* \cong 2 + \frac{2(M+2)}{c}} = 46 \checkmark$$

Also
$$\Delta = 0 \rightarrow k_1^* \equiv 1 + \frac{(M+2) - 2\sqrt{c(1+c)(1+M)}}{c} > 1 + \frac{M+2-2(1+M)}{c} = 1 - \frac{M}{c} = -\left(\frac{M-c}{c}\right)$$

$$= -1$$

$$k_1^* < 1 + \frac{(M+2)-2(1+c)}{c} = 1 + \frac{M-2c}{c} = -1 + \frac{M}{c} = \left(\frac{M-c}{c}\right) = 1$$

$$\underset{(-1)}{-\left(\frac{M-c}{c}\right)} < k_1^* < \underset{(1)}{\left(\frac{M-c}{c}\right)}$$

This is too weak a set of bounds for lower k^*, since $\frac{M-c}{c}$ is a large number

5/9/86

3 ...

(ii). Second Critical Point: $\bar{E}_1 = -\left(\frac{1+M}{M}\right)$, $\bar{Z}_1 = -\left(\frac{c}{1+c}\right)\left(\frac{1+M}{M}\right)$

$$J(\bar{E}_2, \bar{Z}_2) = A_2 \equiv \begin{bmatrix} -kc & k(1+c) \\ \left(\frac{c}{1+c}\right)(M-c) & 1+c \end{bmatrix}$$

$$\mathrm{TR}\,A_2 = -kc + 1 + c = 1 + c(1-k) \gtrless 0 \text{ as } k \lesseqgtr 1 + \frac{1}{c}$$

$$\mathrm{Det}\,A_2 = -kc(1+c) - k\cancel{(1+c)}\left(\frac{c}{\cancel{1+c}}\right)(M-c) = -kc\left[1+c+M-c\right]$$
$$= -kc\left[1+M\right] < 0$$

Note that since $\mathrm{Det}\,A_2 < 0$ for all $k, c, M > 0$, the discriminant Δ is positive (Hirsch & Smale, p. 96)

$$\Delta \equiv (\mathrm{TR}\,A)^2 - 4\,\mathrm{Det}\,A = \left[1+c(1-k)\right]^2 + 4kc(1+M) > 0$$

It follows that all eigenvalues are real and distinct

$$\lambda_1, \lambda_2 = \tfrac{1}{2}\left(\mathrm{TR}\,A \pm \sqrt{\Delta}\right)$$

Now, $\mathrm{Det}\,A_2 = \lambda_1 \lambda_2 < 0$, so λ_1, λ_2 have opposite signs. This means that the second critical point is a saddle point (Hirsch & Smale, p. 96, since Det < 0; Sanchez, p. 75, Case I, b)

— Thus the second critical point is unstable for all values of k, c, M

$\frac{a}{x^2}$

$< \frac{1}{2}\left[x + \sqrt{x^2 + a}\right] = \frac{1}{2}\left[x + \sqrt{x^2(1+\frac{a}{x^2})}\right] = \frac{1}{2}(x + x\sqrt{\ }$

$= \frac{1}{2}\left(2x \pm x(\sqrt{1+\%} - 1)\right) = x \pm \frac{x}{2}\sqrt{\ }$

$\lambda_1 = \frac{1}{2}\left(\left[1+c(1-k)\right] + \sqrt{\left[1+c(1-k)\right]^2 + 4kc(1+M)}\right) > \frac{1}{2}\left(1+c(1-k) + \sqrt{\left[1+c(1-k)\right]^2}\right)$

$\lambda_1 > 1 + c(1-k) \geq 0$. Similarly, $\lambda_2 = \frac{1}{2}(x - \sqrt{x^2+a}) < \frac{1}{2}(x - x) = 0$.

that for $c = 0$, $\lambda_1 = 1$ and $\lambda_2 = 0$ and $\lambda_1 \uparrow$ as $c \uparrow$, $\lambda_2 \downarrow$ as

9/86

3 ...

(III) Summarizing Stability

- System has two critical points : (0, 0) and $\left(-\frac{1+M}{M}, -\frac{c}{k+c}\frac{1+M}{M}\right)$

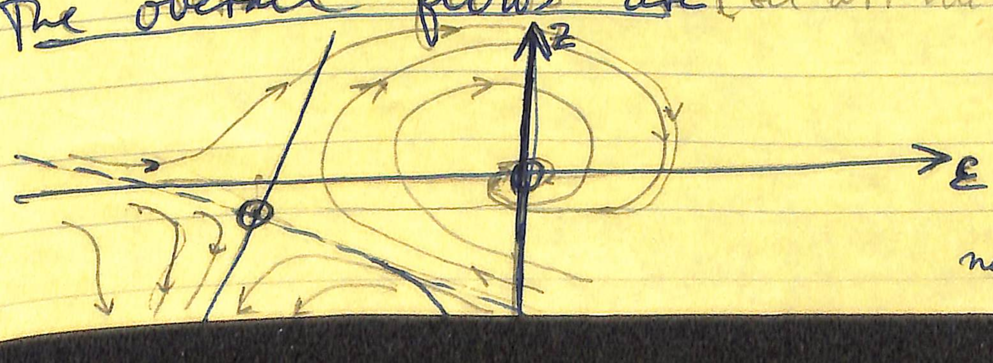
- First critical point is stable for all $k, c, M > 0$, if $M > c$
- Second critical point is unstable for all $k, c, M > 0$
(Moreover, it is economically unsustainable, since
$\varepsilon_2 = -\left(\frac{1+M}{M}\right) \longrightarrow \frac{\dot{kc}}{kc} = -100\% \longrightarrow$ inventory depletion ??)

(IV) The stable critical point is hyperbolic, since $TR\,A_1 < 0$ for all k, c, M, so that A_1 has no zero or purely imaginary eigenvalues [$\lambda_1, \lambda_2 = \frac{1}{2}(TR\,A \pm \sqrt{\Delta})$, and $TR \neq 0$; Guckenheimer and Holmes, p. 13]

- Thus asymptotic behavior of nonlinear solution near (0,0) is determined by stability of the linearization A_1 (G+H, 13)

(V) The unstable critical point is a saddle point.

(VI) The overall flows are (see attached derivation)



Thus any point entering the $z > 0$ space [$\varepsilon > b$ space] is necessarily stable

Phase Diagram : SR Model 9/19/86

1.

$$\dot{\varepsilon} = -KC\varepsilon + K(1+C)Z$$
$$\dot{Z} = -C\varepsilon - (M-C)Z - M\varepsilon Z$$

$$\frac{dZ}{d\varepsilon} = \frac{-C\varepsilon + (1+C)Z - (1+C)Z - (M-C)Z - M\varepsilon Z}{K(-C\varepsilon + (1+C)Z)}$$

$$= \frac{1}{K}\left[1 + \frac{-\cancel{Z} - \cancel{CZ} - MZ + \cancel{CZ} - M\varepsilon Z}{(1+C)Z - C\varepsilon}\right]$$

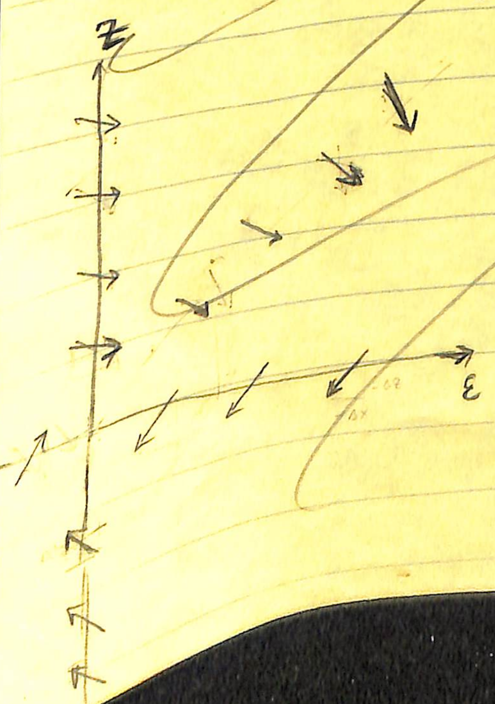
$$= \frac{1}{K}\left[1 - \left\{\frac{(1+M)Z + M\varepsilon Z}{(1+C)Z - C\varepsilon}\right\}\right]$$

$$\boxed{\frac{dZ}{d\varepsilon} = \frac{1}{K}\left[1 - \left(\frac{1+M}{1+C}\right)\left\{\frac{Z + \left(\frac{M}{1+M}\right)\varepsilon Z}{Z - \left(\frac{C}{1+C}\right)\varepsilon}\right\}\right] = \frac{1}{K}\left[1 - \alpha_1 X\right]}$$

where $\alpha_1 = \frac{1+M}{1+C} > 1$ and $\boxed{X \equiv \frac{Z + \left(\frac{M}{1+M}\right)\varepsilon Z}{Z - \left(\frac{C}{1+C}\right)\varepsilon}}$

We will always assume $k > 0$. Then

2. ~~(1) Consider positive quadrant: $\varepsilon \geq 0$, $Z \geq 0$~~



→ For $Z = 0$ (ε-axis), $X = 0$, $\frac{dZ}{d\varepsilon} = \frac{1}{K} > 0$

→ For $\varepsilon = 0$ (Z-axis), $X = 1$, $\frac{dZ}{d\varepsilon} = \frac{1}{K}(1 - \alpha_1) < 0$ since $\alpha_1 > 1$

→ For all $Z > 0$, $\varepsilon > 0$, numerator of X is $> Z$ and denominator is $< Z$, so $X > 1 \rightarrow \alpha_1 X > \alpha_1 > 1$

so that $\frac{dZ}{d\varepsilon} < 0$ (E.G. For $\varepsilon = Z > 0$,

$$X = \frac{\varepsilon\left(1 + \frac{M}{1+M}\varepsilon\right)}{\varepsilon\left(1 - \frac{C}{1+C}\right)} = \frac{1+M+M\varepsilon}{1}\,\frac{1+C}{1+M} = (1+M+M\varepsilon)\frac{1}{\alpha_1}$$

$$\rightarrow \frac{dZ}{d\varepsilon} = \frac{1}{K}(1 - \alpha_1 X) = \frac{1}{K}(1 - 1 - M - M\varepsilon) = -\frac{M(1+\varepsilon)}{K} < 0$$

Phase Diagram: SR Model

9/9/86

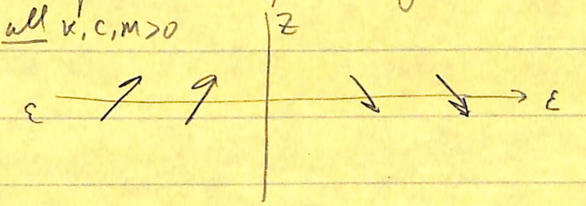
$$\frac{dz}{d\varepsilon} = \frac{1}{k}\left[1 - \frac{(1+M)z + M\varepsilon z}{(1+C)z - C\varepsilon}\right] = \frac{1}{k}\left[1 - \alpha_1 x\right]$$

where $\alpha_1 = \dfrac{1+M}{1+C}$ $\quad x = \dfrac{z + \left(\frac{M}{1+M}\right)\varepsilon z}{z - \left(\frac{C}{1+C}\right)\varepsilon}$, $k > 0$

(i) Along ε-axis, $\underline{\underline{z=0}} \rightarrow \boxed{\dfrac{dz}{d\varepsilon} = \dfrac{1}{k}} > 0$

where $\dot{\varepsilon} = -kC\varepsilon \gtrless 0$ as $\varepsilon \lessgtr 0$ and $\dot{z} = -C\varepsilon$

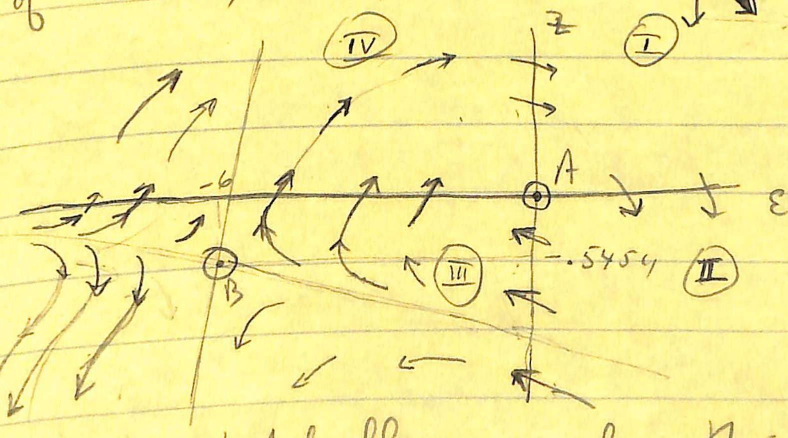
Thus in negative part of ε-axis, $\dot{\varepsilon} > 0$, $\dot{z} > 0$ so arrow point upward, while in positive part of ε-axis, $\dot{\varepsilon} < 0$, $\dot{z} < 0$ so arrow point downward, for all $k, C, M > 0$



(ii) Along z-axis, $\underline{\underline{\varepsilon = 0}} \rightarrow x = 1 \rightarrow \dfrac{dz}{d\varepsilon} = \dfrac{1-\alpha_1}{k} = -\left(\dfrac{M-C}{1+M}\right)\dfrac{1}{k} < 0$

for all $k, C, M > 0$, $M > C$. Also,

$\dot{\varepsilon} = k(1+C)z$ and $\dot{z} = -(M-C)z$, so for positive part of z-axis, $\dot{\varepsilon} > 0$, $\dot{z} < 0 \Rightarrow$ and for negative part

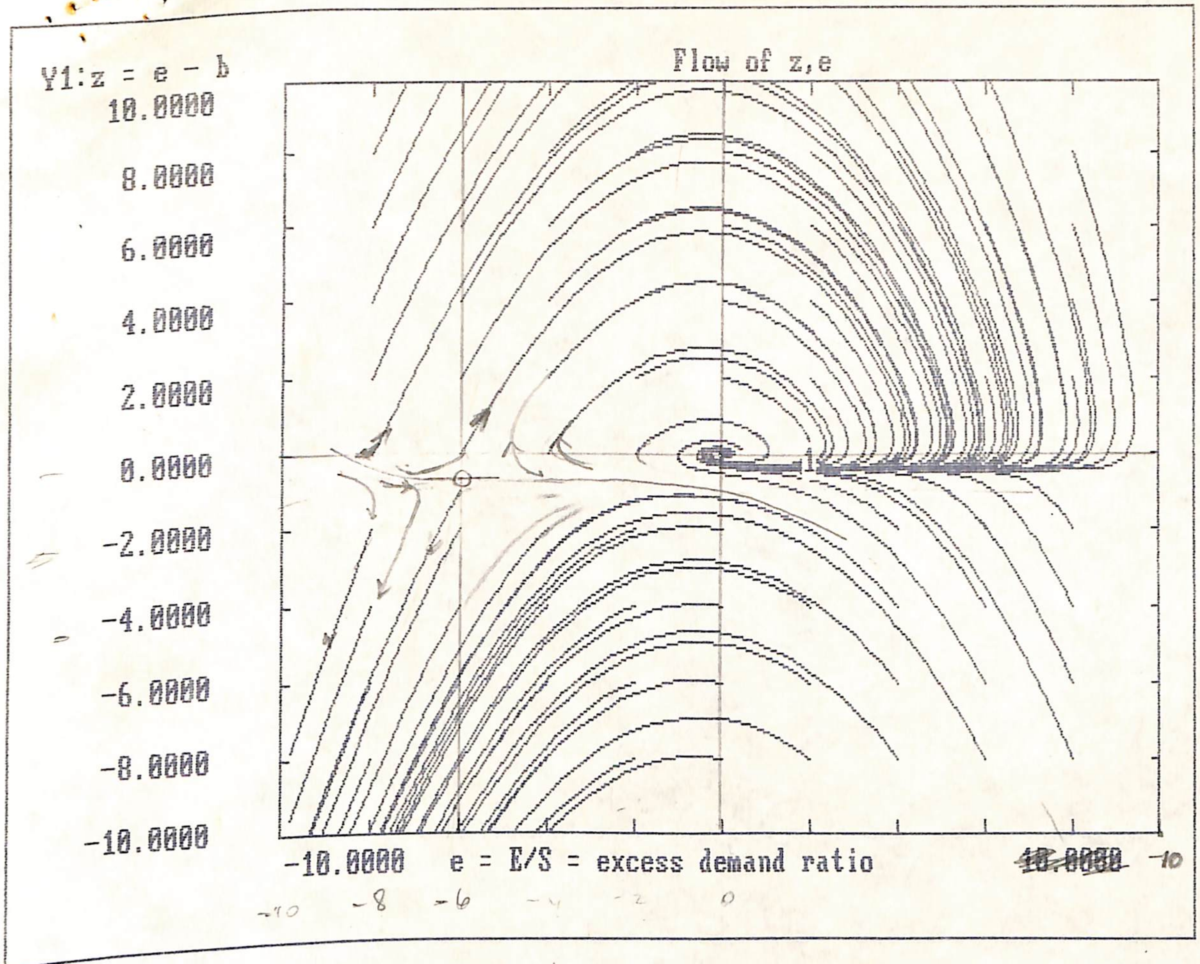


① In region I and IV, slope decreases throughout (prove this)

② Point B is a local saddle point

The simulated flow verifies this

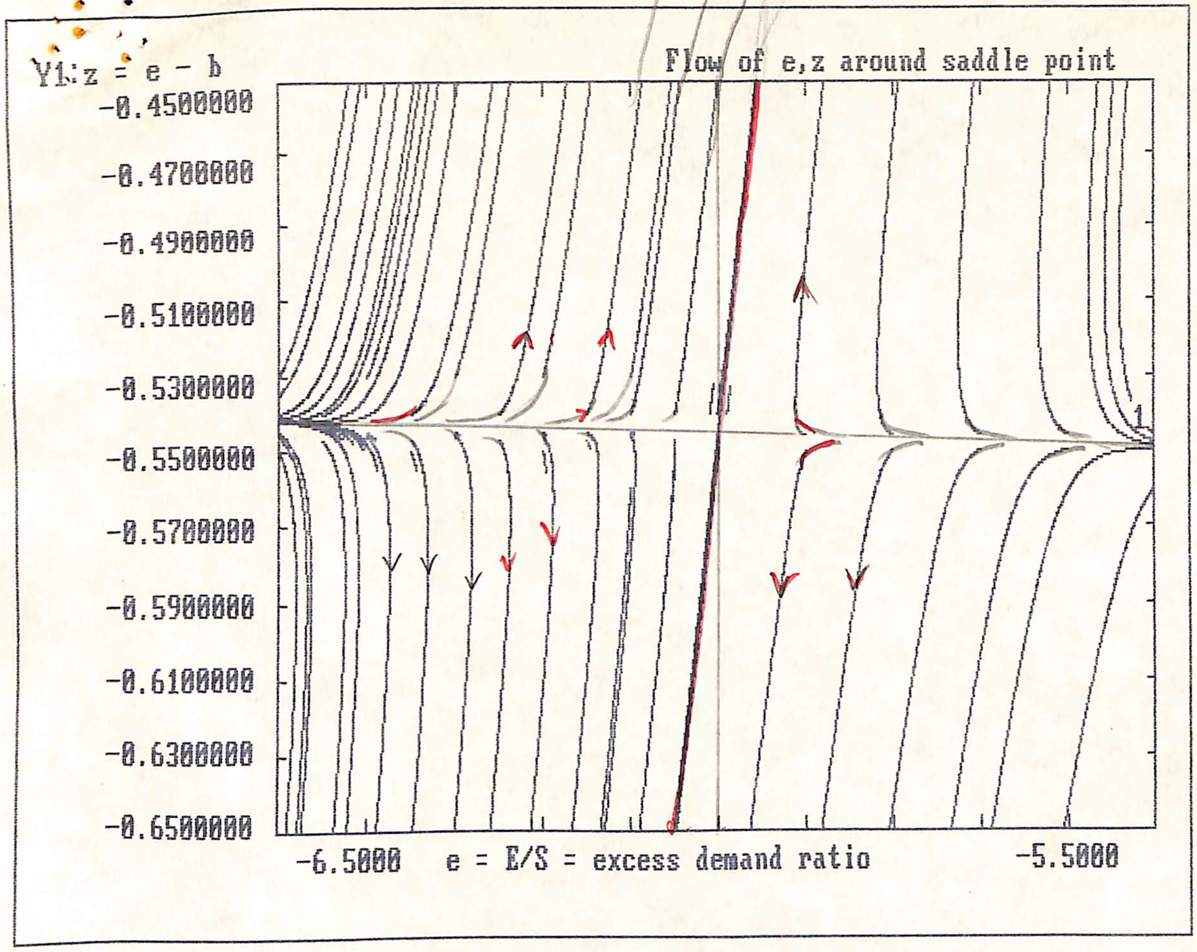
m=.2, c=.1 k=.5



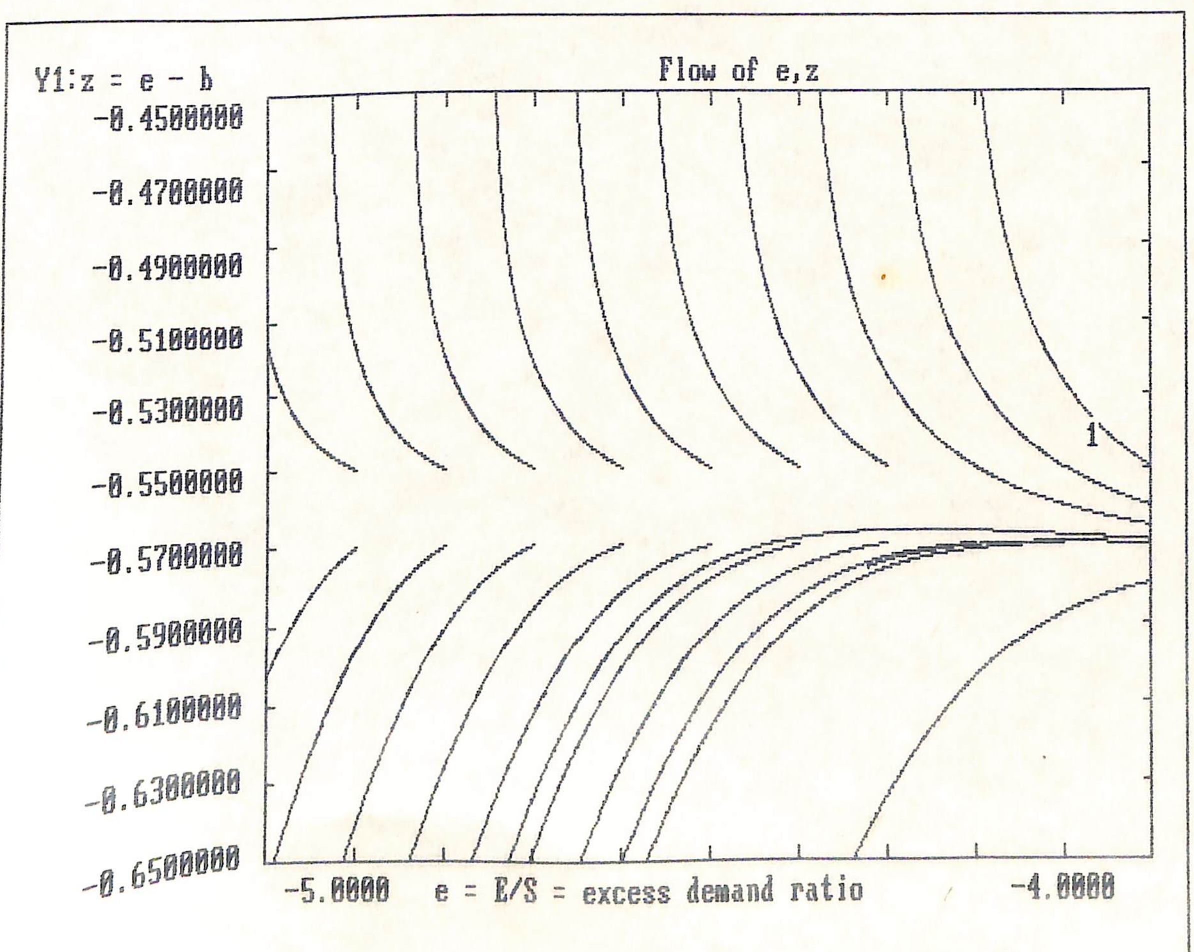
-6, -.5

-6, -.6

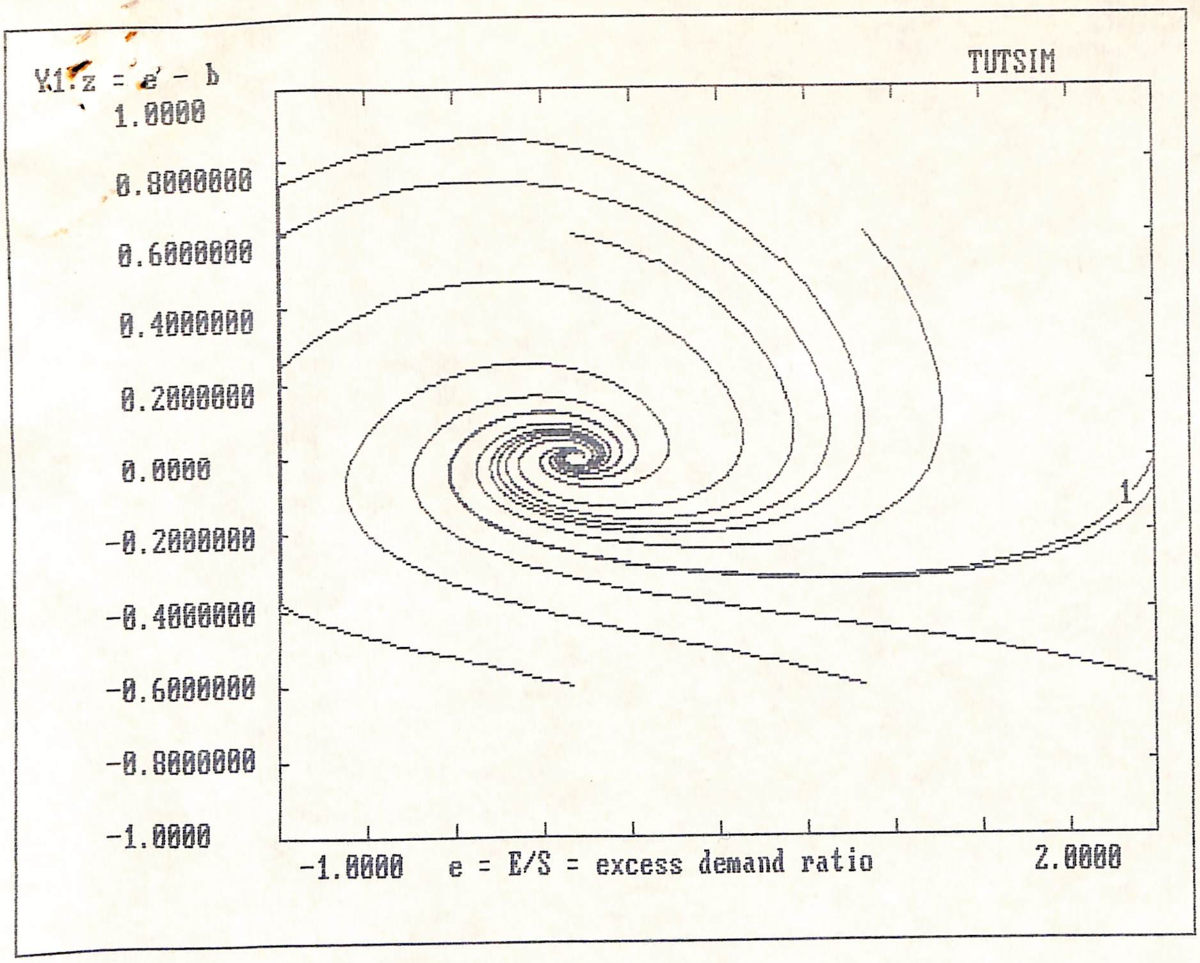
-5.9, -.6



Flow of e,z around saddle point
Y1:z = e - b
-0.4500000
-0.4700000
-0.4900000
-0.5100000
-0.5300000
-0.5500000
-0.5700000
-0.5900000
-0.6100000
-0.6300000
-0.6500000
-6.5000
e = E/S = excess demand ratio
-5.5000



Flow of e,z
Y1:z = e - b
-0.4500000
-0.4700000
-0.4900000
-0.5100000
-0.5300000
-0.5500000
-0.5700000
-0.5900000
-0.6100000
-0.6300000
-0.6500000
-5.0000
e = E/S = excess demand ratio
-4.0000


TUTSIM
Y1:z = e' - b
1.0000
0.8000000
0.6000000
0.4000000
0.2000000
0.0000
-0.2000000
-0.4000000
-0.6000000
-0.8000000
-1.0000
1
-1.0000
e = E/S = excess demand ratio
2.0000

SUMMARY

PER Model: Short-Run Model Stability 5/28/87

I Pure Circulating Capital Model (PER 7)

. Stability condition is $m > i$, and since $m = \frac{P}{K_c} = \frac{\dot{P}}{\dot{K}_c}$

this boils down to requiring that the system be capable of earning a rate of return greater than i

II Circul + Fixed Capital, with $I/P = \theta$ = const in Short-run

Stability now requires $m(1-\theta) > i$, which can be reinterpreted $\dot{\rho}^* = m(1-\theta) \equiv$ equil. ICOR $\equiv \frac{\dot{P}^*}{\dot{K}_c^* + \dot{K}_F} > i$
-(marginal efficiency of invest > i !)

→ boils down to requiring that system be capable in equilibrium of positive incremental profit of enterprise
($\dot{P}^* > i\dot{K}^*$)

→ NOTE THAT ρ^* declines as θ increases → "MEI" declines with invest. share !!

→ Because θ is constant in S.R., by assumption, a positive ~~incremental~~ profit of enterprise implies $\boxed{\theta < \theta^*_m = \frac{m-i}{m}}$

~~Thus~~ the short-run ~~"propensity~~ to invest" θ
is constrained to $\theta < \theta^*$ by profitability requirements

→ ~

III Circul + Fixed Capital + Savings as const. ~~prop~~ of surplus value, in S.R

~~[illegible]~~ Rate of growth of Total ~~invest~~ > i, since $E = 0 \to A + I = \Delta P$, $\dot{A} + \dot{I} = \Delta\dot{P}$

Stability → $m(\delta - \theta) > i$ → interpretation unclear: $\rho^* > i/\delta$
→ restriction on $\delta + \theta$ →

Note that $\rho^* = m\left(\frac{\delta - \theta}{\delta}\right)$, so $\rho^* > i$ [ICOR > i] implies $m(\delta - \theta) > \delta i$

$\frac{\dot{A} + \dot{I}}{A + I} = \frac{\dot{I}_{TOT}}{I_{TOT}} = \frac{\dot{P}}{P}$

Short Run PER with Fixed ~~Capital~~ Invest As Prop of S.V 5/28/87

I Original Circulating Capital Model (PER 7)

1. $E = A - P$
2. $\dot{P} = MA$
3. $\dot{B} = iB + \dot{E}$
4. $\dot{a} = k\left[\frac{E-(1+i)B}{P}\right]$

A sufficient Stability condition is that $M \equiv \frac{P}{K_c} = \frac{\dot{P}}{\dot{K}_c} > i$ with $m, i > 0$

II. Model with Fixed ~~Cap.~~ Invest as Prop. of S.V.: $I/_P = \theta = \text{const}$

A.

1. $E = A + I - P = \boxed{A - P(1-\theta)}$, where $I/_P = \theta = \text{const}$
2. $\dot{P} = MA \longrightarrow \boxed{\dot{P}(1-\theta) = M(1-\theta)\cdot A}$
3. $\dot{B} = iB + \dot{E}$
4. $\dot{a} \equiv \left(\frac{\dot{A}}{P}\right) = k\left[\frac{E-(1+i)B}{P}\right] \rightarrow \left(\frac{\dot{A}}{P(1-\theta)}\right) = k\left[\frac{E-(1+i)B}{P(1-\theta)}\right]$

(1) Thus, if we define $P' = P(1-\theta)$, $a' = A/_{P'}$, $\varepsilon' = \frac{E}{P'}$, $b' = \frac{B}{P'}$ then

1.' $E = A - P'$

2.' $\dot{P}' = M'A$

3.' $\dot{B} = iB + \dot{E}$

4.' $\dot{a}' = k\left[\frac{E-(1+i)B}{P'}\right]$

where $M' \equiv M(1-\theta) \gtrless 0$ $\boxed{\text{as } \theta \lesseqgtr 1}$

Short Run PER with Fixed Invest as Prop. of S.V. 5/28/87

II

(ii). The new model is structurally identical to the original one, with the sole difference being that now

(a) $p' = p(1-\theta)$ and $m' \equiv m(1-\theta)$

are only positive if $\theta < 1 \iff I < P$!

(b) The new (sufficient) stability requirement is that

$\boxed{m' = m(1-\theta) > i}$ at critical point $\varepsilon = 0$, $b = 0$

(c) Note that at critical point $E=0$, $A = P' = P(1-\theta)$, so since surplus value $P > 0$, $A > 0$ is possible only if $\theta < 1 \longrightarrow$ since $A > 0$ is necessary (but not sufficient) for balanced growth, $\theta < 1$ is necessary for balance growth!

Similarly, since $i > 0$, $m' > i$ is only even possible if $m' \equiv m(1-\theta) > 0 \rightarrow$ i.e. if $\theta < 1$.

(d) Finally, since around critical point $E = 0 \rightarrow \overset{*}{A} = \overset{*}{P}(1-\theta)$ $\rightarrow P^* = A^* + I^*$

$\rightarrow 1 - \theta = a^*$, it follow that stability condition is

$m(1-\theta) = ma^* > i$. But $ma^* = \frac{\dot{P}^*}{\dot{K}_c^*}\frac{A^*}{P^*} = \frac{\dot{p}^*}{P^*} = \frac{\dot{p}^*}{A^* + I^*} = \frac{\dot{p}^*}{\dot{K}_c^* \dot{K}_f^*} = \frac{\dot{p}^*}{\dot{K}^*}$

$\therefore ma^* = \frac{\dot{p}^*}{\dot{K}} = p^* \equiv$ Equil. ICOR $\rightarrow$ $\boxed{\text{S.R. Stability REQUIRES } p^* > i}$

5/28/87

II. B

(iii) Summary

a) Short-run model with fixed invest. as a const. prop. of surplus value in short-run is stable if $\rho^* \equiv$ Equil. ICOR $= \frac{\dot{p}^*}{\dot{K}_c + \dot{K}_F} > i$

→ system is stable whenever system is capable of earning a positive incremental profit rate of enterprise in equilibrium.

b) Because $\rho^* = Ma^* = M(1-\theta)$, this further implies that system is capable of earning a positive profit rate of enterprise only if $\boxed{\theta < \frac{m-i}{m}}$ →

→ that $\frac{I}{P} < \frac{P/K_c - i}{P/K_c} = \frac{P - iK_c}{P}$

→ $\boxed{I < P - iK_c}$ → System is capable of earning $\rho^* > i$ only if aggregate planned invest. expenditures are less than anticipated excess of surplus value over interest on circulating capital invested?

c) Note that $\rho^* \equiv m(1-\theta) =$ "MEI" in ~~Keynesian~~ ~~Classical~~ Keynesian sense (since based on produced profit ≠ realized profit) and $\rho^* \downarrow$ as $\theta \uparrow$ ⇒ MEI falls as investment share rises (Keynes? Kalecki?)

Short-Run PER: Circul of Fixed Capital, & Savings 5/28/87

III Model with Fixed Capital & Savings: $I/p = \bar{\theta}$, $\frac{CONR}{P} = \bar{c}$, $s \equiv 1-\bar{c}$

A. 1. $E = A + I + CONR - P = A - P[1-(c+\theta)]$

2. $\dot{P} = mA$

3. $\dot{B} = iB + \dot{E}$

4. $\dot{a} = \left(\frac{\dot{A}}{P}\right) = k\left[\frac{E-(1+i)B}{P}\right]$

(1) If we now define $P'' = P[1-(c+\theta)]$, we get

1''. $E = A - P''$, where $P'' \equiv P[1-(c+\theta)]$

2''. $\dot{P}'' = m''A$, where $m'' \equiv m[1-(c+\theta)]$

3''. $\dot{B} = iB + \dot{E}$

4''. $\left(\frac{\dot{A}}{P''}\right) = k\left[\frac{E-(1+i)B}{P''}\right]$

(11) Once again, this is structurally identical to pure circul. capital model. But now, stability condition is

$m'' \equiv m[1-(c+\theta)] > \hat{i}$ around critical pt. $E=0$, $B=0$

a) Since around $E=0$, $A^* = P''^* = P^*[1-(c+\theta)] \rightarrow \boxed{1-(c+\theta) = \frac{A^*}{P^*} = a^*}$

$\therefore m'' = m[1-(c+\theta)] = ma^* \cong \frac{\dot{P}^*}{\dot{K}_c^*}\,\frac{A^*}{P^*} = \frac{\dot{P}^*}{P^*}$

But now, $E = A + I + CONR - P = 0$ implies $P^* - CONR^* = A^* + I^*$

$\rightarrow P^* = \frac{A^*+I^*}{1-c} = \frac{A^*+I^*}{s} \rightarrow m'' = \frac{\dot{P}^*}{P^*} = \frac{\dot{P}}{\left(\frac{A^*+I^*}{s}\right)} = s\frac{\dot{P}}{\dot{K}} = s\rho^*$

→ incremental equil ROP $= \rho^* > i/s$, for stability !?

S.R. PER: Circul & Fixed Cap + Savings 5/28/87

III (11) a) .–

So first difficulty arises in interpreting stability requirement (that $s\rho^* = m a^* > i$

Note that. $\rho^* \equiv \frac{\dot{P}^*}{K^*} = \frac{\dot{P}^*}{A+I^*} = \frac{\dot{P}^*}{sP^*} = \left(\frac{m}{s}\right) a^* = \left(\frac{m}{s}\right)[s - \theta]$

$\therefore \boxed{\rho^* = m\left[\frac{s-\theta}{s}\right]} \rightarrow \rho^* \downarrow$ as $\theta \uparrow$, $\rho^* \uparrow$ as $s \uparrow$ (or $m \uparrow$)

$\rho^* = m\left[1 - \frac{\theta}{s}\right] = m\left[1 - \frac{I_f}{SAVR}\right]$

→ Equilibrium ICOR is raised by higher "savings propensity" s, and lowered by higher investment propensity" θ.

b) Second difficulty arises with restriction implied on $c+\theta$: $m'' \equiv m[1-(c+\theta)] > i$ implies

$$c + \theta < \frac{m - i}{m}$$

$$CONR^* + I^* < P^* - iK_c^*$$

$$I^* < P^* + R^* - iK_c^* + R - CONR = \overbrace{\left[P^* - (R^* + iK_c^*)\right]}^{\text{Retained Earnings?}} + SAVR$$

→ In Equilibrium, $I^* <$ $\underbrace{\text{"Retained Earnings"} + \text{New Stock Issue}}$

$\boxed{\text{Increase in Fixed Cap.} < \text{Increase in Equity}} \Rightarrow \frac{\Delta Debt}{\Delta Equity} < 0$!?!?

– This basically highlights the difficulty of Keynesian assumption that dividend payout rate R/P and cap. personal savings rate $\frac{SAVR}{PR}$ are all "arbitrarily" constant.

A. Shaikh
Eco. 205
Sp. '86
(Revised)
9/86

HARRODIAN INSTABILITY

1. The Harrodian question can be broken down into three distinct parts

   i) The effect of investment-as-demand on the level of actual output. This is the question of the multiplier in Keynes.

   Given 1) $Z = Y$ (demand=supply, in equilibrium)
   2) $Z = C + I = (1 - s)\,Y + I$ [Harrod assumes $S = sY$, which is equivalent to $C = (1 - s)Y$]

   we get:

   $$1')\quad Z = Y = \frac{I}{s}$$ [equilibrium demand (and supply) are multiples of investment]

   ii) The effect of investment in expanding capacity (potential output). Note that this is not an accelerator effect, since the accelerator is a feedback effect between changes in demand (and hence actual output, Y) and the induced change in potential output (and hence investment)

   The effect of investment in expanding potential output is simply the other aspect of investment - its effect in increasing the stock of capital. In the simplest case, where there is a constant ratio of capital to potential output N, we have:

   3) $K/N = v = \text{constant}$ (behavioral)
   4) $I = \dfrac{dK}{dt}$ (definitional)

   Then, we get $I = dK/dt = v(dN/dt) = (K/N)\,dN/dt$

   therefore, 2') $g_K \equiv I/K = g_N$

   where $g_K = I/K = \dfrac{dK}{dt}\dfrac{1}{K}$ = rate of accumulation of fixed capital

   $g_N = \dfrac{dN}{dt}\dfrac{1}{N}$ = rate of growth of potential output (capacity)

   iii) The response of accumulation to changes in capacity utilization. As we noted earlier, microeconomic behavior implies:

   $$5)\quad \tilde{g}_k \equiv \frac{dg_k}{dt}\frac{1}{g_k} \gtreqless 0 \text{ as } u_t \gtreqless 1$$, where $u = \dfrac{Y}{N} = \dfrac{\text{Actual Output}}{\text{Potential Output}}$ = rate of capacity utilization

   and $u_N = \dfrac{N}{N} = 1$ = normal rate of utiliz

   That is, the rate of accumulation g_k will rise (accumulation will accelerate) when $u > 1$, i.e. when actual utilization levels u_t are above normal, and fall when $u_t < 1$.

HARRODIAN INSTABILITY

Anwar Shaikh
Eco. 205
Sp. '86

2. Taken together, these three components give us the result that, according to basic Keynesian theory, capitalistic balanced growth is inherently unstable (Harrod's Knife Edge):

i) From 1'), we get:

3') $g_Z = g_I$ where g_Z = rate of growth of demand = $\frac{dZ}{dt}\frac{1}{Z}$

g_I = rate of growth of investment = $\frac{dI}{dt}\frac{1}{I}$

ii) From the definition of the rate of accumulation $g_K = I/K$, we ~~get~~ define the rate of change of g_K as $\tilde{g}_K \equiv \frac{dg_K}{dt}\frac{1}{g_K}$, and get

$$\tilde{g}_K = \tilde{I} - \tilde{K} = g_I - g_K \longrightarrow \quad 4')\ g_I = \tilde{g}_K + g_K$$

iii) Combining 3'), 4') and 2'), in that order, we get

$$g_Z = g_I = \tilde{g}_K + g_K = \tilde{g}_K + g_N$$

Thus,

$$g_Z - g_N = \tilde{g}_K$$

Equation 5') tells us that demand (Z) grows faster than potential output when accumulation is accelerating, and the converse when accumulation decelerates.

$$5')\quad g_Z \gtreqless g_N \quad \text{as} \quad \tilde{g}_K \gtreqless 0$$

iv) The Harrodian system can therefore be summarized by 5'). which describes the response of demand and capacity to the acceleration of accumulation, and by 5) from 1 iii) earlier, which describes the response of the acceleration of accumulation to the level of utilization.

5') $g_Z \gtreqless g_N$ as $\tilde{g}_K \gtreqless 0$ (aggregate)

5) $\tilde{g}_K \gtreqless 0$ as $u_t \gtreqless 1$ (micro-behavior)

Suppose initially the system is growing at a steady rate of accumulation g_K.

Then $\tilde{g}_K = 0$ (i.e. the rate of accumulation is not changing), $g_Z = g_N \longrightarrow$

HARRODIAN INSTABILITY

Anwar Shaikh
Eco. 205
Sp. '86

→ demand grows as fast as potential supply, so the level of utilization remains constant. Assume this constant initial level of utilization is the normal level : $u_t^o = 1$, $\tilde{g}_K^o = 0$, $g_Z^o = g_N^o$.

Now suppose that for any reason, the level of capacity utilization u_t rises above normal: $u_t' >$. Then from the microeconomic behavior described in 5), accumulation will accelerate ($\tilde{g}_K' > 0$) as capitalists try to expand capacity (potential supply) in order to bring utilization levels back to normal.

From 5'), however, the acceleration in accumulation ($\tilde{g}_K' > 0$) will result in aggregate demand growing even faster than aggregate capacity (potential) output, because of the multiplier effect.

Thus, capitalists will find that the actual level of utilization u_t will rise even further above normal, which will induce them to try to catch up by accelerating expansion of capacity even more, which will in turn cause aggregate demand to even further outdistance aggregate capacity, etc. The system will explode upward (or downward, in the opposite case) - - it is unstable in the aggregate, around balanced growth.

3. None of the above is particularly altered by introducing variable savings ratios (Kaldor-Pasinetti) and/or variable capital-output ratios (Solow & Neoclassical aggregate production functions). * The knife edge instability remains as long as the multiplier theory is accepted.
   (* See Growth Economics: Selected Readings, Penguin, 1970, edited by A.K. Sen, "Introduction", p.23)

4. Lastly, an alternate route to the same conclusions as above can be traced by combining 1') and 3) and 4) to get the level of utilization explicitly:

i) $$Y = \frac{I}{s} = \frac{dK}{dt}\frac{1}{s} = \left(\frac{1}{K}\frac{dK}{dt}\right)\frac{K}{s} = g_K \frac{K}{s}$$

therefore, $\frac{Y}{N} \equiv u_t$ = the actual level of utilization $= g_K \frac{K}{N}\frac{1}{s} = g_K \frac{v}{s}$

Therefore, at any time t, the actual rate of accumulation is

$$g_{K_t} = \left(\frac{s}{v}\right) u_t$$

6') therefore, $$g_{K_t} = g_K^w u_t$$

where u_t = actual rate of capacity utilization

g_{K_t} = the actual rate of accumulation

g_K^w = Harrodian Warranted rate of accumulation $\equiv \left(\frac{s}{v}\right)$

HARRODIAN INSTABILITY

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Eco. 205
Sp. '86

ii) when $g_{K_t} = g_K^w$ (the actual rate = the warranted rate), then

$u_t = 1$ (actual capacity utilization = normal capacity utilization)

But when $g_{K_t} > g_K^w$, then $u_t > 1$

(since, $u_t = \left(\frac{g_{K_t}}{g_K^w}\right) > 1$, when $g_{K_t} > g_K^w$), and from the micro-behavior described in 5), $\tilde{g}_{K_t} > 0 \longrightarrow$ i.e. g_{K_t} will rise as firms accelerate expansion (and hence accelerate accumulation to try and bring u_t down to normal). But, this will cause u_t to rise even further above 1, from 6') and so on.

0

SUMMARY: Harrod's Contribution & Its Implications for Modern Growth Theory 12/6/84
(A.K. Sen editor, Growth Economics, Penguin, 1970)

I Harrod raised 3 major questions [which recapitulate history of Marxian Pol Econ]

1. Possibility + properties of Normal Capacity Growth
↔ MARX
– Developed into enormous literature: Champernowne, J.R., etc (p.1, notes)

2. Relation of Normal Capacity to Full Employment Growth
↔ BAUER (see HOPE, 1986?)
– Harrod found a "knife-edge" here, since $g_w = s/v$ & $g_N = n + m$ were both fixed → either FE with growing excess capacity or FC with growing unemployment (p.2, not

– Huge literature tried to eliminate this knife-edge via
– variable s: Cambridge Growth Models, π/w changeable (p.3)
– variable v: Neoclassical variable prop. production function (3)
– variable technological change m: Kaldor, Arrow, $y_K\uparrow \rightarrow m\uparrow$ (Economies (4-5)

– All of these suffer from restrictions which need to be imposed
(i) variable s only varies over certain range ($s_w < s < s_\pi$) (p.3)
(ii) variable v, and endogenous technical change requires only "Harrod-Neutral" technical change (p.3-4)
(iii) All assume actual growth is kept at warranted level (hence SAY's law, pre-Keynesian economics) (5)
(iv) Hence not descriptive of actual economy, but explorations of certain properties of full capacity/FE paths (6)

3. Relation of Actual Growth to Normal Capacity Growth ↔ HARROD'INSTABI.
↔ LUXEMBURG

– Once actual investment is not simply assumed to be that required for current full capacity utilization, then all models quickly exhibit Harrod instability (p 7)

– Crucial step is to assume invest. is called forth by expected increase in demand over current capacity level → I_t to fill expected gap
If current capacity utilization $u_t < 1$, then if this feeds back on expected demand & lowers it, $I_t \downarrow$, $u_t \downarrow$ even more, etc –

(*) No modelling of scrapping rate as u_t, $I_t \downarrow$ → (no destruction of real or financial capital value)

12/6/84

Harrod Instability as due to an Independent Invest. Function
(A. Sen *Growth Economics*, Penguin, 1970)

I Modern Growth Theory (1970)

1. "Growth economics ~~—~~ now is an integral part of modern economic theory..." (9)

(i) "This was to a considerable extent the result of an immense practical concern with growth after the Second World War..." (9)

(ii) But even though it had "this immensely practical motivation" it did not "take a fairly practice-oriented shape". Instead, "much of modern growth theory is concerned with rather esoteric issues" and its "link with public polic[y] is often very remote" (10)

* II Harrod — Recapitulates the themes of Marxist Theory (but on the former foundation of Keynes' theory of effective demand)

1. Harrod's "seminal paper" of 1939 raised three major sets of issu[es]
   1. Possibility of steady growth (balanced growth) path (10) [MAR
   2. Instability of this path of steady growth (11) [LUXEMBU
   3. Relation of steady growth path to full employment of labor growth path (determined by pop growth + technical change) (15) [BAU

2. 1 = Possibility of Balanced Growth → Expanded into "an enormous volume of literature including some practical growth plans" (11)
   - Literature studies properties of balanced growth [Joan Robinson, Kahn, Champernowne, Pasinetti] (18-20)

3. 3 = Relation of Steady (Warranted) Growth to Full Employment Path

   (i) Harrod's original concern here was that the warranted

Harrod Instability . . . 12/6/84

II 3.

(i)... rate was <u>fixed</u> by s/v while the full employment growth path was <u>fixed</u> by $n+m$, where n = growth in potential labor force and m = rate of labor-saving technical progress (15).

a) Thus $g_w = s/v \neq g_N = n+m$, except by sheer accident.

b) If $g_w > g_N$, then even assuming actual rate $g = g_w$ at some point the <u>actual</u> growth rate will get stuck at $g = g_N$ because of labor supply constraint, while g_w will be greater → Harrod downward <u>instability</u> of warranted path will ensue because actual rate < warranted (15

<u>Thus full employment</u>, <u>if achieved</u>, <u>would produce growing excess capacity</u> → <u>collapse of accumulation</u>

c) If $g_w < g_N$, then even assuming actual rate $g = g_w$ throughout, <u>unemployment</u> would grow continuously (15)

<u>Thus full capacity</u>, <u>if maintained</u>, <u>would produce growing unemployment</u> → <u>social collapse</u>

(ii) The thrust of the subsequent literature was therefore to focus on <u>this</u> "knife-edge" between full capacity & full employment, by considering various adjustments of $s, v, n, \& m$ which might <u>maintain</u> $g_w = g_N$

12/6/84

II - 3.

(1) - The basic types of solutions proposed for the warranted/natural growth "knife-edge" were

a) Adjustment through a <u>variable savings rate</u> due to shifting wage/profit shares — CAMBRIDGE GROWTH MODELS (16)

— At best, e.g. Pasinetti's contribution, long-run convergence of $g_w \rightarrow g_N$ will occur <u>if the parameters</u> fall within certain ranges. Since s can only vary between $s_w < s < s_\pi$, g_K, v must fall in appropriate ranges.

b) Adjustment through variable capital-output ratio [Neo-Classical Models

At full employment barrier, $w \uparrow \Rightarrow$ "shift to $\boxed{\text{labor}}$ saving techniques" → <u>this "raises the "capital-$\boxed{\text{output}}$ ratio</u>" (21). Converse if $g_w < g_N$ & $w \downarrow$

— Assumes <u>ex ante</u> substitutability (21, ff10) perfect foresight, + no constraints on real wage or interest rate (21)

— <u>Requires all technological progress to be</u> "<u>entirely labour-augmenting</u>" (<u>Harrod-neutr</u> <u>since</u> "<u>other kinds of technical progress are shown to be incompatible with steady growth</u>' " Not even other kinds of 'neutral' technological progress, e.g. 'Hicks neutral' progress, will do, except in the special case of Cobb-Douglas prod. functions where the different types of neutrality coincide": Harrod-neutrality, though required theoretically, has no <u>convincing empirical justification</u> (21)

12/6/84

III . 3 . (ii) . .

b) . . .

- There is "no investment function and investment is $\boxed{\text{assumed}}$ simply to be determined by" "ex ante savings" (22)

- lastly, both real wages and real rate of interest must be completely flexible. Thus no wage rigidity or MINIMUM, & no Keynesian interest rate MINIMUM of liquidity trap. (22)

c) Adjustment through ~~[illegible]~~ rate of technical change (&) [and rate of labor force participation (e.g. women, blacks, youth, immig.)

Kaldor, Arrow, et al have emphasized that the rate of technological progress is endogenous (25)

- This still requires the progress itself to be Harrod-neutral alone (25)

- Arrow points to the growth of knowledge with learning, which depends on the sum of past gross ~~[illegible]~~ investment. ~~Kaldor~~ Kaldor & Mirrlees make learning a function of rate of growth of investment. In both cases, "the genesis of technological progress" is tied to "capital accumulation" itself (25)

- Solow's "VINTAGES" model also gives furth.

importance to capital accumulation, which now acts as "the vehicle of technical progress" through its embodiment in new machines (27)

- lastly, immigration is cited as a way of raising m (26)

4. Finally, on ② = Instability of Warranted (Normal Capacity) Growth Path

(i) Both the discussion of the properties of balanced growth paths (Robinson, et al) and of the relation between full capacity utilization paths and full employment paths simply assume that SAY'S LAW HOLDS: ex ante savings are assumed to determine investment (22, 30)

(ii) The crucial difficulty with these above models of growth is therefore that they completely neglect the relation of potential supply and actual demand in the accumulation process. Investment is assumed to be forthcoming in the necessary proportions to maintain full capacity (warranted) growth.

Thus all adjustments studied are assumed to take place without any departures from full capacity.

(iii) However, in all these models, "once an independent investment function" which assigns "a major role to entrepreneurial expectations about the future" is introduced, "the instability problem of Harrod quickly reappears" (even in Solow-Swan variable proportion models) (23)

(*)

12/5/84

(iv) This means that the growth models cannot be interpreted as descriptive of the actual functioning of capitalist economies, but rather as statements of what might (should) happen.

a) if through "judicious government intervention and planning, ex ante investment and ex ante savings are brought in line with each other" (23)

— "As Swan puts it, 'either the Authorities have read The General Theory or ... they are socialists who don't need to'" (23, ff 14)

b) if we take these models as merely "descriptions of the consequences over time of maintaining full employment, rather than a causal model of what would actually happen ... in a capitalist economy with or without [government] control" (24)

12/6/84

III The Mechanism of Harrod Instability

1. The tracing of the properties of balanced growth paths & their relation to full-employment paths is based on the assumption "that investment is determined entirely by planned savings and there is no independent investment function based on the expectations of the future" (11)

"This is, of course, rather pre-Keynesian in its approach" & does "not make much sense" in a capitalist eco. (2)

2. Consequently, Harrod explicitly considered an investment function in which:

" Expectations of [additional] effective demand determines the level of [current] investment" (23)

— Note that this is a straightforward extension of Keynes & business literature, which measure the viability of potential investments by the stream of expected additional profits, generated by expected additional effective (profitable) demand, & discounted by some rate of return (PV, IRR).

— Note also that this implies that actual investment grows over time only if expected demand grows over time — i.e. only if capitalists expect demand to grow ("animal spirits" of Joan Robinson (p. 237)

12/6/84

3. Harrod considers two questions:

① Under what conditions "will the investor's expectation be realized"? (11)

② "what will happen if they are not realized" (11)

(1) <u>when will investor's expectations be realized?</u>

<u>Only if current level of invest = warranted level I_w</u>

(Hence current rate of accum = warranted rate g_w)

$$\boxed{I_w = s N_t \quad \leftrightarrow \quad g_w = \frac{s}{v}}$$

a) Past period ($t-1$) expectation for current period led them to build up capacity N_t to present level. Thus N_t is an index of ~~past~~ past expectations about current demand

$$\boxed{N_t \cong (D^e_t)_{t-1}}$$

b) For these <u>past</u> expectations to be <u>realized</u>, it is necessary for <u>current</u> level of demand to match current capacity (which in turn embodies past expectations

let ∴ D_{w_t} = ~~current~~ current demand <u>warranted</u> by past expectations

Then $D_{w_t} = N_t$

But since $D_t = \frac{I_t}{s}$ from Keynesian theory, this implies that ~~current~~ current investment must be at a warranted level <u>if</u> expectations are to be realized. Thus

$$\boxed{I_{w_t} = s D_{w_t} = s N_t} = \text{Warranted level of current investment (Sen, p. 227)}$$

∴ $g_{w_t} = \frac{s N_t}{K_t} = \frac{s}{v}$, where $v = \frac{K_t}{N_t}$ = Capital-~~Capacity~~ Capacity Ratio = const.

<u>For expectations to be realized, need</u> ∴ $\boxed{I_{w_t} = s N_t, \quad g_{w_t} = s/v}$

12/6/81

III . 3 ...

(ii) <u>What happens if expectations are not realized</u> ? (<u>i.e. $I_t \neq I_w$</u>

a) By assumption, actual investment seeks <u>solely</u> to match future capacity to expected future demand

$$\therefore \quad \boxed{I_t = v\left(Y^e_{t+1} - N_t\right)}$$

<u>Pure Accelerator</u>
(MYOPIC INVEST.)

b) Thus <u>actual</u> current investment will only coincidentally match current <u>warranted</u> investment.

If $I_t > I_{w_t}$, then $u_t > 1$. This means that current <u>actual</u> demand $D_t = \frac{I_t}{s}$ is <u>greater</u> than previous expectations (as reflected in current capacity N_t). Thus $I_t > I_{w_t}$ means <u>previous expectations</u> were <u>too</u> <u>low</u>.

Now, if this in turn feeds back onto currently held expectations about ~~the~~ future demand then $(D^e_{t+1})_t$ might also be <u>now</u> considered too low, and hence <u>revised</u> upward.

If so, I_t will rise <u>further</u>, making capitalists even more optimistic about the future, etc ... $\longrightarrow$ <u>Harrod instability</u>.

No modelling of scrapping decision and hence of changes in N_t as I_t changes. (Destruction of <u>capital value</u>)

c) Sen claims that "Harrod's instability ~~analysis~~ analysis over-stresses a local problem near equilibrium" (14). He also notes elsewhere that "<u>somewhat surprisingly, irrationality in the form of static expectations seems to make the dynamic path stable</u>" (24-25, ff. 17)

But then $u_t \neq 1$

12/6/84

III . 3. (ii) ...

d) He goes on to note that even the assumption of robustness of investment plans does not really help. Suppose investment is "robust" and actual rate of accum $\cong$ warranted rate. Now suppose s and/or v are changing, so warranted rate itself is changing. Then the stability of the actual rate g_K (because of stability of expectations) will not prevent Harrodian instability.

In fact, insofar as a changing s and/or v is used to justify convergence of g_W and g_N (i.e. full cap. growth + full employment growth), the ~~then robustness of expect~~ g_W must change and robustness of expectations inevitably gives rise to Harrodian instability (229)

III ... 4.

The basic problem arises immediately once we introduce an independent investment function (227)

(i) It is "not very clear" "what kind of an invest. function we should introduce". This "deals with one of the most untraceable elements in a capitalist economy" (227)

(ii) Once an independent assumption "is given a fair play", it is easy to recognize that anything which reduces the 'knife-edge' balance between G_N and G_W, will tend to highlight the 'knife edge' balance between G and G_W" (230, 231)

cat / Baran

A. Shaikh
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Anwar Shaikh
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i) From 1'), we get:

3') $g_Z = g_I$ where g_Z = rate of growth of demand = $\frac{dZ}{dt}\frac{1}{Z}$

g_I = rate of growth of investment = $\frac{dI}{dt}\frac{1}{I}$

ii) From the definition of the rate of accumulation $g_K = I/K$, we ~~set~~ define the rate of change of g_K as $\tilde{g}_K \equiv \frac{dg_K}{dt}\frac{1}{g_K}$, and get

$$\tilde{g}_K = \tilde{I} - \tilde{K} = g_I - g_K \longrightarrow \quad 4')\ g_I = \tilde{g}_K + g_K$$

iii) Combining 3'), 4') and 2'), in that order, we get

$$g_Z = g_I = \tilde{g}_K + g_K = \tilde{g}_K + g_N$$

Thus,

$$g_Z - g_N = \tilde{g}_K$$

Equation 5') tells us that demand (Z) grows faster than potential output when accumulation is accelerating, and the converse when accumulation decelerates.

$$5')\quad g_Z \gtreqless g_N \quad \text{as} \quad \tilde{g}_K \gtreqless 0$$

iv) The Harrodian system can therefore be summarized by 5'). which describes the response of demand and capacity to the acceleration of accumulation, and by 5) from 1 iii) earlier, which describes the response of the acceleration of accumulation to the level of utilization.

$$5')\quad g_Z \gtreqless g_N \quad \text{as} \quad \tilde{g}_K \gtreqless 0 \quad \text{(aggregate)}$$

$$5)\quad \tilde{g}_K \gtreqless 0 \quad \text{as} \quad u_t \gtreqless 1 \quad \text{(micro-behavior)}$$

Suppose initially the system is growing at a steady rate of accumulation g_K.

Then $\tilde{g}_K = 0$ (i.e. the rate of accumulation is not changing), $g_Z = g_N \longrightarrow$

HARRODIAN INSTABILITY

Anwar Shaikh
Eco. 205
Sp. '86

→ demand grows as fast as potential supply, so the level of utilization remains constant. Assume this constant initial level of utilization is the normal level : $u_t^o = 1$, $\tilde{g}_K^o = 0$, $g_Z^o = g_N^o$.

Now suppose that for any reason, the level of capacity utilization u_t rises above normal: $u'_t >$. Then from the microeconomic behavior described in 5), accumulation will accelerate ($\tilde{g}'_K > 0$) as capitalists try to expand capacity (potential supply) in order to bring utilization levels back to normal.

From 5'), however, the acceleration in accumulation ($\tilde{g}'_K > 0$) will result in aggregate demand growing even faster than aggregate capacity (potential) output, because of the multiplier effect.

Thus, capitalists will find that the actual level of utilization u_t will rise even further above normal, which will induce them to try to catch up by accelerating expansion of capacity even more, which will in turn cause aggregate demand to even further outdistance aggregate capacity, etc. The system will explode upward (or downward, in the opposite case) - - it is unstable in the aggregate, around balanced growth.

3. None of the above is particularly altered by intorducing variable savings ratios (Kaldor-Pasinetti) and/or variable capital-output ratios (Solow & Neoclassical aggregate production functions). * The knife edge instability remains as long as the multiplier theory is accepted.
(* See Growth Economics: Selected Readings, Penguin, 1970, edited by A.K. Sen, "Introduction", p.23)

4. Lastly, an alternate route to the same conclusions as above can be traced by combining 1') and 3) and 4) to get the level of utilization explicitly:

i) $$Y = \frac{I}{s} = \frac{dK}{dt}\frac{1}{s} = \left(\frac{1}{K}\frac{dK}{dt}\right)\frac{K}{s} = g_K \frac{K}{s}$$

therefore, $\frac{Y}{N} \equiv u_t$ = the actual level of utilization $= g_K \frac{K}{N}\frac{1}{s} = g_K \frac{v}{s}$

Therefore, at any time t, the actual rate of accumulation is

$$g_{K_t} = \left(\frac{s}{v}\right) u_t$$

6') therefore, $g_{K_t} = g_K^w u_t$

where u_t = actual rate of capacity utilization

g_{K_t} = the actual rate of accumulation

g_K^w = Harrodian Warranted rate of accumulation $\equiv \left(\frac{s}{v}\right)$

HARRODIAN INSTABILITY

Anwar Shaikh
Eco. 205
Sp. '86

ii) when $g_{K_t} = g_K^w$ (the actual rate = the warranted rate), then

$u_t = 1$ (actual capacity utilization = normal capacity utilization)

But when $g_{K_t} > g_K^w$, then $u_t > 1$

$\left(\text{since, } u_t = \left(\frac{g_{K_t}}{g_K^w}\right) > 1, \text{ when } g_{K_t} > g_K^w\right)$, and from the micro-behavior described in 5), $\tilde{g}_{K_t} > 0 \longrightarrow$ i.e. g_{K_t} will rise as firms accelerate expansion (and hence accelerate accumulation to try and bring u_t down to normal). But, this will cause u_t to rise even further above 1, from 6') and so on.