Spring 2023

Spotting k-TriCaps in SPOT IT!

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A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

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Annandale-on-Hudson, New York
May, 2023

Abstract

The card game "SPOT IT!" consists of 55 cards, with 8 symbols appearing on each card. Every pair of cards has exactly one symbol in common, and the goal of the game is to be the first person to find this symbol. An alternate way to play the game is to find sets of 3 cards that have the same symbol in common. We will use combinatorics, probability, and finite projective geometry to analyze the structure of the game. The game "SPOT IT!" can be viewed as the projective plane of order 7. However, we can construct a similar game for any prime number n. The goal of this project is to determine the probability that in a given k-card layout there will be at least one triple. Our results will determine the ideal number of cards to lay out when finding triples in "SPOT IT!" games of order n.

Contents

Dedication

To Andrew Lara, my wittle bother.

A.K.A the other 50% for me.

Congratulations, we made it just like you said we would ;)

Acknowledgments

Dios primero y lo demás que haga fila. Gracias a Dios por llegar al final.

THANK YOU COFFEE AND PANDAS. Y'all make life so sweet.

Thank you Mother. Aside from giving birth to me, you have dedicated your life to making me the strong, independent woman I am today. Has hecho posible lo imposible. Espero darte algún día todo lo que te mereces y lo que el mundo te ha quitado. I love you until Earth's gravitational pull stops pulling.

Thank you Papi Chiki. Thank you for believing in me from the moment you held me to your last day with us. Just like I promised I will cross the stage as if you were here. I hope you are watching from heaven making terrible dad jokes and telling Mom to shut up. I love you, this one's for you!

Thank you Melanie Sickler. Aside from not giving birth to me, you are a home away from home. Thank you for trusting me, seeing my potential and believing in me. You inspire other women to be just as great as you are. I will forever be grateful for your existence.

Thank you Dr.Rose. Thank you for always motivating me. Thank you for teaching me to have fun at math conferences, introducing me to an amazing community of women mathematicians, and giving me the Quad bug. It was a pleasure working with you.

Thank you John Cullinan. You have been a huge support and an amazing adviser. Thank you for all you have done.

Thank you ResLife Gang! and Dorothy Albertini. You guys bent over backwards to get me here. It literally took a whole village. Here's to yelling down the hallway at the office... Shak, Lucas and John would agree.

THANK YOU ANDREW! I am so proud to be your sister, without you there would be no me. To see you grow, learn and thrive is what motivates me. Thank you for being my best friend.

Thank you to my friends, you know who you are, for always reminding me that I be "MATH-ING" and that's awesome. For supporting POC Women in STEM when no one saw us.

1 Introduction

This project is about the math behind a popular card game called SPOT IT! The first chapter is an introduction to the card game SPOT IT!, also known as DOBBLE in the UK. It discusses how the card game was inspired by a word problem Thomas Kirkman created called the Schoolgirl Problem, with a description of the game and what led to the creation of SPOT IT! The card games features and significant qualities are stated, as well as an example of what the cards look like. In addition, we give a brief explanation of Twins, The Tower, The Well, and Hot Potato, which are different games to play using a SPOT IT! deck. However, we note that the most interesting game is Triplets. The goal of this game is to collect triples, i.e. 3 cards that have one symbol in common! We end the chapter with a sneak peak at some of the math behind the card game.

In the second chapter we define finite projective planes of prime order n by stating the axioms, for example the Fano plane. When looking at the lines and points that make up the projective plane order 2, we see similarities between a SPOT IT! deck of order 2 and the Fano plane. This results in axioms of SPOT IT! where points correspond to symbols and lines correspond to cards in a SPOT IT! deck. The rest of the chapter focuses on constructing finite projective planes of prime order n that will later correspond to SPOT IT! Decks of prime order n. We can algebraically find the points and lines that construct a finite projective plane of order n.

Therefore, for every prime order n, a SPOT IT! deck can be created. In this project we will mainly focus on $n = 2$, 3, 5, or 7.

In the third chapter, we will discuss the ways to count and choose cards in a SPOT IT! deck of prime order n. We give an overview on combinatorics formulas and use these to count triples. Using definitions of probability and the various SPOT IT! decks of order n we determine the probability of getting a triple when 3 random cards are chosen from a SPOT IT! deck.

In the final chapter, we focus on trying to find the maximum number of cards that need to be laid out will guarantee at least one triple. To accomplish this we introduce k-TriCap, which are k-card layouts with no triples. We think of k-card layouts as k different slots that a card chosen at random will fill. However, our objective is to never form a triple among the cards as they are randomly chosen. We introduce counting techniques for how to remove cards from the deck while avoiding triples. In the previous chapter we developed combinatorics formulas in order to find all possible ways to choose k cards from a SPOT IT! deck of order $n = 2$, 3, 5, or 7, such that no triples exists. This gives us the number of ways of choosing a TriCap of size k. We also find the probability that a triple is formed when k cards are chosen. This chapter ends with the final results for the number of k cards that need to be laid out to guarantee a triple for a SPOT IT! deck of order n and some suggestions for future research on SPOT IT!

2
SPOT IT!

SPOT IT! is a card game that was created by Blue Orange Games. It is designed to be played with two to eight people. The card game made its first appearance during 1850 in Great Britain. The game focuses on speed and fun. It was initially inspired by a word problem The Reverend Thomas Kirkman created, which he named the "Schoolgirl Problem." He posed the following in a recreational mathematics journal: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast". Here "abreast" means "in a group", this means that a set of 3 girls are walking out while each pair of girls in the set only appear as a pair once. In other words, Kirkman extrapolated this as a question of unrepeated pairs in triplets, asking from a certain number of elements, how many unique sets of triplets can you have before you start repeating pairs? Jacques Cottereau later figured out every possible way to solve the schoolgirl problem. Then he decided to design a "game of insects". This later became what we know as SPOT IT! after being found by Denis Blanchot. Cottereau is Blanchot's sister-in-law's father. SPOT IT! was released initially in France in 2009 under publishers Play Factory, then in Germany in 2010 as "DOBBLE" (a play on word "double"), and eventually released in the U.K. and North America, as SPOT IT!, in 2011 [6].

2.1 What the cards look like

A SPOT IT! deck consists of 55 cards. There are 57 total different symbols and each card has exactly 8 symbols printed on them. It is guaranteed that if you choose any two random cards from a SPOT IT! deck, they will always have a symbol in common. That leads us to question if it is a coincidence that the cards have the same number of symbols on them! Also, if there is any relationship between the numbers 8, 55, 57.

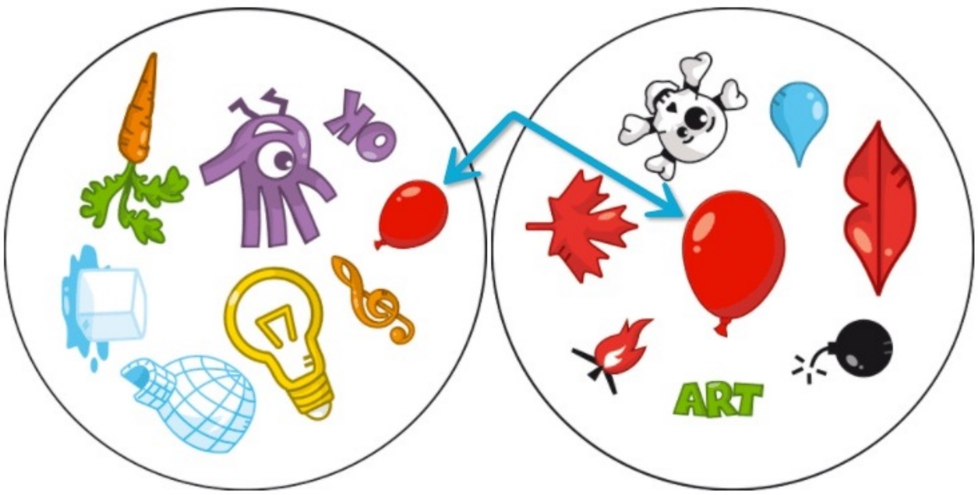

Figure 2.1.1: Example of a matched pair of Spot It! cards

2.2 How to play

There are 5 different games you can play with a SPOT IT! deck. All games require the players to find the card that shares a symbol in common the fastest. Skills that are significant while playing would be visual scanning and visual perception in order to quickly match a symbol on your chosen card to the target card.

In Figure 2.1.1 you can see two cards from a SPOT IT! deck. Notice that each card displays 8 symbols, and there are only 15 distinct pictures displayed on the two cards. The cards have exactly one symbol in common, the red balloon.

In the first version of the game, *Twins*, a player begins by drawing two random cards from the deck, placing them face up on the table. The goal is to examine the cards the fastest and identify the matching symbol on each of the cards. The player who states the symbol first wins that round of the game. The winning player then draws two more cards for the next round.

The second version of the game, *The Tower*, consists of each player being dealt a card faced down. The rest of the cards are faced up in a tower in the center. The players flip their cards over at the same time and try to spot the symbol in common between their card and the top card of the tower. The first to identify the symbol is the winner, they take the center middle card and start their personal pile. The new card gained becomes the new card to spot the symbol in common between that card and the top card of the tower. The processes repeats until there are no cards left in the center and the player with the most cards in their tower wins.

In the third version of the game, *The Well*, one card is placed face-up in the middle. The rest of the cards are dealt face-down to the players. These cards form their personal draw piles. Then all players flip their cards face-up. As the player finds the matching symbol between the first card in their pile and the center card, they are able to place their card on top of the center card if they are the first to identify the matching symbol. This process repeats until one player has no cards. The first player to get rid of all of their cards wins.

The fourth version, *Hot Potato*, consists of each player being dealt a card faced down. The rest of the cards are set aside. At the same time players flip their card over. If they are the first to identify a matching symbol, they proceed to add their card to the top of the other players card. The processes repeats until one player ends up with all the cards. The player with no cards wins.

2.3 Playing *Triplet*

The final way t o play SPOT IT!, *Triplet*, is the most interesting way to play. A player places 9 cards in a 3 by 3 grid. The rest of the cards are face-down in a pile. Then, at the same time, the players try to identify a matching symbol on three cards, to make a matching set. A matching set sharing one symbol in common is known as a Triple. The player who identifies the triple takes the 3 cards and adds them to their personal pile. Then 3 new cards are added to complete the original 9 card lay out. When there are fewer than 9 cards left in the deck and no more sets of matching cards, the game ends and the player with the most cards wins.

The *Triplet* game in SPOT IT! is different from the standard ways of playing the card game. The card game is known for always being able to find a pair of cards that will always have a matching symbol. However, triples may form among 3 cards. The instructions that pertain to playing this game state that 9 cards need to be placed face up before looking for triples. For example, consider Figure 2.3.1. A player can choose from 3 different triples among the 9 cards. Suppose that there are fewer than 9 cards placed down at the beginning of the game, would that still guarantee a Triple? Another question is how many cards will guarantee a Triple? What is the relationship between the symbols on the cards and the number of cards that share a symbol in common? It turns out there is some cool math behind this.

Figure 2.3.1: The Triples formed are circled in red, blue, and yellow

2.4 Math behind SPOT IT!

It turns out that SPOT IT! cards are a model for a finite projective plane of order 7. Since projective planes can be constructed of any prime order n, this means that we can construct are other SPOT IT! decks of prime order n. We will construct these algebraically. The smaller SPOT IT! decks, when n is 2, 3, and 5, will help us analyze the game characteristics.

3
SPOT IT! and PROJECTIVE PLANES

A finite projective plane of prime order n is a geometric structure that extends the concept of a plane. It is defined as a system (or set) that consists of points and lines, where each line is a subset of points that satisfy a set of axioms given below. For example, every two points lie on exactly one line and every two lines intersect at exactly one point. What is interesting about finite projective planes is not only their graphical representation, but how they are algebraically constructed.

3.1 Finite Projective Planes of prime order n

In this section, we take a look at the graphical representation of finite projective planes of prime order n, as well as their components. First we state the axioms.

Definition 3.1.1. [3] <u>Axioms of Finite Projective Planes of prime order n</u>:

1. There are $n + 1$ points on each line.

2. There are $n + 1$ lines that contain given a point.

3. There are a total of $n^2 + n + 1$ distinct lines and points.

4. Any 2 lines intersect in at least 1 point.

5. Any 2 points have at least 1 line in common.

△

3.2 SPOT IT! as a Finite Projective Plane

It turns our that the axioms of finite projective geometry can be applied to a SPOT IT! deck of prime order n. This identifies fundamental properties of a SPOT IT! deck of prime order n. It is significant to notice that the points in a finite projective plane represent symbols, while the lines represent cards in a SPOT IT! deck order n [8].

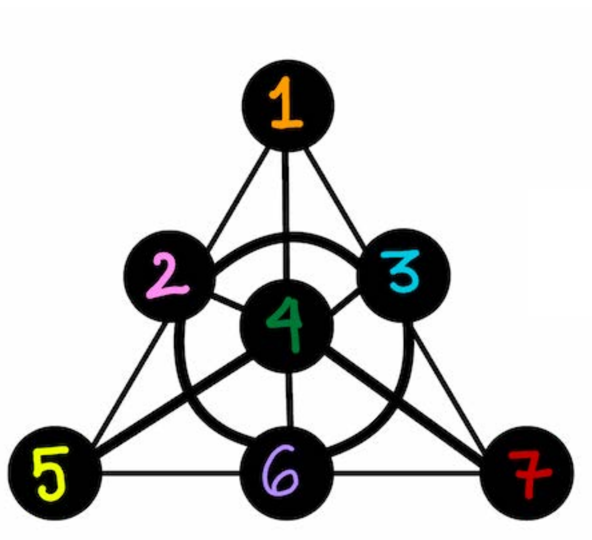

Figure 3.2.1: The Fano PLane

Example 3.2.1. Consider the smallest possible finite projective plane order $n = 2$, the Fano plane, shown in Figure 3.2.1. One each line there are exactly 3 points. The lines:

1. (1, 2, 5)

2. (1, 3, 7)

3. (2, 3, 6)

4. (1, 4, 6)

5. (5, 6, 7)

6. (2, 4, 7)

7. (3, 4, 5)

There are $2^2 + 2 + 1 = 7$ points on the graph, 7 lines, and 3 points on each line. In addition, any 2 points lie on exactly one line. As one can see there are no two lines that have two points in common. Similarly any 2 points have at least 1 line in common, and any two lines intersect in exaclty one point in common.

We can interpret the Fano plane as a SPOT IT! deck of order $n = 2$. Each point on the graph corresponds to a symbol, and each line corresponds to a card. Hence, the Fano Plane can be interpreted as a deck of SPOT IT! cards, seen in Figure 3.2.2. There are exactly $n^2 + n + 1 = 7$ cards in this deck. Notice that each card has exactly $n + 1 = 2 + 1 = 3$ symbols on it, and each symbol shows up on exactly $n + 1 = 2 + 1 = 3$ cards.

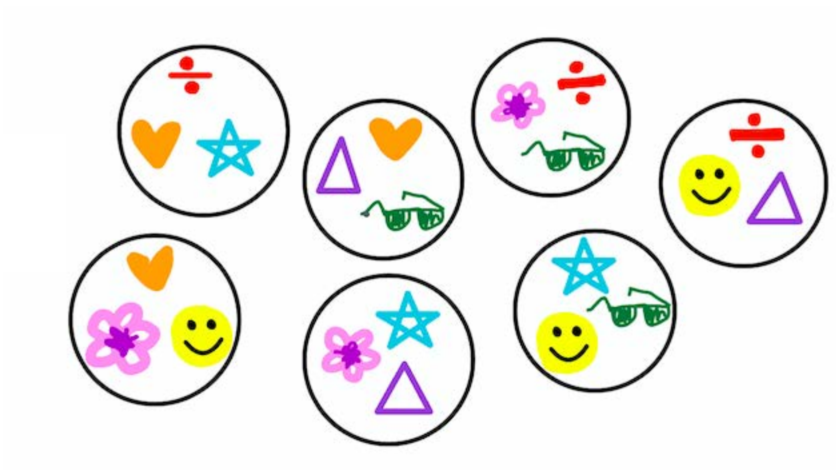

Figure 3.2.2: SPOT IT! deck when n=2

Another example, shown in Figure 3.3.2, is a SPOT IT! deck order $n = 3$. Here it can be seen that are exactly $n^2 + n + 1 = 3^2 + 3 + 1 = 13$ cards in the deck, shown in Figure 3.2.3 13 distinct symbols. Each card has exactly $n + 1 = 3 + 1 = 4$ distinct symbols on each. There are $n + 1 = 3 + 1 = 4$ cards in the deck in which one particular symbol appears. ♢

Proposition 3.2.2. *Properties of SPOT IT! deck of cards of prime order n are as follows:*

1. *There are $n + 1$ symbols on each card*

2. *Each symbol lies on exactly $n + 1$ cards*

Figure 3.2.3: SPOT IT! deck for n=3

3. *There are a total of* $n^2 + n + 1$ *distinct symbols in total and* $n^2 + n + 1$ *cards in a deck*

4. *Any 2 cards share exactly 1 symbol*

5. *Any pair of symbol lies on at least one line*

Proof. Recall that points on a plane correspond to symbols and lines correspond to cards. By Axiom 1, we know that there are $n + 1$ points on each line. Similarly in a SPOT IT! deck of order n, when n is prime, is that there are a total of $n + 1$ symbols on each card.

By Axiom 2 we know there are $n+1$ lines that lie on a particular point. Thus in a SPOT IT! would be deck, each symbol appears in exactly $n + 1$ cards.

By Axiom 3 we know that in general a finite projective plane will have $n^2 + n + 1$ points and $n^2 + n + 1$ lines. Then there are a total of $n^2 + n + 1$ distinct symbols and $n^2 + n + 1$ total cards in a SPOT IT! deck.

By Axiom 4, any 2 lines intersect at one point. This means that any two distinct cards have exactly one symbol in common.

By Axiom 5 any 2 points have at least 1 line in common. Similarly, any pair of symbols lie on at least 2 cards. □

Note that it is a fact if n is a prime power then there exists a finite projective plane of prime order n. Therefore, it is a fact that a SPOT IT! deck of any prime order n can be constructed as well.

3.3 Constructing Finite Projective Planes

Homogeneous coordinates or projective coordinates is a system of coordinates used in projective geometry. In a finite projective plane over $\mathbb{Z}_n$ for n of prime order, points have 3 coordinates (a, b, c), where a, b, c are in $\{0, 1, ..., n-1\}$, but $(0, 0, 0)$ is not included. A point represented by a given set of homogeneous coordinates is unchanged if the coordinates are multiplied by an element of $\mathbb{Z}_n - \{0\}$. The distinct coordinates are the points that lie on at least one line. Give the all the distinct points of a finite projective plane of prime order n, we can find the span of two points to find the unique line which contains these two points. Since points in a finite projective plane are correspond to distinct symbols in a SPOT IT! deck, we can determine how many total symbols and distinct cards make up a SPOT IT! deck order n [7].

Definition 3.3.1. [4] Let $V_n = \mathbb{Z}_n^3 - \{0\}$, all triples of elements of $\mathbb{Z}_n$ except (0,0,0), where n is prime. Let $\overline{V_n}$ denote the set of equivalence classes of V_n △

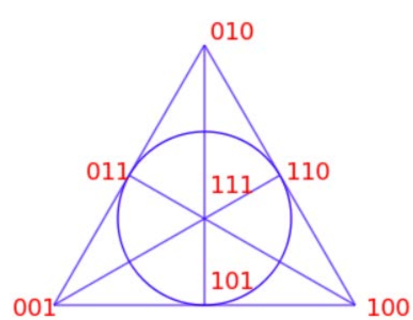

Figure 3.3.1: coordinates of the Fano plane

Definition 3.3.2. We define an equivalence relation on V_n, by $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if $(b_1, b_2, b_3) = c(a_1, a_2, a_3)$, for some $c \in \mathbb{Z}_n - \{0\}$. $\triangle$

Example 3.3.3. Consider V_2. The points have 3 coordinates (a, b, c), where a, b, c can be either 0 or 1. The resulting finite projective plane is known as the Fano Plane, shown in Figure 3.3.1. It has exactly 7 distinct points. Therefore, $\overline{V_2} = \{(0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$. $\diamond$

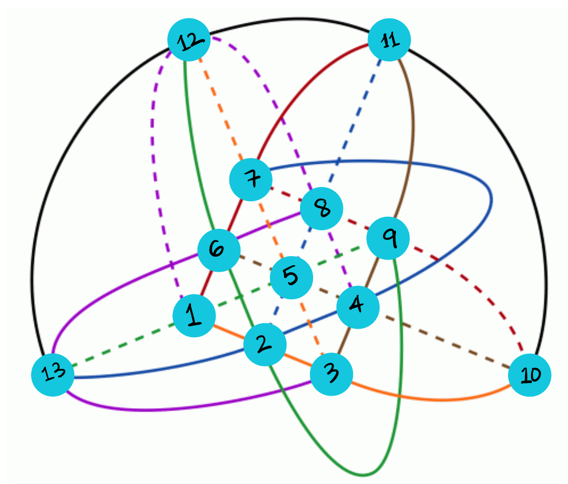

Figure 3.3.2: 13 distinct points of Projective Plane order 3

Example 3.3.4. Consider V_3. The elements of V_3 are of the form (a_1, a_2, a_3) where a_1, a_2, and a_3 are either $0, 1$, or 2. Then, for the first coordinate there are 3 possible choices, for the second coordinate there are 3 choices, and for the third coordinate there are 3 choices. Thus, $3 \cdot 3 \cdot 3 = 27$, but $(0, 0, 0)$ must be removed. Hence there are 26 elements in V_3:

$$\{(0,0,1), (0,0,2), (0,1,0), (0,2,0), (1,0,0), (2,0,0), (0,1,1), (0,2,2), (1,0,1),$$
$$(2,0,2), (1,1,0), (2,2,0), (1,1,1), (2,2,2), (0,1,2), (0,2,1), (1,0,2), (2,0,1),$$
$$(1,2,0), (2,1,0), (1,1,2), (2,2,1), (1,2,1), (2,1,2), (2,1,1), (1,2,2)\}.$$

To make things easier, we will remove the commas in points, so for example $(0, 0, 1) = (001)$. So, consider the element $(001) \in V_3$. We compute $c(001)$ for all $c \in \mathbb{Z}_3 - \{0\}$, so $c = 1$ or

$c = 2$. When $c = 1$, then $c(a_1, a_2, a_3) = 1(001) \mod 3 = (001)$. This means that $(001) \sim (001)$. When $c = 2$, $c(a_1, a_2, a_3) = 2(001) \mod 3 = (002)$. This means that $(001) \sim (002)$. Then, $[001] = \{(001), (002)\}$, is the equivalence class of [001]. Therefore, [001] is an element of $\overline{V_3}$.

Consider another element $(002) \in V_3$, which follows similarly. When $c = 1$, $c(a_1, a_2, a_3) = 1(002) = (002)$. When $c = 2$, $c(a_1, a_2, a_3) = 2(002) \equiv (004)(\mod 3) = (001)$. Thus, $[002] = \{(002), (001)\}$. Note that $[001] = [002]$ since $(001) \sim (002)$.

In general the following are elements are equivalent in V_3:

$(001) \sim (002)$,

$(100) \sim (200)$,

$(111) \sim (222)$,

$(011) \sim (022)$,

$(101) \sim (202)$,

$(120) \sim (210)$,

$(012) \sim (021)$,

$(221) \sim (112)$,

$(010) \sim (020)$,

$(110) \sim (220)$,

$(102) \sim (201)$,

$(122) \sim (211)$,

and $(102) \sim (201)$.

This results in the following 13 equivalence classes and their elements:

$$[001] = \{(001), (002)\}$$

$$[010] = \{(010), (020)\}$$

$$[100] = \{(100), (200)\}$$

$$[011] = \{(011), (022)\}$$

$$[101] = \{(101), (202)\}$$

$$[110] = \{(110), (220)\}$$

$$[111] = \{(111), (222)\}$$

$$[012] = \{(012), (021)\}$$

$$[102] = \{(102), (201)\}$$

$$[120] = \{(120), (210)\}$$

$$[112] = \{(112), (221)\}$$

$$[121] = \{(121), (212)\}$$

$$[211] = \{(211), (122)\}$$

Thus, the distinct elements of $\overline{V_3}$ are the following:

$$\{[1,0,0], [1,0,1], [1,0,2], [1,1,0], [1,1,1], [1,1,2], [1,2,0], [1,2,1], [1,2,2], [0,1,0], [0,1,1], [0,1,2], [0,0,1]\}.$$

$\Diamond$

Example 3.3.5. When $n = 5$, one can follow a similar approach. The elements are in the form (a_1, a_2, a_3) where a_1, a_2, and a_3 lie in $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Then, by definition of the equivalence relation on V_5, $c \in \mathbb{Z}_5 - \{0\}$, can be 0,1,2,3 or 4 as well. One can multiple each element of V_5 by c and remove "duplicates", i.e. the elements that are already represented. The 31 equivalence

1. [100]	2. [130]
3. [101]	4. [131]
5. [102]	6. [132]
7. [103]	8. [133]
9. [104]	10. [134]
11. [110]	12. [140]
13. [111]	14. [141]
15. [112]	16. [142]
17. [113]	18. [143]
19. [114]	20. [144]
21. [120]	22. [010]
23. [121]	24. [011]
25. [122]	26. [012]
27. [123]	28. [013]
29. [124]	30. [014]
31. [001]	

Figure 3.3.3: equivalence classes of V_5

classes of V_5 are shown in Figure 3.3.3, similarly the 31 points on the projective plane order 5 are shown in Figure 3.3.4.

◊

Example 3.3.6. The finite projective plane of order 7 has 57 distinct points as shown in Figure 3.3.5. The equivalence classes of V_7 are shown in Figure 3.3.6.

◊

3.4 Lines in Finite Projective Planes

Definition 3.4.1. [2] Let x_1 and x_2 be two vectors in a vector space over a field F. The "span" of the set $\{x_1, x_2\}$, denoted Span$\{x_1, x_2\}$, is the set of all linear combinations of x_1 and x_2, i.e.

$$\text{Span}\{x_1, x_2\} = \{k_1x_1 + k_2x_2 | k_1, k_2 \in F\}$$

△

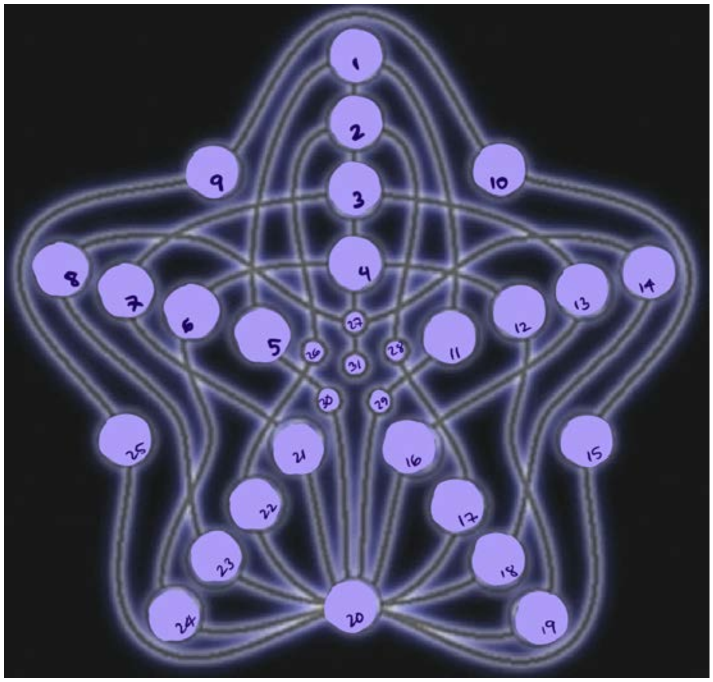

Figure 3.3.4: 31 distinct points in finite projective plane order 5

Example 3.4.2. Let [A] and [B] be distinct equivalence classes in $\overline{V}_2$, where

$$\overline{V}_2 = \{[100], [101], [110], [111], [010], [011], [001]\}.$$

Therefore $\text{Span}\{[A], [B]\} = \{[A], [B], [A] + [B]\} \mod 2$ when $n = 2$. Then, $\text{Span}\{[A], [B]\}$ consists of linear combinations of any of the distinct equivalence classes in $\mathbb{Z}_2$.

Figure 3.4.1 shows a table of [A], [B], and [A]+[B] in $\mathbb{Z}_2$. Note that each element of V_2 lies on exactly 3 lines. We have:

L([100],[110])= {[100], [110], [010]}

L([100], [001])= {[100], [001], [101]}

L([100], [011])= {[100], [011], [111]}

L([011], [110])= {[011], [110], [101]}

L([011], [010])= {[011], [010], [001]}

L([010], [111])= {[010], [111], [101]}

L([110], [111])= {[110], [111], [001]}

are the 7 distinct lines in $\overline{V}_2$. ◊

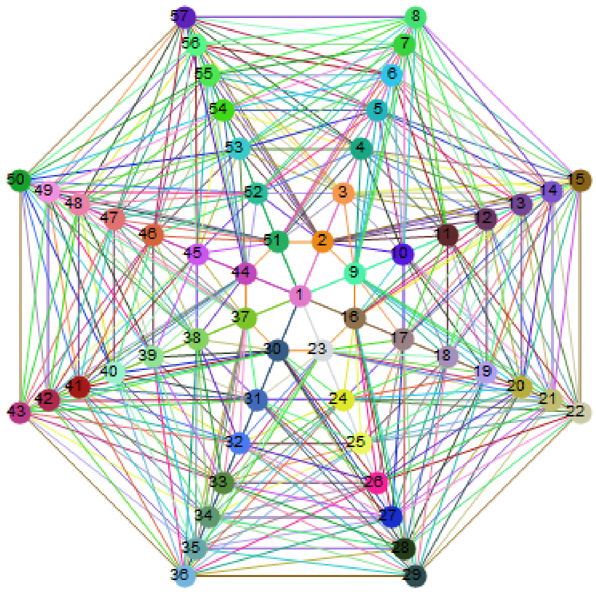

Figure 3.3.5: distinct points on Projective plane order 7

Example 3.4.3. Suppose there are two points, $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, where $[A], [B] \in \overline{V_3}$. Every element in the Span$\{A, B\}$ has the form $cA + dB$ where $c, d \in \mathbb{Z}_n$. When $n = 3$ there are 9 possible combinations since c, d can each be 0 or 1 or 2. Then, $(c, d) = \{(0,0), (0,1), (0,2), (1,1), (1,0), (1,2), (2,1), (2,2), (2,0)\}$. Hence it follows that $c[A] + d[B]$ consists of:

1. [100]	2. [120]	3. [140]	4. [161]
5. [101]	6. [121]	7. [141]	8. [162]
9. [102]	10.[122]	11.[142]	12.[163]
13.[103]	14.[123]	15.[143]	16.[164]
17.[104]	18.[124]	19.[144]	20.[165]
21.[105]	22.[125]	23.[145]	24.[166]
25.[106]	26.[126]	27.[146]	28.[010]
29.[110]	30.[130]	31.[150]	32.[011]
33.[111]	34.[131]	35.[151]	36.[012]
37.[112]	38.[132]	39.[152]	40.[013]
41.[113]	42.[133]	43.[153]	44.[014]
45.[114]	46.[134]	47.[154]	48.[015]
49.[115]	50.[135]	51.[155]	52.[016]
53.[116]	54.[136]	55.[156]	56.[001]
57.[160]			

Figure 3.3.6: equivalence classes of V_7

$$0[A] + 1[B] = [B]$$

$$0[A] + 2[B] = 2[B]$$

$$1[A] + 1[B] = [A] + [B]$$

$$1[A] + 0[B] = [A]$$

$$1[A] + 2[B] = [A] + 2[B]$$

$$2[A] + 1[B] = 2[A] + [B]$$

$$2[A] + 2[B] = 2[A] + 2[B]$$

$$2[A] + 0[B] = 2[A]$$

[A] =	[B]=	[A]+[B] =
[101]	[100]	[101] + [100] = [201] mod 2 = [001]
[110]	[100]	[110] + [100] = [210] mod 2 = [010]
[111]	[100]	[111] + [100] = [211] mod 2 = [011]
[010]	[100]	[010] + [100] = [110] mod 2 = [110]
[001]	[100]	[001] + [100] = [101] mod 2 = [101]
[011]	[100]	[011] + [100] = [111] mod 2 = [111]
[101]	[110]	[101] + [110] = [211] mod 2 = [011]
[101]	[111]	[101] + [111] = [212] mod 2 = [010]
[101]	[010]	[101] + [010] = [111] mod 2 = [111]
[101]	[011]	[101] + [011] = [112] mod 2 = [110]
[101]	[001]	[101] + [001] = [102] mod 2 = [100]
[110]	[111]	[110] + [111] = [221] mod 2 = [001]
[110]	[010]	[110] + [010] = [120] mod 2 = [100]
[110]	[011]	[110] + [011] = [121] mod 2 = [101]
[110]	[001]	[110] + [001] = [111] mod 2 = [111]
[111]	[010]	[111] + [010] = [121] mod 2 = [101]
[111]	[011]	[111] + [011] = [122] mod 2 = [100]
[111]	[001]	[111] + [001] = [112] mod 2 = [110]
[010]	[011]	[010] + [011] = [021] mod 2 = [001]
[010]	[001]	[010] + [001] = [011] mod 2 = [011]
[011]	[001]	[011] + [001] = [012] mod 2 = [010]

Figure 3.4.1: [A], [B], [A]+[B] in $\overline{V_2}$

However, we must remove $(0, 0, 0)$ and the duplicates. By definition of equivalence relation we know we can multiply each of by $k \in \mathbb{Z}_3$, so by 0, 1, or 2. Then the following is true,

$[B] \sim 2[B]$

$[A] + [B] \sim 2[A] + 2[B]$

$[A] \sim 2[A]$

$2(2[A]+[B]) \mod 3 = [A]+2[B]$, so $[A]+2[B] \sim 2[A]+[B]$. Thus, over $\mathbb{Z}_3$, the Span$\{[A], [B]\}$ using the equivalence relation, L([A], [B]) $\mod 3 = \{[A], [B], [A] + [B], [A] + 2[B]\}$. $\Diamond$

Example 3.4.4. Consider $\overline{V_3}$. There are 13 distinct points (equivalence classes). To find the line determined by $[A]$ and $[B]$, we must find the Span$\{[A], [B]\}$. Now suppose that $[A] = [010]$ and $[B] = [121]$. From the previous example we know that, $L(A, B) = \{[A], [B], [A]+[B], [A]+2[B]\}$ in $\mathbb{Z}_3$. Then it follows that, $[A] = [010]$,

$2[A] = 2 \cdot [010] = [020]$,

$[B] = [121]$,

$2[B] = 2 \cdot [121] = [242] \mod 3 = [212]$,

$[A] + [B] = [010] + [121] = [131] \mod 3 = [101]$,

$2[A] + 2[B] = (2 \cdot [010]) + (2 \cdot [121]) = ([020] + [242]) \mod 3 = [202]$,

$2[A] + [B] = (2 \cdot [010]) + [121] = [141] \mod 3 = [111]$,

$[A] + 2[B] = [010] + (2 \cdot [121]) = [252] \mod 3 = [222]$.

Observe that $[B] \sim 2[B]$,

$[A] + [B] \sim 2[A] + 2[B]$,

$[A] \sim 2[A]$, and

$[A] + 2[B] \sim 2[A] + [B]$.

Therefore, L([010], [121])= {[010],[121],[101],[222]}. ◊

Theorem 3.4.5. *In general for order n, all possible linear combinations of A and B can be represented by,*

$$L = \{[A], [B], [A + B], \ldots, [A] + [(n-1)B]\}$$

Proof. Let Span$\{A, B\} = L$ Consider $P = cA + dB$.

Case 1: Suppose that $c \neq 0$. Then $c^{-1} \in \mathbb{Z}_n$, and $c^{-1}(P) = c^{-1}(cA + dB) = 1A + c^{-1}dB$. Therefore, $1A + c^{-1}dB \in L$. So, $c^{-1}P \in L$ and $c(c^{-1}P) = P$. It follows that $P \sim c^{-1}P$. Thus, $[P] = [c^{-1}P]$, so P lies in an equivalence class of L.

Case 2: Suppose $c = 0$. Then $P = (0)A + dB = dB$, so $dB \in L$. It follows that $dB \sim B$, so $[dB] = [B]$ representing the same equivalence class. Thus, P lies in an equivalence class of L. □

Example 3.4.6. When $n = 3$ there are 13 distinct equivalence classes. Therefore there are 13 distinct lines, listed in Figure 3.4.2.

◊

Example 3.4.7. When $n = 5$ there are 31 distinct equivalence classes. Therefore there are 31 distinct lines listed in Figure 3.4.4. Similarly there are 31 cards in a SPOT IT! deck order 5 as shown in Figure 3.4.3.

L_1 =	{ [1, 2, 2], [0, 1, 2], [1, 1, 0], [1, 0, 1] }
L_2 =	{ [0, 1, 2], [0, 1, 1], [0, 1, 0], [0, 0, 1] }
L_3 =	{ [1, 2, 2], [1, 2, 1], [1, 2, 0], [0, 0, 1] }
L_ 4 =	{ [0, 1, 2], [1, 0, 2], [1, 2, 0], [1, 1, 1] }
L_ 5 =	{ [1, 2, 2], [1, 0, 2], [0, 1, 0], [1, 1, 2] }
L_ 6 =	{ [1, 0, 0], [1, 2, 1], [0, 1, 2], [1, 1, 2] }
L_ 7 =	{ [1, 0, 0], [1, 1, 0], [1, 2, 0], [0, 1, 0] }
L_ 8 =	{ [1, 0, 0], [1, 2, 2], [1, 1, 1], [0, 1, 1] }
L_ 9 =	{ [1, 2, 1], [1, 0, 2], [1, 1, 0], [0, 1, 1] }
L_ 10=	{ [0, 1, 1], [1, 2, 0], [1, 1, 2], [1, 0, 1] }
L_ 11=	{ [1, 0, 0], [1, 0, 2], [0, 0, 1], [1, 0, 1] }
L_ 12=	{ [1, 2, 1], [0, 1, 0], [1, 1, 1], [1, 0, 1] }
L_ 13=	{ [1, 1, 0], [1, 1, 1], [0, 0, 1], [1, 1, 2] }

Figure 3.4.2: Lines of Projective plane order 3

$\Diamond$

Example 3.4.8. When $n = 7$ there are 57 distinct equivalence classes. Therefore there are 57 distinct lines listed in Figure 3.4.5. $\Diamond$

Figure 3.4.3: SPOT IT! deck when $n = 5$

L_1 =	{ [1, 3, 0], [1, 2, 2], [1, 1, 4], [0, 1, 3], [1, 4, 3], [1, 0, 1] }	L_ 16 =	{ [0, 1, 1], [1, 4, 0], [1, 3, 4], [1, 1, 2], [1, 2, 3], [1, 0, 1] }
L_2 =	{ [1, 1, 0], [1, 3, 0], [1, 4, 0], [1, 0, 0], [1, 2, 0], [0, 1, 0] }	L_ 17=	{ [0, 1, 2], [1, 3, 2], [1, 1, 3], [1, 4, 4], [1, 2, 0], [1, 0, 1] }
L_3 =	{ [0, 1, 1], [1, 3, 0], [1, 2, 4], [1, 1, 3], [1, 0, 2], [1, 4, 1] }	L_ 18=	{ [1, 3, 2], [0, 1, 4], [1, 0, 0], [1, 4, 1], [1, 1, 4], [1, 2, 3] }
L_ 4 =	{ [1, 0, 3], [1, 3, 0], [0, 1, 4], [1, 2, 1], [1, 4, 4], [1, 1, 2] }	L_ 19=	{ [1, 3, 0], [1, 3, 2], [1, 3, 4], [1, 3, 1], [0, 0, 1], [1, 3, 3] }
L_ 5 =	{ [1, 0, 4], [1, 1, 2], [1, 2, 0], [1, 4, 1], [1, 3, 3], [0, 1, 3] }	L_ 20=	{ [1, 3, 2], [1, 4, 2], [1, 2, 2], [0, 1, 0], [1, 0, 2], [1, 1, 2] }
L_ 6 =	{ [0, 1, 1], [1, 0, 4], [1, 3, 2], [1, 2, 1], [1, 1, 0], [1, 4, 3] }	L_ 21=	{ [1, 2, 1], [0, 1, 0], [1, 4, 1], [1, 3, 1], [1, 1, 1], [1, 0, 1] }
L_ 7 =	{ [1, 0, 3], [1, 0, 4], [1, 0, 0], [1, 0, 2], [0, 0, 1], [1, 0, 1] }	L_ 22=	{ [1, 0, 3], [1, 1, 3], [0, 1, 0], [1, 4, 3], [1, 3, 3], [1, 2, 3] }
L_ 8 =	{ [1, 0, 4], [1, 2, 4], [1, 3, 4], [1, 4, 4], [0, 1, 0], [1, 1, 4] }	L_ 23=	{ [0, 1, 1], [0, 1, 4], [0, 1, 2], [0, 1, 0], [0, 0, 1], [0, 1, 3] }
L_ 9 =	{ [1, 0, 3], [1, 3, 4], [1, 2, 2], [1, 1, 0], [0, 1, 2], [1, 4, 1] }	L_ 24=	{ [1, 3, 0], [1, 0, 4], [1, 4, 2], [0, 1, 2], [1, 1, 1], [1, 2, 3] }
L_ 10=	{ [1, 1, 1], [0, 1, 4], [1, 3, 4], [1, 0, 2], [1, 2, 0], [1, 4, 3] }	L_ 25 =	{ [1, 2, 1], [1, 1, 3], [1, 4, 2], [1, 3, 4], [1, 0, 0], [0, 1, 3] }
L_ 11=	{ [1, 1, 0], [1, 1, 3], [1, 1, 1], [0, 0, 1], [1, 1, 4], [1, 1, 2] }	L_ 26=	{ [0, 1, 1], [1, 4, 4], [1, 2, 2], [1, 0, 0], [1, 1, 1], [1, 3, 3] }
L_ 12=	{ [1, 4, 0], [1, 4, 2], [1, 4, 4], [1, 4, 1], [0, 0, 1], [1, 4, 3] }	L_ 27=	{ [1, 0, 3], [1, 4, 2], [0, 1, 1], [1, 2, 0], [1, 3, 1], [1, 1, 4] }
L_ 13=	{ [1, 2, 1], [1, 2, 4], [1, 2, 2], [1, 2, 0], [0, 0, 1], [1, 2, 3] }	L_ 28=	{ [1, 0, 3], [1, 4, 0], [1, 3, 2], [1, 2, 4], [1, 1, 1], [0, 1, 3] }
L_ 14 =	{ [1, 2, 1], [1, 4, 0], [0, 1, 2], [1, 0, 2], [1, 1, 4], [1, 3, 3] }	L_ 29=	{ [1, 1, 3], [1, 0, 4], [0, 1, 4], [1, 2, 2], [1, 4, 0], [1, 3, 1] }
L_ 15=	{ [1, 1, 0], [1, 4, 4], [1, 0, 2], [0, 1, 3], [1, 3, 1], [1, 2, 3] }	L_ 30=	{ [1, 1, 0], [0, 1, 4], [1, 2, 4], [1, 4, 2], [1, 3, 3], [1, 0, 1] }
		L_ 31=	{ [0, 1, 2], [1, 2, 4], [1, 0, 0], [1, 3, 1], [1, 4, 3], [1, 1, 2] }

Figure 3.4.4: Lines of Projective plane order 5

L_	1=	{[1, 4, 5], [0, 1, 1], [1, 6, 0], [1, 3, 4], [1, 1, 2], [1, 5, 6], [1, 0, 1], [1, 2, 3]}
L_	2=	{[0, 1, 1], [1, 6, 5], [1, 3, 2], [1, 0, 6], [1, 2, 1], [1, 1, 0], [1, 4, 3], [1, 5, 4]}
L_	3=	{[1, 1, 1], [1, 5, 3], [1, 3, 2], [1, 6, 0], [1, 0, 4], [1, 4, 6], [1, 2, 5], [0, 1, 4]}
L_	4=	{[1, 1, 6], [0, 1, 6], [1, 3, 4], [1, 0, 0], [1, 6, 1], [1, 5, 2], [1, 4, 3], [1, 2, 5]}
L_	5=	{[1, 4, 0], [1, 4, 2], [1, 4, 4], [1, 4, 6], [1, 4, 1], [0, 0, 1], [1, 4, 3], [1, 4, 5]}
L_	6=	{[1, 5, 4], [1, 5, 3], [1, 5, 1], [1, 5, 5], [1, 5, 2], [0, 0, 1], [1, 5, 0], [1, 5, 6]}
L_	7=	{[1, 0, 3], [1, 4, 0], [1, 6, 2], [1, 5, 1], [0, 1, 1], [1, 3, 6], [1, 2, 5], [1, 1, 4]}
L_	8=	{[1, 2, 1], [1, 2, 6], [1, 2, 4], [1, 2, 2], [1, 2, 0], [0, 0, 1], [1, 2, 5], [1, 2, 3]}
L_	9=	{[1, 6, 5], [1, 5, 3], [1, 2, 4], [1, 0, 0], [1, 3, 6], [0, 1, 2], [1, 4, 1], [1, 1, 2]}
L_	10=	{[1, 0, 3], [1, 2, 4], [0, 1, 4], [1, 1, 0], [1, 6, 6], [1, 5, 2], [1, 3, 1], [1, 4, 5]}
L_	11=	{[1, 3, 5], [1, 1, 0], [0, 1, 6], [1, 5, 3], [1, 6, 2], [1, 4, 4], [1, 2, 6], [1, 0, 1]}
L_	12=	{[1, 1, 0], [1, 0, 2], [1, 2, 5], [1, 6, 4], [1, 4, 1], [0, 1, 5], [1, 3, 3], [1, 5, 6]}
L_	13=	{[1, 6, 0], [0, 1, 6], [1, 4, 2], [1, 5, 1], [1, 1, 5], [1, 0, 6], [1, 2, 4], [1, 3, 3]}
L_	14=	{[1, 1, 2], [1, 0, 6], [1, 5, 0], [1, 4, 4], [1, 6, 3], [1, 3, 1], [1, 2, 5], [0, 1, 3]}
L_	15=	{[1, 1, 5], [0, 1, 4], [1, 5, 0], [1, 2, 2], [1, 3, 6], [1, 6, 4], [1, 4, 3], [1, 0, 1]}
L_	16=	{[1, 1, 1], [1, 5, 4], [0, 1, 6], [1, 3, 6], [1, 0, 2], [1, 6, 3], [1, 2, 0], [1, 4, 5]}
L_	17=	{[1, 3, 5], [1, 0, 4], [1, 5, 1], [1, 6, 6], [1, 2, 0], [1, 1, 2], [0, 1, 5], [1, 4, 3]}
L_	18=	{[1, 0, 3], [1, 5, 3], [1, 1, 3], [0, 1, 0], [1, 6, 3], [1, 4, 3], [1, 3, 3], [1, 2, 3]}
L_	19=	{[1, 3, 5], [1, 2, 1], [1, 4, 2], [0, 1, 4], [1, 0, 0], [1, 6, 3], [1, 1, 4], [1, 5, 6]}
L_	20=	{[0, 1, 2], [1, 6, 3], [1, 5, 1], [1, 3, 4], [1, 2, 2], [1, 1, 0], [1, 4, 6], [1, 0, 5]}
L_	21=	{[1, 1, 3], [1, 2, 6], [1, 3, 2], [1, 5, 1], [1, 0, 0], [1, 6, 4], [0, 1, 3], [1, 4, 5]}
L_	22=	{[1, 2, 6], [1, 0, 6], [1, 6, 6], [1, 3, 6], [1, 4, 6], [1, 1, 6], [0, 1, 0], [1, 5, 6]}
L_	23=	{[0, 1, 1], [1, 5, 2], [1, 2, 6], [1, 1, 5], [1, 0, 4], [1, 4, 1], [1, 3, 0], [1, 6, 3]}
L_	24=	{[0, 1, 1], [0, 1, 6], [0, 1, 4], [0, 1, 2], [0, 1, 0], [0, 0, 1], [0, 1, 5], [0, 1, 3]}
L_	25=	{[1, 5, 4], [1, 6, 2], [1, 1, 5], [1, 0, 0], [1, 4, 6], [1, 3, 1], [0, 1, 5], [1, 2, 3]}
L_	26=	{[1, 0, 3], [1, 0, 4], [1, 0, 6], [1, 0, 0], [1, 0, 2], [1, 0, 5], [0, 0, 1], [1, 0, 1]}
L_	27=	{[1, 5, 4], [1, 3, 0], [1, 1, 3], [1, 4, 2], [1, 6, 6], [0, 1, 2], [1, 2, 5], [1, 0, 1]}
L_	28=	{[1, 2, 1], [1, 0, 5], [1, 1, 3], [1, 6, 0], [1, 4, 4], [1, 3, 6], [1, 5, 2], [0, 1, 5]}
L_	29=	{[1, 1, 0], [1, 1, 5], [1, 1, 3], [1, 1, 1], [1, 1, 6], [0, 0, 1], [1, 1, 4], [1, 1, 2]}
L_	30=	{[0, 1, 1], [1, 4, 4], [1, 2, 2], [1, 0, 0], [1, 5, 5], [1, 6, 6], [1, 1, 1], [1, 3, 3]}
L_	31=	{[1, 6, 5], [1, 3, 0], [1, 5, 1], [0, 1, 4], [1, 4, 4], [1, 0, 2], [1, 1, 6], [1, 2, 3]}
L_	32=	{[1, 6, 5], [1, 1, 4], [0, 1, 3], [1, 4, 6], [1, 5, 2], [1, 2, 0], [1, 3, 3], [1, 0, 1]}
L_	33=	{[1, 0, 3], [1, 3, 0], [0, 1, 6], [1, 5, 5], [1, 2, 1], [1, 6, 4], [1, 4, 6], [1, 1, 2]}
L_	34=	{[1, 2, 1], [1, 6, 2], [0, 1, 2], [1, 0, 4], [1, 1, 6], [1, 5, 0], [1, 3, 3], [1, 4, 5]}
L_	35=	{[1, 0, 3], [1, 2, 6], [1, 4, 2], [1, 6, 5], [1, 1, 1], [1, 3, 4], [0, 1, 5], [1, 5, 0]}
L_	36=	{[1, 3, 0], [1, 6, 2], [1, 2, 4], [0, 1, 3], [1, 1, 1], [1, 0, 5], [1, 4, 3], [1, 5, 6]}
L_	37=	{[1, 3, 0], [1, 5, 3], [1, 0, 6], [1, 2, 2], [1, 1, 4], [1, 6, 1], [0, 1, 5], [1, 4, 5]}
L_	38=	{[1, 3, 5], [1, 3, 1], [1, 3, 0], [1, 3, 2], [1, 3, 4], [1, 3, 6], [0, 0, 1], [1, 3, 3]}
L_	39=	{[1, 6, 5], [1, 4, 0], [1, 1, 3], [1, 2, 2], [0, 1, 6], [1, 0, 4], [1, 3, 1], [1, 5, 6]}
L_	40=	{[1, 0, 2], [1, 2, 6], [1, 6, 0], [0, 1, 2], [1, 5, 5], [1, 3, 1], [1, 4, 3], [1, 1, 4]}
L_	41=	{[1, 6, 2], [1, 0, 6], [0, 1, 4], [1, 3, 4], [1, 1, 3], [1, 5, 5], [1, 4, 1], [1, 2, 0]}
L_	42=	{[1, 2, 1], [1, 5, 3], [1, 1, 5], [1, 3, 4], [1, 6, 6], [1, 4, 0], [1, 0, 2], [0, 1, 3]}
L_	43=	{[1, 4, 1], [0, 1, 6], [1, 3, 2], [1, 6, 6], [1, 1, 4], [1, 0, 5], [1, 5, 0], [1, 2, 3]}
L_	44=	{[1, 6, 2], [1, 3, 2], [1, 4, 2], [1, 2, 2], [1, 0, 2], [1, 5, 2], [0, 1, 0], [1, 1, 2]}
L_	45=	{[1, 3, 5], [1, 6, 5], [1, 1, 5], [1, 2, 5], [0, 1, 0], [1, 0, 5], [1, 5, 5], [1, 4, 5]}
L_	46=	{[1, 3, 5], [1, 4, 0], [1, 0, 6], [0, 1, 2], [1, 6, 4], [1, 5, 2], [1, 1, 1], [1, 2, 3]}
L_	47=	{[1, 3, 5], [1, 0, 3], [1, 1, 6], [1, 6, 0], [1, 5, 4], [1, 2, 2], [1, 4, 1], [0, 1, 3]}
L_	48=	{[1, 6, 5], [1, 6, 2], [1, 6, 0], [1, 6, 6], [1, 6, 4], [1, 6, 3], [0, 0, 1], [1, 6, 1]}
L_	49=	{[0, 1, 1], [1, 1, 6], [1, 5, 3], [1, 4, 2], [1, 6, 4], [1, 0, 5], [1, 3, 1], [1, 2, 0]}
L_	50=	{[1, 1, 0], [1, 0, 4], [1, 4, 2], [1, 3, 6], [1, 5, 5], [0, 1, 3], [1, 6, 1], [1, 2, 3]}
L_	51=	{[1, 1, 0], [1, 3, 0], [1, 4, 0], [1, 6, 0], [1, 0, 0], [0, 1, 0], [1, 2, 0], [1, 5, 0]}
L_	52=	{[1, 0, 3], [1, 3, 2], [1, 1, 5], [1, 4, 4], [0, 1, 2], [1, 6, 1], [1, 2, 0], [1, 5, 6]}
L_	53=	{[1, 3, 5], [0, 1, 1], [1, 0, 2], [1, 2, 4], [1, 1, 3], [1, 4, 6], [1, 6, 1], [1, 5, 0]}
L_	54=	{[1, 1, 1], [1, 2, 1], [1, 5, 1], [0, 1, 0], [1, 4, 1], [1, 3, 1], [1, 6, 1], [1, 0, 1]}
L_	55=	{[1, 1, 6], [1, 4, 0], [1, 3, 2], [1, 2, 4], [1, 5, 5], [1, 6, 3], [0, 1, 5], [1, 0, 1]}
L_	56=	{[1, 5, 4], [1, 4, 0], [0, 1, 4], [1, 2, 6], [1, 0, 5], [1, 6, 1], [1, 3, 3], [1, 1, 2]}
L_	57=	{[1, 5, 4], [1, 0, 4], [1, 2, 4], [1, 3, 4], [1, 4, 4], [1, 6, 4], [0, 1, 0], [1, 1, 4]}

Figure 3.4.5: Lines of Projective plane order 7

4
TRIPLES in SPOT IT!

Recall that the fifth way of playing SPOT IT! is "Triplets", finding triples. The game suggests a placement of 9 cards before starting the game. The definition of a triple in SPOT IT! is just a set of 3 cards that have one symbol in common. However, one question that arises is how likely it is for a player to be able to find a triple in a SPOT IT! deck of prime order n. We start by finding the overall likelihood of a triple in 3 random cards from a SPOT IT! deck. Based of the axioms of SPOT IT! we know that for every given symbol there are $n+1$ cards in a deck that contain the same symbol.

Combinatorics is a be useful mathematical technique to determine the number of possible arrangements in a collection of items where the order of the selection does not matter. When calculating probability, we divide the number of ways the event can occur by the total number of possible outcomes. Therefore we can use the SPOT IT! axioms and properties to create a general formula for the probability that when 3 cards are randomly chosen from a SPOT IT! deck of prime order n, they are a triple.

4.1 Combinations

The number of cards in a SPOT IT! deck order n will be represented by $S(n)$. This is the number of objects in our sample space. We are counting the number of sets of 3-element cards.

Therefore, $\binom{S(n)}{3}$ represents all possible combinations of 3 cards that can be chosen from a SPOT IT! deck of order n .

Definition 4.1.1. [9] The number of subsets of r objects from a total of n objects is given by $n(n-1)...(n-r+1)$. When the order of selection is not relevant, each group of r objects will be counted $r!$ times, so we get $\frac{n(n-1)...(n-r+1)}{r!}$. Thus, we define $\binom{n}{r}$, for $r \geq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

to be the number of possible combinations of n objects taken r at a time. △

4.2 Probability

The probability of 3 random cards being a triple when randomly drawn from the deck has mutually exclusive outcomes. This means either the set of 3 cards are a triple or they are not, so we can proceed to find the probability of this event occurring. In the previous section we compute the number of the possible outcomes of choosing 3 random cards to be $\binom{S(n)}{3}$, where $S(n)$ is the number of total cards in a SPOT IT! deck of order n. The event we are interested in is when a set of 3 cards chosen form a triple. From the axioms of SPOT IT!, recall that $n+1$ total cards share a given symbol and, from these $n+1$ total cards we are choosing 3 that will ensure our set of 3 cards to be triple. Note that, there are $S(n)$ total cards and $S(n)$ total distinct symbols in a SPOT IT! deck order n, and $S(n) = n^2 + n + 1$.

Definition 4.2.1. [9] Probability is a function, $\mathbb{P} : S \to [0, 1]$, that assigns to each event A in the sample space S, a number $\mathbb{P}(A)$, the probability of the event A, defined by $\mathbb{P}(A) = \frac{E_A}{|S|}$, where E_A is the frequency at which event A occurs. △

Definition 4.2.2. A SPOT IT! triple is a set of 3-cards that have a symbol in common. △

Lemma 4.2.3. *The total number of possible triples for a SPOT IT! deck order n is $S(n) \cdot \binom{n+1}{3}$.*

Proof. By the second Axiom 3.2.2 for an order n SPOT IT! deck each symbol appears on $n+1$ total cards. To form a triple that share one symbol, we must choose 3 cards from the group

Finite Projective Plane order n	Total # of Cards = n^2+n+1= # of symbols/cards in Deck	Total # of Triples per symbol = combin(n+1,3)	Total # of Triples	P(triple) = # of triples / # of ways to choose 3 cards	Probability(Triple)
2	7	1	7	0.2	20%
3	13	4	52	0.1818181818	18%
5	31	20	620	0.1379310345	14%
7	57	56	3192	0.1090909091	11%
11	133	220	29260	0.07633587786	8%
13	183	364	66612	0.06629834254	7%
17	307	816	250512	0.05245901639	5%
19	381	1140	434340	0.04749340369	5%
23	553	2024	1119272	0.03992740472	4%
29	871	4060	3536260	0.03222094361	3%
31	993	4960	4925280	0.03027245207	3%

Figure 4.2.1: Probability that 3 cards are a Triple in an order n SPOT IT!

of cards where that symbol appears on, $n+1$. By the third Axiom 3.2.2 there are n^2+n+1 distinct symbols in total for an order n SPOT IT! deck. Therefore, $S(n) \cdot \binom{n+1}{3}$ represents the number of possible triples for any given symbol in a SPOT IT! deck. □

Example 4.2.4. In a projective plane of order $n = 3$, there are 4 points per line. This means that a SPOT IT! deck of order 3 has $n^2+n+1 = 9+3+1 = 13$ distinct symbols in total and 13 cards in a deck. Each card has $n+1 = 3+1 = 4$ symbols displayed on them, and any particular symbol shows up on exactly 4 cards. Then it follows that the number of triples with a particular symbol is $\binom{4}{3}$ (since there are only 4 cards that share the same symbol and we are choosing 3 of them for our desired triple). There are 13 distinct symbols, so $13 \cdot \binom{4}{3}$ is the total number of triples in the deck (52 total triples). It follows that,

$$\mathbb{P}(3 \text{ cards a triple}) = \frac{(3^2+3+1)\binom{3+1}{3}}{\binom{3^2+3+1}{3}} = \frac{(13)(4)}{286} = 0.181818$$

This means that a set of 3 cards is a triple about 18 percent of the time.

When $n = 5$, a SPOT IT! deck has $n^2+n+1 = 5^2+5+1 = 31$ distinct symbols and cards. Each card has $n+1 = 5+1 = 6$ symbols displayed on them. Any particular symbol shows up on exactly 6 cards. Then it follows that the number of triples with a particular symbol is $\binom{6}{3}$ (since there are only 6 cards that share the same symbol and we are choosing 3 of them for our desired triple). There are 31 distinct symbols, so $31 \cdot \binom{6}{3} = 31 \cdot 20 = 620$ total number of triples.

Then it follows that,

$$\mathbb{P}(\text{3 cards a triple}) = \frac{(5^2+5+1)\binom{5+1}{3}}{\binom{5^2+5+1}{3}} = \frac{(31)(20)}{4495} = .13793$$

This means that a set of 3 cards is a triple is about 14 percent of the time.

When $n = 7$, a SPOT IT! deck of order 7 there are $n^2 + n + 1 = 7^2 + 7 + 1 = 57$ distinct symbols and cards. Each card has $n+1 = 7+1 = 8$ symbols displayed on them. Any particular symbol shows up on exactly 8 cards. Then it follows that the number of triples with a particular symbol is $\binom{8}{3}$ (since there are only 8 cards that share the same symbol and we are choosing 3 of them for our desired triple). There are 57 distinct symbols, so $57 \cdot \binom{8}{3} = 57 \cdot 56 = 3192$ total number of triples. Then it follows that,

$$\mathbb{P}(\text{3 cards a triple}) = \frac{(7^2+7+1)\binom{7+1}{3}}{\binom{7^2+7+1}{3}} = \frac{(57)(56)}{29260} = .1090909$$

This means that a set of 3 cards is a triple is about 11 percent.

When $n = 11$, a SPOT IT! deck of order 11 has $n^2+n+1 = 11^2+11+1 = 133$ distinct symbols and cards. Each card has $n+1 = 11+1 = 12$ symbols displayed on them. Any particular symbol shows up on exactly 12 cards. Then it follows that the number of triples with a particular symbol is $\binom{12}{3}$ (since there are only 12 cards that share the same symbol and we are choosing 3 of them for our desired triple). There are 133 distinct symbols, so $133 \cdot \binom{12}{3} = 133 \cdot 220 = 29260$ total number of triples. Then it follows that,

$$\mathbb{P}(\text{3 cards a triple}) = \frac{(11^2+11+1)\binom{11+1}{3}}{\binom{11^2+11+1}{3}} = \frac{(133)(220)}{383306} = 0.07634$$

This means that a set of 3 cards is a triple about 8 percent of the time.

Figure 4.2.1 presents a chart where for a given n, the probability of a triple when choosing 3 random cards is given for a SPOT IT! deck order n. ◊

Theorem 4.2.5. *In a SPOT IT! deck of order n, the probability that 3 cards will form a SPOT IT! triple is*

$$\frac{(n^2+n+1)\binom{n+1}{3}}{\binom{n^2+n+1}{3}}.$$

Proof. By Lemma 4.2.3, $(n^2+n+1)\binom{n+1}{3}$, is total number of triples for a given symbol for order n. Then the number of ways there are to choose 3 cards for a triple from the total number of cards in a deck is represented by $\binom{n^2+n+1}{3}$. The probability of a triple of a particular symbol is the number of triples divided by the different ways one can choose 3 cards from the deck, so $\mathbb{P}(3 \text{ cards are a Triple}) = \frac{(n^2+n+1)\binom{n+1}{3}}{\binom{n^2+n+1}{3}}$. □

5
K CARD LAYOUTS

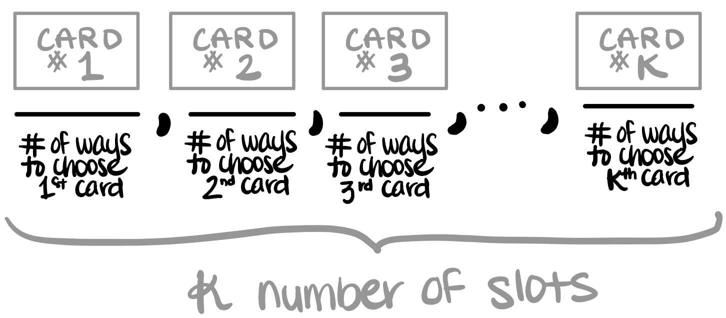

Figure 5.0.1: slots for a k-card layout

When trying to count the numbers of possible triples that can be formed among k SPOT IT! cards laid out it is easier to count the possible ways that a triple will not be formed among k SPOT IT! cards laid out. This is because we know that it takes 3 cards that share 1 symbol in common to complete a triple.

From the axioms of an order n SPOT IT! deck we know that in general $n+1$ cards will have a fixed symbol in common. This is true for each symbol, $S(n)$ in total. In a k-card layout we consider the placement of the cards as slots, as shown in Figure 5.0.1. As each card is chosen to fill each slot notice that as the cards are chosen there are cards removed from the deck before proceeding to fill the next slot. After a card is chosen, from the leftover cards all cards that share a symbol with 2 cards already in a given slot need to be removed to avoid a triple to be formed if that card were to be chosen. Overall, the goal is to be able to find all the possible ways that the cards could be chosen where no triples form amongst the k cards chosen.

After finding the number of possible ways of not choosing a triple when k cards are chosen from a SPOT IT! deck order n we can find the probability of not getting a triple in a k card layout.

Recall that the game *Triplets* suggests that 9 cards are placed when playing. In section 5.2 we will answer the question of whether 9 cards have to be laid out to guarantee a triple, as well as what is the probability of choosing a triple in a k card layout.

5.1 TriCaps

Definition 5.1.1. Let $S(n)$ denote the total number of cards in a SPOT IT! deck of order n, where $S(n) = n^2 + n + 1$.

1. A **TriCap** in $S(n)$ is a collection of cards where no three cards have a symbol in common.
2. A k-TriCap is a TriCap with k elements.
3. A k-subset is a subset with k elements.

△

Example 5.1.2. Consider a SPOT IT! deck order 2. Then, there are $S(n) = 7$ cards total. Suppose we are trying to find all the possible 1-TriCaps. In a 1 card layout there is only one slot to fill with a random chosen card from the deck. Therefore, there are 7 possible cards to choose from to fill the slot. Thus, there are 7 possible TriCaps for a SPOT IT! deck of order 2. ♢

Example 5.1.3. Consider a SPOT IT! deck order 3. Then, there are $S(n) = 13$ cards total. Suppose we are trying to find all the possible 1-TriCaps. In a 1 card layout there is only one slot to fill with a random chosen card from the deck. Therefore, there are 13 possible cards to choose from to fill the slot. Thus, there are 13 possible 1-TriCaps for a SPOT IT! deck of order 3. ♢

Example 5.1.4. Consider a SPOT IT! deck order 5. Then, there are $S(n) = 31$ cards total. Suppose we are trying to find all the possible 1-TriCaps. In a 1 card layout there is only one slot to fill with a random chosen card from the deck. Therefore, there are 31 possible cards to choose from to fill the slot. Thus, there are 31 possible 1-TriCaps for a SPOT IT! deck of order 5. ♢

Example 5.1.5. Consider a SPOT IT! deck order 7. Then, there are $S(n) = 57$ cards total. Suppose we are trying to find all the possible 1-TriCaps. In a 1 card layout there is only one slot to fill with a random chosen card from the deck. Therefore, there are 57 possible cards to choose from to fill the slot. Thus, there are 57 possible 1-TriCaps for a SPOT IT! deck of order 7. ♢

Definition 5.1.6. Let $T_k(n)$ denote a k-TriCaps in a SPOT IT! deck of order n △

Theorem 5.1.7. *The number of 1-TriCaps in a SPOT IT! deck of order n is given by*

$$T_1(n) = \frac{S(n)}{1!}.$$

Example 5.1.8. Consider a SPOT IT! deck order 2. Then, there are $S(n) = 7$ cards total. Suppose we are trying to find all the possible 2-TriCaps. Then, there are 2 slots. When choosing the first card, there are 7 cards to choose from. Since there is already a card chosen for slot 1, there is one less card to choose from, so $S(n) - 1$ cards left. Then for slot 2 there are 6 cards to choose from. Since the order in which the cards were chosen does not affect our final outcome we divide by $k!$ for every slot to be filled, so 2!. Then, $\frac{7 \cdot 6}{2!} = 21$. Therefore there are 21 possible 2-TriCaps for a SPOT IT! deck of order 2. ♢

Example 5.1.9. Consider a SPOT IT! deck order 5. Then, there are $S(n) = 31$ cards total. Suppose we are trying to find all the possible 2-TriCaps. Then, there are 2 slots. When choosing the first card, there are 31 cards to choose from. Since there is already a card chosen for slot 1, there is one less card to choose from, so $S(n) - 1$ cards left. Then for slot 2 there are 30 cards to choose from. Since the order in which the cards were chosen does not affect our final outcome we divide by $k!$ for every slot to be filled, so 2!. Then, $\frac{31 \cdot 30}{2!} = 465$. Therefore there are 465 possible 2-TriCaps for a SPOT IT! deck of order 5. ◊

Theorem 5.1.10. *The number of 2-TriCaps in a SPOT IT! deck of order n is given by*

$$T_2(n) = \frac{S(n) \cdot S(n) - 1}{2!}.$$

Example 5.1.11. Consider a SPOT IT! deck order 2. Recall that $S(n) = 7$ cards in the deck. Suppose we are trying to find all the possible 3-TriCaps. Then, there are 3 slots when $k = 3$. There are 7 cards to choose from in order to fill the first slot. Suppose the first card is chosen at random. Then when choosing the second card there are $S(n) - 1 = 7 - 1 = 6$ cards to choose from for slot k_2. An example of the cards for the first two slots is given in Figure 5.1.1.

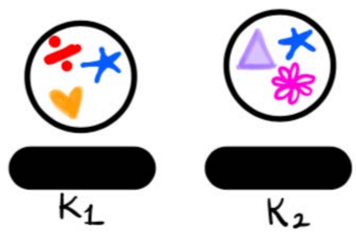

Figure 5.1.1: chosen cards for slots k_1 and k_2

Figure 5.1.2: Card removed before slot k_3 is filled

When choosing the third card notice that cards in slot k_1 and k_2 have the symbol "star" in common. This means that from the leftover cards shown in Figure 5.1.2, there is 1 card that needs to be removed before choosing a card to fill slot k_3. Then the card with the symbols star, green glassses, and smiley face gets removed and the third card can now be chosen at random. Since the order at which the cards were chosen in did not matter we divide by $k! = 3!$. Note that the axioms of SPOT IT! state that there will be $n+1$ cards that share one symbol in common. Therefore, after choosing the first two slots we knew there will be $n+1$ total cards removed before choosing the third card to avoid forming a triple. For slot k_3 the number of cards to choose from is $S(n)-(n+1)$. Therefore, $S(n)-(n+1) = 7-(2+1) = 7-3 = 4$ cards left to choose from for slot k_3. Thus, $T_3(2) = \frac{7 \cdot 6 \cdot 4}{3!} = 28$, so there are 28 possible 3-TriCaps for a SPOT IT! deck order 2. ♢

Theorem 5.1.12. *The number of 3-TriCaps in a SPOT IT! deck of order n is given by*

$$T_3(n) = \frac{S(n) \cdot S(n) - 1 \cdot S(n) - (n+1)}{3!}.$$

Example 5.1.13. Consider the Figure 3.2.2 SPOT IT! deck of order 2. Recall there are 7 total cards in the entire deck and 4 distinct symbols appear on each individual card.

Figure 5.1.3: 4 card layout

Now, the goal is to find the number of TriCaps in a 4 card layout, shown in Figure 5.1.3. For the first card there are $S(2) = 7$ total cards to choose from. Supposed the first card in Figure 5.1.4 is the first pick for the first slot.

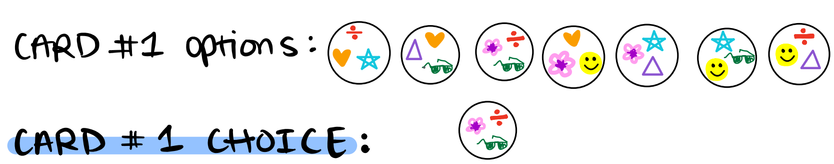

Figure 5.1.4: Slot 1 for card 1

For the second open slot, considering 1 card has been already chosen, there is one fewer card to choose from. Then there are $S(2)-1 = 6$ cards to choose from for the second slot. A random card is chosen for slot 2.

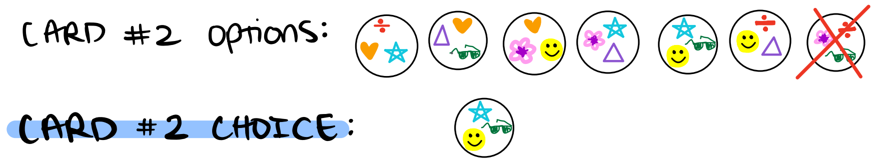

Figure 5.1.5: Slot 2 for card 2

From the axioms for SPOT IT! we know there are $n + 1$ cards that all share one symbol in common. From Figure 5.1.5 the green sunglasses appear twice already on card 1 and card 2. Therefore, any cards with green sunglasses have to be removed, as well as the already 2 chosen cards for slot 1 and 2. Notice how for slot 3 the $n+1$ cards that all share one symbol in common had to be removed to avoid a triple. Then there are $S(2)-3 = S(2)-(n+1) = S(2)-(2+1) = 4$ cards to choose from.

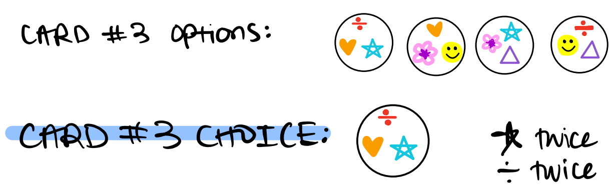

Figure 5.1.6: Slot 3 for card 3

When choosing the last card recall what the card chosen for slot 3 was. The third card shares a symbol with card 2, a blue star. Then the last card where a blue star appears on must be removed. Additionally, it shares a red division sign with card 1. Any cards sharing this symbol must also be removed, including the original cards filling previous slots. This is shown in Figure 5.1.6. Then, out of $S(n)$ cards to choose from we are removing the 3 groups of cards that shared a particular symbol in common. In this case, the 2 cards with division signs, 2 cards with sunglasses, and 2 cards with blue stars on them. Therefore, there are $S(n) - 3n$ choices for the last slot. The last card chosen is the only card one can choose to make a TriCap, shown in Figure 5.1.7.

Since the order in which the cards are picked for the particular slot does not matter, we know that if we multiply the possible choices for each slot it would be an overcount, as shown in

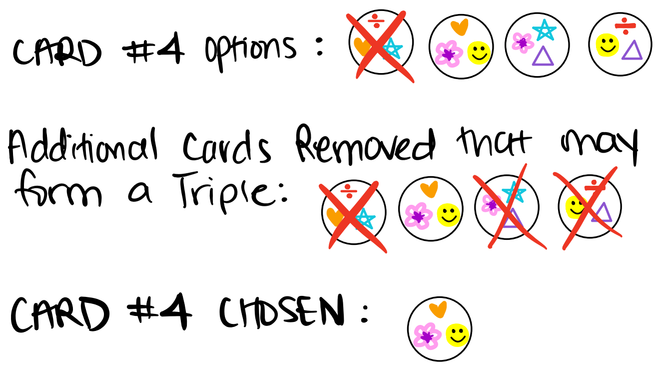

Figure 5.1.7: Slot 4 for card 4

Figure 5.1.8. Each combination can appear in 4! possible orders, which correspond to the same combination. Therefore we have to divide by 4! to find the number of combinations.

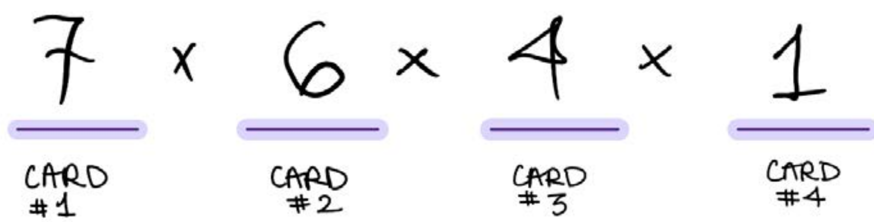

Figure 5.1.8: Slot 4 for card 4

Thus, $T_4(2) = \frac{7*6*4*1}{4!}$, so there are 7 possible 4-TriCaps . ◊

Theorem 5.1.14. *When $k = 4$, the number of k-card layouts having no triples can be expressed as the following:*

$$T_4(n) = \frac{(S(n))(S(n)-1)(S(n)-(n+1))(S(n)-3n)}{4!}$$

Proof. Let $S(n) = n^2 + n + 1$, the total number of cards in a deck when n is prime. Let $S(n)$ be the total number of cards in a SPOT IT! deck of order n.

The first card is chosen at random. Therefore there are $S(n)$ cards to choose from. When choosing the second card there is one fewer card in the deck to choose from. Hence there are $S(n)-1$ cards to choose from. When choosing the third card, the goal is to eliminate all the cards that share a symbol with the original two cards already chosen. Hence there are $S(n)-(n+1)$

cards to choose from. This way any card that would complete a triple with the first two cards is eliminated.

When choosing the last card for a 4 card layout, to eliminate all the possible cards with matching symbols considering the first 3 cards already chosen. Let A, B, C represent the first 3 cards chosen for the first 3 slots. Then we know that cards A and B share a symbol x. Cards B and C share a symbol y. Cards A and C share a symbol z. For symbols x, y, z there are $n+1$ cards of each. Therefore $3(n+1)$ total cards display any of these symbols mentioned. However, this is an over-count because it includes the 3 cards previously chosen. So there are $3(n+1)-3$ cards to be removed and $3(n+1)-3 = 3n+3-3 = 3n$. Thus, there are $S(n)-3n$ cards to choose from when choosing the fourth card. □

Example 5.1.15. Suppose that $n = 3$. Then there are $S(3) = 13$ cards in a SPOT IT! deck. When choosing the first card for the first slot in the k card layout there are 13 cards to choose from. Therefore there are 13 possible 1-TriCaps to choose the first card.

Now we are going to count the number of 2-Tricaps. There are 13 ways to choose the first card. When choosing the second card there are 12 ways to choose a card for the second slot. Since order does not matter, we must divide by 2!. Therefore, $\frac{13 \cdot 12}{2!} = 78$. Thus, there are 78 possible 2-TriCaps.

When counting the number of 3-TriCaps. There are 13 ways to choose the first card. There are 12 ways to choose the second card. However, when choosing the third card notice that the first and second card share a symbol, a, there are $n+1 = 3+1 = 4$ cards that contain symbol a. Therefore, we must removed 4 cards from the 13 card deck, leaving us with 9 cards that may be chosen for the third slot. Since there are 3 slots and order does not matter, we must divide by 3!. Thus, $\frac{13 \cdot 12 \cdot 9}{3!} = 234$, so there are 234 possible 3-TriCaps.

When counting the number of 4-TriCaps, there are 13 ways to choose the first card. There are 12 ways to choose the second card. There are 9 possible ways to choose the third card. When choosing the fourth card, recall that the first and second card share a symbol, a, there are $n+1 = 3+1 = 4$ cards that contain symbol a. However, the first and third card share a symbol, b. While the second and third card share a symbol, c. Since 3 cards are already chosen and 2 cards of each symbol, a, b, c, are left 9 cards from the original deck of cards have to be removed, leaving us with 4 possible ways to choose the fourth card. Since there are 4 slots and order does not matter, we must divide by 4!. Therefore $\frac{13 \cdot 12 \cdot 9 \cdot 4}{4!} = 234$, so there are 234 possible 4-TriCaps.

When counting the number of 5-TriCaps, there are 13 ways to choose the first card. There are 12 ways to choose the second card. There are 9 possible ways to choose the third card. There are 4 ways of choosing the fourth card. When choosing the fifth card, recall that the first and second card share a symbol, a. The first and third card share a symbol, b. The second and third card share a symbol, c. The third and fourth card share a symbol, d. The first and fourth card share a symbol, e. The second and fourth card share a symbol, f. There are 2 cards of each symbol left, so $2 \cdot 6 = 12$ that have to be removed, in addition to the first 4 cards already chosen, so 16 cards need to be removed from the original deck of cards. However, when $n = 3$ there are only 13 cards, yet 16 cards have to be removed when choosing the fifth card.

Thus, there are no possible 5-TriCaps when $n = 3$. $\diamondsuit$

Example 5.1.16. Consider a SPOT IT! deck for $n = 5$, as shown in Figure 3.4.3. It follows that there are $5^2 + 5 + 1 = 31$ cards in the deck with $5 + 1 = 6$ symbols on each card. Suppose the symbols on the cards are just the numbers 1 through 30. There are a 6 slots for each card in a 6 card layout.

For the first slot there are a total of 31 cards to choose from. Choose the first card at random. For the second slot there are 30 cards to choose from, given that the first card was already chosen.

Suppose that the first 2 cards that were chosen for slot 1 and 2 are those in Figure 5.1.9. Notice that the only symbol they share is 27.

For the third slot all of the cards that share a 27 symbol must be removed from the deck to avoid a triple being formed.

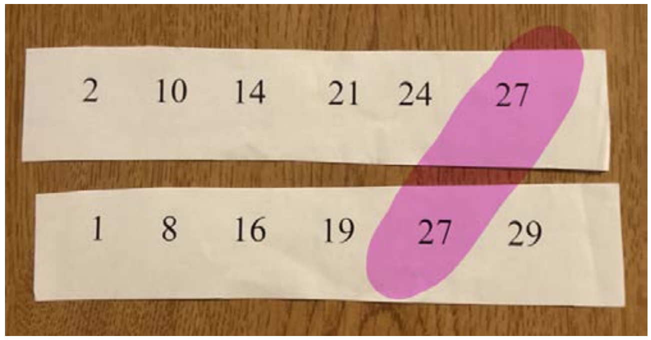

Figure 5.1.9: First two cards chosen

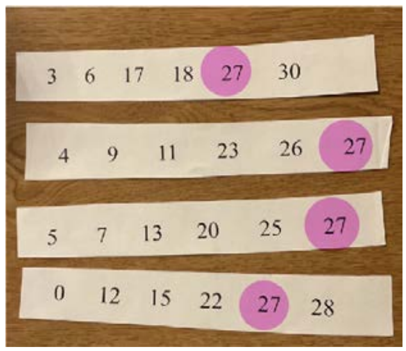

Figure 5.1.10: cards removed

This means that a total of 6 cards have been removed from the original deck of cards before choosing the third card, shown in Figure 5.1.10. The two original cards in slot 1 and 2, plus the 4 new cards that share a symbol with the first two cards. Following that for slot 3, $S(n)-(n+1)$ cards are left to choose from for the third slot.

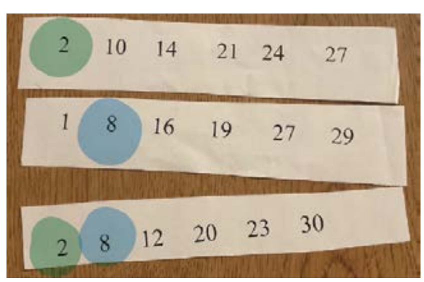

Figure 5.1.11: Cards in slots 1, 2, and 3

Suppose that the first 3 slots of randomly chosen cards is represented in Figure 5.1.11. Notice that now between these 3 cards there are two symbols, 2 and 8, that have to be removed from the deck before choosing a card for slot 4. Recall that the cards that also share the symbol 12 were removed previously.

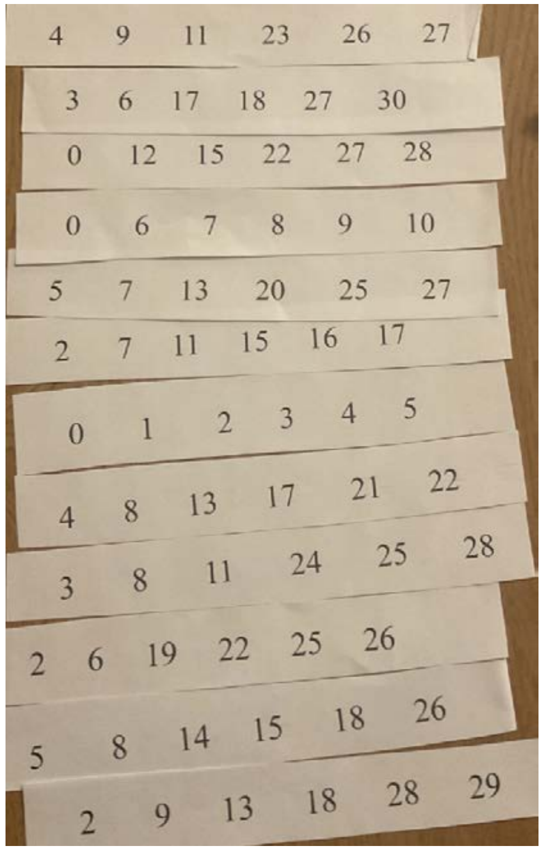

Figure 5.1.12: Cards removed after choosing card 3

Then the cards in Figure 5.1.12 are all the cards that share a symbol 2, 8, or 27. Therefore, 12 cards have been removed in addition to the 3 cards that are already chosen. A total of 15 cards have been removed. For the fourth slot $S(n) - 3n$ cards are left to choose from, since $3n = 3(5) = 15$ cards have been removed before choosing the fourth card. Therefore, $S(n) - 3n = 31 - 15 = 16$ cards left to choose from to fill the 4^{th} slot.

Let the fourth card be the one as shown in Figure 5.1.13. Notice that now after filling the fourth slot there are 6 symbols that need to be removed from the deck before choosing a fifth card.

In Figure 5.1.14 one can see that 21 cards have to to be removed that would complete a triple. Therefore there have been 25 cards removed considering the first 4 cards already chosen. In

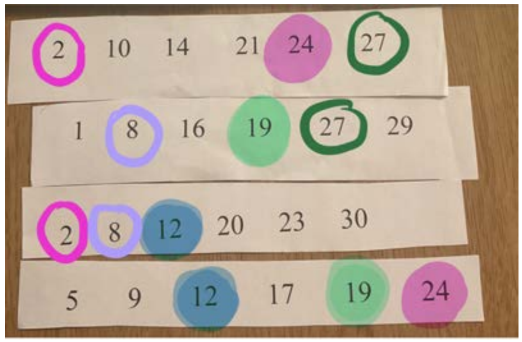

Figure 5.1.13: Cards in slots 1 through 4

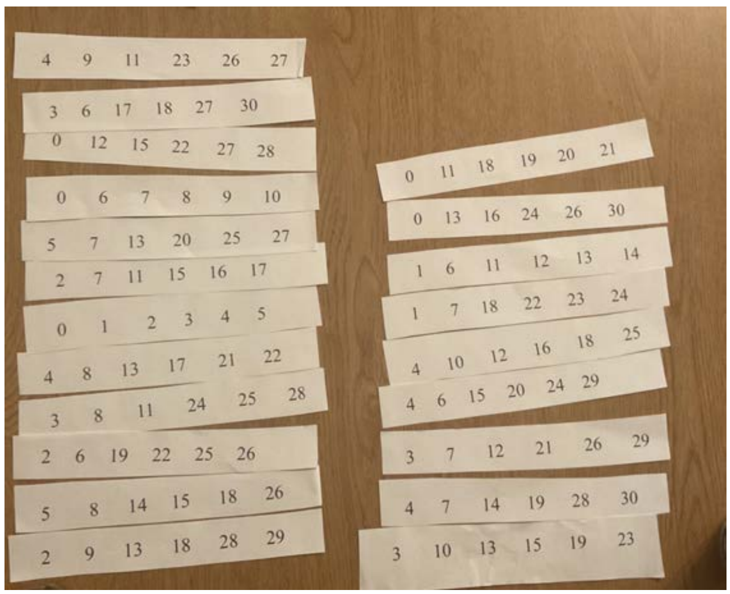

Figure 5.1.14: Cards removed after card 4 chosen

Figure 5.1.17 are the total cards that need to be removed to prevent a triple from forming with any of the previous cards already chosen. Therefore, there are only 3 cards to choose from when filling the 5^{th} slot.

Suppose that the last card is chosen for the fifth slot. Then as shown Figure 5.1.15 the symbols that need to be removed after the fifth card are 1, 2, 8, 9, 12, 19, 27, and 30. However, there are not enough cards in the deck to remove cards to fill a 6^{th} slot. Therefore, there are no possible 6-TriCaps when $n = 5$. Consider Figure 5.1.16 which shows the possible ways to choose a card for each slot in a k-card layout. Since there are 5 slots and order does not matter, we must divide by 5!. Thus, $\frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 3}{5!} = 9300$, so there are 9300 possible 5-Tricap when $n = 5$.

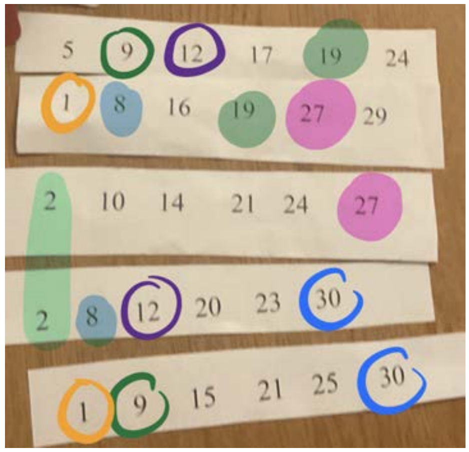

Figure 5.1.15: Cards in slots 1 through 5

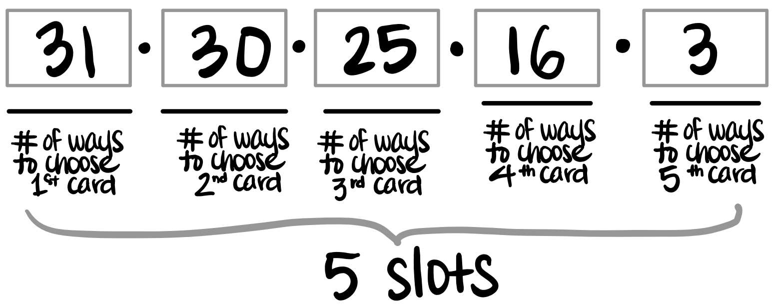

Figure 5.1.16: 5-card layout when $n = 5$

◊

Theorem 5.1.17. *The number of possible k-card layouts with no triples for a SPOT IT! deck of prime order n, where S(n) is the total number of symbols and cards in the deck, can be expressed as the following:*

$$T_k(n) = \frac{S(n)\cdot(S(n)-1)\cdot(S(n)-2-\binom{2}{2}(n-1))\cdot(S(n)-3-\binom{3}{2}(n-1))\cdot\ldots\cdot(S(n)-(k-1)-\binom{k-1}{2}(n-1))}{k!}$$

Proof. Given a finite projective plane of order n, where n is prime, there are $S(n) = n^2 + n + 1$ distinct points and lines. Since points correspond to symbols in a SPOT IT! deck and lines correspond to cards, then $S(n) = n^2 + n + 1$ is the number of total cards in a SPOT IT! deck of order n. Recall that a triple is defined as 3 cards that share one particular symbol in common. Suppose we have a k-card layout (meaning k cards are placed down at a time), then we have exactly k slots to fill with SPOT IT! cards that will not contain a triple. Recall that

Figure 5.1.17: Cards removed after card 5 chosen

there are k slots in a k-card layout as shown in Figure 5.0.1. For the first slot, we can freely choose any card from the total $S(n)$ cards in a SPOT IT deck. Therefore there are $S(n)$ cards to choose from. For the second slot, since the first card has been picked and there are not enough cards to form a triple, there are $S(n) - 1$ cards to choose from considering that first card has already been chosen. For the third slot, since 2 cards have already been chosen they are removed from the total amount of cards in the deck. Then we get $\binom{2}{2}$ since there are 2 cards and you are choosing 2 to find out how many symbols occur as pairs among any 2 cards that are already chosen. We multiply by the number of leftover cards of that particular symbol which is $(n-1)$ since 2 instances of the symbol have already shown up on the 2 chosen cards. Then, $S(n) - 2 - (\binom{2}{2}(n-1))$ is the number of cards left to choose from when slots 1 and 2 are already chosen.

Now assume we have chosen $k-1$ cards. The cards we are looking to remove are those that would complete a triple with any pair of previously chosen cards. First, the number of previous cards already chosen have to be taken into account, so we remove the $k-1$ cards that have already been chosen from $S(n)$ cards. Then, to prevent a triple we must find all the pairs already created among the already chosen cards and kick out the cards left that have the matching symbol from these pairs. There are $\binom{k-1}{2}$ total pairs, so each of the $\binom{k-1}{2}$ matching symbols must be avoided. We also know by axioms of SPOT IT! that there are n+1 cards with a particular symbol in common. Since our goal is to avoid triples we remove 2 cards, hence, $n+1-2=n-1$, leftover cards with each symbol that we are trying to avoid. The total number of additional cards that need to be kicked out would be $\binom{k-1}{2}(n-1)$, the total number of pairs that share a matching symbol times the number of leftover cards with that particular symbol. Adding this to the $(k-1)$ cards chosen, these are $(S(n)-(k-1)-\binom{k-1}{2}(n-1))$ ways to choose the k^{th} card without forming a triple. □

Example 5.1.18. Consider $n=2$, $S(n)=7$. Then by Theorem 5.1.17,

$$T_1(2)=\frac{7}{1!}=7$$
$$T_2(2)=\frac{7\cdot 6}{2!}=21$$
$$T_3(2)=\frac{7\cdot 6\cdot 4}{3!}=28$$
$$T_4(2)=\frac{7\cdot 6\cdot 4\cdot 1}{4!}=7$$
$$T_5(2)=\frac{7\cdot 6\cdot 4\cdot 1\cdot -3}{5!}=-4.2$$

For a SPOT IT! deck order 2, $T_4(2)=7$. Notice that $T_5(2)=-4.2$, so $T_k(2)=0$ for all $k\geq 5$. This means that for a SPOT IT! deck order 3 that there aren't enough cards in the original deck when trying to remove all cards that would form a triple. When $k<4$ we can choose cards so that a triple will not form, but when $k=5$, there will always be a triple.

◊

Example 5.1.19. Consider $n=3$, $S(n)=13$. Then by Theorem 5.1.17,

$$T_1(3)=\frac{13}{1!}=13$$
$$T_2(3)=\frac{13\cdot 12}{2!}=78$$
$$T_3(3)=\frac{13\cdot 12\cdot 9}{3!}=234$$
$$T_4(3)=\frac{13\cdot 12\cdot 9\cdot 4}{4!}=234$$
$$T_5(3)=\frac{13\cdot 12\cdot 9\cdot 4\cdot -3}{5!}=-140.4$$

For a SPOT IT! deck order 3, $T_4(3) = 234$. Notice that $T_5(3) = -140.4$, so $T_k(3) = 0$ for all $k \geq 5$. This means that for a SPOT IT! deck order 3 that there aren't enough cards in the original deck when trying to remove all cards that would form a triple. When $k < 4$ we can choose cards so that a triple will not form, in fact there are 234 possible ways to choose 4 cards that do not form a triple. But, when $k = 5$, there will always be a triple. ◊

Example 5.1.20. Consider $n = 5$, $S(n) = 31$. Then by Theorem 5.1.17,

$$T_1(5) = \frac{31}{1!} = 31$$
$$T_2(5) = \frac{31 \cdot 30}{2!} = 465$$
$$T_3(5) = \frac{31 \cdot 30 \cdot 25}{3!} = 3875$$
$$T_4(5) = \frac{31 \cdot 30 \cdot 25 \cdot 16}{4!} = 15500$$
$$T_5(5) = \frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 3}{5!} = 9300$$
$$T_6(5) = \frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 3 \cdot -14}{6!} = -21700$$

For a SPOT IT! deck order 5, $T_5(5) = 9300$. Notice that $T_6(5) = -21700$, so $T_k(5) = 0$ for all $k \geq 6$. This means that for a SPOT IT! deck order 5 that there aren't enough cards in the original deck when trying to remove all cards that would form a triple. When $k < 5$ we can choose cards so that a triple will not form, but when $k = 6$, there will always be a triple.

◊

Example 5.1.21. Consider $n = 7$, $S(n) = 57$. Then by Theorem 5.1.17,

$$T_1(5) = \frac{57}{1!} = 57$$
$$T_2(5) = \frac{57 \cdot 56}{2!} = 1596$$
$$T_3(5) = \frac{57 \cdot 56 \cdot 49}{3!} = 26068$$
$$T_4(5) = \frac{57 \cdot 56 \cdot 49 \cdot 36}{4!} = 234612$$
$$T_5(5) = \frac{57 \cdot 56 \cdot 49 \cdot 36 \cdot 17}{5!} = 797680.8$$
$$T_6(5) = \frac{57 \cdot 56 \cdot 49 \cdot 36 \cdot 17 \cdot -8}{6!} = -1063574.4$$

For a SPOT IT! deck order 7, $T_5(7) = 97680.8$. Notice that $T_6(7) = -1063574.4$, so $T_k(7) = 0$ for all $k \geq 6$. This means that for a SPOT IT! deck order 7 that there aren't enough cards in the original deck when trying to remove all cards that would form a triple. When $k < 5$ we can choose cards so that a triple will not form, but when $k = 6$, there will always be a triple. ◊

5.2 Probability of a TriCap

In the previous section, we have found the number of k-Tricaps in a SPOT IT! deck. We know that $T_k(n)$ is the number of ways to choose k-cards that do not have a triple. Note that when there are negative number of $T_k(n)$, then $T_k(n) = 0$ since there are no possible k-TriCaps in a SPOT IT! deck. Recall that in order to find all possible ways to choose r objects from a sample space of size n is $\binom{n}{r}$. Thus, the number of ways to choose k cards from a SPOT IT! deck of order n is $\binom{S(n)}{k}$.

Lemma 5.2.1. *The probability of a k-TriCap in a SPOT IT! deck of order n is given by,*

$\mathbb{P}(k\text{-}TriCap) = \frac{T_k(n)}{\binom{S(n)}{k}}$.

Example 5.2.2. Consider the SPOT IT! deck of order 2. Then, $S(n) = 7$. The number of ways of picking k cards from $S(n)$ are as follows:

$$\binom{S(2)}{1} = \binom{7}{1} = 7$$
$$\binom{S(2)}{2} = \binom{7}{2} = 21$$
$$\binom{S(2)}{3} = \binom{7}{3} = 35$$
$$\binom{S(2)}{4} = \binom{7}{4} = 35$$
$$\binom{S(2)}{5} = \binom{7}{5} = 21$$

From Theorem 5.1.17, the numbers $T_k(n)$ are:

$$T_1(2) = \frac{7}{1!} = 7$$
$$T_2(2) = \frac{7 \cdot 6}{2!} = 21$$
$$T_3(2) = \frac{7 \cdot 6 \cdot 4}{3!} = 28$$
$$T_4(2) = \frac{7 \cdot 6 \cdot 4 \cdot 1}{4!} = 7$$
$$T_5(2) = \frac{7 \cdot 6 \cdot 4 \cdot 1 \cdot -3}{5!} = 0$$

Then, the probability of a not getting a Triple for each k card layout is the following,

$$\mathbb{P}(\frac{T_1(2)}{\binom{7}{1}}) = \frac{7}{7} = 1$$
$$\mathbb{P}(\frac{T_2(2)}{\binom{7}{2}}) = \frac{21}{21} = 1$$
$$\mathbb{P}(\frac{T_3(2)}{\binom{7}{3}}) = \frac{28}{35} = .8$$
$$\mathbb{P}(\frac{T_4(2)}{\binom{7}{4}}) = \frac{7}{35} = .2$$
$$\mathbb{P}(\frac{T_5(2)}{\binom{7}{5}}) = \frac{0}{21} = 0$$

Table 5.2.1: Probability Table for $S(2)$

k	$\binom{S(2)}{k}$	$T_k(2)$	$\mathbb{P}(\text{TriCap}) = \frac{T_k(2)}{\binom{S(2)}{k}}$	$\mathbb{P}$(Triple)=1-$\mathbb{P}$(TriCap)
1	7	7	1	0
2	21	21	1	0
3	35	28	0.8	0.2
4	35	7	0.2	0.8
5	21	0	0	1

Let $\mathbb{P}(A)$ be the probability of a TriCap in a k-card layout. It follows that $1 - \mathbb{P}(A)$ is the probability of getting a triple in a k-card layout.

Then, these are the probabilities of getting a triple for a SPOT IT! deck order 2 when k card layout are as following,

When k=1, $1 - (\frac{T_1(2)}{\binom{7}{1}}) = 1 - 1 = 0$

When k=2, $1 - (\frac{T_2(2)}{\binom{7}{2}}) = 1 - 1 = 0$

When k=3, $1 - (\frac{T_3(2)}{\binom{7}{3}}) = 1 - 0.8 = 0.2$

When k=4, $1 - (\frac{T_4(2)}{\binom{7}{4}}) = 1 - 0.2 = 0.8$

When k=5, $1 - (\frac{T_5(2)}{\binom{7}{5}}) = 1 - (-0.2) = 1.2$

Table 5.2.1 shows the probabilities of a triple occurring in a k card layout. Recall that when $k = 5$ there are not enough cards in a SPOT IT! deck order, when removing cards that will form triples. This means $k > 4$, meaning more than 4 cards are laid out when playing, there is a 100 percent chance of getting a triple when choosing cards at random. ♢

Definition 5.2.3. [9] Let A_k be the event in which a k-TriCap exists when k cards are randomly chosen from a SPOT IT! deck order n and $\mathbb{P}(A_k)$ be the probability of event A_k occurring. Then,

$1 - \mathbb{P}(A_k) = \mathbb{P}(B_k)$,

where B_k is the event in which a Triple exists when k cards are randomly chosen from a SPOT IT! deck order n and $\mathbb{P}(B_k)$ is the probability of event B_k occurring.

△

The probability Table 5.2.2 shows how many possible ways there are to choose k cards when $n = 3$. The third column shows how many possible k-TriCaps there are for a SPOT IT! deck

Table 5.2.2: Probability Table for $S(3)$

k	$\binom{S(3)}{k}$	$T_k(3)$	$\mathbb{P}(\text{TriCap}) = \frac{T_k(3)}{\binom{S(3)}{k}}$	$\mathbb{P}$(Triple)=1-$\mathbb{P}$(TriCap)
1	13	13	1	0
2	78	78	1	0
3	286	234	0.8181818182	0.1818181818
4	715	234	0.3272727273	0.6727272727
5	1287	0	0	1

Table 5.2.3: Probability Table for $S(5)$

$k =$	$\binom{S(5)}{k}$	$T_k(5)$	$\mathbb{P}(\text{TriCap}) = \frac{T_k(5)}{\binom{S(5)}{k}}$	$\mathbb{P}$(Triple)=1-$\mathbb{P}$(TriCap)
1	31	31	1	0
2	465	465	1	0
3	4495	3875	0.8620689655	0.1379310345
4	31465	15500	0.4926108374	0.5073891626
5	169911	9300	0.05473453749	0.9452654625
6	736281	0	0	1

order 3, followed by the probability of a k-Tricap when choosing k cards at random from the deck in the 4^{th} column. The last column expresses the probability of there being a triple when k cards are laid out. The table shows that when 4 cards are laid out from a SPOT IT! deck of order 3, there is a 67.3% chance of a Triple. However, notice that the probability of $T_5(3)$ is 0%, meaning that there are no possible ways of choosing a 5-TriCap. Hence, the probability of a triple of a triple in a 5 card layout is 100%. Then it follows that when playing with a SPOT IT! deck of order 3 only 5 cards need to be laid out to guarantee a triple. Thus, the best way to play *Triplets* with a SPOT IT! deck order 3, with 13 cards is to lay out 5 cards, to guarantee at least 1 triple to choose from.

The probability Table 5.2.3 shows how many possible ways there are to choose k cards when $n = 5$. The third column shows how many possible k-TriCaps there are for a SPOT IT! deck order 5, followed by the probability of a k-TriCap when choosing k cards at random from the deck in the 4^{th} column. The last column expresses the probability of there being a triple when k cards are laid out. The table shows that when 5 cards are laid out from a SPOT IT! deck order 5, there is a 94.5% chance of a Triple. However, notice that the probability of $T_6(5)$ is 0%, meaning that there are no ways of choosing a 6-TriCap. Hence, the probability of a triple of a

Table 5.2.4: Probability Table for $S(7)$

$k =$	$\binom{S(7)}{k}$	$T_k(7)$	$\mathbb{P}(\text{TriCap}) = \frac{T_k(7)}{\binom{S(7)}{k}}$	$\mathbb{P}$(Triple)=1-$\mathbb{P}$(TriCap)
1	57	57	1	0
2	1596	1596	1	0
3	29260	26068	0.8909090909	0.1090909091
4	395010	234612	0.5939393939	0.4060606061
5	4187106	797680.8	0.1905088622	0.8094911378
6	36288252	-1063574.4	-0.02930905572	1.029309056

triple in a 6 card layout is 100%. Then it follows that when playing with a SPOT IT! deck order 5, only 6 cards need to be laid out to guarantee a triple to appear amongst the cards. Thus, the best way to play *Triplets* with a SPOT IT! deck order 5, with 31 cards is to lay out 6 cards, to guarantee at least 1 triple to choose from.

The probability Table 5.2.4 shows how many possible ways there are to choose k cards when $n = 7$. The third column shows how many possible k-TriCaps there are for a SPOT IT! deck order 7, followed by the probability of a k-Tricap when choosing k cards at random from the deck in the fourth column. The last column expresses the probability of there being a triple when k cards are laid out. The table shows that when 5 cards are laid out from a SPOT IT! deck order 7, there is a 80.9% chance of a Triple. However, notice that the probability of $T_6(7)$ is 0%, meaning that there are no possible ways of choosing a 6-TriCap. Hence, the probability of a triple of a triple in a 6 card layout is 100%. Then it follows that when playing with a SPOT IT! deck of order 7, only 6 cards need to be laid out to guarantee a triple to appear amongst the cards. Thus, the best way to play *Triplets* with a SPOT IT! deck of order 7, with 57 cards is to lay out 6 cards, to guarantee at least 1 triple to choose from.

Recall that the SPOT IT! instructions stated that 9 cards should be placed at the beginning of a Triplet game. However, one can place fewer than 9 cards and still guarantee a triple. For a SPOT IT! deck of order 2 and 3, only 5 cards need to be placed to guarantee a triple. For a SPOT IT! deck order 5 and 7, only 6 cards need to be placed to guarantee a triple. Recall Figure 2.3.1 among a 9 card layout there were about 3 triples to choose from. A 5 or 6 cards layout, depending which SPOT IT! deck order n you are playing with, will be more challenging for players to find a triple amongst a smaller number of cards laid out.

5.3 Future Work

From the final results we are now sure of how many cards need to be placed down in a k card layout to guarantee a triple for any SPOT IT! deck of prime order n. When placing 6 cards for a SPOT IT! deck order 7 there will always be at least one triple to choose. Notice again Figure 2.3.1. When 9 cards are placed there are at least 2 independent triples (notice that some triples actually rely on the same card to be a triple but there are 2 triples that stand independently from each other in the 9 card lay out). The following questions could be studied in a future research project expanding current research on the math behind SPOT IT!:

1. How can we find the number of cards that will guarantee more than just one triple.

2. What is the relationship between two triples that share one card in common to complete their individual sets?

3. Considering every symbol shows up on $n+1$ cards could it be possible to play or manipulate the game looking for other sets of cards that share the same symbol?

4. Can we produce various examples of SPOT IT! decks which do not correspond to finite projective planes? Does the symmetry of the deck of cards still hold after all?

Appendix A
Constructing Finite Projective Planes

A.1 Python Code

Listing A.1: Constructing Points and Lines in $\mathbb{Z}_n$

```
def Eclasseslist(p):
    Eclasses=[]
    for x in range(0 ,p):
        for y in range(0 ,p):
            T=(1,x,y)
            Eclasses.append(T)

    for y in range(0 ,p):
        T=(0,1,y)
        Eclasses.append(T)

    T=(0,0,1)
    Eclasses.append(T)
    return Eclasses

def isRep(x):
    if x[0] == 1:
        return True
    if x[0] != 0:
        return False
    if x[1] == 1:
        return True
    if x[1] != 0:
        return False
    if x[2] == 1:
        return True
    return False

x = (1,0,2)
print(isRep(x))

def representative(x, p):
    for i in range(1, p):
        maybe_rep = multstuple(x, i, p)
        if isRep(maybe_rep):
```

```
        return maybe_rep

def addtuples(A,B,p):
    return ((A[0] + B[0]) %p , (A[1] + B[1]) %p , (A[2] + B[2]) %p)

def multstuple(B,i,p):
    return (B[0]*i %p, B[1]*i %p, B[2]*i %p)


def ELine(A,B,p):
    line = frozenset(representative(addtuples(A,multstuple(B,i,p),p), p) for i in range(p)) |
    frozenset({B})
    return line


p = 5
Eclasses = Eclasseslist(p)
for T in Eclasses:
    print(T)
print(len(Eclasses))

line_set = set()
for A in Eclasses:
    for B in Eclasses:
        if A == B:
            continue
        line_set.add(ELine(A,B,p))
print(line_set)
for x in line_set:
    print(list(x))
print(len(line_set))
```

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