An Exploration on Condorcet-Approval-Range Voting Function with limits

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Abstract

In contrast to most social choice methods, which use ranked ballots, range voting is a well-known social choice method that offers the voters more choices in the form of an allowed range of possible scores. In this project, by allowing voters to give positive and negative scores, we hope to find a way that can explicitly show how voters disapprove, feel neutral, or approve of the alternatives instead of just giving ranking orders. Also, by applying a function to constrain the scores given in range voting, each voter will have the same influence when they give scores. After combining these conditions with Condorcet method by transferring scores into ranked ballot, we get a new voting function that involves Condorcet, approval and range voting. In this project, we explore how this new voting function behaves with respect to certain voting criteria.

Contents

Dedication

To the fading days.

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1

Introduction

1.1 Social Choice Procedure

Voting theory is commonly known as the mathematical study of voting systems. There are a wide variety of voting systems and each has its advantages and disadvantages. Some of the most common differences among voting systems is how to decide who is the winner, and whether or not they involve a multi-step ranking process.

In a ranked voting system, voters rank the alternatives in the order they prefer it, ranking their most preferred alternative as the first choice and their least preferred alternative as the last choice. Also, tied votes sometimes are allowed in the ranking, depended on the different voting systems.

Definition 1.1.1. A set of **alternative** is $A = \{a_1, \ldots, a_m\}$ such that there are m alternatives in A. △

Definition 1.1.2. A set of **voter** is $V = \{v_1, \ldots, v_n\}$ such that there are n voters in V. △

It is assumed that each voter arrives at some ordering of the alternatives in accordance with his preferences.

The following definitions are from [1].

Definition 1.1.3. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. A **preference order** of A, written as p_A, is a linear order on A. We shall represent p_A by a column vector

$$p_A = \begin{pmatrix} a_{\sigma(1)} \\ a_{\sigma(2)} \\ \vdots \\ a_{\sigma(m)} \end{pmatrix}$$

where $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ is a permutation. The top alternative is most preferred and the bottom alternative is least preferred. We say voter i **prefer** $a_{\sigma(p)}$ to $a_{\sigma(q)}$ for any $p, q \in \{1, \ldots, m\}$ if $p < q$. △

Definition 1.1.4. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let P_A be the set of preference orders of A. A **preference profile** is the function $f : V \to P_A$. We define $X_{V,A}$ to be the set of preference profiles. △

Definition 1.1.5. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\mathcal{P}(A)$ be the power set of A. A **social choice procedure** is a function $\Psi : X_{V,A} \to \mathcal{P}(A)$. △

The image of Ψ is always a subset of $\mathcal{P}(A)$. The output $\Psi(f)$ for some $f \in X_{V,A}$ is an element or several elements of $\mathcal{P}(A)$, which is a subset of A (which could be the empty set or a set with a single element).

1.2 Examples of Social Choice Procedures

Definition 1.2.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p, q \in \{1, \ldots, m\}$ be such that $p \neq q$. Let $T = \{a_p, a_q\}$. Let $f \in X_{V,A}$. Let $s = |\{i \in \{1, \ldots, n\} | a_{\sigma(p)} > a_{\sigma(q)}\}|$ and $t = |\{i \in \{1, \ldots, n\} | a_{\sigma(p)} < a_{\sigma(q)}\}|$. We define a_p **wins the pairwise comparison** if $s > t$. △

Definition 1.2.2. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p \in \{1, \ldots, m\}$. Let $f \in X_{V,A}$. The alternative a_p is a **Condorcet winner** if a_p wins its pairwise comparison with a_q for all $q \in \{1, \ldots, m\}$ such that $q \neq p$. △

The following statements are from [2].

For example, let $A = \{a, b, c\}$ and suppose the preference orders are

$$p_1 = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \quad p_2 = \begin{pmatrix} b \\ a \\ c \end{pmatrix} \quad p_3 = \begin{pmatrix} b \\ c \\ a \end{pmatrix} \quad p_4 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad p_5 = \begin{pmatrix} c \\ a \\ b \end{pmatrix}.$$

Then b defeats a by a score of 3 to 2, since the first three voters prefer b to a, while the last two voters prefer a to b. We can also check that b defeats c by a score of 3 to 2, and c defeats a by a score of 3 to 2. Because b defeats each of the other alternatives it is the social choice for this profile when Condorcet's method is used. Here, b is the Condorcet Winner.

The social choice procedure **Plurality voting** declares that the alternative(s) who gets the largest number of first place rankings in the preference order should be in the social choice set.

The social choice procedure **Borda Count** uses each preference order to award "score" to each of m alternatives as follows: for each voter, the alternative at the bottom of the order gets zero points, the alternative at the next to the bottom spot gets one point, the next one up gets two points and so on up to the top alternative which gets $m - 1$ points. For each alternative, we add the score awarded it from each of the individual preference orders. The alternative(s) with the highest "Borda score" is declared to be the social choice.

The social choice procedure **Instant runoff voting** is based on the idea of arriving at a social choice by successive deletions of less desirable alternatives. We begin by deleting the alternative or alternatives who get the smallest number of first place rankings. At this stage we have orders that are at least one alternative fewer than that with which we started. Now, we simply repeat this process of deleting the least desirable alternative or alternatives (as measured by the number of preference orders on top of which it, or they, appear). The alternative(s) deleted last is declared the social choice.

The social choice procedure **Dictatorship** ignores all the preference orders except that of the dictator v_d. The alternative in first place rankings of v_d is now declared to be the social choice.

1.3 Properties of Social Choice Procedures

An ideal social choice procedure demands as many as possible of the following conditions be satisfied.

Definition 1.3.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Always-A-Winner Condition** if $\Psi(f) \neq \emptyset$ for all $f \in X_{V,A}$. $\triangle$

Definition 1.3.2. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Majority Criterion** if the following condition holds: for each $f \in X_{V,A}$, if alternative a_j for some $j \in \{1, \ldots, m\}$, is most preferred by more than $\frac{n}{2}$ the voters, then a_j should be in $\Psi(f)$. $\triangle$

Definition 1.3.3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Pareto condition** if the following condition holds: for each $f \in X_{V,A}$, if $j, k \in \{1, \ldots, m\}$, and if every voters prefers a_j to a_k, then a_k cannot be in $\Psi(f)$. $\triangle$

Definition 1.3.4. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Condorcet Criterion** if it always chooses the Condorcet Winner to be the only element in $\Psi(f)$ when one exists. $\triangle$

Definition 1.3.5. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Monotonicity Criterion** if the following holds: for each $f \in X_{V,A}$, if a_j is in $\Psi(f)$ for some $j \in \{1, \ldots, m\}$ and one voter changes his/her preference order by moving a_j up one spot, then a_j should still be in $\Psi(f)$. △

Definition 1.3.6. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Independence of irrelevant alternatives** if the following condition holds: if $a_j \in \Psi(f)$ but $a_k \notin \Psi(f)$ for some $j, k \in \{1, \ldots, m\}$, and one or more voters change their preference orders, but no one changes about whether a_j is preferred to a_k or a_k to a_j, then a_k should still not be in $\Psi(f)$. △

Definition 1.3.7. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say Ψ satisfies the **Participation Criterion** if the following condition holds: if $a_j \in \Psi(f)$ but $a_k \notin \Psi(f)$ for some $j, k \in \{1, \ldots, m\}$, and we add one voter with a_j preferred to a_k, then a_k should still not be in $\Psi(f)$. △

2

Range Voting with Limits

2.1 Range Voting

Here, we introduce another social choice procedure, called as Range Voting, that offers the voter more choices in the form of an allowed range of possible scores.

Definition 2.1.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. A **ranging profile** is the function $f : V \to B^m$. △

Definition 2.1.2. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. We define $R_{V,A,B}$ as the set of ranging profiles. △

Definition 2.1.3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. Let $i \in \{1, \ldots, n\}$. The alternative a_k for some $k \in \{1, \ldots, m\}$ is **most preferred** by v_i if $f(v_i)_k > f(v_i)_j$ for all $j \in \{1, \ldots, m\}$. △

Definition 2.1.4. Let $V = \{v_1, \dots, v_n\}$ be a set of voters such that $n \in \mathbb{N}$ and $A = \{a_1, \dots, a_m\}$ be a set of alternatives such that $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For all $f \in R_{V,A,B}$, the **scoring function** for V and A is the function $S_f : A \to \mathbb{R}$ such that

$$S_f(a_j) = \sum_{i=1}^{n} f(v_i)_j,$$

for all $j \in \{1, \dots, m\}$. $\triangle$

Definition 2.1.5. Let $V = \{v_1, \dots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \dots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let S_f be the scoring function for V and A. The **range voting function** is the function $\phi : R_{V,A,B} \to \mathcal{P}(A)$ defined by letting $\phi(f)$ be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \dots, m\}$, for all $f \in R_{V,A,B}$. $\triangle$

As we mentioned above, one of the most commonly used social choice procedures is the plurality voting.

In the plurality voting each voter casts one vote for his most preferred alternative, and the alternative(s) with the largest total number of votes constitute the social choice set. We may think of this procedure as assigning a score of 1 to each voter's most preferred alternative, a score of 0 to the others such that $B = \{0, 1\}$ where $\{1\}$ can only be assigned once, and selecting the alternative(s) with highest total score, summed over all voters.

Another well-known social choice procedure is the Borda Count. It asks each voter to assign score $m-1$ to his most preferred alternative, score $m-2$ to his second most preferred alternative, and in general score $m - i$ to his ith most preferred alternative such that $B = \{0, 1, \dots, m - 1\}$ where each element can only be assigned once. Then the alternative(s) with highest total score define the social choice set for Borda Count Voting. Therefore, these two social choice procedures can be described as special cases for range voting function.

Another special case of range voting is approval voting, where $B = \{0, 1\}$.

2.2 Properties of Social Choice Procedures for Range Voting

Some of the criteria for voting methods mentioned above can be applied as stated to range voting, somehow we need to have them reformulated for regular Range Voting.

2.2.1 Always-A-Winner Condition for Range Voting

Definition 2.2.1. Let $V = \{v_1, \dots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \dots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Always-A-Winner Condition for range voting** if $\phi(f) \neq \emptyset$, for all $f \in R_{V,A,B}$. △

Theorem 2.2.2. *Range voting satisfies Always-A-Winner Condition for range voting.*

Proof. Let $V = \{v_1, \dots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \dots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V,A,B}$. Let S_f be the scoring function for V and A.

For any f, it is clear that Range Voting satisfies Always-A-Winner Criterion, since for some $k \in \{1, \dots, n\}$, there always exists one or some of alternatives a_k that lie in $\phi(f)$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \dots, n\}$. □

2.2.2 Majority Criterion for Range Voting

Definition 2.2.3. Let $V = \{v_1, \dots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \dots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Majority Criterion for range voting** if the following condition holds: for all $f \in R_{V,A,B}$, if for any $k \in \{1, \dots, m\}$, the alternative a_k is most preferred by more than $\frac{n}{2}$ voters, then $a_k \in \phi(f)$. △

I will show that range voting does not satisfy the Majority Criterion by giving the following example.

Example 2.2.4. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V,A,B}$ shown as below.

	v_1	v_2	v_3
a_1	5	3	-9
a_2	2	2	3
a_3	-8	-10	3

Let S_f be the scoring function for V and A. We observe that alternative a_1 is most preferred by the majority of voters since there are 2 voters out of 3 who choose a_1 as their most preferred alternative. However, we have $S_f(a_1) = 5 + 3 + (-9) = -1$, and $S_f(a_2) = 2 + 2 + 3 = 7$, and $S_f(a_3) = -8 + (-10) + 3 = -1$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A$, then we know that $\phi(f) = \{a_2\}$ instead of $\{a_1\}$. $\Diamond$

2.2.3 Condorcet Criterion for Range Voting

Definition 2.2.5. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p, q \in \{1, \ldots, m\}$ be such that $p \neq q$. Let $T = \{a_p, a_q\}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. Let $s = |\{i \in \{1, \ldots, n\} | f(v_i)_p > f(v_i)_q\}|$ and $t = |\{i \in \{1, \ldots, n\} | f(v_i)_p < f(v_i)_q\}|$. We define a_p **wins the pairwise comparison for range voting** if $s > t$. $\triangle$

Definition 2.2.6. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p \in \{1, \ldots, m\}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. The alternative a_p is a **Condorcet winner for range voting** if a_p wins its pairwise comparison with a_q for all $q \in \{1, \ldots, m\}$ such that $q \neq p$. $\triangle$

Definition 2.2.7. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Condorcet Winner Criterion for range voting** if the following condition holds: for each $f \in R_{V,A,B}$, if there exists a Condorcet winner for range voting a_p for f for some $p \in \{1, \ldots, m\}$, then a_p alone is in $\phi(f)$. △

In order to check if Range Voting satisfies the Condorcet Winner Criterion, we need to convert B^m from range profile to preference orders, and then determine the Condorcet Winner with pairwise comparisons.

I will show that range voting function does not satisfy the Condorcet Criterion for range voting by giving the following example.

Example 2.2.8. Let $V = \{v_1, \ldots, v_3\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V,A,B}$ shown as below.

	v_1	v_2	v_3
a_1	-7	5	6
a_2	8	4	2
a_3	0	-6	-7

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_2	a_1	a_1
a_3	a_2	a_2
a_1	a_3	a_3

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following. The alternative with square is the one who wins the pairwise comparison.

$\boxed{a_1}$ vs a_2 2:1
$\boxed{a_1}$ vs a_3 2:1
$\boxed{a_2}$ vs a_3 2:1

As shown in the table, a_1 wins each pairwise comparison with a_2 and a_3, therefore a_1 is the Condorcet winner.

Let S_f be the scoring function for V and A. We have $S_f(a_1) = (-7) + 5 + 6 = 4$, and $S_f(a_2) = 8 + 4 + 2 = 14$, and $S_f(3) = 0 + (-6) = (-7) = -13$. Since $S_f(a_1) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$, then $\phi(f) = \{a_2\}$. $\Diamond$

2.2.4 Pareto Criterion for Range Voting

Definition 2.2.9. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Pareto Criterion for range voting** if the following condition holds: for each $f \in R_{V,A,B}$, if $k, j \in \{1, \ldots, m\}$, and if $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$, then alternative $a_j \notin \phi(f)$. $\triangle$

Theorem 2.2.10. *Range voting satisfies Pareto Criterion for range voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $k, j \in \{1, \ldots, m\}$. Let S_f be the scoring function for V and A.

Suppose $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, ...n\}$. Then we know $S_f(a_k) > S_f(a_j)$. By the definition of the range voting function, we know that a_j can never be in $\phi(f)$. Therefore, Range Voting satisfies the Pareto Criterion. $\square$

2.2.5 Monotonicity Criterion for Range Voting

Definition 2.2.11. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Monotonicity Criterion for Range Voting** if the following condition holds: for each $f \in R_{V,A,B}$, if a_j is in $\phi(f)$ and $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$, then a_j should still be in $\phi(f)$. $\triangle$

Theorem 2.2.12. *Range voting satisfies Monotonicity Criterion for range voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $j \in \{1, \ldots, m\}$. Let S_f be the scoring function for V and A.

Suppose a_j is in $\phi(f)$. Then we know $S_f(a_j) \geq S_f(a_k)$ for all $k \in \{1, \ldots, m\}$.

If $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$, then we have $S_f(a_j) > S_f(a_k)$ for all $k \in \{1, \ldots, m\}$. Hence, by the definition of range voting function, $a_j \in \phi(f)$. □

2.2.6 *Independence of Irrelevant Alternative Criterion for Range Voting*

Definition 2.2.13. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Independence of Irrelevant Alternative Criterion for Range Voting** if the following condition holds: for each $f \in R_{V,A,B}$, if $j, k \in \{1, \ldots, m\}$ and $a_j \in \phi(f)$ and $a_k \notin \phi(f)$, and if $f(v_i)j$ and $f(v_i)_k$ are changed for some $i \in \{1, \ldots, n\}$, but remaining in $f(v_i)_j > f(v_i)_k$, then $\phi(f)$ should not change so as to include a_k. △

I will show range voting function does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 2.2.14. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{\ V,A,B} \to \mathcal{P}(A)$ be the range voting function.

Let $f \in R_{\ V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	7	-7	2
a_2	2	5	-6
a_3	-6	3	7

Let S_f be the scoring function for V and A. We have $S_f(a_1) = 7 + (-7) + 2 = 2$, and $S_f(a_2) = 2 + 5 + (-6) = 1$, and $S_f(a_3) = -6 + 3 + 7 = 4$. Since $S_f(a_3) \geq S_f(a_j)$ for all $j \in \{1, 2, 3\}$, then $\phi(f) = \{a_3\}$.

We change $f(v_3)_1$ and $f(v_3)_3$ but keep $f(v_3)_1 < f(v_3)_3$. Let $f' \in R_{V,A,B}$ be given as in the following table.

	v_1	v_2	v_3
a_1	7	-7	4
a_2	2	5	-6
a_3	-6	3	5

Before we change $f(v_3)_1$ and $f(v_3)_3$, we had $\phi(f) = \{a_3\}$. However, in the second table, since we have $S_{f'}(a_1) = 7+(-7)+4 = 4$ and $S_{f'}(a_2) = 2+5+(-6) = 1$, while $S_{f'}(a_3) = -6+3+5 = 2$. Since $S_{f'}(a_1) \geq S_{f'}(a_j)$ for all $j \in \{1, 2, 3\}$, then $\phi(f') = \{a_1\} \neq \{a_3\}$. $\Diamond$

2.2.7 Intensity of Independence of Irrelevant Alternative Criterion for Range Voting

Definition 2.2.15. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $j, k \in \{1, \ldots, m\}$. If $i \in \{1, \ldots, n\}$ and $f \in R_{V,A,B}$, **intensity of preference** of voter v_i for a_j over a_k is $f(v_i)_j - f(v_i)_k$. $\triangle$

Note that for voter v_i, the intensity of preference for a_j over candidate a_k could be positive, 0 or negative. Also, the intensity of preference of voter v_i for a_k over a_j is the negative of the intensity of preference of voter v_i for a_j over a_k.

Definition 2.2.16. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting** if the following holds: for each $f \in R_{V,A,B}$, if $j, k \in \{1, \ldots, m\}$, and if $a_j \in \phi(f)$ and $a_k \notin \phi(f)$, and if for some $i \in \{1, \ldots, n\}$, $f(v_i)_p$ changes for some $p \in \{1, \ldots, m\}$, and the intensity of preference of v_i for a_j over a_k does not change, then still $a_k \notin \phi(f)$. $\triangle$

Theorem 2.2.17. *Range voting satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let S_f be the scoring function for V and A.

Suppose that $a_j \in \phi(f)$ and $a_k \notin \phi(f)$, and that $f(v_i)_p$ changes for some $i \in \{1, \ldots, n\}$ and for some $p \in \{1, \ldots, m\}$ while $f(v_i)_j - f(v_i)_k$ remains the same for some $i \in \{1, \ldots, n\}$. Since none of the voters change their intensity of preference for a_j over a_k, which is $f(v_i)_j - f(v_i)_k$ does not change for all $i \in \{1, \ldots, n\}$, and so $S_f(a_j) - S_f(a_k)$ does not change as well. Therefore, by the definition of range voting, we conclude that it is always the case that $S_f(a_j) > S_f(a_k)$, and thus $a_k \notin \phi(f)$. $\square$

2.2.8 Participation Criterion for Range Voting

Definition 2.2.18. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Participation Criterion** if the following condition holds: for each $f \in R_{V,A,B}$, suppose $a_j \in \phi(f)$ and $a_k \notin \phi(f)$ for some $j, k \in \{1, \ldots, m\}$. If one more voter v_i is added to V, who gives $f(v_i)_j > f(v_i)_k$, then it should not be the case that $a_k \in \phi(f')$ and $a_j \notin \phi(f')$. $\triangle$

Theorem 2.2.19. *Range voting satisfies the Participation Criterion for Range Voting.*

Proof. Suppose that $a_j \in \phi(f)$ and $a_k \notin \phi(f)$. By the definition of ϕ, we know that $S_f(a_j) \geq S_f(a_k)$. After adding v_i, we get new $S'_f(a_j) = S_f(a_j) + f(v_i)_j$ and $S'_f(a_k) = S_f(a_k) + f(v_i)_k$. Since $f(v_i)_j > f(v_i)_k$, then we get $S'_f(a_j) > S'_f(a_k)$. Hence $a_k \notin \phi(f)$. □

The following definitions are from [3] .

2.2.9 Consistent property for range voting

Definition 2.2.20. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such the $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$, we call f_1 and f_2 **disjoint** profiles. Then $f_1 + f_2$ denotes the profile with voter set $V_1 \cup V_2$, which when restricted to V_i agrees with f_i for each $i \in \{1, 2\}$. △

Definition 2.2.21. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such that $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let ϕ be the range voting function. We define the social choice function ϕ satisfies **consistent** property if the following condition is satisfied: for disjoint profiles $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$, if $\phi(f_1) \cap \phi(f_2) \neq \emptyset$ then $\phi(f_1) \cap \phi(f_2) = \phi(f_1 + f_2)$. △

Theorem 2.2.22. *The range voting function satisfies consistent property.*

Proof. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such that $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let ϕ be the range voting function. Let $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$. Let S_{f_1} be the score function for f_1; let S_{f_2} be the score function for f_2.

Suppose $\phi(f_1) \cap \phi(f_2) \neq \emptyset$.

Let $a_k \in \phi(f_1) \cap \phi(f_2)$. Then by the definition of ϕ we know that $S_{f_1}(a_k) \geq S_{f_1}(a_j)$ for all $j \in \{1, \ldots, m\}$, as well as $S_{f_2}(a_k) \geq S_{f_2}(a_j)$ for all $j \in \{1, \ldots, m\}$. Then we get $S_{f_1}(a_k) + S_{f_2}(a_k) \geq S_{f_1}(a_j) + S_{f_2}(a_j)$ for all $j \in \{1, \ldots, m\}$. Since f_1 and f_2 are disjoint profiles, then $S_{f_1+f_2}(a_k) \geq S_{f_1+f_2}(a_j)$ for all $j \in \{1, \ldots, m\}$, hence $a_k \in \phi(f_1 + f_2)$. Therefore $\phi(f_1) \cap \phi(f_2) \subseteq \phi(f_1 + f_2)$.

I will prove the other direction by contradiction.

Let $a_k \in \phi(f_1 + f_2)$. Suppose $a_k \notin \phi(f_1) \cap \phi(f_2)$. There are three cases.

First, suppose $a_k \notin \phi(f_1)$ and $a_k \in \phi(f_2)$. Let $a_j \in \phi(f_1) \cap \phi(f_2)$. Then it has to be the case that $S_{f_1}(a_j) > S_{f_1}(a_k)$. Since $a_k \in \phi(f_2)$, then we have $S_{f_2}(a_j) = S_{f_2}(a_k) \geq S_{f_2}(a_i)$ for all $i \in \{n+1, \ldots, q\}$. Then $S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k)$. Since f_1 and f_2 are disjoint profiles, then $S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k)$. Hence $a_k \notin \phi(f_1 + f_2)$. A contradiction.

In the second case, suppose $a_k \in \phi(f_1)$ and $a_k \notin \phi(f_2)$. Let $a_j \in \phi(f_1) \cap \phi(f_2)$. Then it has to be the case that $S_{f_2}(a_j) > S_{f_2}(a_k)$. Since $a_k \in \phi(f_1)$, then we have $S_{f_1}(a_j) = S_{f_1}(a_k) \geq S_{f_1}(a_i)$ for all $i \in \{1, \ldots, n\}$. Then $S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k)$. Since f_1 and f_2 are disjoint profiles, then $S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k)$. Hence $a_k \notin \phi(f_1 + f_2)$. A contradiction.

In the third case, suppose $a_k \notin \phi(f_1)$ and $a_k \notin \phi(f_2)$. Let $a_j \in \phi(f_1) \cap \phi(f_2)$. Then it has to be the case that $S_{f_1}(a_j) > S_{f_1}(a_k)$ and $S_{f_2}(a_j) > S_{f_2}(a_k)$. Then $S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k)$. Since f_1 and f_2 are disjoint profiles, then $S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k)$. Hence $a_k \notin \phi(f_1 + f_2)$. A contradiction.

So, $\phi(f_1 + f_2) \subseteq \phi(f_1) \cap \phi(f_2)$.

Therefore $\phi(f_1) \cap \phi(f_2) = \phi(f_1 + f_2)$. □

2.2.10 Faithful Property for range voting

Definition 2.2.23. Let $V = \{v\}$ be a set of one voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. We define the social choice function ϕ satisfies **faithful** property if the following condition is satisfied: for any profile $f \in R_{V,A,B}$, if $f(v)_j > f(v)_k$ for some $j, k \in \{1, \ldots, m\}$, then $a_k \notin \phi(f)$. $\triangle$

Theorem 2.2.24. *The range voting function satisfies faithful property.*

Proof. Let $V = \{v\}$ be a set of voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function.

Let $f \in R_{V,A,B}$. I will prove this by contrapositive. Suppose $a_k \in \phi(f)$. Let S_f be the scoring function for V and A. Then by the definition of ϕ, we have $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$. Since $|V| = 1$, then we know that $f(v)_k \geq f(v)_j$ for all $j \in \{1, \ldots, m\}$.

Therefore, by contrapositive, if $f(v)_j > f(v)_k$ for some $j, k \in \{1, \ldots, m\}$, then $a_k \notin \phi(f')$. $\square$

2.2.11 Cancellation property for range voting

Definition 2.2.25. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. For a profile $f \in R_{V,A,B}$ and $j, k \in \{1, \ldots, m\}$ such that $j \neq k$, let $\pi_{a_j a_k}(f) = |\{i \in \{1, \ldots, n\} | f(v_i)_j > f(v_i)_k\}|$. We define the social choice function ϕ satisfies **cancellation property** if the following condition is satisfied: for any $f \in R_{V,A,B}$, if $\pi_{a_j a_k}(f) = \pi_{a_k a_j}(f)$ for all $j, k \in \{1, \ldots, m\}$ such that $j \neq k$, then $\phi(f) = A$. $\triangle$

I will show that the range voting function does not satisfy the cancellation property by giving the following example.

Example 2.2.26. Let $V = \{v_1, v_2\}$ be a set of voter and $A = \{a_1, a_2\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function.Let $f \in R_{V,A,B}$ be given as the following chart.

	v_1	v_2
a_1	8	3
a_2	2	7

As we can see from the chart that $f(v_1)_1 > f(v_1)_2$. So $\pi_{a_1a_2}(f) = 1$. Also $f(v_2)_2 > f(v_2)_1$, which means $\pi_{a_2a_1}(f) = 1$. Thus, in this case $\pi_{a_1a_2}(f) = \pi_{a_2a_1}(f) = 1$.

Let S_f be the scoring function for V and A. We have $S_f(a_1) = 8 + 3 = 11$ and $S_f(a_2) = 2 + 7 = 9$. Since $S_f(a_1) > S_f(a_2)$, then $\phi(f) = \{a_1\}$.

However, we notice that $\phi(f) \neq A$. Hence, range voting function does not satisfy the cancellation property. $\diamondsuit$

2.3 Range Voting Function with Limits

As we see in several examples of range voting function, generally speaking, in a ranging profile, the range of B can be restricted in different ways. However, more than setting restrictions on B, we hope to let voters distribute their scores in a fairer way. Hence, I try to find a limited function that can be applied to the general range voting, and make our voters have an equal influence on the alternatives.

It takes me a while to determine our limited function. At first, we tried to set $\xi(f(v_i)_1 + \ldots + f(v_i)_m) = f(v_i)_1 + \cdots + f(v_i)_m - k = 0$. However, we realized that if we set $k > 0$, we are forcing our voters to give positive score to at least one alternative; similarly, if we set $k < 0$, we are forcing our voters to give negative score to at least one alternative. If we set $k = 0$, then we are forcing our voters either give 0 scores to all alternative, either they have to give both positive and negative scores to different alternative in order to balance their total score. Therefore, I decide it is better to use absolute value as a constraint to the total score instead of simply use sum.

Also, I took some time to think about a reasonable value for k such that $|f(v_i)_1| + \cdots + |f(v_i)_m)| - k = 0$. At first, I tried to set a range for k, i.e. $k \in (0, mb)$, so that voters cannot give 0 score or b score to all voters; otherwise, anything in between is allowed. However, that is a problem as we now see. For example, when $k = 1$, means that b has to be somehow equal or smaller than 1, which leaves a really limited space for voter to assign their scores; when $k = mb - 1$, means that each alternative has to receive a score that has an absolute value pretty close to b, which seems not to be a rational assignment as well. And then, I got some inspired by [3]. In this paper, the author mentions constant total weight condition for classic Borda and the total score is actually $\frac{m(m-1)}{2}$; so this idea of $\frac{mb}{2}$ comes to my mind and seems to be a reasonable condition to impose on our range voting function to avoid favoring certain preference rankings over others.

Definition 2.3.1. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. We define the **limited function** to be the function $\xi : B^m \to \mathbb{R}$ defined by $\xi(x_1, \ldots, x_m) = |x_1| + \cdots + |x_m| - \frac{mb}{2}$ for all $(x_1, \ldots, x_m) \in B^m$. △

Definition 2.3.2. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$. Let $B = [-b, b]$. Then we define

$$R^{\xi}_{V,A,B} = \{f \in R_{V,A,B} | \xi(f(v_i)_1, \ldots, f(v_i)_m) = 0 \text{ for all } i \in \{1, \ldots, n\}\} .$$

△

Note that $R^{\xi}_{V,A,B} \subseteq R_{V,A,B}$.

Definition 2.3.3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$. Let $B = [-b, b]$. Let S_f be the scoring function for V and A. The **range voting function with limit** is a function $\Phi : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ defined by letting $\Phi(f)$ be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$, for all $f \in {R^\xi}_{V,A,B}$. $\triangle$

Furthermore, it is worth to point out that in our following example, I will mostly set $B = [-10, 10]$. Since our $\frac{mb}{2}$ is a fixed value based on score b and the number of alternative m, it is clear that my example can be generalized by applying different value of b proportionately with same outcome when m keeps the same for the voting function. To illustrate this, we can compare two following examples.

Example 2.3.4. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\Phi : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limit.

We have $\frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$. Then we suppose $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| - 15 = 0$ for all $i \in \{1, 2, 3\}$. Let us consider the following example. Let $f \in {R^\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	10	8	8
a_2	0	4	-4
a_3	5	-3	3

Let S_f be the scoring function for V and A. Here, we can see that $S_f(a_1) = 10 + 8 + 8 = 26$ while $S_f(a_2) = 0 + 4 + (-4) = 0$, and $S_f(a_3) = 5 + (-3) + 3 = 5$. So $\Phi(f) = \{a_2\}$. $\Diamond$

Example 2.3.5. Let $V = \{v_1, \ldots, v_3\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Now we set $b = 5$, then $B = [-5, 5]$. Let $\Phi : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limit.

We have $\frac{mb}{2} = \frac{3 \cdot 5}{2} = 7.5$. Then we suppose $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| - 7.5 = 0$ for all $i \in \{1, 2, 3\}$. Let us consider the following example. Let $f \in {R^\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	5	4	4
a_2	0	2	-2
a_3	2.5	-1.5	1.5

Let S_f be the scoring function for V and A. Here, we can see that $S_f(a_1) = 5 + 4 + 4 = 13$ while $S_f(a_2) = 0 + 2 + (-2) = 0$, and $S_f(a_3) = 2.5 + (-1.5) + 1.5 = 2.5$. So $\Phi(f) = \{a_2\}$. $\diamondsuit$

As we see from Example 2.3.4 and Example 2.3.5, no matter what value b is, the example I give can be always applied to the general case and have the same winner set since our limited function does not change.

Also, from now on, instead of writing in the form of $\xi(f(v_i)_1, \ldots, f(v_i)_m) = |f(v_i)_1| + \cdots + |f(v_i)_m| - \frac{mb}{2} = 0$ for our limited function, I will write $|f(v_i)_1| + \cdots + |f(v_i)_m| = \frac{mb}{2}$, where they expresses the same thing but the second one looks more familiar.

In the next part, we will check if range voting function with limit works better in satisfying the criterions we showed in Section 2.2, and therefore to see if this restriction actually helps to improve Range Voting, whether it matters if we impose some restrictions on $R_{V,A,B}$.

2.4 Properties of Social Choice Procedures for Range Voting Function with Limits

2.4.1 Always-A-Winner Condition for Range Voting Function with Limits

Theorem 2.4.1. *Range voting function with limits satisfies Always-A-Winner Condition.*

The proof is the same as for Theorem 2.2.2 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits.

2.4.2 Majority Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Majority Criterion for Range Voting. The counter example is the same as in Example 2.2.4 except we let $\Phi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits, and we observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}$ for all $i \in \{1, 2, 3\}$.

2.4.3 Condorcet Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Majority Criterion for Range Voting. The counter example is the same as in Example 2.2.8 except we let $\Phi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits, and we observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}$ for all $i \in \{1, 2, 3\}$.

2.4.4 Pareto Criterion for Range Voting Function with Limits

Theorem 2.4.2. *Range voting function with limits satisfies Pareto Criterion for range voting.*

The proof is the same as for Theorem 2.2.10 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits.

2.4.5 Monotonicity Criterion for Range Voting Function with Limits

Theorem 2.4.3. *Range voting function with limits satisfies Monotonicity Criterion for range voting.*

The proof is the same as for Theorem 2.2.12 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits.

2.4.6 Independence of Irrelevant Alternative Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterionfor Range Voting. The counter example is the same as in Example 2.2.14 except we let $\Phi : {R^{\xi}}_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function with limits, and we observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}$ for all $i \in \{1, 2, 3\}$.

2.4.7 Intensity of Independence of Irrelevant Alternative Criterion for Range Voting Function with Limits

Theorem 2.4.4. *The range voting function with limits satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.*

The proof is the same as for Theorem 2.2.17 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : {R^{\xi}}_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function with limits.

2.4.8 Participation Criterion for Range Voting Function with limits

Theorem 2.4.5. *Range voting function with limits satisfies participation criterion.*

The proof is the same as for Theorem 2.2.19 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : {R^{\xi}}_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function with limits.

2.4.9 Consistent Property for range voting with limits

Theorem 2.4.6. *The range voting function with limits satisfies consistent property.*

The proof is the same as for Theorem 2.2.22 except we $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$, and we let $\Phi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits, and $f_1 \in {R^{\xi}}_{V_1,A,B}$ and $f_2 \in {R^{\xi}}_{V_2,A,B}$.

2.4.10 Faithful Property for range voting with limits

Theorem 2.4.7. *The range voting function with limits satisfies faithful property.*

The proof is the same as for Theorem 2.2.24 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$, and we let $\Phi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits.

2.4.11 Cancellation Property for range voting with limits

The range voting function with limits does not satisfy the cancellation property. The counter example is the same as in Example 2.2.26 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and $\Phi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits, and we observe that $|f(v_i)_1| + |f(v_i)_2| = 10 = \frac{mb}{2} = \frac{2 \cdot 10}{2}$ for all $i \in \{1, 2\}$.

3

A-R Voting Function With Limits

3.1 Approval voting function

Definition 3.1.1. The **approval voting function** $\alpha : R_{V,A,B} \to \mathcal{P}(A)$ is the same as the range voting function $\phi : R\ _{V,A,B} \to \mathcal{P}(A)$ in the case where $B = \{0, 1\}$. $\triangle$

Here is an example for approval voting function.

Example 3.1.2. Let $V = \{v_1, \ldots, v_4\}$ be a set of voters and $A = \{a_1, \ldots, a_4\}$ be a set of alternatives. Let $B \in \{0, 1\}$. Let $\alpha : R_{V,A,B} \to \mathcal{P}(A)$ be the approval voting function.

Let $f \in R\ _{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3	v_4
a_1	0	0	0	1
a_2	1	1	0	0
a_3	1	0	0	1
a_4	1	1	1	0

Let S_f be the scoring function for V and A. We notice that $S_f(a_1) = 0 + 0 + 0 + 1 = 1$, $S_f(a_2) = 1 + 1 + 0 + 0 = 2$, $S_f(a_3) = 1 + 0 + 0 + 1 = 2$ and $S_f(a_4) = 1 + 1 + 1 + 0 = 3$. Since $S_f(a_4) \geq S_f(a_j)$ for all $j \in \{1, \ldots, 4\}$, then $\alpha(f) = \{a_4\}$. $\lozenge$

It's obvious that the approval voting function satisfies the same properties mentioned above that are satisfied by range voting function. More than this, we notice that approval voting plays a perfect role in showing voters' attitude towards alternatives on whether he/she approves the alternative or not.

Inspired by this, we hope to combine approval voting function with our range voting function with limits. In the approval voting function, voters show their disapproval attitude towards the alternative by giving 0 score, as well as show their approval by giving 1 score. In the previous chapter, we allows negative scores in our range voting system, since we set $B = [-b, b]$; so it comes to our mind that if we can use negative scores to Exampleress disapproval and positive score for approval. In this way, we can observe whether a voter like or dislike the alternative, eliminate the alternatives who are not approved by the majority of voters as well as determine the winner(s) by summing the score, which is actually the same process in range voting function besides the approval voting part.

3.2 Approval-Range Voting function

Definition 3.2.1. 1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For all $f \in R_{V,A,B}$, the alternative $a_k \in A$ is **approved** by v_i if $f(v_i)_k \geq 0$ for some $i \in \{1, \ldots, v_n\}$.

2. An alternative $a_k \in A$ is **majority-approved** if a_k is approved by v_i for at least $\frac{n}{2}$ values of i. The **approved alternative set** A_f^p of A is the subset of A such that a_k is majority-approved.

3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let A_f^p be the approved alternative set for A. Let S_f be the scoring function for V and A. The **A-R voting function** is a

function $\epsilon : R_{V,A,B} \to \mathcal{P}(A)$ defined by letting $\epsilon(f)$ be the subset of all $a_k \in A_f^p$ such that $S_f(a_k) \geq S_f(a_j)$ for all $a_j \in A_f^p$, for all $f \in R_{V,A,B}$.

△

Here, let us look at an example to see how A-R voting function differs from regular range voting function. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $f \in R_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	-2	-1	10
a_2	1	-3	3
a_3	6	10	-10

Let S_f be the scoring function for V and A. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. By regular range voting function, we have $S_f(a_1) = -2 + (-1) + 10 = 7$, $S_f(a_2) = 1 + (-3) + 3 = 1$, $S_f(a_3) = 6 + 10 + (-10) = 6$. Since $S_f(a_1) \geq S_f(a_j)$ for all $j \in \{1, 2, 3\}$, for all $f \in R_{V,A,B}$, then we know that $\phi(f) = \{a_1\}$.

Now, let us consider A-R voting function. Let $\epsilon : R_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function. Here we can see alternative a_1 is given positive score by 1 voters v_3, a_2 is given positive scores by 2 voters v_1 and v_3, and a_3 is given positive scores by 2 voters v_1 and v_2. Therefore, we know that a_2 and a_3 are approved by the majority of voters. Hence, the approved alternative set for A is $A_f^p = \{a_2, a_3\}$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then $\epsilon(f) = \{a_3\}$ instead of $\{a_1\}$ in the regular range voting function.

3.3 Approval-Range voting function with limits

This A-R voting function also works with limited function.

Definition 3.3.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. Let A_f^p be the approved alternative set for A. Let S_f be the scoring function for V and A. The **A-R voting function with limits** is a function $\varepsilon : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ defined by letting $\varepsilon(f)$ be the subset of A^p_f such that $S_f(a_k) \geq S_f(a_j)$ for all $a_k \in \varepsilon(f)$ and $a_j \in A^p_f$, for all $f \in R^{\xi}_{V,A,B}$. $\triangle$

Here is an example for Approval-Range voting function with limits.

Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\varepsilon : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limit.

Let us consider the following situation. Let $f \in R^{\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	-1	-1	10
a_2	8	-10	2
a_3	6	4	-3

Let S_f be the scoring function for V and A. As we can see from the chart, $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$. Let $\Phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits. By regular range voting function, we have $S_f(a_1) = (-1) + (-1) + 10 = 8$, and $S_f(a_2) = 8 + (-10) + 2 = 0$, and $S_f(a_3) = 6 + 4 + (-3) = 7$. Since $S_f(a_1) \geq S_f(a_j)$ for all $j \in \{1, 2, 3\}$, for all $f \in R_{V,A,B}$, then we know that $\Phi(f) = \{a_1\}$.

Now, let us consider A-R voting function with limits. Let $\varepsilon : R_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function. Here we can see alternative a_1 is given positive score by 1 voters v_3, a_2 is given positive scores by 2 voters v_1 and v_3, and a_3 is given positive scores by 2 voters v_1 and v_2. Therefore, we know that a_2 and a_3 are approved by the majority of voters. Hence, the approved alternative set for A is $A^p_f = \{a_2, a_3\}$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then $\varepsilon(f) = \{a_3\}$ instead of $\{a_1\}$ in the regular range voting function.

3.4 Properties for A-R Voting Function with Limits

3.4.1 Always-A-Winner Condition for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting by illustrating a counter example.

Example 3.4.1. Let $V = \{v_1, \ldots, v_3\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	-5	5	-5
a_2	-5	-5	5
a_3	5	-5	-5

As we can see from the chart, $|f(v_i)_1|+|f(v_i)_2|+|f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Here we know alternative a_1 is given positive score by 1 voters v_2, a_2 is given positive score by 1 voters v_3, and a_3 is given positive score by 1 voters v_1. Since none of them is majority-approved, then the approved alternative set $A^p_f = \emptyset$. Therefore, $\varepsilon(f) = \emptyset$. $\Diamond$

Thus, A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting.

However, considering that this condition does not usually happen in the reality, we can somehow improve our A-R voting function to avoid this violation of always-a-winner condition.

Definition 3.4.2. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. Let A^p_f be the approved alternative set for A. Let S_f be the scoring function for V and A. Let $\varepsilon : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. The **improved A-R voting function with limits** is a function $\varepsilon' : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ such that for all $f \in R^{\xi}_{V,A,B}$, and if $A^p_f \neq \emptyset$, then $\varepsilon'(f) = \varepsilon(f)$; If $A^p_f(f) = \emptyset$, then $\varepsilon'(f)$ is be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$.

△

In this way, the improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting as well as regular range voting function does.

The following theorem is trivial.

Theorem 3.4.3. *The improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting.*

In the reality world, although it barely happens that in an election, all the alternatives are not liked by most voters, which means the people who hold such an election should probably reconsider the legitimacy of their alternatives, mathematically we need to consider the case that every alternative is not approved by most of voters. If such an election is for a small group of people, for example club election, then when this situation takes place, people should consider if they pick the suitable alternatives and therefore may hold another election with different alternatives; if this happens in politics, with a great amount of voters, it may be unrealistic to regather our voters and ask them to vote for a new group of alternatives, then our improved A-R voting function with limits can be used in such condition, in order to choose the one(s) that our voters disapproves the least.

3.4.2 Majority Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Majority Criterion for Range Voting by giving the following example.

Example 3.4.4. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in R^{\xi}_{V,A,B}$ be given as the following chart. (Here, all f are the same as the ones for Example 2.2.4.)

	v_1	v_2	v_3
a_1	5	3	-9
a_2	2	2	3
a_3	-8	-10	3

As we can see from the chart, $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Here we can see alternative a_1 is given positive score by 2 voters v_1 and v_2, a_2 is given positive score by all 3 voters, and a_3 is given positive or 0 score by 1 voter v_3. Therefore $A^p_f = \{a_1, a_2\}$. Let S_f be the scoring function for V and A. We observe that alternative a_1 is most preferred by the majority of voters since there are 2 voters out of 3 who choose a_1 as their most preferred alternative. However, we have $S_f(a_1) = 5 + 3 + (-9) = -1$, and $S_f(a_2) = 2 + 2 + 3 = 7$, $S_f(a_3) = -8 + (-10) + 3 = -1$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_2\}$ instead of $\{a_1\}$. $\Diamond$

3.4.3 Condorcet Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Condorcet Criterion for Range Voting by giving the following example.

Example 3.4.5. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in R^{\xi}_{V,A,B}$ be given as the following chart. (Here, all f are the same as the ones for Example 2.2.8.)

As we can see from the chart, $|f(v_i)_1|+|f(v_i)_2|+|f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

	v_1	v_2	v_3
a_1	-7	5	6
a_2	8	4	2
a_3	0	-6	-7

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_2	a_1	a_1
a_3	a_2	a_2
a_1	a_3	a_3

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following. The alternative with square is the one who wins the pairwise comparison.

$\boxed{a_1}$ vs a_2 2:1
$\boxed{a_1}$ vs a_3 2:1
$\boxed{a_2}$ vs a_3 2:1

As shown in the table a_1 wins each pairwise comparison with a_2 and a_3, therefore a_1 is the Condorcet winner.

However, we observe that alternative a_1 is given positive score by 2 voters v_2 and v_3, a_2 is given positive score by all 3 voters, and a_3 is given positive or 0 score by only 1 voters v_1. Therefore, a_1 and a_2 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = (-7) + 5 + 6 = 4$ and $S_f(a_2) = 8 + 4 + 2 = 14$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f) = \{a_2\}$ instead of $\{a_1\}$. $\diamondsuit$

3.4.4 Pareto Criterion for A-R Voting Function with Limits

Theorem 3.4.6. *A-R voting function with limits satisfies Pareto Criterion for range voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, -b]$. Let S_f be the score function for

V and A. Let $\varepsilon : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limit. Let $f \in {R^\xi}_{V,A,B}$. Let $A^p_f \subseteq A$ be the approved alternative set of A. Let a_k, $a_j \in A$. Suppose we have $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$.

There are three cases for this situation. First suppose $a_k \notin A^p_f$ and $a_j \notin A^p_f$. Then $a_j \notin \varepsilon(f)$.

Second, suppose $a_k \in A^p_f$ and $a_j \notin A^p_f$. Then $a_j \notin \varepsilon(f)$.

Third, suppose $a_k \in A^p_f$ and $a_j \in A^p_f$. Then we have $S_f(a_k) > S_f(a_j)$. By the definition of the A-R voting function with limits, we have $a_j \notin \varepsilon(f)$.

It cannot be the case that $a_k \notin A^p_f$ and $a_j \in A^p_f$, since $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, ...n\}$.

Therefore a_j can never be in $\varepsilon(f)$. $\square$

3.4.5 Monotonicity Criterion for A-R Voting Function with Limits

Theorem 3.4.7. *A-R voting function with limits satisfies Monotonicity Criterion for range voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, -b]$. For $f \in {R^\xi}_{V,A,B}$, let S_f be the score function for V and A. Let $\Phi : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limit. Let $A^p_f \subseteq A$ be the approved alternative set of A. Let $j \in \{1, \ldots, m\}$.

Suppose a_j is in $\varepsilon(f)$. Then we know $a_j \in A^p_f$ and $S_f(a_j) \geq S_f(a_k)$ for all $a_k \in A^p_f$. If $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$, then we have $a_j \in A^p_f$ still and $S_f(a_j) > S_f(a_k)$ for all $a_k \in A^p_f$. Hence, by the definition of range voting function with limits, we have $a_j \in \varepsilon(f)$. $\square$

3.4.6 Independence of Irrelevant Alternative Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 3.4.8. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart. (Here, all f are the same as the ones for Example 2.2.14.)

	v_1	v_2	v_3
a_1	7	-7	2
a_2	2	5	-6
a_3	-6	3	7

As we can see from the chart, $|f(v_i)_1|+|f(v_i)_2|+|f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Let S_f be the scoring function for V and A. We can see alternative a_1 is given positive or 0 scores by 2 voters, v_1 and v_3, while a_2 is given positive or 0 scores by 2 voters v_1 and v_2, and a_3 is given positive or 0 scores by 2 voters v_2 and v_3. Therefore, the approved alternative set $A^p_f = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_3\}$.

We change $f(v_3)_1$ and $f(v_3)_3$ but keep $f(v_3)_1 < f(v_3)_3$, we get new $f' \in {R^{\xi}}_{V,A,B}$ as in the following table.

	v_1	v_2	v_3
a_1	7	-7	4
a_2	2	5	-6
a_3	-6	3	5

After changing $f(v_3)_1$ and $f(v_3)_3$, let $S_{f'}$ be the score function for new V and A. We can see alternative a_1 is given positive or 0 scores by 2 voters, v_1 and v_3, while a_2 is given positive or 0 scores by 2 voters v_1 and v_2, and a_3 is given positive or 0 scores by 2 voters v_2 and v_3. Therefore, the approved alternative set $A^p_{f'} = \{a_1, a_2, a_3\}$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A^p_{f'}$, then we know that $\varepsilon(f') = \{a_1\}$ instead of $\{a_3\}$. $\Diamond$

3.4.7 Intensity of Independence of Irrelevant Alternative Criterion for A-R Voting Function with Limits

Theorem 3.4.9. *A-R voting function with limits satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\varepsilon : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limit. Let $f \in R^{\xi}{}_{V,A,B}$ and S_f be the score function for V and A. Let $j, k \in \{1, \ldots, m\}$.

Suppose that $a_j \in \varepsilon(f)$ and $a_k \notin \varepsilon(f)$, and for some $i \in \{1, \ldots, n\}$, and $f(v_i)_p$ changes for some $p \in \{1, \ldots, m\}$ while $f(v_i)_j - f(v_i)_k$ remains the same.

There are three cases for this situation. First, suppose that after $f(v_i)_p$ changes, we have $a_j \notin A^p_{f'}$ and $a_k \notin A^p_{f'}$ where $f' \in R^{\xi}{}_{V,A,B}$ as our new profile. Then $a_k \notin \varepsilon(f')$.

Second, suppose that $a_j \in A^p_{f'}$ and $a_k \notin A^p_{f'}$. Then $a_k \notin \varepsilon(f')$.

Third, suppose that $a_j \in A^p_{f'}$ and $a_k \in A^p_{f'}$. Since none of the voters change their intensity of preference for a_j over a_k, which means $f(v_i)_j - f(v_i)_k$ does not change for all $i \in \{1, \ldots, n\}$, then $S_f(a_j) - S_f(a_k)$ does not change as well. So we always have $S_f(a_j) > S_f(a_k)$. By the definition of the A-R voting function with limits, we know $a_k \notin \varepsilon(f)$.

It cannot be the case that $a_j \notin A^p_f$ and $a_k \in A^p_f$, since $f(v_i)_j - f(v_i)_k$ does not change.

Therefore, we conclude that it is always the case that $a_k \notin \varepsilon(f)$. □

3.4.8 Consistency Property for A-R Voting Function with Limits

I will show that the A-R Voting function with limits does not satisfy consistency property by giving the following example.

Example 3.4.10. Let $V = \{v_1, \ldots, v_6\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\varepsilon : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in R^{\xi}{}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3	v_4	v_5	v_6
a_1	10	-1	-1	8	5	3
a_2	0	-6	10	-2	7	4
a_3	-5	8	4	5	3	8

As we can see from the chart, $|f(v_i)_1|+|f(v_i)_2|+|f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, \ldots, 6\}$.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$.

Here we can get disjoint profiles $f_1 \in R_{V_1,A,B}$ as below.

	v_1	v_2	v_3
a_1	10	-1	-1
a_2	0	-6	10
a_3	-5	8	4

In f_1,we can see alternative a_1 is given positive score or zero by 1 voter v_1, and a_2 is given positive or 0 score by 2 voters v_1, v_3, and a_3 is given positive or 0 score by 2 voters, v_2, v_3. Therefore, a_2 and a_3 are majority-approved, then the approved alternative set $A^p_f = \{a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_2) = 0 + (-6) + 10 = 4$, $S_f(a_3) = (-5) + 8 + 4 = 7$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f_1) = \{a_3\}$.

Then let us look at $f_2 \in R_{V_2,A,B}$ as given.

	v_4	v_5	v_6
a_1	8	5	3
a_2	-2	7	4
a_3	5	3	8

In f_2, we can see alternative a_1 is given positive or 0 score by all 3 voters, a_2 is given positive score by 2 voters v_5 and v_6, and a_3 is given positive or 0 score by all 3 voters. Therefore, a_1, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = 8+5+3 = 16$, $S_f(a_2) = (-2)+7+4 = 9$, $S_f(a_3) = 5+3+8 = 16$. Since $S_f(a_1) = S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f_2) = \{a_1, a_3\}$.

Then we observe that $\varepsilon(f_1) \cap \varepsilon(f_2) = \{a_3\}$.

However, let us take a look at $\varepsilon(f_1 + f_2)$. In $f_1 + f_2$, we can see alternative a_1 is given positive score by 4 voters v_1, v_4, v_5 and v_6, a_2 is given positive or 0 score by 4 voters v_1, v_3, v_5 and v_6, and a_3 is given positive or 0 score by 5 voters except for v_1. Therefore, the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = 10 + (-1) + (-1) + 8 + 5 + 3 = 24$, $S_f(a_2) = 0 + (-6) + 10 + (-2) + 7 + 4 = 13$, $S_f(a_3) = (-5) + 8 + 4 + 5 + 3 + 8 = 23$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f_1 + f_2) = \{a_1\}$.

Here, we notice that $\varepsilon(f_1 + f_2) = \{a_1\}$ while $\varepsilon(f_1) \cap \varepsilon(f_2)$ does exist and is actually $\{a_3\}$. Hence $\varepsilon(f_1 + f_2) \neq \varepsilon(f_1) \cap \varepsilon(f_2)$. $\diamondsuit$

3.4.9 Faithful Property for A-R Voting Function with Limits

Theorem 3.4.11. *A-R voting function with limits satisfies faithful property.*

Proof. Let $V = \{v\}$ be a set of voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\varepsilon : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let S_f be the scoring function for V and A.

Suppose $f(v)_j > f(v)_k$ for some $j, k \in \{1, \ldots, m\}$. There are three cases for this situation.

First, suppose that $a_j \notin A^p_f$ and $a_k \notin A^p_f$. Then $a_k \notin \varepsilon(f)$.

Second, suppose that $a_j \in A^p_f$ and $a_k \notin A^p_f$. Then $a_k \notin \varepsilon(f)$.

Third, suppose that $a_j \in A^p_f$ and $a_k \in A^p_f$. Since $f(v)_j > f(v)_k$, then $S_f(a_j) > S_f(a_k)$ with only one voter. By the definition of the A-R voting function with limits, we know $a_k \notin \varepsilon(f)$.

It cannot be the case that $a_j \notin A^p_f$ and $a_k \in A^p_f$, since $f(v)_j > f(v)_k$.

Therefore, we always have $a_k \notin \varepsilon(f)$. $\square$

3.4.10 Cancellation Property for A-R Voting Function with limits

I will show that the A-R voting function does not satisfy the cancellation property by giving the following example.

Example 3.4.12. Let $V = \{v_1, v_2\}$ be a set of voters and $A = \{a_1, a_2\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\Phi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the range voting function with limits. Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart. (Here, all f are the same as the ones for Example 2.2.26.)

	v_1	v_2
a_1	8	3
a_2	2	7

As we can see from the chart $|f(v_i)_1| + |f(v_i)_2| = \frac{mb}{2} = \frac{2 \cdot 10}{2} = 10$ for all $i \in \{1, 2\}$. And we observe from the chart that $f(v_1)_1 > f(v_1)_2$. So $\pi_{a_1a_2}(f) = 1$. Also $f(v_2)_2 > f(v_2)_1$, which means $\pi_{a_2a_1}(f) = 1$. Thus, in this case, $\pi_{a_1a_2}(f) = \pi_{a_2a_1}(f) = 1$.

Let S_f be the scoring function for V and A. Alternative a_1 is given positive score by all 2 voters v_1, a_2 is given positive score by all 2 voters. Therefore, the approved alternative set $A^p_f = \{a_1, a_2\}$. We see that $S_f(a_1) = 8 + 3 = 11$ and $S_f(a_2) = 2 + 7 = 9$. Since $S_f(a_1) > S_f(a_2)$, then $\varepsilon(f) = \{a_1\}$.

However, we notice that $\varepsilon(f) \neq A$.

Hence, A-R voting function with limits does not satisfy the cancellation property. $\Diamond$

3.4.11 Participation Criterion for A-R Voting Function with limits

I will show that A-R voting function with limits does not satisfy Participation Criterion by giving a counter example.

Example 3.4.13. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in R^{\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	10	-1	-1
a_2	2	7	-6
a_3	3	7	-8

From this chart, we can see alternative a_1 is given positive score by 1 voter v_1, a_2 is given positive or 0 score by 2 voters v_1, v_2, and a_3 is given positive or 0 score by 2 voter v_1, v_2. Therefore, a_2 and a_3 are majority-approved, then the approved alternative set $A^p_f = \{a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_2) = 2 + 7 + (-6) = 3$, $S_f(a_3) = 3 + 7 + (-8) = 2$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_2\}$.

Now add another v_4 to our voter set, with $f(v_4)_1 < f(v_4)_2$. Then we obtain $f' \in {R^{\xi}}_{V,A,B}$ as follow.

	v_1	v_2	v_3	v_4
a_1	10	-1	-1	5
a_2	2	7	-6	6
a_3	3	7	-8	4

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, \ldots, 4\}$.

From this new chart, we can see alternative a_1 is given positive score by 2 voter v_1, v_4, a_2 is given positive or 0 score by 3 voters v_1, v_2, v_4, and a_3 is given positive or 0 score by 3 voters v_1, v_2, v_4. Therefore, a_1, a_2 and a_3 are majority-approved, then the approved alternative set $A^p_f = \{a_1, a_2, a_3\}$. Then we have $S_f(a_1) = 10+(-1)+(-1)+5 = 13$, and $S_f(a_2) = 2+7+(-6)+6 = 9$, and $S_f(a_3) = 3 + 7 + (-8) + 6 = 8$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_1\}$.

We observe that before v_4 is added, $a_2 \in \varepsilon(f)$ and $a_1 \notin \varepsilon(f)$. However, we get $a_1 \in \varepsilon(f)$ and $a_2 \notin \varepsilon(f)$.

Hence, A-R voting function with new limits does not satisfy participation criterion. $\diamondsuit$

Since here is the case that a_1 gets 2 approved and 2 disapproved and hence become an element of the approved alternative set, which is similar to a tied condition; I manage to add another v_5 to see what happens if we add two voters for Participation Criterion. Same as above, we let $f(v_5)_1 < f(v_5)_2$. And we obtain our new $f \in {R^{\xi}}_{V,A,B}$ as follow.

Example 3.4.14. Let $V = \{v_1, \ldots, v_5\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3	v_4	v_5
a_1	10	-1	-1	5	5
a_2	2	7	-6	6	6
a_3	3	7	-8	4	4

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, \ldots, 5\}$.

We can see alternative a_1 is given positive score by 3 voter v_1, v_4, v_5; and a_2 is given positive or 0 score by 4 voters v_1, v_2, v_4, v_5, and a_3 is given positive or 0 score by 4 voters v_1, v_2, v_4 and v_5. Therefore, a_1, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Then we have $S_f(a_1) = 10+(-1)+(-1)+5+5 = 18$, $S_f(a_2) = 2+7+(-6)+6+6 = 16$, $S_f(a_3) = 3 + 7 + (-8) + 6 + 4 = 14$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f) = \{a_1\}$.

Hence, we still get $a_1 \in \varepsilon(f)$ and $a_2 \notin \varepsilon(f)$.

Therefore, A-R voting function with new limits does not satisfy participation criterion as well. $\Diamond$

4

C-A-R Voting Function With Limits

4.1 Condorcet-Approval-Range voting function with limits

As we introduced Condorcet voting method earlier, we can review our definition for Condorcet winner for range voting through Definition 2.2.5 to Definition 2.2.6.

Definition 4.1.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let A_f^p be the approved alternative set for A. Let S_f be the scoring function for V and A. Let $\varepsilon : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. The **C-A-R voting function with limits** is a function $\psi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ defined by two conditions. First, determine whether there exists a Condorcet winner a_p in A; if so, let $\psi(f) = \{a_p\}$. If there does not exist a Condorcet winner, let $\psi(f) = \varepsilon(f)$. $\triangle$

I will illustrate this C-A-R voting function by showing the following example.

Example 4.1.2. Let $V = \{v_1, \ldots, v_4\}$ be a set of voters and $A = \{a_1, \ldots, a_4\}$ be a set of alternatives. Let $b > 0$, then $B = [-10, 10]$. Let $\psi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3	v_4
a_1	-5	-5	-5	10
a_2	4	4	4	10
a_3	-1	-1	-1	0
a_4	10	10	10	0

We observe that $|f(v_i)_1| + \cdots + |f(v_i)_4| = \frac{mb}{2} = \frac{4 \cdot 10}{2} = 20$ for all $i \in \{1, \ldots, 4\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3	v_4
a_4	a_4	a_4	a_1, a_2(tied)
a_2	a_2	a_2	a_3, a_4 (tied)
a_3	a_3	a_3	
a_1	a_1	a_1	

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

a_1 vs $\boxed{a_2}$ 0:3	$\boxed{a_2}$ vs a_3 4:0	a_3 vs $\boxed{a_4}$ 0:3
a_1 vs $\boxed{a_3}$ 1:3	a_2 vs $\boxed{a_4}$ 1:3	
a_1 vs $\boxed{a_4}$ 1:3		

As shown in the graph, a_4 wins each pairwise comparison with a_2, a_3 and a_4, therefore a_4 is the Condorcet winner. Hence $\psi(f) = \{a_4\}$. $\Diamond$

Now let us look at another example that there is no Condorcet winner.

Example 4.1.3. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, v_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	-1	10	-1
a_2	8	5	-4
a_3	6	0	10

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_2	a_1	a_3
a_3	a_2	a_1
a_1	a_3	a_2

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the chart, no alternative a_p wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

Here we can see alternative a_1 is given positive score by 1 voters v_3, a_2 is given positive scores by 2 voters v_1 and v_2, and a_3 is given positive or 0 scores by all 3 voters. Therefore, we know that a_2 and a_3 are approved by the majority of voters. Hence, the approved alternative set for A is $A_f^p = \{a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_2) = 8 + 5 + (-4) = 9$ and $S_f(a_3) = 6 + 0 + 10 = 16$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then $\psi(f) = \{a_3\}$. $\diamondsuit$

4.2 Properties for C-A-R Voting Function with Limits

4.2.1 Always-A-Winner Condition for C-A-R Voting Function with Limits

I will show that C-A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting. by giving the following example.

Example 4.2.1. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	-1	10	-3
a_2	-5	-1	10
a_3	10	-4	-2

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_3	a_1	a_2
a_1	a_2	a_3
a_2	a_3	a_1

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the graph, there is no Condorcet winner. Now we check the approval alternative set. Here we can see alternative a_1 is given positive score by 1 voters v_2, a_2 is given positive score by 1 voters v_3, and a_3 is given positive score by 1 voters v_1. Since none of them is majority-approved, then the approved alternative set $A_f^p = \emptyset$. Therefore, $\psi(f) = \emptyset$. Hence, there is no winner for this example. $\Diamond$

Thus, C-A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting.

However, considering that this condition does not usually happen in the reality, similarly we can do what we did for A-R voting to improve C-A-R voting function and avoid this violation of always-a-winner condition.

Definition 4.2.2. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. Let A_f^p be the approved alternative set for A. Let S_f be the scoring function for V and A. Let $\psi : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. The **improved C-A-R voting function with limits** is a function $\psi' : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ such that for all $f \in {R^\xi}_{V,A,B}$, if there exists a Condorcet winner or $A_f^p \neq \emptyset$, then $\psi'(f) = \psi(f)$, and if $A_f^p(f) = \emptyset$, then $\psi'(f)$ is the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$. $\triangle$

In this way, improved C-A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting as well as regular range voting function does.

The following theorem is trivial.

Theorem 4.2.3. *The improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting.*

4.2.2 Majority Criterion for C-A-R Voting Function with Limits

Theorem 4.2.4. *C-A-R voting function with limits satisfies the Majority Criterion for Range Voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Suppose $k \in \{1, \ldots, m\}$ and a_k is most preferred by more than $\frac{n}{2}$ voters, which says that a_k wins each pairwise comparison for all $j \in \{1, \ldots, m\}$ such that $k \neq j$.

Then by the definition of C-A-R voting function, we know that we do have a Condorcet winner in A which is a_k. Hence, $a_k \in \psi(f)$ for all $f \in {R^{\xi}}_{V,A,B}$. $\square$

4.2.3 Condorcet Criterion for C-A-R Voting Function with Limits

Theorem 4.2.5. *C-A-R voting function with limits satisfies the Condorcet Criterion for Range Voting.*

The proof for this theorem is trivial.

4.2.4 Pareto Criterion for C-A-R Voting Function with Limits

Theorem 4.2.6. *C-A-R voting function with limits satisfies the Pareto Criterion for Range Voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$. Suppose $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$. There are actually two cases depended on Condorcet winner. If there exists a Condorcet winner in A, by the definition of C-

A-R voting function, since a_j cannot win pairwise comparison with a_k, then a_j naturally cannot be that Condorcet winner.

Suppose there is no Condorcet winner. Let S_f be the scoring function for V and A. There are three cases. First, suppose a_j, $a_k \in A_p^f$. However, since we know that it is always the case that $f(v_i)_k > f(v_i)_j$, then we have $S_f(a_k) > S_f(a_j)$, then by the definition of C-A-R voting function with limits, so $a_j \notin \psi(f)$. Second, suppose $a_k \in A_p^f$ and $a_j \notin A_p^f$. Then $a_j \notin \psi(f)$. Third, suppose a_k, $a_j \notin A_p^f$. Then still $a_j \notin \psi(f)$. Note that it cannot be the case that $a_j \in A_p^f$ but $a_k \notin A_p^f$, since $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$.

Therefore, we always have $a_j \notin \psi(f)$. □

4.2.5 Monotonicity Criterion for C-A-R Voting Function with Limits

Theorem 4.2.7. *C-A-R voting function with limits satisfies the Monotonicity Criterion for Range Voting.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\psi : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

For all $f \in R^{\xi}_{V,A,B}$, suppose a_j is in $\psi(f)$ and $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$. There are two cases. First, suppose there exists a Condorcet winner before $f(v_i)_j$ increases, which is a_j. By the definition of C-A-R voting function, since a_j wins each pairwise comparison, then a_j still wins each pairwise comparison when $f(v_i)_j$ increases. Hence a_j is still in $\psi(f)$. Second, suppose there is no Condorcet winner before $f(v_i)_j$ increases. Let S_f be the scoring function for V and A. Since $a_j \in \psi(f)$, then from the definition of C-A-R voting function, we know that $S_f(a_j) \geq S_f(a_k)$ for all $a_k \in A_f^p$. When $f(v_i)_j$ increases, we get that $S_f(a_j) > S_f(a_k)$ for all

$a_k \in A_f^p$. By the definition of C-A-R voting function, a_j is still in $\psi(f)$. Therefore, under both cases, $a_j \in \psi(f)$. □

4.2.6 Independence of Irrelevant Alternative Criterion for C-A-R Voting Function with Limits

I will show C-A-R voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 4.2.8. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\psi : R^{\xi}{}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R^{\xi}{}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	7	-7	2
a_2	2	5	-6
a_3	-6	3	7

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the chart, no alternative a_p wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

Here we can see alternative a_1 is given positive or 0 scores by 2 voters, v_1 and v_3, a_2 is given positive or 0 scores by 2 voters v_1 and v_2, and a_3 is given positive or 0 scores by 2 voters v_2 and v_3. Therefore, we know that a_1, a_2 and a_3 are approved by the majority of voters. Hence, the approved alternative set for A is $A^p_f = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = 7 + (-7) + 2 = 2$, $S_f(a_2) = 2 + 5 + (-6) = 1$, and $S_f(a_3) = -6 + 3 + 7 = 4$. Since$S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\psi(f) = \{a_3\}$.

We change $f(v_3)_1$ and $f(v_3)_3$ but keep $f(v_3)_1 < f(v_3)_3$, then we have new $f' \in R^{\xi}{}_{V,A,B}$ as in the following table.

	v_1	v_2	v_3
a_1	7	-7	4
a_2	2	5	-6
a_3	-6	3	5

Same as above, we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the graph, no alternative a_p wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

Let $S_{f'}$ be the score function for our new V and A. Before we change $f(v_3)_2$ and $f(v_3)_2$, $\psi(f) = \{a_3\}$. However, in the second table, since $S_{f'}(a_1) = 7 + (-7) + 4 = 4$ while $S_{f'}(a_3) = -6 + 3 + 5 = 2$, $S_{f'}(a_1) > S_{f'}(a_3)$, then $\psi(f') = \{a_1\}$. Therefore, C-A-R voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting. $\diamondsuit$

4.2.7 Intensity of Independence of Irrelevant Alternative Criterion for C-A-R Voting Function with Limits

I will show that C-A-R voting function with limits does not satisfy the Intensity of Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 4.2.9. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\psi : R^{\xi}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R^{\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	5	5	5
a_2	4	10	4
a_3	-6	0	6

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_1	a_1
a_3	a_3	a_2

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1 | $\boxed{a_2}$ vs a_3 2:1
$\boxed{a_1}$ vs a_3 2:1 |

As shown in the chart, alternative a_1 wins each pairwise comparison for all $k \in \{2, 3\}$ and therefore a_1 is the Condorcet winner.

We change $f(v_2)_1$, as well as $f(v_2)_2$ and $f(v_2)_3$ but keep $f(v_2)_1 - f(v_2)_2$ the same as before, then we have new $f' \in R^{\xi}_{V,A,B}$ as in the following table.

	v_1	v_2	v_3
a_1	5	2	5
a_2	4	7	4
a_3	-6	6	6

Same as above, we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1 | $\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2 |

As shown in the graph, no alternative a_p wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

Let $S_{f'}$ be the score function for our new V and A. We can see alternative a_1 is given positive or 0 scores by all 3 voters, and a_2 is given positive or 0 scores by all 3 voters, and a_3 is given positive or 0 scores by 2 voters v_2 and v_3. Hence, the approved alternative set for A is $A^p_{f'} = \{a_1, a_2, a_3\}$. Since $S_{f'}(a_1) = 5+2+5 = 12$, and $S_{f'}(a_2) = 4+7+4 = 15$, and $S_{f'}(a_3) = -6+6+6 = 6$. Since $S_{f'}(a_2) \geq S_{f'}(a_k)$ for all $a_k \in A^p_{f'}$, then $\psi(f') = \{a_2\}$. Before we change $f(v_2)_1$, and $f(v_2)_2$ and

$f(v_2)_3$, we have $\psi(f) = \{a_1\}$. Therefore, C-A-R voting function with limits does not satisfy the Intensity of Independence of Irrelevant Alternative Criterion for Range Voting. $\Diamond$

4.2.8 Participation Criterion for C-A-R Voting Function with limits

I will show C-A-R voting function with limits does not satisfy Participation Criterion by giving a counter example.

Example 4.2.10. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\psi : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3
a_1	10	-1	-1
a_2	2	10	-9
a_3	-3	4	5

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the table, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

From this chart, we can see alternative a_1 is given positive score by 1 voter v_1, a_2 is given positive or 0 score by 2 voters v_1, v_2; a_3 is given positive or 0 score by 2 voters v_2, v_3 . Therefore, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_2) = 2 + 10 + (-9) = 3$, $S_f(a_3) = (-3) + 4 + 5 = 6$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f) = \{a_3\}$.

Then add another v_4 to our voter set such that $f(v_4)_1 < f(v_4)_3$. Then we may obtain $f' \in R^{\xi}{}_{V,A,B}$ as follow.

	v_1	v_2	v_3	v_4
a_1	10	-1	-1	4
a_2	2	10	-10	5
a_3	-3	4	4	5

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_1	v_2	v_3	v_4
a_1	a_2	a_3	a_2, a_3 tied
a_2	a_3	a_1	a_1
a_3	a_1	a_2	

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$a_1$vs a_2 2:2	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 3:1	

As shown in the table, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

From this new chart, we can see alternative a_1 is given positive or 0 score by 2 voter v_1, v_4, a_2 is given positive score by 3 voters v_1, v_2, v_4, and a_3 is given positive or 0 score by 3 voters v_2, v_3, v_4. Therefore, a_1, a_2, and a_3 are majority-approved, then the approved alternative set $A^p_{f'} = \{a_1, a_2, a_3\}$. Then we have $S_{f'}(a_1) = 10+(-1)+(-1)+4 = 12$, $S_{f'}(a_2) = 2+10+(-10)+5 = 7$, $S_{f'}(a_3) = (-3) + 4 + 4 + 5 = 10$. Since $S_{f'}(a_1) \geq S_{f'}(a_j)$ for all $a_j \in A^p_{f'}$, then we know that $\psi(f') = \{a_1\}$.

We observe that before v_4 is added, we had $a_3 \in \psi(f)$ and $a_1 \notin \psi(f)$. However, after v_4 is added, we get $a_1 \in \psi(f)$ and $a_3 \notin \psi(f)$. Hence, C-A-R voting function with limits does not satisfy the participation criterion. $\Diamond$

4.2.9 Consistency Property for C-A-R Voting Function with Limits

I will show C-A-R Voting function with limits does not satisfy consistency property by giving a counter example.

Example 4.2.11. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\varepsilon : {R^\xi}_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in {R^\xi}_{V,A,B}$ be given as the following chart.

	v_1	v_2	v_3	v_4	v_5	v_6
a_1	10	-1	-1	8	5	3
a_2	0	-6	10	-2	7	4
a_3	-5	8	4	5	3	8

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, \ldots, 6\}$.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$. Here we can get disjoint profiles $f_1 \in R_{V_1,A,B}$ as below.

	v_1	v_2	v_3
a_1	10	-1	-1
a_2	0	-6	10
a_3	-5	8	4

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_1	v_2	v_3
a_1	a_3	a_2
a_2	a_1	a_3
a_3	a_2	a_1

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\boxed{a_1}$ vs a_2 2:1	$\boxed{a_2}$ vs a_3 2:1
a_1 vs $\boxed{a_3}$ 1:2	

As shown in the graph, In f_1, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

From the table, we can see alternative a_1 is given positive score by all 1 voter v_1, a_2 is given positive or 0 score by all 2 voters v_1, v_3, and a_3 is given positive or 0 score by 2 voters, v_2, v_3. Therefore, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_2) = 0 + (-6) + 10 = 4$, $S_f(a_3) = (-5) + 8 + 4 = 7$. Since $S_f(a_3)) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\psi(f_1) = \{a_3\}$.

Then let us look at $f_2 \in R_{V_2,A,B}$ as given.

	v_4	v_5	v_6
a_1	8	5	3
a_2	-2	7	4
a_3	5	3	8

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

v_4	v_5	v_6
a_1	a_2	a_3
a_3	a_1	a_2
a_2	a_3	a_1

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$a_1$vs $\boxed{a_2}$ 1:2	a_2 vs $\boxed{a_3}$ 1:2
$\boxed{a_1}$ vs a_3 2:1	

As shown in the graph, In f_2, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

In f_2, we can see alternative a_1 is given positive or 0 score by all 3 voters, a_2 is given positive score by 2 voters v_5 and v_6, and a_3 is given positive or 0 score by all 3 voters. Therefore, a_1, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = 8 + 5 + 3 = 16$, $S_f(a_2) = (-2) + 7 + 4 = 9$, $S_f(a_3) = 5 + 3 + 8 = 16$. Since $S_f(a_1) = S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\psi(f_2) = \{a_1, a_3\}$.

Then we observe that $\psi(f_1) \cap \psi(f_3) = \{a_3\}$.

Let us take a look at $\psi(f_1 + f_2)$.

Let us convert $f_1 + f_2$ to preference orders by looking at each column separately and obtain the following table.

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

v_1	v_2	v_3	v_4	v_5	v_6
a_1	a_3	a_2	a_1	a_2	a_3
a_2	a_1	a_3	a_3	a_1	a_2
a_3	a_2	a_1	a_2	a_3	a_1

a_1 vs a_2 3:3	a_2 vs a_3 3:3
a_1 vs a_3 3:3	

As shown in the graph, In $f_1 + f_2$, no alternative $a_p \in$ wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

In $f_1 + f_2$, we can see alternative a_1 is given positive score by 4 voters v_1, v_4, v_5 and v_6, a_2 is given positive or 0 score by 4 voters v_1, v_3, v_5 and v_6, and a_3 is given positive or 0 score by 5 voters except for v_1. Therefore, a_1, a_2 and a_3 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Let S_f be the scoring function for V and A. We have $S_f(a_1) = 10 + (-1) + (-1) + 8 + 5 + 3 = 24$, $S_f(a_2) = 0 + (-6) + 10 + (-2) + 7 + 4 = 13$, $S_f(a_3) = (-5) + 8 + 4 + 5 + 3 + 8 = 23$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\psi(f_1 + f_2) = \{a_1\}$.

Here, we notice that $\psi(f_1+f_2) = \{a_1\}$ while $\psi(f_1) \cap \psi(f_2)$ does exist and is actually $\{a_2\}$. Hence $\psi(f_1 + f_2) \neq \psi(f_1) \cap \psi(f_2)$. Therefore, C-A-R voting function does not satisfy the consistency property. $\diamondsuit$

4.2.10 Faithful Property for C-A-R Voting Function with Limits

Theorem 4.2.12. *C-A-R voting function with limits satisfies the faithful property.*

Proof. Let $V = \{v\}$ be a set of voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, -b]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R_{V,A,B}$ and $a_k, a_j \in A$. Let S_f be the scoring fucntion for V and A. Suppose $f(v)_j > f(v)_k$.

By the definition of ψ, we have pairwise comparison a_j vs a_k: $1:0$. Hence a_k is defeated by a_j in pairwise comparison, then $a_k \notin \psi(f)$. □

4.2.11 Cancellation Property for A-R Voting Function with limits

I will show that C-A-R voting function does not satisfy the cancellation property by showing the following example.

Example 4.2.13. Let $V = \{v_1, v_2\}$ be a set of voters and $A = \{a_1, a_2\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\psi : {R^{\xi}}_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in {R^{\xi}}_{V,A,B}$ be given as the following chart.

	v_1	v_2
a_1	8	4
a_2	2	6

As we can see from the chart $|f(v_i)_1| + |f(v_i)_2| = \frac{mb}{2} = \frac{2 \cdot 10}{2} = 10$ for all $i \in \{1, 2\}$.

We observe that there is one voter v_1 that $f(v_1)_1 > f(v_1)_2$. So, $\pi_{a_1 a_2}(f) = 1$. Also, there is one voter v_2 that $f(v_2)_2 > f(v_2)_1$, which means $\pi_{a_2 a_1}(f) = 1$. Thus, in this case, $\pi_{a_1 a_2}(f) = \pi_{a_2 a_1}(f) = 1$.

After converting the table to preference orders, we obtain the following table.

v_1	v_2
a_1	a_2
a_2	a_1

Now we do pairwise comparisons among the alternatives. Then we get a_1 vs a_2: $1:1$. Hence there is no Condorcet winner. Then we check A-R part.

Let S_f be the scoring function for V and A. Alternative a_1 is given positive score by all 2 voters v_1, a_2 is given positive score by all 2 voters. Therefore, a_1, a_2 are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2\}$. We see that $S_f(a_1) = 8+4 = 12$ and $S_f(a_2) = 2+6 = 8$. Since $S_f(a_1) > S_f(a_2)$, then $\psi(f) = \{a_1\}$.

However, we notice that $\psi(f) \neq A$.

Hence, C-A-R voting function with limits does not satisfy the cancellation property. $\Diamond$

5

Result and further discussion

5.1 Result

Here is the table for what we found in regular range voting function, range voting function with limits, A-R voting function with limits and C-A-R voting function with limits.

	R/ R WL	A-R WL	C-A-R WL
Always A Winner	Yes (Thm 2.2.2)	Yes* (Thm 3.4.3)	Yes* (Thm 4.2.3)
Majority	No (Exp 2.2.4)	No (Exp 3.4.4)	Yes (Thm 4.2.4)
Condorcet	No (Exp 2.2.8)	No (Exp 3.4.5)	Yes (Thm 4.2.5)
Pareto	Yes (Thm 2.2.10)	Yes (Thm 3.4.6)	Yes (Thm 4.2.6)
Monotonicity	Yes (Thm 2.2.12)	Yes (Thm 3.4.7)	Yes (Thm 4.2.7)
Independence of Irrelevant Alternative	No (Exp 2.2.14)	No (Exp 3.4.8)	No (Exp 4.2.8)
Intensity of IIA	Yes (Thm 2.2.17)	Yes (Thm 3.4.9)	No (Exp 4.2.9)
Participation	Yes (Thm 2.2.19)	No (Exp 3.4.13)	No (Exp 4.2.10)
Consistency	Yes (Thm 2.2.22)	No (Exp 3.4.10)	No (Exp 4.2.11)
Faithful	Yes (Thm 2.2.24)	Yes (Thm 3.4.11)	Yes (Thm 4.2.12)
Cancellation	No (Thm 2.2.26)	No (Exp 3.4.12)	No (Exp 4.2.13)

As we can see from the chart, we check on 12 Criteria in total. Range voting and range voting with limits satisfies 8 of them, and A-R voting function with limits satisfies 6 of them, while C-A-R voting function with limits satisfies 8 of them. In some way our C-A-R voting function with limits does improve and behave better than original range voting function, since it satisfies the

Majority and Condorcet Criteria which are not satisfied by original range voting function; but it fails in satisfying the Independence of Irrelevant Alternative, Participation and Consistency Criteria, since the ranking ballots and choosing the approval set leave impacts on how function behaves. However, I think it is still reasonable to say that the process of determining whether an alternative is approved by the majority of voters is beneficial. We don't want our voting function completely relies on alternative's total score; firstly voters should express their overall opinions towards the alternatives with positive or negative score, then we may know how the majority of voters regards our alternatives and only those who are approved by the majority are qualified to compete for the winner spot.

5.2 Further thoughts

In our definition for range voting function with limits, we mention that it does not matter what value b is since we can always use decimal score and therefore scale up or scale down according to a fixed ratio. However, in the reality world, it is not always the case that people are willing to give decimal scores. If we give a range for B such that $B = [-10, 10]$, then people are inclined to give integer scores such as $-8, -5, -1, 0, 3, 6, 10$, etc. If we only allow integer for B, then it does matter what value b we choose, because now we cannot apply our example to any general case. If we set $B = [-1, 1]$, then people can only give $-1, 0$ and 1 to the alternatives; if we set $B = [-100, 100]$, then it gives our voters a great freedom to Exampleress their altitudes to our alternatives. We don't know if $[-10, 10]$ is a rational choice for our score range then. The difference of range may psychologically influence voters' rating as well.

Furthermore, I am also thinking about if it is possible that we don't impose a limited function to our voting function at all: instead of doing that, for each voter v_i, we may calculate the $|f(v_i)_1| + \cdots + |f(v_i)_m|$ and then have their score divided by $|f(v_i)_1| + \cdots + |f(v_i)_m|$. For example, if there are three alternatives, and voter v_i gives a_1 -2 points, a_2 -10 points, and a_3

5 points. Then we calculate the total points of the absolute value is $|-2|+|-10|+|5|=17$, and a_1 gets $\frac{-2}{17}=-0.11$, a_2 gets $\frac{-10}{17}=-0.59$, and $a_3=\frac{5}{17}=0.29$. So the absolute value of their total score is 1, and it seems that each voters have a same influence on the score as well: they can give whatever points they would like to give, with no restriction, both mathematically and psychologically.

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