Spring 2021

An Exploration on Condorcet-Approval-Range Voting Function with limits

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An Exploration on Condorcet-Approval-Range Voting Function with Limits

A Senior Project submitted to
The Division of Science, Mathematics, and Computing of
Bard College

by
Jiangli Liu

Annandale-on-Hudson, New York
May, 2021
Abstract

In contrast to most social choice methods, which use ranked ballots, range voting is a well-known social choice method that offers the voters more choices in the form of an allowed range of possible scores. In this project, by allowing voters to give positive and negative scores, we hope to find a way that can explicitly show how voters disapprove, feel neutral, or approve of the alternatives instead of just giving ranking orders. Also, by applying a function to constrain the scores given in range voting, each voter will have the same influence when they give scores. After combining these conditions with Condorcet method by transferring scores into ranked ballot, we get a new voting function that involves Condorcet, approval and range voting. In this project, we explore how this new voting function behaves with respect to certain voting criteria.
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Dedication

To the fading days.
Acknowledgments

Deepest thanks to Ethan Bloch for your patience and guidance as my senior project advisor. To my parents for your unconditional love and support; and to my friends at Bard for your warm accompany during the pandemic.
1

Introduction

1.1 Social Choice Procedure

Voting theory is commonly known as the mathematical study of voting systems. There are a wide variety of voting systems and each has its advantages and disadvantages. Some of the most common differences among voting systems is how to decide who is the winner, and whether or not they involve a multi-step ranking process.

In a ranked voting system, voters rank the alternatives in the order they prefer it, ranking their most preferred alternative as the first choice and their least preferred alternative as the last choice. Also, tied votes sometimes are allowed in the ranking, depended on the different voting systems.

Definition 1.1.1. A set of alternative is $A = \{a_1, \ldots, a_m\}$ such that there are $m$ alternatives in $A$. △

Definition 1.1.2. A set of voter is $V = \{v_1, \ldots, v_n\}$ such that there are $n$ voters in $V$. △

It is assumed that each voter arrives at some ordering of the alternatives in accordance with his preferences.
The following definitions are from [1].

**Definition 1.1.3.** Let \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). A **preference order** of \( A \), written as \( p_A \), is a linear order on \( A \). We shall represent \( p_A \) by a column vector

\[
p_A = \begin{pmatrix}
    a_{\sigma(1)} \\
    a_{\sigma(2)} \\
    \vdots \\
    a_{\sigma(m)}
\end{pmatrix}
\]

where \( \sigma : \{1, \ldots, m\} \to \{1, \ldots, m\} \) is a permutation. The top alternative is most preferred and the bottom alternative is least preferred. We say voter \( i \) **prefer** \( a_{\sigma(p)} \) to \( a_{\sigma(q)} \) for any \( p, q \in \{1, \ldots, m\} \) if \( p < q \). △

**Definition 1.1.4.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( P_A \) be the set of preference orders of \( A \). A **preference profile** is the function \( f : V \to P_A \). We define \( X_{V,A} \) to be the set of preference profiles. △

**Definition 1.1.5.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( P(A) \) be the power set of \( A \). A **social choice procedure** is a function \( \Psi : X_{V,A} \to P(A) \).

The image of \( \Psi \) is always a subset of \( P(A) \). The output \( \Psi(f) \) for some \( f \in X_{V,A} \) is an element or several elements of \( P(A) \), which is a subset of \( A \) (which could be the empty set or a set with a single element).

### 1.2 Examples of Social Choice Procedures

**Definition 1.2.1.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( p, q \in \{1, \ldots, m\} \) be such that \( p \neq q \). Let \( T = \{a_p, a_q\} \). Let \( f \in X_{V,A} \). Let \( s = |\{i \in \{1, \ldots, n\} | a_{\sigma(p)} > a_{\sigma(q)}\}| \) and \( t = |\{i \in \{1, \ldots, n\} | a_{\sigma(p)} < a_{\sigma(q)}\}| \). We define \( a_p \) **wins the pairwise comparison** if \( s > t \). △
Definition 1.2.2. Let \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( p \in \{1, \ldots, m\} \). Let \( f \in X_{V,A} \). The alternative \( a_p \) is a Condorcet winner if \( a_p \) wins its pairwise comparison with \( a_q \) for all \( q \in \{1, \ldots, m\} \) such that \( q \neq p \). \( \square \)

The following statements are from [2].

For example, let \( A = \{a, b, c\} \) and suppose the preference orders are

\[
\begin{align*}
p_1 &= \begin{pmatrix} c \\ b \\ a \end{pmatrix} & p_2 &= \begin{pmatrix} b \\ a \\ c \end{pmatrix} & p_3 &= \begin{pmatrix} b \\ c \\ a \end{pmatrix} & p_4 &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} & p_5 &= \begin{pmatrix} c \\ a \\ b \end{pmatrix}.
\end{align*}
\]

Then \( b \) defeats \( a \) by a score of 3 to 2, since the first three voters prefer \( b \) to \( a \), while the last two voters prefer \( a \) to \( b \). We can also check that \( b \) defeats \( c \) by a score of 3 to 2, and \( c \) defeats \( a \) by a score of 3 to 2. Because \( b \) defeats each of the other alternatives it is the social choice for this profile when Condorcet’s method is used. Here, \( b \) is the Condorcet Winner.

The social choice procedure **Plurality voting** declares that the alternative(s) who gets the largest number of first place rankings in the preference order should be in the social choice set.

The social choice procedure **Borda Count** uses each preference order to award “score” to each of \( m \) alternatives as follows: for each voter, the alternative at the bottom of the order gets zero points, the alternative at the next to the bottom spot gets one point, the next one up gets two points and so on up to the top alternative which gets \( m - 1 \) points. For each alternative, we add the score awarded it from each of the individual preference orders. The alternative(s) with the highest “Borda score” is declared to be the social choice.

The social choice procedure **Instant runoff voting** is based on the idea of arriving at a social choice by successive deletions of less desirable alternatives. We begin by deleting the alternative or alternatives who get the smallest number of first place rankings. At this stage we have orders that are at least one alternative fewer than that with which we started. Now, we simply repeat this process of deleting the least desirable alternative or alternatives (as measured by the number
of preference orders on top of which it, or they, appear). The alternative(s) deleted last is declared
the social choice.

The social choice procedure Dictatorship ignores all the preference orders except that of the
dictator \( v_d \). The alternative in first place rankings of \( v_d \) is now declared to be the social choice.

1.3 Properties of Social Choice Procedures

An ideal social choice procedure demands as many as possible of the following conditions be
satisfied.

**Definition 1.3.1.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \)
be a set of alternatives for some \( m \in \mathbb{N} \). Let \( \Psi : X_{V,A} \rightarrow \mathcal{P}(A) \) be a social choice procedure. We
say \( \Psi \) satisfies the **Always-A-Winner Condition** if \( \Psi(f) \neq \emptyset \) for all \( f \in X_{V,A} \). △

**Definition 1.3.2.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \)
be a set of alternatives for some \( m \in \mathbb{N} \). Let \( \Psi : X_{V,A} \rightarrow \mathcal{P}(A) \) be a social choice procedure.
We say \( \Psi \) satisfies the **Majority Criterion** if the following condition holds: for each \( f \in X_{V,A} \),
if alternative \( a_j \) for some \( j \in \{1, \ldots, m\} \), is most preferred by more than \( \frac{n}{2} \) the voters, then \( a_j \)
should be in \( \Psi(f) \). △

**Definition 1.3.3.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \)
be a set of alternatives for some \( m \in \mathbb{N} \). Let \( \Psi : X_{V,A} \rightarrow \mathcal{P}(A) \) be a social choice procedure.
We say \( \Psi \) satisfies the **Pareto condition** if the following condition holds: for each \( f \in X_{V,A} \), if
\( j, k \in \{1, \ldots, m\} \), and if every voters prefers \( a_j \) to \( a_k \), then \( a_k \) cannot be in \( \Psi(f) \). △

**Definition 1.3.4.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \). Let \( A = \{a_1, \ldots, a_m\} \)
be a set of alternatives for some \( m \in \mathbb{N} \). Let \( \Psi : X_{V,A} \rightarrow \mathcal{P}(A) \) be a social choice procedure. We
say \( \Psi \) satisfies the **Condorcet Criterion** if it always chooses the Condorcet Winner to be the
only element in \( \Psi(f) \) when one exists. △
Definition 1.3.5. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say $\Psi$ satisfies the Monotonicity Criterion if the following holds: for each $f \in X_{V,A}$, if $a_j$ is in $\Psi(f)$ for some $j \in \{1, \ldots, m\}$ and one voter changes his/her preference order by moving $a_j$ up one spot, then $a_j$ should still be in $\Psi(f)$. \hfill \triangle

Definition 1.3.6. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say $\Psi$ satisfies the Independence of irrelevant alternatives if the following condition holds: if $a_j \in \Psi(f)$ but $a_k \notin \Psi(f)$ for some $j,k \in \{1, \ldots, m\}$, and one or more voters change their preference orders, but no one changes about whether $a_j$ is preferred to $a_k$ or $a_k$ to $a_j$, then $a_k$ should still not be in $\Psi(f)$. \hfill \triangle

Definition 1.3.7. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\Psi : X_{V,A} \to \mathcal{P}(A)$ be a social choice procedure. We say $\Psi$ satisfies the Participation Criterion if the following condition holds: if $a_j \in \Psi(f)$ but $a_k \notin \Psi(f)$ for some $j,k \in \{1, \ldots, m\}$, and we add one voter with $a_j$ preferred to $a_k$, then $a_k$ should still not be in $\Psi(f)$. \hfill \triangle
INTRODUCTION
2

Range Voting with Limits

2.1 Range Voting

Here, we introduce another social choice procedure, called as Range Voting, that offers the voter more choices in the form of an allowed range of possible scores.

**Definition 2.1.1.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. A **ranging profile** is the function $f : V \to B^m$. △

**Definition 2.1.2.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$. Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. We define $R_{V,A,B}$ as the set of ranging profiles. △

**Definition 2.1.3.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. Let $i \in \{1, \ldots, n\}$. The alternative $a_k$ for some $k \in \{1, \ldots, m\}$ is **most preferred** by $v_i$ if $f(v_i)_k > f(v_i)_j$ for all $j \in \{1, \ldots, m\}$. △
**Definition 2.1.4.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters such that $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives such that $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For all $f \in R_{V,A,B}$, the **scoring function** for $V$ and $A$ is the function $S_f : A \to \mathbb{R}$ such that

$$S_f(a_j) = \sum_{i=1}^{n} f(v_i)_j,$$

for all $j \in \{1, \ldots, m\}$. \(\triangle\)

**Definition 2.1.5.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $S_f$ be the scoring function for $V$ and $A$. The **range voting function** is the function $\phi : R_{V,A,B} \to \mathcal{P}(A)$ defined by letting $\phi(f)$ be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$, for all $f \in R_{V,A,B}$. \(\triangle\)

As we mentioned above, one of the most commonly used social choice procedures is the plurality voting.

In the plurality voting each voter casts one vote for his most preferred alternative, and the alternative(s) with the largest total number of votes constitute the social choice set. We may think of this procedure as assigning a score of 1 to each voter’s most preferred alternative, a score of 0 to the others such that $B = \{0, 1\}$ where \{1\} can only be assigned once, and selecting the alternative(s) with highest total score, summed over all voters.

Another well-known social choice procedure is the Borda Count. It asks each voter to assign score $m - 1$ to his most preferred alternative, score $m - 2$ to his second most preferred alternative, and in general score $m - i$ to his $i$th most preferred alternative such that $B = \{0, 1, \ldots, m - 1\}$ where each element can only be assigned once. Then the alternative(s) with highest total score define the social choice set for Borda Count Voting. Therefore, these two social choice procedures can be described as special cases for range voting function.

Another special case of range voting is approval voting, where $B = \{0, 1\}$. 
2.2 Properties of Social Choice Procedures for Range Voting

Some of the criteria for voting methods mentioned above can be applied as stated to range voting, somehow we need to have them reformulated for regular Range Voting.

2.2.1 Always-A-Winner Condition for Range Voting

Definition 2.2.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Always-A-Winner Condition for range voting** if $\phi(f) \neq \emptyset$, for all $f \in R_{V,A,B}$.

\[\square\]

Theorem 2.2.2. Range voting satisfies Always-A-Winner Condition for range voting.

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V,A,B}$. Let $S_f$ be the scoring function for $V$ and $A$.

For any $f$, it is clear that Range Voting satisfies Always-A-Winner Criterion, since for some $k \in \{1, \ldots, n\}$, there always exists one or some of alternatives $a_k$ that lie in $\phi(f)$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, n\}$.

\[\square\]

2.2.2 Majority Criterion for Range Voting

Definition 2.2.3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Majority Criterion for range voting** if the following condition holds: for all $f \in R_{V,A,B}$, if for any $k \in \{1, \ldots, m\}$, the alternative $a_k$ is most preferred by more than $\frac{n}{2}$ voters, then $a_k \in \phi(f)$.

\[\square\]
I will show that range voting does not satisfy the Majority Criterion by giving the following example.

**Example 2.2.4.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V,A,B}$ shown as below.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>5</td>
<td>3</td>
<td>-9</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-8</td>
<td>-10</td>
<td>3</td>
</tr>
</tbody>
</table>

Let $S_f$ be the scoring function for $V$ and $A$. We observe that alternative $a_1$ is most preferred by the majority of voters since there are 2 voters out of 3 who choose $a_1$ as their most preferred alternative. However, we have $S_f(a_1) = 5 + 3 + (-9) = -1$, and $S_f(a_2) = 2 + 2 + 3 = 7$, and $S_f(a_3) = -8 + (-10) + 3 = -1$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A$, then we know that $\phi(f) = \{a_2\}$ instead of $\{a_1\}$.

\[\Box\]

### 2.2.3 Condorcet Criterion for Range Voting

**Definition 2.2.5.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p, q \in \{1, \ldots, m\}$ be such that $p \neq q$. Let $T = \{a_p, a_q\}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. Let $s = |\{i \in \{1, \ldots, n\} | f(v_i)_p > f(v_i)_q\}|$ and $t = |\{i \in \{1, \ldots, n\} | f(v_i)_p < f(v_i)_q\}|$. We define $a_p$ wins the pairwise comparison for range voting if $s > t$.

\[\triangle\]

**Definition 2.2.6.** Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $p \in \{1, \ldots, m\}$. Let $B \subseteq \mathbb{R}$. Let $f \in R_{V,A,B}$. The alternative $a_p$ is a Condorcet winner for range voting if $a_p$ wins its pairwise comparison with $a_q$ for all $q \in \{1, \ldots, m\}$ such that $q \neq p$.

\[\triangle\]

**Definition 2.2.7.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting
function. The range voting function is said to satisfy the **Condorcet Winner Criterion for range voting** if the following condition holds: for each \( f \in R_{V,A,B} \), if there exists a Condorcet winner for range voting \( a_p \) for \( f \) for some \( p \in \{1, \ldots, m\} \), then \( a_p \) alone is in \( \phi(f) \).

In order to check if Range Voting satisfies the Condorcet Winner Criterion, we need to convert \( B^m \) from range profile to preference orders, and then determine the Condorcet Winner with pairwise comparisons.

I will show that range voting function does not satisfy the Condorcet Criterion for range voting by giving the following example.

**Example 2.2.8.** Let \( V = \{v_1, \ldots, v_3\} \) be a set of voters and \( A = \{a_1, \ldots, a_3\} \) be a set of alternatives. Let \( B = [-10,10] \). Let \( \phi : R_{V,A,B} \to \mathcal{P}(A) \) be the range voting function. Let \( f \in R_{V,A,B} \) shown as below.

\[
\begin{array}{ccc}
  & v_1 & v_2 & v_3 \\
 a_1 & -7 & 5 & 6 \\
 a_2 & 8 & 4 & 2 \\
 a_3 & 0 & -6 & -7 \\
\end{array}
\]

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

\[
\begin{array}{ccc}
  & v_1 & v_2 & v_3 \\
 a_2 & a_1 & a_1 & a_1 \\
 a_3 & a_2 & a_2 & a_2 \\
 a_1 & a_3 & a_3 & a_3 \\
\end{array}
\]

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following. The alternative with square is the one who wins the pairwise comparison.

\[
\begin{array}{c}
 a_1 \text{ vs } a_2 : 2:1 \\
 a_1 \text{ vs } a_3 : 2:1 \\
 a_2 \text{ vs } a_3 : 2:1 \\
\end{array}
\]

As shown in the table, \( a_1 \) wins each pairwise comparison with \( a_2 \) and \( a_3 \), therefore \( a_1 \) is the Condorcet winner.
Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_1) = (-7) + 5 + 6 = 4$, and $S_f(a_2) = 8 + 4 + 2 = 14$, and $S_f(3) = 0 + (-6) = (-7) = -13$. Since $S_f(a_1) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$, then $\phi(f) = \{a_2\}$. 

2.2.4 Pareto Criterion for Range Voting

**Definition 2.2.9.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Pareto Criterion for range voting** if the following condition holds: for each $f \in R_{V,A,B}$, if $k, j \in \{1, \ldots, m\}$, and if $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$, then alternative $a_j \notin \phi(f)$. ◇

**Theorem 2.2.10.** Range voting satisfies Pareto Criterion for range voting.

**Proof.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. Let $k, j \in \{1, \ldots, m\}$. Let $S_f$ be the scoring function for $V$ and $A$.

Suppose $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots n\}$. Then we know $S_f(a_k) > S_f(a_j)$. By the definition of the range voting function, we know that $a_j$ can never be in $\phi(f)$. Therefore, Range Voting satisfies the Pareto Criterion. ◐

2.2.5 Monotonicity Criterion for Range Voting

**Definition 2.2.11.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the **Monotonicity Criterion for Range Voting** if the following condition holds: for each $f \in R_{V,A,B}$, if $a_j$ is in $\phi(f)$ and $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$, then $a_j$ should still be in $\phi(f)$. ◇
2.2. PROPERTIES OF SOCIAL CHOICE PROCEDURES FOR RANGE VOTING

Theorem 2.2.12. Range voting satisfies Monotonicity Criterion for range voting.

Proof. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. Let $j \in \{1, \ldots, m\}$. Let $S_f$ be the scoring function for $V$ and $A$.

Suppose $a_j$ is in $\phi(f)$. Then we know $S_f(a_j) \geq S_f(a_k)$ for all $k \in \{1, \ldots, m\}$.

If $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$, then we have $S_f(a_j) > S_f(a_k)$ for all $k \in \{1, \ldots, m\}$.

Hence, by the definition of range voting function, $a_j \in \phi(f)$. \qed

2.2.6 Independence of Irrelevant Alternative Criterion for Range Voting

Definition 2.2.13. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function. The range voting function is said to satisfy the Independence of Irrelevant Alternative Criterion for Range Voting if the following condition holds: for each $f \in R_{V,A,B}$, if $j, k \in \{1, \ldots, m\}$ and $a_j \in \phi(f)$ and $a_k \notin \phi(f)$, and if $f(v_i)_j$ and $f(v_i)_k$ are changed for some $i \in \{1, \ldots, n\}$, but remaining in $f(v_i)_j > f(v_i)_k$, then $\phi(f)$ should not change so as to include $a_k$. \triangle

I will show range voting function does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 2.2.14. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V,A,B} \to \mathcal{P}(A)$ be the range voting function.

Let $f \in R_{V,A,B}$ be given as the following chart.
Let \( S_f \) be the scoring function for \( V \) and \( A \). We have \( S_f(a_1) = 7 + (-7) + 2 = 2 \), and \( S_f(a_2) = 2 + 5 + (-6) = 1 \), and \( S_f(a_3) = -6 + 3 + 7 = 4 \). Since \( S_f(a_3) \geq S_f(a_j) \) for all \( j \in \{1, 2, 3\} \), then \( \phi(f) = \{a_3\} \).

We change \( f(v_3)_1 \) and \( f(v_3)_3 \) but keep \( f(v_3)_1 < f(v_3)_3 \). Let \( f' \in R_{V,A,B} \) be given as in the following table.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>7</td>
<td>-7</td>
<td>4</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>5</td>
<td>-6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-6</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Before we change \( f(v_3)_1 \) and \( f(v_3)_3 \), we had \( \phi(f) = \{a_3\} \). However, in the second table, since we have \( S_{f'}(a_1) = 7 + (-7) + 4 = 4 \) and \( S_{f'}(a_2) = 2 + 5 + (-6) = 1 \), while \( S_{f'}(a_3) = -6 + 3 + 5 = 2 \). Since \( S_{f'}(a_1) \geq S_{f'}(a_j) \) for all \( j \in \{1, 2, 3\} \), then \( \phi(f') = \{a_1\} \neq \{a_3\} \).

\[ \Box \]

### 2.2.7 Intensity of Independence of Irrelevant Alternative Criterion for Range Voting

**Definition 2.2.15.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( B \subseteq \mathbb{R} \). Let \( j, k \in \{1, \ldots, m\} \). If \( i \in \{1, \ldots, n\} \) and \( f \in R_{V,A,B} \), **intensity of preference** of voter \( v_i \) for \( a_j \) over \( a_k \) is \( f(v_i)_j - f(v_i)_k \).

Note that for voter \( v_i \), the intensity of preference for \( a_j \) over candidate \( a_k \) could be positive, 0 or negative. Also, the intensity of preference of voter \( v_i \) for \( a_k \) over \( a_j \) is the negative of the intensity of preference of voter \( v_i \) for \( a_j \) over \( a_k \).

**Definition 2.2.16.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( B \subseteq \mathbb{R} \). Let \( \phi : R_{V,A,B} \to \mathcal{P}(A) \) be the range
voting function. The range voting function is said to satisfy the **Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting** if the following holds: for each \( f \in R_{V,A,B} \), if \( j,k \in \{1,\ldots,m\} \), and if \( a_j \in \phi(f) \) and \( a_k \notin \phi(f) \), and if for some \( i \in \{1,\ldots,n\} \), \( f(v_i)_p \) changes for some \( p \in \{1,\ldots,m\} \), and the intensity of preference of \( v_i \) for \( a_j \) over \( a_k \) does not change, then still \( a_k \notin \phi(f) \).

**Theorem 2.2.17.** Range voting satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.

**Proof.** Let \( V = \{v_1,\ldots,v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1,\ldots,a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( B \subseteq \mathbb{R} \). Let \( \phi : R_{V,A,B} \to \mathcal{P}(A) \) be the range voting function. Let \( S_f \) be the scoring function for \( V \) and \( A \).

Suppose that \( a_j \in \phi(f) \) and \( a_k \notin \phi(f) \), and that \( f(v_i)_p \) changes for some \( i \in \{1,\ldots,n\} \) and for some \( p \in \{1,\ldots,m\} \) while \( f(v_i)_j - f(v_i)_k \) remains the same for some \( i \in \{1,\ldots,n\} \). Since none of the voters change their intensity of preference for \( a_j \) over \( a_k \), which is \( f(v_i)_j - f(v_i)_k \) does not change for all \( i \in \{1,\ldots,n\} \), and so \( S_f(a_j) - S_f(a_k) \) does not change as well. Therefore, by the definition of range voting, we conclude that it is always the case that \( S_f(a_j) > S_f(a_k) \), and thus \( a_k \notin \phi(f) \).

2.2.8 Participation Criterion for Range Voting

**Definition 2.2.18.** Let \( V = \{v_1,\ldots,v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1,\ldots,a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( B \subseteq \mathbb{R} \). Let \( \phi : R_{V,A,B} \to \mathcal{P}(A) \) be the range voting function. The range voting function is said to satisfy the **Participation Criterion** if the following condition holds: for each \( f \in R_{V,A,B} \), suppose \( a_j \in \phi(f) \) and \( a_k \notin \phi(f) \) for some \( j,k \in \{1,\ldots,m\} \). If one more voter \( v_i \) is added to \( V \), who gives \( f(v_i)_j > f(v_i)_k \), then it should not be the case that \( a_k \in \phi(f') \) and \( a_j \notin \phi(f') \).

**Theorem 2.2.19.** Range voting satisfies the Participation Criterion for Range Voting.
Proof. Suppose that $a_j \in \phi(f)$ and $a_k \notin \phi(f)$. By the definition of $\phi$, we know that $S_f(a_j) \geq S_f(a_k)$. After adding $v_i$, we get new $S'_f(a_j) = S_f(a_j) + f(v_i)_j$ and $S'_f(a_k) = S_f(a_k) + f(v_i)_k$. Since $f(v_i)_j > f(v_i)_k$, then we get $S'_f(a_j) > S'_f(a_k)$. Hence $a_k \notin \phi(f)$. □

The following definitions are from [3].

2.2.9 Consistent property for range voting

Definition 2.2.20. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such that $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$, we call $f_1$ and $f_2$ disjoint profiles. Then $f_1 + f_2$ denotes the profile with voter set $V_1 \cup V_2$, which when restricted to $V_i$ agrees with $f_i$ for each $i \in \{1, 2\}$. △

Definition 2.2.21. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such that $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $\phi$ be the range voting function. We define the social choice function $\phi$ satisfies consistent property if the following condition is satisfied: for disjoint profiles $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$, if $\phi(f_1) \cap \phi(f_2) \neq \emptyset$ then $\phi(f_1) \cap \phi(f_2) = \phi(f_1 + f_2)$. △

Theorem 2.2.22. The range voting function satisfies consistent property.

Proof. Let $V_1 = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $V_2 = \{v_{n+1}, \ldots, v_q\}$ be another set of voters for some $q \in \mathbb{N}$ such that $q < n$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi$ be the range voting function. Let $f_1 \in R_{V_1,A,B}$ and $f_2 \in R_{V_2,A,B}$. Let $S_{f_1}$ be the score function for $f_1$; let $S_{f_2}$ be the score function for $f_2$.

Suppose $\phi(f_1) \cap \phi(f_2) \neq \emptyset$. 

□
Let \( a_k \in \phi(f_1) \cap \phi(f_2) \). Then by the definition of \( \phi \) we know that \( S_{f_1}(a_k) \geq S_{f_1}(a_j) \) for all \( j \in \{1,\ldots,m\} \), as well as \( S_{f_2}(a_k) \geq S_{f_2}(a_j) \) for all \( j \in \{1,\ldots,m\} \). Then we get \( S_{f_1}(a_k) + S_{f_2}(a_k) \geq S_{f_1}(a_j) + S_{f_2}(a_j) \) for all \( j \in \{1,\ldots,m\} \). Since \( f_1 \) and \( f_2 \) are disjoint profiles, then \( S_{f_1+f_2}(a_k) \geq S_{f_1+f_2}(a_j) \) for all \( j \in \{1,\ldots,m\} \), hence \( a_k \in \phi(f_1 + f_2) \). Therefore \( \phi(f_1) \cap \phi(f_2) \subseteq \phi(f_1 + f_2) \).

I will prove the other direction by contradiction.

Let \( a_k \in \phi(f_1 + f_2) \). Suppose \( a_k \notin \phi(f_1) \cap \phi(f_2) \). There are three cases.

First, suppose \( a_k \notin \phi(f_1) \) and \( a_k \in \phi(f_2) \). Let \( a_j \in \phi(f_1) \cap \phi(f_2) \). Then it has to be the case that \( S_{f_1}(a_j) > S_{f_1}(a_k) \). Since \( a_k \in \phi(f_2) \), then we have \( S_{f_2}(a_j) = S_{f_2}(a_k) \geq S_{f_2}(a_i) \) for all \( i \in \{n + 1,\ldots,q\} \). Then \( S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k) \). Since \( f_1 \) and \( f_2 \) are disjoint profiles, then \( S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k) \). Hence \( a_k \notin \phi(f_1 + f_2) \). A contradiction.

In the second case, suppose \( a_k \in \phi(f_1) \) and \( a_k \notin \phi(f_2) \). Let \( a_j \in \phi(f_1) \cap \phi(f_2) \). Then it has to be the case that \( S_{f_2}(a_j) > S_{f_2}(a_k) \). Since \( a_k \in \phi(f_1) \), then we have \( S_{f_1}(a_j) = S_{f_1}(a_k) \geq S_{f_1}(a_i) \) for all \( i \in \{1,\ldots,n\} \). Then \( S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k) \). Since \( f_1 \) and \( f_2 \) are disjoint profiles, then \( S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k) \). Hence \( a_k \notin \phi(f_1 + f_2) \). A contradiction.

In the third case, suppose \( a_k \notin \phi(f_1) \) and \( a_k \notin \phi(f_2) \). Let \( a_j \in \phi(f_1) \cap \phi(f_2) \). Then it has to be the case that \( S_{f_1}(a_j) > S_{f_1}(a_k) \) and \( S_{f_2}(a_j) > S_{f_2}(a_k) \). Then \( S_{f_1}(a_j) + S_{f_2}(a_j) > S_{f_1}(a_k) + S_{f_2}(a_k) \). Since \( f_1 \) and \( f_2 \) are disjoint profiles, then \( S_{f_1+f_2}(a_j) > S_{f_1+f_2}(a_k) \). Hence \( a_k \notin \phi(f_1 + f_2) \). A contradiction.

So, \( \phi(f_1 + f_2) \subseteq \phi(f_1) \cap \phi(f_2) \).

Therefore \( \phi(f_1) \cap \phi(f_2) = \phi(f_1 + f_2) \).

\( \square \)
2.2.10 Faithful Property for range voting

**Definition 2.2.23.** Let $V = \{v\}$ be a set of one voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. We define the social choice function $\phi$ satisfies **faithful** property if the following condition is satisfied: for any profile $f \in R_{V,A,B}$, if $f(v)_j > f(v)_k$ for some $j, k \in \{1, \ldots, m\}$, then $a_k \notin \phi(f)$. △

**Theorem 2.2.24.** The range voting function satisfies faithful property.

**Proof.** Let $V = \{v\}$ be a set of voter and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function.

Let $f \in R_{V,A,B}$. I will prove this by contrapositive. Suppose $a_k \in \phi(f)$. Let $S_f$ be the scoring function for $V$ and $A$. Then by the definition of $\phi$, we have $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$. Since $|V| = 1$, then we know that $f(v)_k \geq f(v)_j$ for all $j \in \{1, \ldots, m\}$.

Therefore, by contrapositive, if $f(v)_j > f(v)_k$ for some $j, k \in \{1, \ldots, m\}$, then $a_k \notin \phi(f')$. □

2.2.11 Cancellation property for range voting

**Definition 2.2.25.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. For a profile $f \in R_{V,A,B}$ and $j, k \in \{1, \ldots, m\}$ such that $j \neq k$, let $\pi_{a_j a_k}(f) = |\{i \in \{1, \ldots, n\} | f(v_i)_j > f(v_i)_k\}|$. We define the social choice function $\phi$ satisfies **cancellation** property if the following condition is satisfied: for any $f \in R_{V,A,B}$, if $\pi_{a_j a_k}(f) = \pi_{a_k a_j}(f)$ for all $j, k \in \{1, \ldots, m\}$ such that $j \neq k$, then $\phi(f) = A$. △

I will show that the range voting function does not satisfy the cancellation property by giving the following example.
Example 2.2.26. Let $V = \{v_1, v_2\}$ be a set of voter and $A = \{a_1, a_2\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $\phi : R_{V, A, B} \to \mathcal{P}(A)$ be the range voting function. Let $f \in R_{V, A, B}$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

As we can see from the chart that $f(v_1)_1 > f(v_1)_2$. So $\pi_{a_1 a_2}(f) = 1$. Also $f(v_2)_2 > f(v_2)_1$, which means $\pi_{a_2 a_1}(f) = 1$. Thus, in this case $\pi_{a_1 a_2}(f) = \pi_{a_2 a_1}(f) = 1$.

Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_1) = 8 + 3 = 11$ and $S_f(a_2) = 2 + 7 = 9$. Since $S_f(a_1) > S_f(a_2)$, then $\phi(f) = \{a_1\}$.

However, we notice that $\phi(f) \neq A$. Hence, range voting function does not satisfy the cancellation property.

2.3 Range Voting Function with Limits

As we see in several examples of range voting function, generally speaking, in a ranging profile, the range of $B$ can be restricted in different ways. However, more than setting restrictions on $B$, we hope to let voters distribute their scores in a fairer way. Hence, I try to find a limited function that can be applied to the general range voting, and make our voters have an equal influence on the alternatives.

It takes me a while to determine our limited function. At first, we tried to set $\xi(f(v_i)_1 + \ldots + f(v_i)_m) = f(v_i)_1 + \ldots + f(v_i)_m - k = 0$. However, we realized that if we set $k > 0$, we are forcing our voters to give positive score to at least one alternative; similarly, if we set $k < 0$, we are forcing our voters to give negative score to at least one alternative. If we set $k = 0$, then we are forcing our voters either give 0 scores to all alternative, either they have to give both positive and negative scores to different alternative in order to balance their total score. Therefore, I
decide it is better to use absolute value as a constraint to the total score instead of simply use sum.

Also, I took some time to think about a reasonable value for $k$ such that $|f(v_1) + \cdots + f(v_m)| - k = 0$. At first, I tried to set a range for $k$, i.e. $k \in (0, mb)$, so that voters cannot give 0 score or $b$ score to all voters; otherwise, anything in between is allowed. However, that is a problem as we now see. For example, when $k = 1$, means that $b$ has to be somehow equal or smaller than 1, which leaves a really limited space for voter to assign their scores; when $k = mb - 1$, means that each alternative has to receive a score that has an absolute value pretty close to $b$, which seems not to be a rational assignment as well. And then, I got some inspired by [3]. In this paper, the author mentions constant total weight condition for classic Borda and the total score is actually $\frac{m(m-1)}{2}$; so this idea of $\frac{mb}{2}$ comes to my mind and seems to be a reasonable condition to impose on our range voting function to avoid favoring certain preference rankings over others.

**Definition 2.3.1.** Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. We define the **limited function** to be the function $\xi : B^m \to \mathbb{R}$ defined by $\xi(x_1, \ldots, x_m) = |x_1| + \cdots + |x_m| - \frac{mb}{2}$ for all $(x_1, \ldots, x_m) \in B^m$. △

**Definition 2.3.2.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$. Let $B = [-b, b]$. Then we define

$$R^\xi_{V,A,B} = \{ f \in R_{V,A,B} | \xi(f(v_1), \ldots, f(v_m)) = 0 \text{ for all } i \in \{1, \ldots, n\} \} .$$

△

Note that $R^\xi_{V,A,B} \subseteq R_{V,A,B}$.

**Definition 2.3.3.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$. Let $B = [-b, b]$. Let $S_f$ be the scoring function for $V$ and $A$. The **range voting function with limit** is a function $\Phi : R^\xi_{V,A,B} \to \mathcal{P}(A)$ defined
by letting $\Phi(f)$ be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$, for all $f \in \mathcal{R}_{V,A,B}^\xi$.

Furthermore, it is worth to point out that in our following example, I will mostly set $B = [-10, 10]$. Since our $\frac{mb}{2}$ is a fixed value based on score $b$ and the number of alternative $m$, it is clear that my example can be generalized by applying different value of $b$ proportionately with same outcome when $m$ keeps the same for the voting function. To illustrate this, we can compare two following examples.

**Example 2.3.4.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\Phi : \mathcal{R}_{V,A,B}^\xi \rightarrow \mathcal{P}(A)$ be the range voting function with limit.

We have $\frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$. Then we suppose $|f(v_i)_{1}| + |f(v_i)_{2}| + |f(v_i)_{3}| - 15 = 0$ for all $i \in \{1, 2, 3\}$. Let us consider the following example. Let $f \in \mathcal{R}_{V,A,B}^\xi$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>10</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>$a_3$</td>
<td>5</td>
<td>-3</td>
<td>3</td>
</tr>
</tbody>
</table>

Let $S_f$ be the scoring function for $V$ and $A$. Here, we can see that $S_f(a_1) = 10 + 8 + 8 = 26$ while $S_f(a_2) = 0 + 4 + (-4) = 0$, and $S_f(a_3) = 5 + (-3) + 3 = 5$. So $\Phi(f) = \{a_2\}$.

**Example 2.3.5.** Let $V = \{v_1, \ldots, v_3\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Now we set $b = 5$, then $B = [-5, 5]$. Let $\Phi : \mathcal{R}_{V,A,B}^\xi \rightarrow \mathcal{P}(A)$ be the range voting function with limit.

We have $\frac{mb}{2} = \frac{3 \cdot 5}{2} = 7.5$. Then we suppose $|f(v_i)_{1}| + |f(v_i)_{2}| + |f(v_i)_{3}| - 7.5 = 0$ for all $i \in \{1, 2, 3\}$. Let us consider the following example. Let $f \in \mathcal{R}_{V,A,B}^\xi$ be given as the following chart.
Let $S_f$ be the scoring function for $V$ and $A$. Here, we can see that $S_f(a_1) = 5 + 4 + 4 = 13$ while $S_f(a_2) = 0 + 2 + (-2) = 0$, and $S_f(a_3) = 2.5 + (-1.5) + 1.5 = 2.5$. So $\Phi(f) = \{a_2\}$. ♦

As we see from Example 2.3.4 and Example 2.3.5, no matter what value $b$ is, the example I give can be always applied to the general case and have the same winner set since our limited function does not change.

Also, from now on, instead of writing in the form of $\xi(f(v_1), \ldots, f(v_m)) = |f(v_1)| + \cdots + |f(v_m)| - \frac{mb}{2} = 0$ for our limited function, I will write $|f(v_1)| + \cdots + |f(v_m)| = \frac{mb}{2}$, where they expresses the same thing but the second one looks more familiar.

In the next part, we will check if range voting function with limit works better in satisfying the criterions we showed in Section 2.2, and therefore to see if this restriction actually helps to improve Range Voting, whether it matters if we impose some restrictions on $R_{V,A,B}$.

2.4 Properties of Social Choice Procedures for Range Voting Function with Limits

2.4.1 Always-A-Winner Condition for Range Voting Function with Limits

**Theorem 2.4.1.** Range voting function with limits satisfies Always-A-Winner Condition.

The proof is the same as for Theorem 2.2.2 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and let $\Phi : R_{V,A,B}^\xi \to \mathcal{P}(A)$ be the range voting function with limits.
2.4. PROPERTIES FOR RANGE VOTING FUNCTION WITH LIMITS

2.4.2 Majority Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Majority Criterion for Range Voting. The counter example is the same as in Example 2.2.4 except we let \( \Phi : \mathbb{R}^\xi_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits, and we observe that
\[
|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}
\]
for all \( i \in \{1, 2, 3\} \).

2.4.3 Condorcet Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Majority Criterion for Range Voting. The counter example is the same as in Example 2.2.8 except we let \( \Phi : \mathbb{R}^\xi_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits, and we observe that
\[
|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}
\]
for all \( i \in \{1, 2, 3\} \).

2.4.4 Pareto Criterion for Range Voting Function with Limits

Theorem 2.4.2. Range voting function with limits satisfies Pareto Criterion for range voting.

The proof is the same as for Theorem 2.2.10 except we let \( B = [-b, b] \) for some \( b \in \mathbb{R} \) such that \( b > 0 \) and let \( \Phi : \mathbb{R}^\xi_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits.

2.4.5 Monotonicity Criterion for Range Voting Function with Limits

Theorem 2.4.3. Range voting function with limits satisfies Monotonicity Criterion for range voting.

The proof is the same as for Theorem 2.2.12 except we let \( B = [-b, b] \) for some \( b \in \mathbb{R} \) such that \( b > 0 \) and let \( \Phi : \mathbb{R}^\xi_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits.
2.4.6 Independence of Irrelevant Alternative Criterion for Range Voting Function with Limits

Range voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting. The counter example is the same as in Example 2.2.14 except we let \( \Phi : R_{V,A,B}^\xi \to \mathcal{P}(A) \) be the range voting function with limits, and we observe that 

\[
|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = 15 = \frac{mb}{2} = \frac{3 \cdot 10}{2}
\]

for all \( i \in \{1, 2, 3\} \).

2.4.7 Intensity of Independence of Irrelevant Alternative Criterion for Range Voting Function with Limits

**Theorem 2.4.4.** The range voting function with limits satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.

The proof is the same as for Theorem 2.2.17 except we let \( B = [-b, b] \) for some \( b \in \mathbb{R} \) such that \( b > 0 \) and let \( \Phi : R_{V,A,B}^\xi \to \mathcal{P}(A) \) be the range voting function with limits.

2.4.8 Participation Criterion for Range Voting Function with limits

**Theorem 2.4.5.** Range voting function with limits satisfies participation criterion.

The proof is the same as for Theorem 2.2.19 except we let \( B = [-b, b] \) for some \( b \in \mathbb{R} \) such that \( b > 0 \) and let \( \Phi : R_{V,A,B}^\xi \to \mathcal{P}(A) \) be the range voting function with limits.

2.4.9 Consistent Property for range voting with limits

**Theorem 2.4.6.** The range voting function with limits satisfies consistent property.
2.4. PROPERTIES FOR RANGE VOTING FUNCTION WITH LIMITS

The proof is the same as for Theorem 2.2.22 except we $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$, and we let $\Phi : R_{V,A,B}^\xi \to \mathcal{P}(A)$ be the range voting function with limits, and $f_1 \in R_{V_1,A,B}^\xi$ and $f_2 \in R_{V_2,A,B}^\xi$.

2.4.10 Faithful Property for range voting with limits

Theorem 2.4.7. The range voting function with limits satisfies faithful property.

The proof is the same as for Theorem 2.2.24 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$, and we let $\Phi : R_{V,A,B}^\xi \to \mathcal{P}(A)$ be the range voting function with limits.

2.4.11 Cancellation Property for range voting with limits

The range voting function with limits does not satisfy the cancellation property. The counter example is the same as in Example 2.2.26 except we let $B = [-b, b]$ for some $b \in \mathbb{R}$ such that $b > 0$ and $\Phi : R_{V,A,B}^\xi \to \mathcal{P}(A)$ be the range voting function with limits, and we observe that $|f(v_i)_1| + |f(v_i)_2| = 10 = \frac{mb}{2} = \frac{2 \cdot 10}{2}$ for all $i \in \{1, 2\}$.
3

A-R Voting Function With Limits

3.1 Approval voting function

**Definition 3.1.1.** The approval voting function \( \alpha : R_{V,A,B} \to \mathcal{P}(A) \) is the same as the range voting function \( \phi : R_{V,A,B} \to \mathcal{P}(A) \) in the case where \( B = \{0, 1\} \).

Here is an example for approval voting function.

**Example 3.1.2.** Let \( V = \{v_1, \ldots, v_4\} \) be a set of voters and \( A = \{a_1, \ldots, a_4\} \) be a set of alternatives. Let \( B \in \{0, 1\} \). Let \( \alpha : R_{V,A,B} \to \mathcal{P}(A) \) be the approval voting function.

Let \( f \in R_{V,A,B} \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( S_f \) be the scoring function for \( V \) and \( A \). We notice that \( S_f(a_1) = 0 + 0 + 0 + 1 = 1 \), \( S_f(a_2) = 1 + 1 + 0 + 0 = 2 \), \( S_f(a_3) = 1 + 0 + 0 + 1 = 2 \) and \( S_f(a_4) = 1 + 1 + 1 + 0 = 3 \). Since \( S_f(a_4) \geq S_f(a_j) \) for all \( j \in \{1, \ldots, 4\} \), then \( \alpha(f) = \{a_4\} \).
It’s obvious that the approval voting function satisfies the same properties mentioned above that are satisfied by range voting function. More than this, we notice that approval voting plays a perfect role in showing voters’ attitude towards alternatives on whether he/she approves the alternative or not.

Inspired by this, we hope to combine approval voting function with our range voting function with limits. In the approval voting function, voters show their disapproval attitude towards the alternative by giving 0 score, as well as show their approval by giving 1 score. In the previous chapter, we allows negative scores in our range voting system, since we set $B = [-b, b]$; so it comes to our mind that if we can use negative scores to Exampleress disapproval and positive score for approval. In this way, we can observe whether a voter like or dislike the alternative, eliminate the alternatives who are not approved by the majority of voters as well as determine the winner(s) by summing the score, which is actually the same process in range voting function besides the approval voting part.

3.2 Approval-Range Voting function

**Definition 3.2.1.** 1. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $B \subseteq \mathbb{R}$. For all $f \in R_{V,A,B}$, the alternative $a_k \in A$ is approved by $v_i$ if $f(v_i)_k \geq 0$ for some $i \in \{1, \ldots, v_n\}$.

2. An alternative $a_k \in A$ is majority-approved if $a_k$ is approved by $v_i$ for at least $\frac{n}{2}$ values of $i$. The approved alternative set $A^p_f$ of $A$ is the subset of $A$ such that $a_k$ is majority-approved.

3. Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $A^p_f$ be the approved alternative set for $A$. Let $S_f$ be the scoring function for $V$ and $A$. The **A-R voting function** is a
3.3 APPROVAL-RANGE VOTING FUNCTION WITH LIMITS

function $\epsilon : R_{V,A,B} \rightarrow \mathcal{P}(A)$ defined by letting $\epsilon(f)$ be the subset of all $a_k \in A_P^f$ such that $S_f(a_k) \geq S_f(a_j)$ for all $a_j \in A_P^f$, for all $f \in R_{V,A,B}$.

Here, let us look at an example to see how A-R voting function differs from regular range voting function. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $B = [-10, 10]$. Let $f \in R_{V,A,B}$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>-2</td>
<td>-1</td>
<td>10</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
<td>-3</td>
<td>3</td>
</tr>
<tr>
<td>$a_3$</td>
<td>6</td>
<td>10</td>
<td>-10</td>
</tr>
</tbody>
</table>

Let $S_f$ be the scoring function for $V$ and $A$. Let $\phi : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the range voting function. By regular range voting function, we have $S_f(a_1) = -2 + (-1) + 10 = 7$, $S_f(a_2) = 1 + (-3) + 3 = 1$, $S_f(a_3) = 6 + 10 + (-10) = 6$. Since $S_f(a_1) \geq S_f(a_j)$ for all $j \in \{1, 2, 3\}$, for all $f \in R_{V,A,B}$, then we know that $\phi(f) = \{a_1\}$.

Now, let us consider A-R voting function. Let $\epsilon : R_{V,A,B} \rightarrow \mathcal{P}(A)$ be the A-R voting function. Here we can see alternative $a_1$ is given positive score by 1 voters $v_3$, $a_2$ is given positive scores by 2 voters $v_1$ and $v_3$, and $a_3$ is given positive scores by 2 voters $v_1$ and $v_2$. Therefore, we know that $a_2$ and $a_3$ are approved by the majority of voters. Hence, the approved alternative set for $A$ is $A_P^f = \{a_2, a_3\}$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_P^f$, then $\epsilon(f) = \{a_3\}$ instead of $\{a_1\}$ in the regular range voting function.

3.3 Approval-Range voting function with limits

This A-R voting function also works with limited function.

**Definition 3.3.1.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$, and $B = [-b, b]$. Let $A_P^f$ be the approved
alternative set for \( A \). Let \( S_f \) be the scoring function for \( V \) and \( A \). The **A-R voting function with limits** is a function \( \varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A) \) defined by letting \( \varepsilon(f) \) be the subset of \( A^p_f \) such that \( S_f(a_k) \geq S_f(a_j) \) for all \( a_k \in \varepsilon(f) \) and \( a_j \in A^p_f \), for all \( f \in R^\xi_{V,A,B} \).

Here is an example for Approval-Range voting function with limits.

Let \( V = \{v_1, v_2, v_3\} \) be a set of voters and \( A = \{a_1, a_2, a_3\} \) be a set of alternatives. Let \( b = 10 \), then \( B = [-10, 10] \). Let \( \varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A) \) be the A-R voting function with limit.

Let us consider the following situation. Let \( f \in R^\xi_{V,A,B} \) be given as the following chart.

\[
\begin{array}{ccc}
  & v_1 & v_2 & v_3 \\
 a_1 & -1 & -1 & 10 \\
a_2 & 8 & -10 & 2 \\
a_3 & 6 & 4 & -3 \\
\end{array}
\]

Let \( S_f \) be the scoring function for \( V \) and \( A \). As we can see from the chart, \(|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15 \) for all \( i \in \{1, 2, 3\} \). Let \( \Phi : R_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits. By regular range voting function, we have \( S_f(a_1) = (-1) + (-1) + 10 = 8 \), and \( S_f(a_2) = 8 + (-10) + 2 = 0 \), and \( S_f(a_3) = 6 + 4 + (-3) = 7 \). Since \( S_f(a_1) \geq S_f(a_j) \) for all \( j \in \{1, 2, 3\} \), for all \( f \in R_{V,A,B} \), then we know that \( \Phi(f) = \{a_1\} \).

Now, let us consider A-R voting function with limits. Let \( \varepsilon : R_{V,A,B} \to \mathcal{P}(A) \) be the A-R voting function. Here we can see alternative \( a_1 \) is given positive score by 1 voters \( v_3 \), \( a_2 \) is given positive scores by 2 voters \( v_1 \) and \( v_3 \), and \( a_3 \) is given positive scores by 2 voters \( v_1 \) and \( v_2 \). Therefore, we know that \( a_2 \) and \( a_3 \) are approved by the majority of voters. Hence, the approved alternative set for \( A \) is \( A^p_f = \{a_2, a_3\} \). Since \( S_f(a_3) \geq S_f(a_j) \) for all \( a_j \in A^p_f \), then \( \varepsilon(f) = \{a_3\} \) instead of \( \{a_1\} \) in the regular range voting function.
3.4 Properties for A-R Voting Function with Limits

3.4.1 Always-A-Winner Condition for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting by illustrating a counter example.

**Example 3.4.1.** Let \( V = \{v_1, \ldots, v_3\} \) be a set of voters and \( A = \{a_1, \ldots, a_3\} \) be a set of alternatives. Let \( b = 10 \), and \( B = [-10, 10] \). Let \( \varepsilon : R^\xi_{V,A,B} \rightarrow \mathcal{P}(A) \) be the A-R voting function with limits.

Let \( f \in R^\xi_{V,A,B} \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>-5</td>
<td>5</td>
<td>-5</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-5</td>
<td>-5</td>
<td>5</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>5</td>
<td>-5</td>
<td>-5</td>
</tr>
</tbody>
</table>

As we can see from the chart, \(|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot10}{2} = 15\) for all \( i \in \{1, 2, 3\} \).

Here we know alternative \( a_1 \) is given positive score by 1 voters \( v_2 \), \( a_2 \) is given positive score by 1 voters \( v_3 \), and \( a_3 \) is given positive score by 1 voters \( v_1 \). Since none of them is majority-approved, then the approved alternative set \( A^p_f = \emptyset \). Therefore, \( \varepsilon(f) = \emptyset \).

Thus, A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting.

However, considering that this condition does not usually happen in the reality, we can somehow improve our A-R voting function to avoid this violation of always-a-winner condition.

**Definition 3.4.2.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \), and \( B = [-b, b] \). Let \( A^p_f \) be the approved alternative set for \( A \). Let \( S_f \) be the scoring function for \( V \) and \( A \). Let \( \varepsilon : R^\xi_{V,A,B} \rightarrow \mathcal{P}(A) \) be
the A-R voting function with limits. The improved A-R voting function with limits is a function $\varepsilon' : R_{V,A,B}^\xi \to \mathcal{P}(A)$ such that for all $f \in R_{V,A,B}^\xi$, and if $A_f^p \neq \emptyset$, then $\varepsilon'(f) = \varepsilon(f)$; If $A_f^p(f) = \emptyset$, then $\varepsilon'(f)$ is be the set of all $a_k \in A$ such that $S_f(a_k) \geq S_f(a_j)$ for all $j \in \{1, \ldots, m\}$.

$\triangle$

In this way, the improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting as well as regular range voting function does.

The following theorem is trivial.

**Theorem 3.4.3.** The improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting.

In the reality world, although it barely happens that in an election, all the alternatives are not liked by most voters, which means the people who hold such an election should probably reconsider the legitimacy of their alternatives, mathematically we need to consider the case that every alternative is not approved by most of voters. If such an election is for a small group of people, for example club election, then when this situation takes place, people should consider if they pick the suitable alternatives and therefore may hold another election with different alternatives; if this happens in politics, with a great amount of voters, it may be unrealistic to regather our voters and ask them to vote for a new group of alternatives, then our improved A-R voting function with limits can be used in such condition, in order to choose the one(s) that our voters disapproves the least.

### 3.4.2 Majority Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Majority Criterion for Range Voting by giving the following example.
Example 3.4.4. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in R^\xi_{V,A,B}$ be given as the following chart. (Here, all $f$ are the same as the ones for Example 2.2.4.)

\begin{center}
\begin{tabular}{c|ccc}
 & $v_1$ & $v_2$ & $v_3$ \\
\hline
$a_1$ & 5 & 3 & -9 \\
$a_2$ & 2 & 2 & 3 \\
$a_3$ & -8 & -10 & 3 \\
\end{tabular}
\end{center}

As we can see from the chart, $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Here we can see alternative $a_1$ is given positive score by 2 voters $v_1$ and $v_2$, $a_2$ is given positive score by all 3 voters, and $a_3$ is given positive or 0 score by 1 voter $v_3$. Therefore $A^p_f = \{a_1, a_2\}$.

Let $S_f$ be the scoring function for $V$ and $A$. We observe that alternative $a_1$ is most preferred by the majority of voters since there are 2 voters out of 3 who choose $a_1$ as their most preferred alternative. However, we have $S_f(a_1) = 5 + 3 + (-9) = -1$, and $S_f(a_2) = 2 + 2 + 3 = 7$, $S_f(a_3) = -8 + (-10) + 3 = -1$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_2\}$ instead of $\{a_1\}$. \hfill \Box

3.4.3 Condorcet Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Condorcet Criterion for Range Voting by giving the following example.

Example 3.4.5. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in R^\xi_{V,A,B}$ be given as the following chart. (Here, all $f$ are the same as the ones for Example 2.2.8.)

As we can see from the chart, $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$. 
After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>-7</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$a_2$</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0</td>
<td>-6</td>
<td>-7</td>
</tr>
</tbody>
</table>

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following. The alternative with square is the one who wins the pairwise comparison.

- $a_1$ vs $a_2$: 2:1
- $a_1$ vs $a_3$: 2:1
- $a_2$ vs $a_3$: 2:1

As shown in the table $a_1$ wins each pairwise comparison with $a_2$ and $a_3$, therefore $a_1$ is the Condorcet winner.

However, we observe that alternative $a_1$ is given positive score by 2 voters $v_2$ and $v_3$, $a_2$ is given positive score by all 3 voters, and $a_3$ is given positive or 0 score by only 1 voters $v_1$. Therefore, $a_1$ and $a_2$ are majority-approved, then the approved alternative set $A_p^f = \{a_1, a_2\}$. Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_1) = (-7) + 5 + 6 = 4$ and $S_f(a_2) = 8 + 4 + 2 = 14$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A_p^f$, then we know that $\varepsilon(f) = \{a_2\}$ instead of $\{a_1\}$.  

\[\diamondsuit\]

3.4.4 Pareto Criterion for A-R Voting Function with Limits

**Theorem 3.4.6.** A-R voting function with limits satisfies Pareto Criterion for range voting.

**Proof.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, -b]$. Let $S_f$ be the score function for
V and A. Let \( \varepsilon : R^{\xi}_{V,A,B} \rightarrow \mathcal{P}(A) \) be the A-R voting function with limit. Let \( f \in R^{\xi}_{V,A,B} \). Let \( A^p_f \subseteq A \) be the approved alternative set of A. Let \( a_k, a_j \in A \). Suppose we have \( f(v_i)_k > f(v_i)_j \) for all \( i \in \{1, \ldots, n\} \).

There are three cases for this situation. First suppose \( a_k \notin A^p_f \) and \( a_j \notin A^p_f \). Then \( a_j \notin \varepsilon(f) \).

Second, suppose \( a_k \in A^p_f \) and \( a_j \notin A^p_f \). Then \( a_j \notin \varepsilon(f) \).

Third, suppose \( a_k \in A^p_f \) and \( a_j \in A^p_f \). Then we have \( S_f(a_k) > S_f(a_j) \). By the definition of the A-R voting function with limits, we have \( a_j \notin \varepsilon(f) \).

It cannot be the case that \( a_k \notin A^p_f \) and \( a_j \in A^p_f \), since \( f(v_i)_k > f(v_i)_j \) for all \( i \in \{1, \ldots, n\} \).

Therefore \( a_j \) can never be in \( \varepsilon(f) \).

\[ \square \]

### 3.4.5 Monotonicity Criterion for A-R Voting Function with Limits

**Theorem 3.4.7.** A-R voting function with limits satisfies Monotonicity Criterion for range voting.

**Proof.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \) and \( B = [-b, -b] \). For \( f \in R^{\xi}_{V,A,B} \), let \( S_f \) be the score function for \( V \) and \( A \). Let \( \Phi : R^{\xi}_{V,A,B} \rightarrow \mathcal{P}(A) \) be the range voting function with limit. Let \( A^p_f \subseteq A \) be the approved alternative set of \( A \). Let \( j \in \{1, \ldots, m\} \).

Suppose \( a_j \) is in \( \varepsilon(f) \). Then we know \( a_j \in A^p_f \) and \( S_f(a_j) \geq S_f(a_k) \) for all \( a_k \in A^p_f \). If \( f(v_i)_j \) increases for some \( i \in \{1, \ldots, n\} \), then we have \( a_j \in A^p_f \) still and \( S_f(a_j) > S_f(a_k) \) for all \( a_k \in A^p_f \). Hence, by the definition of range voting function with limits, we have \( a_j \in \varepsilon(f) \). \[ \square \]
3.4.6 Independence of Irrelevant Alternative Criterion for A-R Voting Function with Limits

I will show that A-R voting function with limits does not satisfy Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

**Example 3.4.8.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : \mathbb{R}_V \times A \times B \to \mathcal{P}(A)$ be the A-R voting function with limits. Let $f \in \mathbb{R}^\mathbb{E}_{V,A,B}$ be given as the following chart. (Here, all $f$ are the same as the ones for Example 2.2.14)

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>7</td>
<td>-7</td>
<td>2</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
<td>5</td>
<td>-6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-6</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

As we can see from the chart, $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

Let $S_f$ be the scoring function for $V$ and $A$. We can see alternative $a_1$ is given positive or 0 scores by 2 voters, $v_1$ and $v_3$, while $a_2$ is given positive or 0 scores by 2 voters $v_1$ and $v_2$, and $a_3$ is given positive or 0 scores by 2 voters $v_2$ and $v_3$. Therefore, the approved alternative set $A_f^p = \{a_1, a_2, a_3\}$. Let $S_f$ be the scoring function for $V$ and $A$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f) = \{a_3\}$.

We change $f(v_3)_1$ and $f(v_3)_3$ but keep $f(v_3)_1 < f(v_3)_3$, we get new $f' \in \mathbb{R}^\mathbb{E}_{V,A,B}$ as in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>7</td>
<td>-7</td>
<td>4</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
<td>5</td>
<td>-6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-6</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

After changing $f(v_3)_1$ and $f(v_3)_3$, let $S_{f'}$ be the score function for new $V$ and $A$. We can see alternative $a_1$ is given positive or 0 scores by 2 voters, $v_1$ and $v_3$, while $a_2$ is given positive or 0
scores by 2 voters \(v_1\) and \(v_2\), and \(a_3\) is given positive or 0 scores by 2 voters \(v_2\) and \(v_3\). Therefore, the approved alternative set \(A_{f'}^p = \{a_1, a_2, a_3\}\). Since \(S_f(a_1) \geq S_f(a_j)\) for all \(a_j \in A_{f'}^p\), then we know that \(\varepsilon(f') = \{a_1\}\) instead of \(\{a_3\}\).

3.4.7 Intensity of Independence of Irrelevant Alternative Criterion for A-R Voting Function with Limits

**Theorem 3.4.9.** A-R voting function with limits satisfies the Intensity of Independence of Irrelevant Alternatives Criterion for Range Voting.

**Proof.** Let \(V = \{v_1, \ldots, v_n\}\) be a set of voters for some \(n \in \mathbb{N}\) and \(A = \{a_1, \ldots, a_m\}\) be a set of alternatives for some \(m \in \mathbb{N}\). Let \(b > 0\) and \(B = [-b, b]\). Let \(\varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A)\) be the A-R voting function with limit. Let \(f \in R^\xi_{V,A,B}\) and \(S_f\) be the score function for \(V\) and \(A\). Let \(j, k \in \{1, \ldots, m\}\).

Suppose that \(a_j \in \varepsilon(f)\) and \(a_k \not\in \varepsilon(f)\), and for some \(i \in \{1, \ldots, n\}\), and \(f(v_i)_p\) changes for some \(p \in \{1, \ldots, m\}\) while \(f(v_i)_j - f(v_i)_k\) remains the same.

There are three cases for this situation. First, suppose that after \(f(v_i)_p\) changes, we have \(a_j \not\in A_{f'}^p\) and \(a_k \not\in A_{f'}^p\) where \(f' \in R^\xi_{V,A,B}\) as our new profile. Then \(a_k \not\in \varepsilon(f')\).

Second, suppose that \(a_j \in A_{f'}^p\) and \(a_k \not\in A_{f'}^p\). Then \(a_k \not\in \varepsilon(f')\).

Third, suppose that \(a_j \in A_{f'}^p\) and \(a_k \in A_{f'}^p\). Since none of the voters change their intensity of preference for \(a_j\) over \(a_k\), which means \(f(v_i)_j - f(v_i)_k\) does not change for all \(i \in \{1, \ldots, n\}\), then \(S_f(a_j) - S_f(a_k)\) does not change as well. So we always have \(S_f(a_j) > S_f(a_k)\). By the definition of the A-R voting function with limits, we know \(a_k \not\in \varepsilon(f)\).

It cannot be the case that \(a_j \not\in A_{f}^p\) and \(a_k \in A_{f}^p\), since \(f(v_i)_j - f(v_i)_k\) does not change.
Therefore, we conclude that it is always the case that $a_k \notin \varepsilon(f)$. \qed

3.4.8 Consistency Property for A-R Voting Function with Limits

I will show that the A-R Voting function with limits does not satisfy consistency property by giving the following example.

**Example 3.4.10.** Let $V = \{v_1, \ldots, v_6\}$ be a set of voters and $A = \{a_1, \ldots, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\varepsilon : \mathbb{R}^{\xi,V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in \mathbb{R}^{\xi,V,A,B}$ be given as the following chart.

\[
\begin{array}{cccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
 a_1 & 10 & -1 & -1 & 8 & 5 & 3 \\
 a_2 & 0 & -6 & 10 & -2 & 7 & 4 \\
 a_3 & -5 & 8 & 4 & 5 & 3 & 8 \\
\end{array}
\]

As we can see from the chart, $|f(v_1)| + |f(v_2)| + |f(v_3)| = \frac{mb}{2} = \frac{3 \times 10}{2} = 15$ for all $i \in \{1, \ldots, 6\}$.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$.

Here we can get disjoint profiles $f_1 \in R_{V_1,A,B}$ as below.

\[
\begin{array}{ccc}
 & v_1 & v_3 \\
 a_1 & 10 & -1 \\
 a_2 & 0 & -6 \\
 a_3 & -5 & 8 \\
\end{array}
\]

In $f_1$, we can see alternative $a_1$ is given positive score or zero by 1 voter $v_1$, and $a_2$ is given positive or 0 score by 2 voters $v_1$, $v_3$, and $a_3$ is given positive or 0 score by 2 voters, $v_2$, $v_3$. Therefore, $a_2$ and $a_3$ are majority-approved, then the approved alternative set $A_f^p = \{a_2, a_3\}$.

Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_2) = 0 + (-6) + 10 = 4$, $S_f(a_3) = (-5) + 8 + 4 = 7$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A_f^p$, then we know that $\varepsilon(f_1) = \{a_3\}$.

Then let us look at $f_2 \in R_{V_2,A,B}$ as given.
3.4. PROPERTIES FOR A-R FUNCTION WITH LIMITS

<table>
<thead>
<tr>
<th></th>
<th>(v_4)</th>
<th>(v_5)</th>
<th>(v_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>8</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>(a_2)</td>
<td>-2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>(a_3)</td>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

In \(f_2\), we can see alternative \(a_1\) is given positive or 0 score by all 3 voters, \(a_2\) is given positive score by 2 voters \(v_5\) and \(v_6\), and \(a_3\) is given positive or 0 score by all 3 voters. Therefore, \(a_1\), \(a_2\) and \(a_3\) are majority-approved, then the approved alternative set \(A_f^p = \{a_1, a_2, a_3\}\). Let \(S_f\) be the scoring function for \(V\) and \(A\). We have \(S_f(a_1) = 8 + 5 + 3 = 16\), \(S_f(a_2) = (-2) + 7 + 4 = 9\), \(S_f(a_3) = 5 + 3 + 8 = 16\). Since \(S_f(a_1) = S_f(a_3) \geq S_f(a_j)\) for all \(a_j \in A_f^p\), then we know that \(\varepsilon(f_2) = \{a_1, a_3\}\).

Then we observe that \(\varepsilon(f_1) \cap \varepsilon(f_2) = \{a_3\}\).

However, let us take a look at \(\varepsilon(f_1 + f_2)\). In \(f_1 + f_2\), we can see alternative \(a_1\) is given positive score by 4 voters \(v_1\), \(v_4\), \(v_5\) and \(v_6\), \(a_2\) is given positive or 0 score by 4 voters \(v_1\), \(v_3\), \(v_5\) and \(v_6\), and \(a_3\) is given positive or 0 score by 5 voters except for \(v_1\). Therefore, the approved alternative set \(A_f^p = \{a_1, a_2, a_3\}\). Let \(S_f\) be the scoring function for \(V\) and \(A\). We have \(S_f(a_1) = 10 + (-1) + (-1) + 8 + 5 + 3 = 24\), \(S_f(a_2) = 0 + (-6) + 10 + (-2) + 7 + 4 = 13\), \(S_f(a_3) = (-5) + 8 + 4 + 5 + 3 + 8 = 23\). Since \(S_f(a_1) \geq S_f(a_j)\) for all \(a_j \in A_f^p\), then we know that \(\varepsilon(f_1 + f_2) = \{a_1\}\).

Here, we notice that \(\varepsilon(f_1 + f_2) = \{a_1\}\) while \(\varepsilon(f_1) \cap \varepsilon(f_2)\) does exist and is actually \(\{a_3\}\). Hence \(\varepsilon(f_1 + f_2) \neq \varepsilon(f_1) \cap \varepsilon(f_2)\). ♦

3.4.9 Faithful Property for A-R Voting Function with Limits

Theorem 3.4.11. A-R voting function with limits satisfies faithful property.
Proof. Let \( V = \{v\} \) be a set of voters and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \) and \( B = [-b, b] \). Let \( \varepsilon : R^k_{V,A,B} \to \mathcal{P}(A) \) be the A-R voting function with limits. Let \( S_f \) be the scoring function for \( V \) and \( A \).

Suppose \( f(v)_j > f(v)_k \) for some \( j, k \in \{1, \ldots, m\} \). There are three cases for this situation.

First, suppose that \( a_j \notin A^p_f \) and \( a_k \notin A^p_f \). Then \( a_k \notin \varepsilon(f) \).

Second, suppose that \( a_j \in A^p_f \) and \( a_k \notin A^p_f \). Then \( a_k \notin \varepsilon(f) \).

Third, suppose that \( a_j \in A^p_f \) and \( a_k \in A^p_f \). Since \( f(v)_j > f(v)_k \), then \( S_f(a_j) > S_f(a_k) \) with only one voter. By the definition of the A-R voting function with limits, we know \( a_k \notin \varepsilon(f) \).

It cannot be the case that \( a_j \notin A^p_f \) and \( a_k \in A^p_f \), since \( f(v)_j > f(v)_k \).

Therefore, we always have \( a_k \notin \varepsilon(f) \). \(\square\)

3.4.10 Cancellation Property for A-R Voting Function with limits

I will show that the A-R voting function does not satisfy the cancellation property by giving the following example.

Example 3.4.12. Let \( V = \{v_1, v_2\} \) be a set of voters and \( A = \{a_1, a_2\} \) be a set of alternatives. Let \( b = 10 \), and \( B = [-10, 10] \). Let \( \Phi : R^k_{V,A,B} \to \mathcal{P}(A) \) be the range voting function with limits. Let \( f \in R^k_{V,A,B} \) be given as the following chart. (Here, all \( f \) are the same as the ones for Example \([2.2.26]\))

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>
As we can see from the chart \( |f(v_1)| + |f(v_2)| = \frac{mb}{2} = \frac{2 \cdot 10}{2} = 10 \) for all \( i \in \{1, 2\} \). And we observe from the chart that \( f(v_1)_1 > f(v_1)_2 \). So \( \pi_{a_1a_2}(f) = 1 \). Also \( f(v_2)_2 > f(v_2)_1 \), which means \( \pi_{a_2a_1}(f) = 1 \). Thus, in this case, \( \pi_{a_1a_2}(f) = \pi_{a_2a_1}(f) = 1 \).

Let \( S_f \) be the scoring function for \( V \) and \( A \). Alternative \( a_1 \) is given positive score by all 2 voters \( v_1, a_2 \) is given positive score by all 2 voters. Therefore, the approved alternative set \( A_f^p = \{a_1, a_2\} \). We see that \( S_f(a_1) = 8 + 3 = 11 \) and \( S_f(a_2) = 2 + 7 = 9 \). Since \( S_f(a_1) > S_f(a_2) \), then \( \varepsilon(f) = \{a_1\} \).

However, we notice that \( \varepsilon(f) \neq A \).

Hence, A-R voting function with limits does not satisfy the cancellation property. \( \diamond \)

3.4.11 Participation Criterion for A-R Voting Function with limits

I will show that A-R voting function with limits does not satisfy Participation Criterion by giving a counter example.

**Example 3.4.13.** Let \( V = \{v_1, v_2, v_3\} \) be a set of voters and \( A = \{a_1, a_2, a_3\} \) be a set of alternatives. Let \( b = 10 \), and \( B = [-10, 10] \). Let \( \varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A) \) be the A-R voting function with limits.

Let \( f \in R^\xi_{V,A,B} \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>7</td>
<td>-6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>3</td>
<td>7</td>
<td>-8</td>
</tr>
</tbody>
</table>

From this chart, we can see alternative \( a_1 \) is given positive score by 1 voter \( v_1 \), \( a_2 \) is given positive or 0 score by 2 voters \( v_1, v_2 \), and \( a_3 \) is given positive or 0 score by 2 voter \( v_1, v_2 \). Therefore, \( a_2 \) and \( a_3 \) are majority-approved, then the approved alternative set \( A_f^p = \{a_2, a_3\} \). Let \( S_f \) be the
scoring function for $V$ and $A$. We have $S_f(a_2) = 2 + 7 + (-6) = 3$, $S_f(a_3) = 3 + 7 + (-8) = 2$. Since $S_f(a_2) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_2\}$.

Now add another $v_4$ to our voter set, with $f(v_4)_1 < f(v_4)_2$. Then we obtain $f' \in R^\xi_{V,A,B}$ as follow.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
<td>5</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
<td>7</td>
<td>-6</td>
<td>6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>3</td>
<td>7</td>
<td>-8</td>
<td>4</td>
</tr>
</tbody>
</table>

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \times 10}{2} = 15$ for all $i \in \{1, \ldots, 4\}$.

From this new chart, we can see alternative $a_1$ is given positive score by 2 voter $v_1$, $v_4$, $a_2$ is given positive or 0 score by 3 voters $v_1$, $v_2$, $v_4$, and $a_3$ is given positive or 0 score by 3 voters $v_1$, $v_2$, $v_4$. Therefore, $a_1$, $a_2$ and $a_3$ are majority-approved, then the approved alternative set $A^p_f = \{a_1, a_2, a_3\}$. Then we have $S_f(a_1) = 10 + (-1) + (-1) + 5 = 13$, and $S_f(a_2) = 2 + 7 + (-6) + 6 = 9$, and $S_f(a_3) = 3 + 7 + (-8) + 6 = 8$. Since $S_f(a_1) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\varepsilon(f) = \{a_1\}$.

We observe that before $v_4$ is added, $a_2 \in \varepsilon(f)$ and $a_1 \notin \varepsilon(f)$. However, we get $a_1 \in \varepsilon(f)$ and $a_2 \notin \varepsilon(f)$.

Hence, A-R voting function with new limits does not satisfy participation criterion.

Example 3.4.14. Let $V = \{v_1, \ldots, v_5\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10, 10]$. Let $\varepsilon : R^\xi_{V,A,B} \rightarrow \mathcal{P}(A)$ be the A-R voting function with limits.
3.4. PROPERTIES FOR A-R FUNCTION WITH LIMITS

Let \( f \in R_{V,A,B}^\xi \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>7</td>
<td>-6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>3</td>
<td>7</td>
<td>-8</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

We observe that \( |f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15 \) for all \( i \in \{1, \ldots, 5\} \).

We can see alternative \( a_1 \) is given positive score by 3 voter \( v_1, v_4, v_5 \); and \( a_2 \) is given positive or 0 score by 4 voters \( v_1, v_2, v_4, v_5 \), and \( a_3 \) is given positive or 0 score by 4 voters \( v_1, v_2, v_4 \) and \( v_5 \). Therefore, \( a_1, a_2 \) and \( a_3 \) are majority-approved, then the approved alternative set \( A^p_f = \{a_1, a_2, a_3\} \). Then we have \( S_f(a_1) = 10 + (-1) + (-1) + 5 + 5 = 18 \), \( S_f(a_2) = 2 + 7 + (-6) + 6 + 6 = 16 \), \( S_f(a_3) = 3 + 7 + (-8) + 6 + 4 = 14 \). Since \( S_f(a_1) \geq S_f(a_j) \) for all \( a_j \in A^p_f \), then we know that \( \varepsilon(f) = \{a_1\} \).

Hence, we still get \( a_1 \in \varepsilon(f) \) and \( a_2 \notin \varepsilon(f) \).

Therefore, A-R voting function with new limits does not satisfy participation criterion as well. \( \Box \)
3. A-R VOTING FUNCTION WITH LIMITS
C-A-R Voting Function With Limits

4.1 Condorcet-Approval-Range voting function with limits

As we introduced Condorcet voting method earlier, we can review our definition for Condorcet winner for range voting through Definition 2.2.5 to Definition 2.2.6.

**Definition 4.1.1.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $A^p_f$ be the approved alternative set for $A$. Let $S_f$ be the scoring function for $V$ and $A$. Let $\varepsilon : R^{\xi}_{V,A,B} \rightarrow \mathcal{P}(A)$ be the A-R voting function with limits. The **C-A-R voting function with limits** is a function $\psi : R^{\xi}_{V,A,B} \rightarrow \mathcal{P}(A)$ defined by two conditions. First, determine whether there exists a Condorcet winner $a_p$ in $A$; if so, let $\psi(f) = \{a_p\}$. If there does not exist a Condorcet winner, let $\psi(f) = \varepsilon(f)$.

I will illustrate this C-A-R voting function by showing the following example.

**Example 4.1.2.** Let $V = \{v_1, \ldots, v_4\}$ be a set of voters and $A = \{a_1, \ldots, a_4\}$ be a set of alternatives. Let $b > 0$, then $B = [-10, 10]$. Let $\psi : R^{\xi}_{V,A,B} \rightarrow \mathcal{P}(A)$ be the C-A-R voting function with limits.
Let $f \in R_{V,A,B}^{\xi}$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>10</td>
</tr>
<tr>
<td>$a_2$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$a_4$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

We observe that $|f(v_i)|_1 + \cdots + |f(v_i)|_4 = \frac{mb}{2} = \frac{4 \cdot 10}{2} = 20$ for all $i \in \{1, \ldots, 4\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_4$</td>
<td>$a_4$</td>
<td>$a_4$, $a_2$ (tied)</td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_2$</td>
<td>$a_2$</td>
<td>$a_3$, $a_4$ (tied)</td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3$</td>
<td>$a_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_1$</td>
<td>$a_1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[
\begin{array}{c|c|c|c}
& a_1 & a_2 & a_3 \\
\hline
a_1 & 0:3 & \underline{a_2} & 4:0 \\
a_2 & 1:3 & \underline{a_2} & 1:3 \\
a_3 & 1:3 & \underline{a_4} & 0:3 \\
\hline
\end{array}
\]

As shown in the graph, $a_4$ wins each pairwise comparison with $a_2$, $a_3$ and $a_4$, therefore $a_4$ is the Condorcet winner. Hence $\psi(f) = \{a_4\}$. \hfill \diamond

Now let us look at another example that there is no Condorcet winner.

**Example 4.1.3.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, v_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\psi : R_{V,A,B}^{\xi} \rightarrow \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R_{V,A,B}^{\xi}$ be given as the following chart.
4.1. CONDORCET-APPROVAL-RANGE VOTING FUNCTION WITH LIMITS

We observe that \(|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15\) for all \(i \in \{1, 2, 3\}\).

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>v1</th>
<th>v2</th>
<th>v3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>-1</td>
<td>10</td>
<td>-1</td>
</tr>
<tr>
<td>a2</td>
<td>8</td>
<td>5</td>
<td>-4</td>
</tr>
<tr>
<td>a3</td>
<td>6</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

<table>
<thead>
<tr>
<th></th>
<th>a1 vs a2 2:1</th>
<th>a2 vs a3 2:1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>a3</td>
<td>a3</td>
</tr>
<tr>
<td>a3</td>
<td>a1</td>
<td>a2</td>
</tr>
</tbody>
</table>

As shown in the chart, no alternative \(a_p\) wins each pairwise comparison for all \(q \in \{1, 2, 3\}\) such that \(q \neq p\), and therefore there is no Condorcet winner. Then we go to A-R part.

Here we can see alternative \(a_1\) is given positive score by 1 voters \(v_3\), \(a_2\) is given positive scores by 2 voters \(v_1\) and \(v_2\), and \(a_3\) is given positive or 0 scores by all 3 voters. Therefore, we know that \(a_2\) and \(a_3\) are approved by the majority of voters. Hence, the approved alternative set for \(A\) is \(A_p^f = \{a_2, a_3\}\). Let \(S_f\) be the scoring function for \(V\) and \(A\). We have \(S_f(a_2) = 8 + 5 + (-4) = 9\) and \(S_f(a_3) = 6 + 0 + 10 = 16\). Since \(S_f(a_3) \geq S_f(a_j)\) for all \(a_j \in A_p^f\), then \(\psi(f) = \{a_3\}\).
4.2 Properties for C-A-R Voting Function with Limits

4.2.1 Always-A-Winner Condition for C-A-R Voting Function with Limits

I will show that C-A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting, by giving the following example.

Example 4.2.1. Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, and $B = [-10,10]$. Let $\psi : R^b_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R^b_{V,A,B}$ be given as the following chart.

\[
\begin{array}{|c|c|c|}
\hline
 & v_1 & v_2 & v_3 \\
\hline
a_1 & -1 & 10 & -3 \\
a_2 & -5 & -1 & 10 \\
a_3 & 10 & -4 & -2 \\
\hline
\end{array}
\]

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1,2,3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

\[
\begin{array}{|c|c|c|}
\hline
 & v_1 & v_2 & v_3 \\
\hline
a_3 & a_1 & a_2 \\
a_1 & a_2 & a_3 \\
a_2 & a_3 & a_1 \\
\hline
\end{array}
\]

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[
\begin{array}{c}
\text{a}_1 \text{ vs } \text{a}_2 \ 2:1 \\
\text{a}_1 \text{ vs } \text{a}_3 \ 1:2 \\
\text{a}_2 \text{ vs } \text{a}_3 \ 2:1
\end{array}
\]

As shown in the graph, there is no Condorcet winner. Now we check the approval alternative set. Here we can see alternative $a_1$ is given positive score by 1 voters $v_2$, $a_2$ is given positive score by 1 voters $v_3$, and $a_3$ is given positive score by 1 voters $v_1$. Since none of them is majority-
4.2. Properties for C-A-R Voting Function with Limits

If the approved alternative set \( A_f^p = \emptyset \). Therefore, \( \psi(f) = \emptyset \). Hence, there is no winner for this example. ∎

Thus, C-A-R voting function with limits does not satisfy the Always-A-Winner Condition for Range Voting.

However, considering that this condition does not usually happen in the reality, similarly we can do what we did for A-R voting to improve C-A-R voting function and avoid this violation of always-a-winner condition.

**Definition 4.2.2.** Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \), and \( B = [-b, b] \). Let \( A_f^p \) be the approved alternative set for \( A \). Let \( S_f \) be the scoring function for \( V \) and \( A \). Let \( \psi : \mathbb{R}_{V,A,B}^\xi \to \mathcal{P}(A) \) be the A-R voting function with limits. The **improved C-A-R voting function with limits** is a function \( \psi' : \mathbb{R}_{V,A,B}^\xi \to \mathcal{P}(A) \) such that for all \( f \in \mathbb{R}_{V,A,B}^\xi \), if there exists a Condorcet winner or \( A_f^p \neq \emptyset \), then \( \psi'(f) = \psi(f) \), and if \( A_f^p(f) = \emptyset \), then \( \psi'(f) \) is the set of all \( a_k \in A \) such that \( S_f(a_k) \geq S_f(a_j) \) for all \( j \in \{1, \ldots, m\} \).

△

In this way, improved C-A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting as well as regular range voting function does.

The following theorem is trivial.

**Theorem 4.2.3.** The improved A-R voting function with limits satisfies the Always-A-Winner Condition for Range Voting.

4.2.2 Majority Criterion for C-A-R Voting Function with Limits

**Theorem 4.2.4.** C-A-R voting function with limits satisfies the Majority Criterion for Range Voting.
Proof. Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \) and \( B = [-b, b] \). Let \( \psi : R_{V,A,B}^\xi \to \mathcal{P}(A) \) be the C-A-R voting function with limits.

Suppose \( k \in \{1, \ldots, m\} \) and \( a_k \) is most preferred by more than \( \frac{n}{2} \) voters, which says that \( a_k \) wins each pairwise comparison for all \( j \in \{1, \ldots, m\} \) such that \( k \neq j \).

Then by the definition of C-A-R voting function, we know that we do have a Condorcet winner in \( A \) which is \( a_k \). Hence, \( a_k \in \psi(f) \) for all \( f \in R_{V,A,B}^\xi \).

4.2.3 Condorcet Criterion for C-A-R Voting Function with Limits

Theorem 4.2.5. C-A-R voting function with limits satisfies the Condorcet Criterion for Range Voting.

The proof for this theorem is trivial.

4.2.4 Pareto Criterion for C-A-R Voting Function with Limits


Proof. Let \( V = \{v_1, \ldots, v_n\} \) be a set of voters for some \( n \in \mathbb{N} \) and \( A = \{a_1, \ldots, a_m\} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \) and \( B = [-b, b] \). Let \( \psi : R_{V,A,B}^\xi \to \mathcal{P}(A) \) be the C-A-R voting function with limits.

Let \( f \in R_{V,A,B}^\xi \). Suppose \( f(v_i)_k > f(v_i)_j \) for all \( i \in \{1, \ldots, n\} \). There are actually two cases depended on Condorcet winner. If there exists a Condorcet winner in \( A \), by the definition of C-
A-R voting function, since $a_j$ cannot win pairwise comparison with $a_k$, then $a_j$ naturally cannot be that Condorcet winner.

Suppose there is no Condorcet winner. Let $S_f$ be the scoring function for $V$ and $A$. There are three cases. First, suppose $a_j, a_k \in A_p^f$. However, since we know that it is always the case that $f(v_i)_k > f(v_i)_j$, then we have $S_f(a_k) > S_f(a_j)$, then by the definition of C-A-R voting function with limits, so $a_j \notin \psi(f)$. Second, suppose $a_k \in A_p^f$ and $a_j \notin A_p^f$. Then $a_j \notin \psi(f)$. Third, suppose $a_k, a_j \notin A_p^f$. Then still $a_j \notin \psi(f)$. Note that it cannot be the case that $a_j \in A_p^f$ but $a_k \notin A_p^f$, since $f(v_i)_k > f(v_i)_j$ for all $i \in \{1, \ldots, n\}$.

Therefore, we always have $a_j \notin \psi(f)$. 

**4.2.5 Monotonicity Criterion for C-A-R Voting Function with Limits**

**Theorem 4.2.7.** C-A-R voting function with limits satisfies the Monotonicity Criterion for Range Voting.

**Proof.** Let $V = \{v_1, \ldots, v_n\}$ be a set of voters for some $n \in \mathbb{N}$ and $A = \{a_1, \ldots, a_m\}$ be a set of alternatives for some $m \in \mathbb{N}$. Let $b > 0$ and $B = [-b, b]$. Let $\psi : R_{V,A,B}^\mathcal{K} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

For all $f \in R_{V,A,B}^\mathcal{K}$, suppose $a_j$ is in $\psi(f)$ and $f(v_i)_j$ increases for some $i \in \{1, \ldots, n\}$. There are two cases. First, suppose there exists a Condorcet winner before $f(v_i)_j$ increases, which is $a_j$. By the definition of C-A-R voting function, since $a_j$ wins each pairwise comparison, then $a_j$ still wins each pairwise comparison when $f(v_i)_j$ increases. Hence $a_j$ is still in $\psi(f)$. Second, suppose there is no Condorcet winner before $f(v_i)_j$ increases. Let $S_f$ be the scoring function for $V$ and $A$. Since $a_j \in \psi(f)$, then from the definition of C-A-R voting function, we know that $S_f(a_j) \geq S_f(a_k)$ for all $a_k \in A_p^f$. When $f(v_i)_j$ increases, we get that $S_f(a_j) > S_f(a_k)$ for all
4. C-A-R VOTING FUNCTION WITH LIMITS

\( a_k \in A^p \). By the definition of C-A-R voting function, \( a_j \) is still in \( \psi(f) \). Therefore, under both cases, \( a_j \in \psi(f) \).

\[ \square \]

4.2.6 Independence of Irrelevant Alternative Criterion for C-A-R Voting Function with Limits

I will show C-A-R voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

Example 4.2.8. Let \( V = \{v_1, v_2, v_3\} \) be a set of voters and \( A = \{a_1, a_2, a_3\} \) be a set of alternatives. Let \( b = 10 \), then \( B = [-10, 10] \). Let \( \psi : R_{V,A,B}^\xi \rightarrow \mathcal{P}(A) \) be the C-A-R voting function with limits.

Let \( f \in R_{V,A,B}^\xi \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>7</td>
<td>-7</td>
<td>2</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>5</td>
<td>-6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-6</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

We observe that \( |f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot10}{2} = 15 \) for all \( i \in \{1, 2, 3\} \).

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td></td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td></td>
</tr>
<tr>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td></td>
</tr>
</tbody>
</table>

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[ a_1 \) vs \( a_2 \) 2:1 \]
\[ a_1 \) vs \( a_3 \) 1:2 \]
\[ a_2 \) vs \( a_3 \) 2:1 \]
As shown in the chart, no alternative \( a_p \) wins each pairwise comparison for all \( q \in \{1, \ldots, m\} \) such that \( q \neq p \), and therefore there is no Condorcet winner. Then we go to A-R part.

Here we can see alternative \( a_1 \) is given positive or 0 scores by 2 voters, \( v_1 \) and \( v_3 \), \( a_2 \) is given positive or 0 scores by 2 voters \( v_1 \) and \( v_2 \), and \( a_3 \) is given positive or 0 scores by 2 voters \( v_2 \) and \( v_3 \). Therefore, we know that \( a_1 \), \( a_2 \) and \( a_3 \) are approved by the majority of voters. Hence, the approved alternative set for \( A \) is \( A^p_f = \{a_1, a_2, a_3\} \). Let \( S_f \) be the scoring function for \( V \) and \( A \).

We have \( S_f(a_1) = 7 + (-7) + 2 = 2 \), \( S_f(a_2) = 2 + 5 + (-6) = 1 \), and \( S_f(a_3) = -6 + 3 + 7 = 4 \). Since \( S_f(a_3) \geq S_f(a_j) \) for all \( a_j \in A^p_f \), then we know that \( \psi(f) = \{a_3\} \).

We change \( f(v_3)_1 \) and \( f(v_3)_3 \) but keep \( f(v_3)_1 < f(v_3)_3 \), then we have new \( f' \in \mathcal{R}^\xi_{V,A,B} \) as in the following table.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>7</td>
<td>-7</td>
<td>4</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>5</td>
<td>-6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-6</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Same as above, we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td></td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td></td>
</tr>
<tr>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td></td>
</tr>
</tbody>
</table>

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[
\begin{array}{c|c|c}
\hline
a_1 & a_2 & 2:1 \\
\hline
a_1 & a_3 & 1:2 \\
\hline
a_2 & a_3 & 2:1 \\
\hline
\end{array}
\]

As shown in the graph, no alternative \( a_p \) wins each pairwise comparison for all \( q \in \{1, \ldots, m\} \) such that \( q \neq p \), and therefore there is no Condorcet winner. Then we go to A-R part.
4. C-A-R VOTING FUNCTION WITH LIMITS

Let $S'_f$ be the score function for our new $V$ and $A$. Before we change $f(v_3)_2$ and $f(v_3)_2$, $\psi(f) = \{a_3\}$. However, in the second table, since $S'_f(a_1) = 7 + (−7) + 4 = 4$ while $S'_f(a_3) = −6 + 3 + 5 = 2$, $S'_f(a_1) > S'_f(a_3)$, then $\psi(f') = \{a_1\}$. Therefore, C-A-R voting function with limits does not satisfy the Independence of Irrelevant Alternative Criterion for Range Voting.

4.2.7 Intensity of Independence of Irrelevant Alternative Criterion for C-A-R Voting Function with Limits

I will show that C-A-R voting function with limits does not satisfy the Intensity of Independence of Irrelevant Alternative Criterion for Range Voting by giving the following example.

**Example 4.2.9.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [−10, 10]$. Let $\psi : R^\xi_{V,A,B} \to \mathcal{P}(A)$ be the C-A-R voting function with limits.

Let $f \in R^\xi_{V,A,B}$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$a_2$</td>
<td>4</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>$a_3$</td>
<td>−6</td>
<td>10</td>
<td>6</td>
</tr>
</tbody>
</table>

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15$ for all $i \in \{1, 2, 3\}$.

After converting the previous table to preference orders by looking at each column separately, we obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$a_1$</td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3$</td>
<td>$a_2$</td>
<td></td>
</tr>
</tbody>
</table>

Same as what we did for original range voting, now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.
4.2. PROPERTIES FOR C-A-R VOTING FUNCTION WITH LIMITS

As shown in the chart, alternative $a_1$ wins each pairwise comparison for all $k \in \{2, 3\}$ and therefore $a_1$ is the Condorcet winner.

We change $f(v_2)_1$, as well as $f(v_2)_2$ and $f(v_2)_3$ but keep $f(v_2)_1 - f(v_2)_2$ the same as before, then we have new $f' \in R^{\xi}_{V,A,B}$ as in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$a_2$</td>
<td>4</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Same as above, we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_1$</td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td></td>
</tr>
</tbody>
</table>

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

As shown in the graph, no alternative $a_p$ wins each pairwise comparison for all $q \in \{1, \ldots, m\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

Let $S_{f'}$ be the score function for our new $V$ and $A$. We can see alternative $a_1$ is given positive or 0 scores by all 3 voters, and $a_2$ is given positive or 0 scores by all 3 voters, and $a_3$ is given positive or 0 scores by 2 voters $v_2$ and $v_3$. Hence, the approved alternative set for $A$ is $A_{f'} = \{a_1, a_2, a_3\}$. Since $S_{f'}(a_1) = 5 + 2 + 5 = 12$, and $S_{f'}(a_2) = 4 + 7 + 4 = 15$, and $S_{f'}(a_3) = -6 + 6 + 6 = 6$. Since $S_{f'}(a_2) \geq S_{f'}(a_k)$ for all $a_k \in A_{f'}$, then $\psi(f') = \{a_2\}$. Before we change $f(v_2)_1$, and $f(v_2)_2$ and
4. C-A-R VOTING FUNCTION WITH LIMITS

\[ f(v_2)_3, \] we have \( \psi(f) = \{a_1\} \). Therefore, C-A-R voting function with limits does not satisfy the Intensity of Independence of Irrelevant Alternative Criterion for Range Voting.

4.2.8 Participation Criterion for C-A-R Voting Function with limits

I will show C-A-R voting function with limits does not satisfy Participation Criterion by giving a counter example.

**Example 4.2.10.** Let \( V = \{v_1, v_2, v_3\} \) be a set of voters and \( A = \{a_1, a_2, a_3\} \) be a set of alternatives. Let \( b = 10 \), then \( B = [-10, 10] \). Let \( \psi : R^{\xi}_{V,A,B} \to \mathcal{P}(A) \) be the C-A-R voting function with limits.

Let \( f \in R^{\xi}_{V,A,B} \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>10</td>
<td>-9</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

We observe that \( |f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3 \cdot 10}{2} = 15 \) for all \( i \in \{1, 2, 3\} \).

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td></td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td></td>
</tr>
<tr>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td></td>
</tr>
</tbody>
</table>

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[ \begin{array}{c|c|c|c|c} a_1 \text{ vs } a_2 \text{ 2:1} \hline a_1 \text{ vs } a_3 \text{ 1:2} & a_2 \text{ vs } a_3 \text{ 2:1} \end{array} \]
4.2. PROPERTIES FOR C-A-R VOTING FUNCTION WITH LIMITS

As shown in the table, no alternative \(a_p \in A\) wins each pairwise comparison for all \(q \in \{1, \ldots, m\}\) such that \(q \neq p\), and therefore there is no Condorcet winner. Then we go to A-R part.

From this chart, we can see alternative \(a_1\) is given positive score by 1 voter \(v_1\), \(a_2\) is given positive or 0 score by 2 voters \(v_1, v_2\); \(a_3\) is given positive or 0 score by 2 voters \(v_2, v_3\). Therefore, \(a_2\) and \(a_3\) are majority-approved, then the approved alternative set \(A_p^f = \{a_2, a_3\}\). Let \(S_f\) be the scoring function for \(V\) and \(A\). We have \(S_f(a_2) = 2 + 10 + (-9) = 3\), \(S_f(a_3) = (-3) + 4 + 5 = 6\). Since \(S_f(a_2) \geq S_f(a_j)\) for all \(a_j \in A_p^f\), then we know that \(\varepsilon(f) = \{a_3\}\).

Then add another \(v_4\) to our voter set such that \(f(v_4)_1 < f(v_4)_3\). Then we may obtain \(f' \in R^\xi_{V,A,B}\) as follow.

<table>
<thead>
<tr>
<th></th>
<th>(v_1)</th>
<th>(v_2)</th>
<th>(v_3)</th>
<th>(v_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>(a_2)</td>
<td>2</td>
<td>10</td>
<td>-10</td>
<td>5</td>
</tr>
<tr>
<td>(a_3)</td>
<td>-3</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th>(v_1)</th>
<th>(v_2)</th>
<th>(v_3)</th>
<th>(v_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_2, a_3) tied</td>
</tr>
<tr>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_1)</td>
<td>(a_1)</td>
</tr>
<tr>
<td>(a_3)</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td></td>
</tr>
</tbody>
</table>

Then we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

\[
\begin{array}{c|c|c|c}
\hline
a_1 & a_2 & a_3 & a_2, a_3 \text{ tied} \\
\hline
a_1 & 2:2 & & \\
\hline
a_1 & 3:1 & a_2, a_3 & 2:1 \\
\hline
\end{array}
\]

As shown in the table, no alternative \(a_p \in A\) wins each pairwise comparison for all \(q \in \{1, \ldots, m\}\) such that \(q \neq p\), and therefore there is no Condorcet winner. Then we go to A-R part.
From this new chart, we can see alternative $a_1$ is given positive or 0 score by 2 voter $v_1, v_4$, $a_2$ is given positive score by 3 voters $v_1, v_2, v_4$, and $a_3$ is given positive or 0 score by 3 voters $v_2, v_3, v_4$. Therefore, $a_1, a_2$, and $a_3$ are majority-approved, then the approved alternative set $A_{f'}^p = \{a_1, a_2, a_3\}$. Then we have $S_{f'}(a_1) = 10 + (-1) + (-1) + 4 = 12$, $S_{f'}(a_2) = 2 + 10 + (-10) + 5 = 7$, $S_{f'}(a_3) = (-3) + 4 + 4 + 5 = 10$. Since $S_{f'}(a_1) \geq S_{f'}(a_j)$ for all $a_j \in A_{f'}^p$, then we know that $\psi(f') = \{a_1\}$.

We observe that before $v_4$ is added, we had $a_3 \in \psi(f)$ and $a_1 \notin \psi(f)$. However, after $v_4$ is added, we get $a_1 \in \psi(f)$ and $a_3 \notin \psi(f)$. Hence, C-A-R voting function with limits does not satisfy the participation criterion.

\[\diamond\]

4.2.9 Consistency Property for C-A-R Voting Function with Limits

I will show C-A-R Voting function with limits does not satisfy consistency property by giving a counter example.

**Example 4.2.11.** Let $V = \{v_1, v_2, v_3\}$ be a set of voters and $A = \{a_1, a_2, a_3\}$ be a set of alternatives. Let $b = 10$, then $B = [-10, 10]$. Let $\varepsilon : R^\xi_{V,A,B} \to \mathcal{P}(A)$ be the A-R voting function with limits.

Let $f \in R^\xi_{V,A,B}$ be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
<td>8</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0</td>
<td>-6</td>
<td>10</td>
<td>-2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-5</td>
<td>8</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

We observe that $|f(v_i)_1| + |f(v_i)_2| + |f(v_i)_3| = \frac{mb}{2} = \frac{3\cdot 10}{2} = 15$ for all $i \in \{1, \ldots, 6\}$.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$. Here we can get disjoint profiles $f_1 \in R_{V_1,A,B}$ as below.
Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>10</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0</td>
<td>-6</td>
<td>10</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-5</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

- $a_1$ vs $a_2$: 2:1
- $a_1$ vs $a_3$: 1:2
- $a_2$ vs $a_3$: 2:1

As shown in the graph, In $f_1$, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

From the table, we can see alternative $a_1$ is given positive score by all 1 voter $v_1$, $a_2$ is given positive or 0 score by all 2 voters $v_1$, $v_3$, and $a_3$ is given positive or 0 score by 2 voters, $v_2$, $v_3$. Therefore, $a_2$ and $a_3$ are majority-approved, then the approved alternative set $A^p_f = \{a_2, a_3\}$.

Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_2) = 0 + (-6) + 10 = 4$, $S_f(a_3) = (-5) + 8 + 4 = 7$. Since $S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\psi(f_1) = \{a_3\}$.

Then let us look at $f_2 \in R_{V_2,A,B}$ as given.

<table>
<thead>
<tr>
<th></th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>8</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>$a_3$</td>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>
Then we convert the previous table to preference orders by looking at each column separately and obtain the following table.

<table>
<thead>
<tr>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_1$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_1$</td>
</tr>
</tbody>
</table>

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.

$\begin{array}{c|c|c|c|c|c|c} & a_1 & a_2 & a_3 \\ \hline a_1 & 1:2 & & \\ a_2 & & 1:2 & \\ a_3 & & & 2:1 & \\ \end{array}$

As shown in the graph, In $f_2$, no alternative $a_p \in A$ wins each pairwise comparison for all $q \in \{1, 2, 3\}$ such that $q \neq p$, and therefore there is no Condorcet winner. Then we go to A-R part.

In $f_2$, we can see alternative $a_1$ is given positive or 0 score by all 3 voters, $a_2$ is given positive score by 2 voters $v_5$ and $v_6$, and $a_3$ is given positive or 0 score by all 3 voters. Therefore, $a_1$, $a_2$ and $a_3$ are majority-approved, then the approved alternative set $A^p_f = \{a_1, a_2, a_3\}$. Let $S_f$ be the scoring function for $V$ and $A$. We have $S_f(a_1) = 8 + 5 + 3 = 16$, $S_f(a_2) = (-2) + 7 + 4 = 9$, $S_f(a_3) = 5 + 3 + 8 = 16$. Since $S_f(a_1) = S_f(a_3) \geq S_f(a_j)$ for all $a_j \in A^p_f$, then we know that $\psi(f_2) = \{a_1, a_3\}$.

Then we observe that $\psi(f_1) \cap \psi(f_3) = \{a_3\}$.

Let us take a look at $\psi(f_1 + f_2)$.

Let us convert $f_1 + f_2$ to preference orders by looking at each column separately and obtain the following table.

Now we do pairwise comparisons among the alternatives. The results of which are summarized in the following.
As shown in the graph, In \( f_1 + f_2 \), no alternative \( a_p \in A_p \) wins each pairwise comparison for all \( q \in \{1, 2, 3\} \) such that \( q \neq p \), and therefore there is no Condorcet winner. Then we go to A-R part.

In \( f_1 + f_2 \), we can see alternative \( a_1 \) is given positive score by 4 voters \( v_1, v_4, v_5 \) and \( v_6 \), \( a_2 \) is given positive or 0 score by 4 voters \( v_1, v_3, v_5 \) and \( v_6 \), and \( a_3 \) is given positive or 0 score by 5 voters except for \( v_1 \). Therefore, \( a_1, a_2 \) and \( a_3 \) are majority-approved, then the approved alternative set \( A_p^f = \{ a_1, a_2, a_3 \} \). Let \( S_f \) be the scoring function for \( V \) and \( A \). We have \( S_f(a_1) = 10 + (-1) + (-1) + 8 + 5 + 3 = 24 \), \( S_f(a_2) = 0 + (-6) + 10 + (-2) + 7 + 4 = 13 \), \( S_f(a_3) = (-5) + 8 + 4 + 5 + 3 + 8 = 23 \). Since \( S_f(a_1) \geq S_f(a_j) \) for all \( a_j \in A_p^f \), then we know that \( \psi(f_1 + f_2) = \{ a_1 \} \).

Here, we notice that \( \psi(f_1 + f_2) = \{ a_1 \} \) while \( \psi(f_1) \cap \psi(f_2) \) does exist and is actually \( \{ a_2 \} \). Hence \( \psi(f_1 + f_2) \neq \psi(f_1) \cap \psi(f_2) \). Therefore, C-A-R voting function does not satisfy the consistency property.

\[ \psi(f_1 + f_2) = \{ a_1 \} \]

\[ \psi(f_1) \cap \psi(f_2) = \{ a_2 \} \]

\[ \psi(f_1 + f_2) \neq \psi(f_1) \cap \psi(f_2) \]

\[ \diamond \]

\section*{Faithful Property for C-A-R Voting Function with Limits}

\textbf{Theorem 4.2.12.} C-A-R voting function with limits satisfies the faithful property.

\textbf{Proof.} Let \( V = \{ v \} \) be a set of voter and \( A = \{ a_1, \ldots, a_m \} \) be a set of alternatives for some \( m \in \mathbb{N} \). Let \( b > 0 \) and \( B = [-b, -b] \). Let \( \psi : R^2_{V,A,B} \rightarrow \mathcal{P}(A) \) be the C-A-R voting function with limits.
4. C-A-R VOTING FUNCTION WITH LIMITS

Let \( f \in R_{V,A,B} \) and \( a_k, a_j \in A \). Let \( S_f \) be the scoring function for \( V \) and \( A \). Suppose \( f(v)_j > f(v)_k \).

By the definition of \( \psi \), we have pairwise comparison \( a_j \) vs \( a_k \): 1 : 0. Hence \( a_k \) is defeated by \( a_j \) in pairwise comparison, then \( a_k \notin \psi(f) \). \( \square \)

4.2.11 Cancellation Property for A-R Voting Function with limits

I will show that C-A-R voting function does not satisfy the cancellation property by showing the following example.

Example 4.2.13. Let \( V = \{v_1, v_2\} \) be a set of voters and \( A = \{a_1, a_2\} \) be a set of alternatives. Let \( b = 10 \), and \( B = [-10, 10] \). Let \( \psi : R^\xi_{V,A,B} \rightarrow \mathcal{P}(A) \) be the C-A-R voting function with limits.

Let \( f \in R^\xi_{V,A,B} \) be given as the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( v_1 )</th>
<th>( v_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

As we can see from the chart \( |f(v_1)_1| + |f(v_i)_2| = \frac{mb}{2} = \frac{240}{2} = 10 \) for all \( i \in \{1, 2\} \).

We observe that there is one voter \( v_1 \) that \( f(v_1)_1 > f(v_1)_2 \). So, \( \pi_{a_1a_2}(f) = 1 \). Also, there is one voter \( v_2 \) that \( f(v_2)_2 > f(v_2)_1 \), which means \( \pi_{a_2a_1}(f) = 1 \). Thus, in this case, \( \pi_{a_1a_2}(f) = \pi_{a_2a_1}(f) = 1 \).

After converting the table to preference orders, we obtain the following table.

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_2 )</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_1 )</td>
</tr>
</tbody>
</table>
Now we do pairwise comparisons among the alternatives. Then we get $a_1$ vs $a_2$: 1 : 1. Hence there is no Condorcet winner. Then we check A-R part.

Let $S_f$ be the scoring function for $V$ and $A$. Alternative $a_1$ is given positive score by all 2 voters $v_1$, $a_2$ is given positive score by all 2 voters. Therefore, $a_1$, $a_2$ are majority-approved, then the approved alternative set $A_f^p = \{a_1, a_2\}$. We see that $S_f(a_1) = 8 + 4 = 12$ and $S_f(a_2) = 2 + 6 = 8$. Since $S_f(a_1) > S_f(a_2)$, then $\psi(f) = \{a_1\}$.

However, we notice that $\psi(f) \neq A$.

Hence, C-A-R voting function with limits does not satisfy the cancellation property. ✷
4. C-A-R VOTING FUNCTION WITH LIMITS
5

Result and further discussion

5.1 Result

Here is the table for what we found in regular range voting function, range voting function with limits, A-R voting function with limits and C-A-R voting function with limits.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Always A Winner</td>
<td>Yes (Thm 2.2.2)</td>
<td>Yes* (Thm 3.4.3)</td>
<td>Yes* (Thm 4.2.3)</td>
</tr>
<tr>
<td>Majority</td>
<td>No (Exp 2.2.4)</td>
<td>No (Exp 3.4.4)</td>
<td>Yes (Thm 4.2.4)</td>
</tr>
<tr>
<td>Condorcet</td>
<td>No (Exp 2.2.8)</td>
<td>No (Exp 3.4.5)</td>
<td>Yes (Thm 4.2.5)</td>
</tr>
<tr>
<td>Pareto</td>
<td>Yes (Thm 2.2.10)</td>
<td>Yes (Thm 3.4.6)</td>
<td>Yes (Thm 4.2.6)</td>
</tr>
<tr>
<td>Monotonicity</td>
<td>Yes (Thm 2.2.12)</td>
<td>Yes (Thm 3.4.7)</td>
<td>Yes (Thm 4.2.7)</td>
</tr>
<tr>
<td>Independence of Irrelevant Alternative</td>
<td>No (Exp 2.2.14)</td>
<td>No (Exp 3.4.8)</td>
<td>No (Exp 4.2.8)</td>
</tr>
<tr>
<td>Intensity of IIA</td>
<td>Yes (Thm 2.2.17)</td>
<td>Yes (Thm 3.4.9)</td>
<td>No (Exp 4.2.9)</td>
</tr>
<tr>
<td>Participation</td>
<td>Yes (Thm 2.2.19)</td>
<td>No (Exp 3.4.13)</td>
<td>No (Exp 4.2.10)</td>
</tr>
<tr>
<td>Consistency</td>
<td>Yes (Thm 2.2.22)</td>
<td>No (Exp 3.4.10)</td>
<td>No (Exp 4.2.11)</td>
</tr>
<tr>
<td>Faithful</td>
<td>Yes (Thm 2.2.24)</td>
<td>Yes (Thm 3.4.11)</td>
<td>Yes (Thm 4.2.12)</td>
</tr>
<tr>
<td>Cancellation</td>
<td>No (Thm 2.2.26)</td>
<td>No (Exp 3.4.12)</td>
<td>No (Exp 4.2.13)</td>
</tr>
</tbody>
</table>

As we can see from the chart, we check on 12 Criteria in total. Range voting and range voting with limits satisfies 8 of them, and A-R voting function with limits satisfies 6 of them, while C-A-R voting function with limits satisfies 8 of them. In some way our C-A-R voting function with limits does improve and behave better than original range voting function, since it satisfies the
Majority and Condorcet Criteria which are not satisfied by original range voting function; but it fails in satisfying the Independence of Irrelevant Alternative, Participation and Consistency Criteria, since the ranking ballots and choosing the approval set leave impacts on how function behaves. However, I think it is still reasonable to say that the process of determining whether an alternative is approved by the majority of voters is beneficial. We don’t want our voting function completely relies on alternative’s total score; firstly voters should express their overall opinions towards the alternatives with positive or negative score, then we may know how the majority of voters regards our alternatives and only those who are approved by the majority are qualified to compete for the winner spot.

5.2 Further thoughts

In our definition for range voting function with limits, we mention that it does not matter what value $b$ is since we can always use decimal score and therefore scale up or scale down according to a fixed ratio. However, in the reality world, it is not always the case that people are willing to give decimal scores. If we give a range for $B$ such that $B = [-10, 10]$, then people are inclined to give integer scores such as $-8, -5, -1, 0, 3, 6, 10$, etc. If we only allow integer for $B$, then it does matter what value $b$ we choose, because now we cannot apply our example to any general case. If we set $B = [-1, 1]$, then people can only give $-1, 0$ and $1$ to the alternatives; if we set $B = [-100, 100]$, then it gives our voters a great freedom to express their altitudes to our alternatives. We don’t know if $[-10, 10]$ is a rational choice for our score range then. The difference of range may psychologically influence voters’ rating as well.

Furthermore, I am also thinking about if it is possible that we don’t impose a limited function to our voting function at all: instead of doing that, for each voter $v_i$, we may calculate the $|f(v_i)_1| + \cdots + |f(v_i)_m|$ and then have their score divided by $|f(v_i)_1| + \cdots + |f(v_i)_m|$. For example, if there are three alternatives, and voter $v_i$ gives $a_1$ $-2$ points, $a_2$ $-10$ points, and $a_3$
5 points. Then we calculate the total points of the absolute value is $| - 2 | + | - 10 | + | 5 | = 17$, and $a_1$ gets $\frac{2}{17} = -0.11$, $a_2$ gets $\frac{-10}{17} = -0.59$, and $a_3 = \frac{5}{17} = 0.29$. So the absolute value of their total score is 1, and it seems that each voters have a same influence on the score as well: they can give whatever points they would like to give, with no restriction, both mathematically and psychologically.
Bibliography


