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Analysing Flow Free with Pairs of Dots In Triangular Graphs

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Abstract

In the puzzle game Flow Free, the player is given a $n \times n$ grid with a number of colored point pairings. In order to solve the puzzle, the player must draw a path connecting each pair of points so that the following conditions are met: each pair of dots is connected by a path, each square of the grid is crossed by a path, and no paths intersect. Based on these puzzles, this project examines pairs of points in triangular grid graphs obtained by hexagons for which Hamiltonian paths exist in order to identify which point configurations have solutions. We show that $n \geq 5$, any pairs of endpoints admit a Hamiltonian path as they do not surround a corner. This is a solution when $n = 2$ fails when $n = 3$ or 4.

Contents

Dedication

Dedicate to all of my friends.

Acknowledgments

Thank you to Steven Simon, for advising me on this project; the other members of my board, Ethan Bloch and Caitlin Leverson; Eliot Roske and Mehreen Kabir, for their work on this topic of of which I have built.

1
Introduction

The game Flow Free is the basis for this project. A grid of pairs of colored dots is presented to the player in the smartphone puzzle game Flow Free. The player must create paths between each pair of dots that satisfy the following conditions in order to solve the puzzle: as can be seen in Figure 1.0.1, a path connects each pair of dots, a path passes through each square of the grid, and no paths cross each other. After converting rectangular grids to triangular grids, finding a method to determine which configuration of pairs of dots in a triangular T_n admits solutions.

For the purpose of this project, the puzzles will be represented as grid graphs. We determine when there is a solution if there is only one pair of dots. The paths will go along the grid's edges rather than through the hexagons in the puzzle, and each triangle will be referred to as vertices

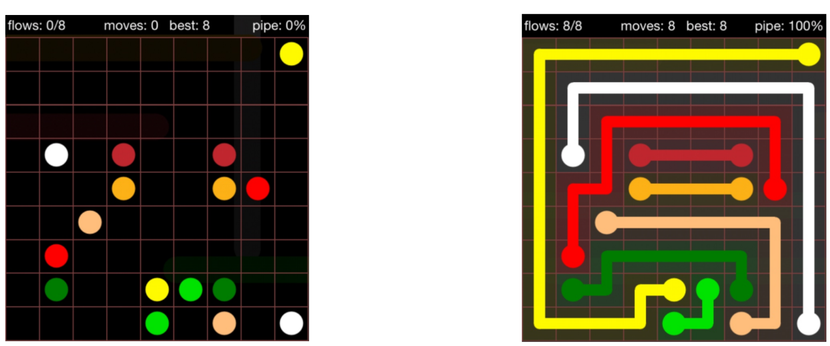

Figure 1.0.1: A Flow Free game and its solution

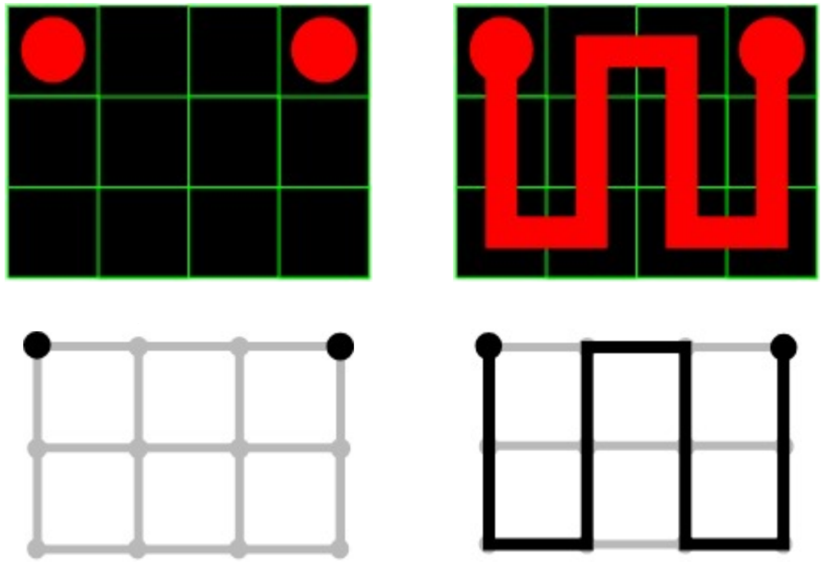

Figure 1.0.2: Transition to grid graph

on the grid. The colored dots will then be on vertices and will be referred to as endpoints. Figure 1.0.2 illustrates this method of representation.

To get triangular grids T_n from a triangular array of hexagons, we find the center of hexagons and connect them as shown in Figure 1.0.3. We have T_3 in this case. Then we can consider the Flow Free game in triangular grid graphs: if we have any two endpoints, can we always find a solution? And how can we achieve this goal? In this project, we will figure out these questions.

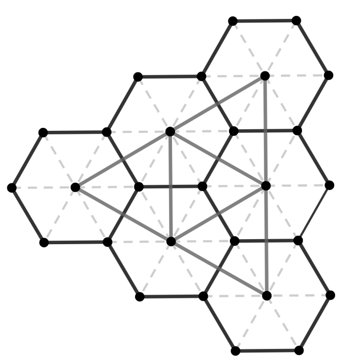

Figure 1.0.3: How to get triangular grid graphs

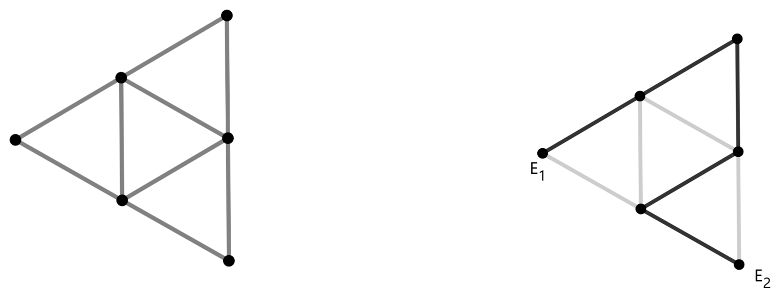

Figure 1.0.4: A T_3 triangular grid graph and its solution

Figure 1.0.4 shows a triangular grid graph T_3 and its solution. Suppose E_1 and E_2 are the two endpoints. The path connects E_1 and E_2 while passing all the other vertices.

2
Preliminary Lemmas

2.1 Background

In this chapter, we will give examples of the puzzle concerning the central question we are exploring in this research. The following definitions will be used throughout this project.

Definition 2.1.1.

1. The regular triangular grid graph T_n is obtained by interpreting the order n triangular grid as a graph, with the intersection of grid lines being the vertices and the line segments between vertices being the edges.
2. The rows of a triangular grid graph are numbered from top to bottom.
3. A vertex is the intersection of edges.

△

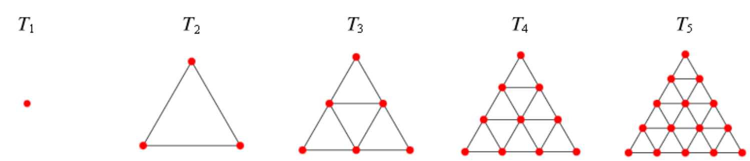

Figure 2.1.1: Triangular graph T_1, T_2, T_3, T_4, and T_5

Definition 2.1.2. Let T be a regular triangular grid graph.

1. A Hamiltonian path or traceable path is a path that visits each vertex of the graph exactly

once.

2. Suppose we are given two distinct vertices in T. A solution to the triangular grid graph (referred to just as a solution) is a Hamiltonian path starting at one of the two vertices and ending at the other.

3. Any two distinct vertices in T are called endpoints whether or not there is a solution for them.

△

2.2 Examples for small T_n

Next, we will show some examples of T_n. Since T_1 is a vertex, we omit the case.

Example 2.2.1. T figure illustrates all the possible positions of vertices in T_2. We can see in every case, there exists a solution in the triangular grid graph. ♢

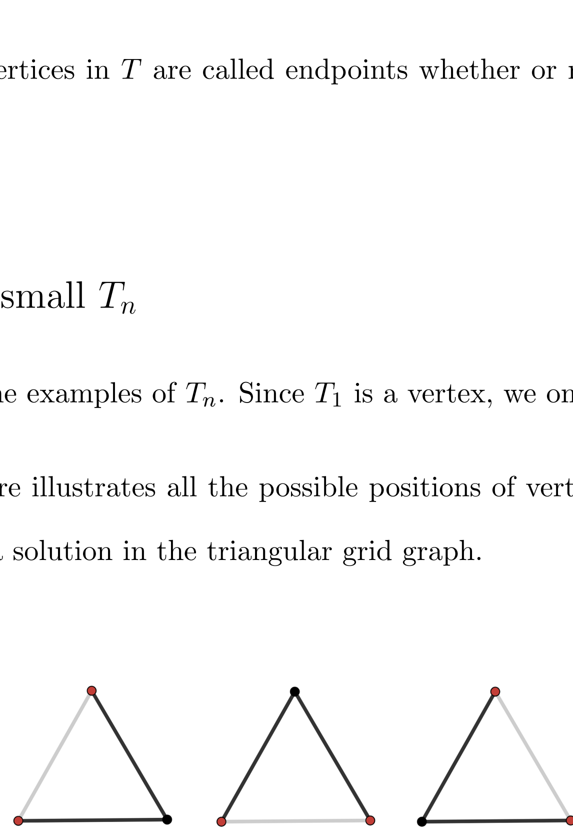

Figure 2.2.1: All possible solutions in T_2

Example 2.2.2. The figure illustrates two cases with a solution in the triangular graph T_3. If two endpoints are on the same edge in T_3, then there is a solution. ♢

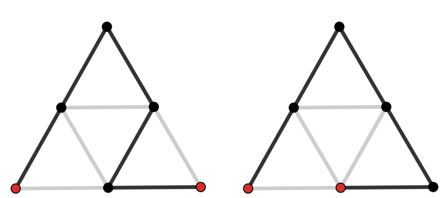

Figure 2.2.2: Some possible solutions in T_3

Example 2.2.3. Figure 2.2.3 illustrates two cases in which there is no solution in the triangular graph T_3. No path exists because no matter where we start up, there is no way to hit all the vertices in the graph.

For the left one, if the path starts at the top endpoint, it can only go down and hit the two vertices in the middle row, then pass one of the corner vertex and finally hit the bottom endpoint. But the path can never hit the other corner vertex, so this graph has no solution.

For the right graph, no matter where we start the path, it cannot hit the left corner vertex because the path hits the endpoint before it gets to the corner. Hence, no solution exists in this triangular grid graph.

$\Diamond$

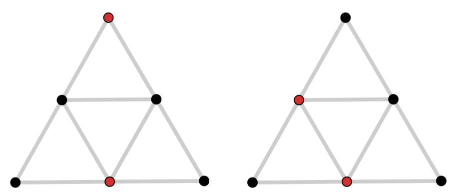

Figure 2.2.3: No solution exists in T_3

Example 2.2.4. The figure illustrates two cases that there is a solution in the triangular graph T_4. $\Diamond$

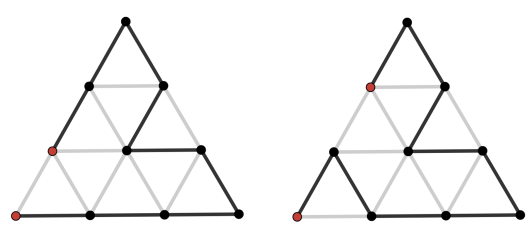

Figure 2.2.4: Some possible solutions in T_4

Example 2.2.5. Figure 2.2.5 illustrates two cases that there is no solution in the triangular graph T_4.

For the left one, there is no solution in this graph for the same reason as we have explained in Example 2.2.3 since the two endpoints surround the corner vertex.

For the right one, no matter which endpoint we start at, the path cannot go through all the vertices on the grids. If we start at the bottom endpoint and let the path go to the left until it hits the corner vertex, it can only go up and right because we cannot reach the endpoint. Now the problem is that the path either goes down to hit the corner vertex or goes up to pass all the other vertices and reach the top endpoint, but neither of the directions can make a solution in this graph. ◊

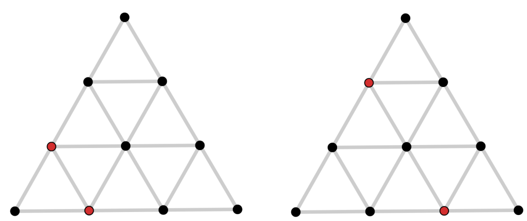

Figure 2.2.5: No solution exists in T_4

All the examples above lead us to have a hypothesis that if the triangular graph is getting bigger, then there we have a higher probability of finding a solution in the graph.

2.3 Before Conjecture

Proposition 2.3.1. *Let $n \in \mathbb{N}$ such that $n \geq 3$. Let T_n be a triangular graph. If two endpoints surround the same corner vertex, then there is no solution in the graph.*

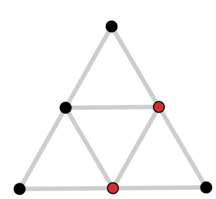

Figure 2.3.1: No solution exists in T_3

Proof. Suppose two endpoints surround the same corner vertex, as shown in Figure 2.2.7. We observe that the path could never hit the corner after it passes all the other vertices, no matter how big the graph is. So, there is no solution in this situation. Therefore, in T_n where $n \geq 3$, if two endpoints surround the same corner vertex, then no solution exists. □

Definition 2.3.2. Let T_n be a triangular grid graph, $n \geq 5$.

1. Figure 2.2.6 shows the **inside zone** is the vertices and edges within the dotted lines. T_5 is the smallest triangular graph that has an inside zone.

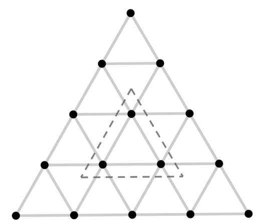

Figure 2.3.2: The inside zone in T_5

2. The **outside zone** is the rest of the graph except for the inside zone. △

2.4 Conjecture

Conjecture 2.4.1. *Let $n \in \mathbb{N}$. Let T_n be a triangular grid graph. If $n \geq 5$, then any two endpoints have at least one solution so long as they don't surround a corner.*

We will prove the conjecture in the next section.

Conjecture 2.4.2. *Let Q_n be the number of pairs of endpoints in T_n. Let S_n be the subset of Q_n that consists of all pairs of endpoints for which there is at least one solution. Then $P_n = \frac{S_n}{Q_n}$ is the probability that a random pair of endpoints has at least one solution.*

$$\lim_{n \to \infty} P_n = 1.$$

Conjecture 2.4.3. $P_n = \frac{Q_n - 3}{Q_n}$ *for all $n \geq 5$, if any two endpoints have at least one solution so long as they don't surround a corner.*

Follow from Conjecture 2.4.3, we are able to prove Conjecture 2.4.2 by doing calculations.

Proof. Based on our examples, we can calculate the probability of finding a solution in a triangular grid graph.

For triangular grid graph T_2, it has three vertices. As shown in Figure, we see three pairs of endpoints in the graph, and all of them have a solution, so $S_2 = Q_2 = 3$. Then the probability of finding a solution, in this case, is $P_2 = \frac{S_2}{Q_2} = 1$.

T_3 has 6 vertices. Then by combination, the number of pairs of endpoints is

$$Q_3 = \frac{6!}{2!(6-2)!} = \frac{6 \times 5 \times \ldots \times 1}{2 \times 1 \times (4 \times 3 \times 2 \times 1)} = \frac{720}{48} = 15.$$

According to our examples, we can identify that 6 pairs of endpoints have no solutions, as shown in Figure.

Then we know that $S_3 = 15 - 6 = 9$, and so $P_3 = \frac{9}{15} = \frac{3}{5}$.

T_4 has 10 vertices. Then by combination, the number of pairs of endpoints is

$$Q_4 = \frac{10!}{2!(10-2)!} = \frac{10 \times 9 \times \ldots \times 1}{2 \times 1 \times (8 \times 7 \times \ldots \times 1)} = \frac{3628800}{80640} = 45.$$

According to our examples, we can identify that 6 pairs of endpoints have no solutions, as shown in Figure.

Then we know that $S_4 = 45 - 6 = 39$, and so $P_4 = \frac{39}{45} = \frac{13}{15}$.

T_5 has 15 vertices. Then by combination, the number of pairs of endpoints is

$$Q_5 = \frac{15!}{2!(15-2)!} = \frac{15 \times 14 \times \ldots \times 1}{2 \times 1 \times (13 \times 12 \times \ldots \times 1)} = \frac{15 \times 14}{2 \times 1} = 105.$$

According to our examples, we can identify that 3 pairs of endpoints have no solutions, as shown in Figure.

Then we know that $S_5 = 105 - 3 = 102$, and so $P_5 = \frac{102}{105} = \frac{34}{35}$.

Let the number of vertices in T_n be V. From the calculation, we see that

$$Q_n = \left(\binom{V!}{2} \atop 2 \right) = \frac{(V \cdot (V-1))}{2} = \frac{V^2 - V}{2}.$$

We have stated in Conjecture 2.3.4 that $P_n = \frac{Q_n - 3}{Q_n}$ for all $n \geq 5$. Thus,

$$P_n = \frac{\frac{V^2 - V}{2} - 3}{\frac{V^2 - V}{2}} = \frac{V^2 - V - 6}{V^2 - V}.$$

Therefore, we can conclude that

$$\lim_{n \to \infty} P_n = \lim_{n \to \infty} \frac{V^2 - V - 6}{V^2 - V} = \lim_{n \to \infty} \frac{1 - \frac{1}{V} - \frac{6}{V^2}}{1 - \frac{1}{V}} = 1.$$

□

Our final goal is to prove Conjecture 2.4.1 is true by using induction. As long as the two endpoints do not surround a corner, we show that there always exists a solution in T_n for $n = 5, 6, 7$. Next, we show that if it's true for n and $n + 1$, then it's true for $n + 3$. Thus, we can conclude that the conjecture is true for all n.

3
Solutions In the T_5 Triangular Graphs

In this chapter, we will show that the conjecture is true for $n = 5$, which is Theorem 3.2.1.

3.1 Cases

According to the position of the endpoints, we have the following separate cases:

(a) both endpoints are in the inside zone;

(b) one endpoint is in the inside zone, and the other is in the outside zone;

(c) both endpoints are in the outside zone;

- (i) endpoints are on the same edge;
- (ii) endpoints are on the different edges.

3.2 Theorem

Theorem 3.2.1. *Let $n \in \mathbb{N}$. Let T_n be a triangular grid graph. If $n = 5$, then any two endpoints have at least one solution so long as they don't surround a corner.*

To prove Theorem 3.2.1, each case is an individual lemma that accounts for every possible pair of endpoints in the triangular grid graph.

3.3 Lemmas

Lemma 3.3.1. *Let T_5 be a triangular grid graph. Suppose both endpoints are in the inside zone. Then there is a solution.*

Proof. We see that the inside zone is T_2. By following the strategy, the path goes through the outside zone first, and the path goes back to hit the endpoint. Starting from E_1, the path goes out and hits the outside zone. Then it goes through the outside zone and finally goes back to hits another endpoint E_2. Therefore, a solution exists when both endpoints are in the inside zone. □

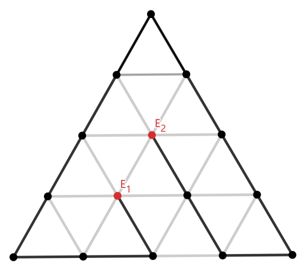

Figure 3.3.1: Lemma 3.3.1

Lemma 3.3.2. *Let T_5 be a triangular grid graph. Suppose one endpoint is in the inside zone, and the other is in the outside zone. Then there is a solution.*

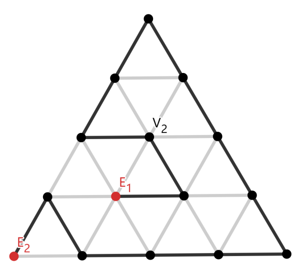

Figure 3.3.2: Case 1 in Lemma 3.3.2

Proof. Case 1: Suppose one endpoint is on the inside zone, and the other is on the outside zone, as shown in Figure 3.3.2. In this case, our strategy is to hit every vertex before the path enters the inside zone. So The path starts at E_2, goes up first, and then passes all the other vertices in the outside zone and stops at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case.

Case 2: Suppose one endpoint is in the inside zone, and the other is in the outside zone. Without loss of generality, suppose the endpoints are located in the triangular graph T_5 as shown in Figure 3.3.3. This case is more general because the path can go through the whole outside zone before it enters the inside zone, so we don't need to worry about the corner vertex. In this case, we can follow the strategy directly. Then the path comes out at E_1 and passes every vertex in the outside zone, and then stops at V_1. Finally, we connect V_1 and V_2, then a solution is found in this case. □

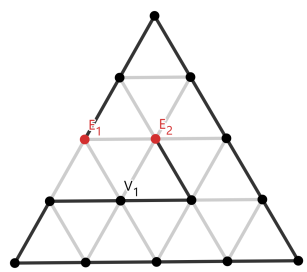

Figure 3.3.3: Case 2 in Lemma 3.3.2

Lemma 3.3.3. *Let T_5 be a triangular grid graph. Suppose both endpoints are in the outside zone. Then there is a solution.*

Proof. Case 1: Suppose the endpoints are located on the outside zone, and they are on the same edge, as shown in Figure 3.3.4. Observe that the inside zone is T_2. We use the same strategy as we used in the previous theorem. From Example 2.2.1, we know where the endpoints are located when the triangular graph has a solution, so our goal is to make sure that the path passes every vertex before it goes back to the inside zone. In this case, starting from E_1, we can simply go through the outside zone and stop at V_2. Then, the path starts from E_2 and passes every vertex left in the graph before it stops at V_1. Finally, we connect V_1 and V_2 then a solution is found in this case. □

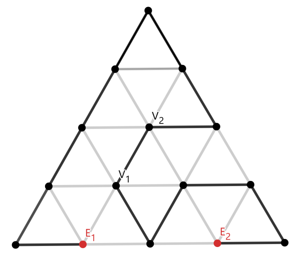

Figure 3.3.4: Case 1 in Lemma 3.3.3

Case 2: Suppose both endpoints are in the outside zone, and they are on the same edge. Without loss of generality, suppose the endpoints are located in the triangular graph T_5 as shown in Figure 3.3.3. This case is more general because the path can go through the outside zone without According to the strategy, the path comes out at E_1, then goes through the outside zone and hits V_1. Next, the path starts at E_2 and passes the vertices left in the outside zone, and ends at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case.

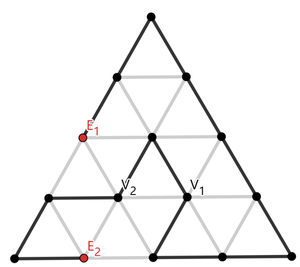

Figure 3.3.5: Case 2 in Lemma 3.3.3

Case 3: Suppose both endpoints are in the outside zone, and they are not on the same edge, as shown in Figure 3.3.6. In this case, we cannot follow the strategy directly, so we need to make adjustments. The path starts at E_1, then passes the bottom left vertices, and stops at V_1. Next, to hits all the vertices left in the graph, the path first hits the top three vertices and passes the third vertex in the inside zone. Then the path hits all the vertices that are left in the graph and end at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case.

Case 4: Suppose both endpoints are in the outside zone, and they are not on the same edge. Without loss of generality, suppose the endpoints are located in the triangular graph T_5 as shown in Figure 3.3.7. According to the strategy, the path comes out at E_1, then goes through the

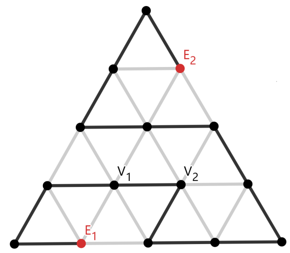

Figure 3.3.6: Case 3 in Lemma 3.3.3

outside zone and hits V_1. Next, the path starts at E_2 and passes the vertices left in the outside zone, and ends at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case. □

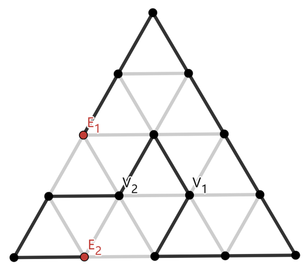

Figure 3.3.7: Case 4 in Lemma 3.3.3

By these lemmas, we have shown that in T_5 any two endpoints have at least one solution so long as they don't surround a corner.

4
Solutions In the T_6 Triangular Grid Graphs

In this chapter, we will show that the conjecture is true for $n = 6$, which is Theorem 4.2.1.

4.1 Cases

According to the position of the endpoints, we have the following separate cases:

(a) both endpoints are in the inside zone;

(b) one endpoint is in the inside zone, and the other is in the outside zone;

(c) both endpoints are in the outside zone;

 (i) endpoints are on the same edge;

 (ii) endpoints are on the different edges.

4.2 Theorem

Theorem 4.2.1. *Let $n \in \mathbb{N}$. Let T_n be a triangular grid graph. If $n = 6$, then any two endpoints have at least one solution so long as they don't surround a corner.*

To prove Theorem 4.2.1, each case is an individual lemma that accounts for every possible pair of endpoints in the triangular grid graph.

4.3 Lemmas

Lemma 4.3.1. *Let T_6 be a triangular grid graph. Suppose the two endpoints are in the inside zone. Then there is a solution.*

Proof. Case 1: Suppose both endpoints are in the inside zone. In Figure 4.3.1, we see that there is no solution in the inside zone T_2. In this case, the path starts at E_1 and hits the corner of T_3, which is the inside zone, and then it enters the outside zone and stops at V_1 in the inside zone. We have a similar strategy for E_2, that is, hit the corner vertex of the inside zone. The path comes out at E_2 and stops at V_2. As we connect V_1 and V_2, we find a solution in this case.

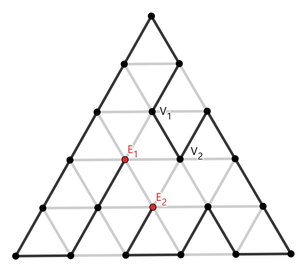

Figure 4.3.1: Case 1 In Lemma 4.3.1

Case 2: Suppose both endpoints are in the inside zone. Figure 4.3.2 shows no solution in the inside zone T_2, but this case differs from Case 1. The path starts at E_1, and it needs to hit the corner of the inside zone before it goes out, and then the path stops at V_1. Then we follow the standard strategy for E_2. Then we find a solution by connecting V_1 and V_2.

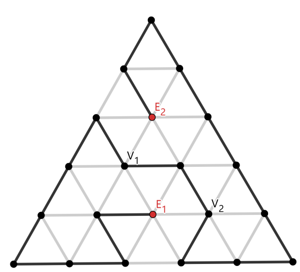

Figure 4.3.2: Case 2 In Lemma 4.3.1

Case 3: Suppose both endpoints are in the inside zone. We know that there is a solution in the inside zone T_2. In this case, the strategy is that the path goes through the outside zone first, and finally, it returns to hit another endpoint. Starting from E_1, the path hits the outside zone. Then it goes through the outside zone and eventually goes back to hits another endpoint E_2. Therefore, a solution exists when both endpoints are in the inside zone.

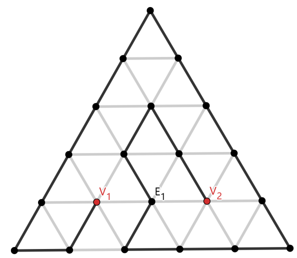

Figure 4.3.3: Case 3 In Lemma 4.3.1

Case 4: Suppose both endpoints are in the inside zone. In this case, the solution is obvious because the path can go through the inside and outside zone without considering anything. The first step is to connect all the vertices in the outside zone. Then delete line segment AB and CD and connect B and C, A and D. Thus, a new path is found in the graph. In this case, we can always follow this strategy to find a solution. □

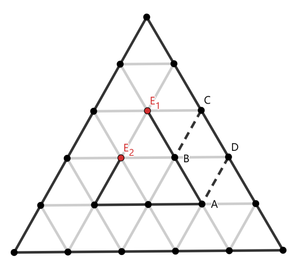

Figure 4.3.4: Case 4 In Lemma 4.3.1

Lemma 4.3.2. *Let T_6 be a triangular grid graph. Suppose one endpoint is in the inside zone, and the other is in the outside zone. Then there is a solution.*

Proof. Case 1: Suppose one endpoint is in the inside zone, and the other is in the outside zone. In this case, our strategy is to hit every vertex before the path enters the inside zone. So The path starts at E_2, goes up first, then passes all the other vertices in the outside zone and stops at V_2. Finally, we connect E_1 and V_2, then a solution is found in this case.

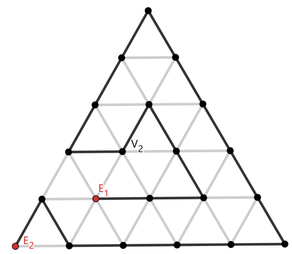

Figure 4.3.5: Case 1 In Lemma 4.3.2

Case 2: Suppose one endpoint is in the inside zone, and the other is in the outside zone. Without loss of generality, suppose they are in T_6 as shown in Figure 4.3.6. In this case, we can follow the strategy directly. Then the path comes out at E_1 and passes every vertex in the outside zone, and then stops at V_1. Finally, we connect V_1 and V_2, then a solution is found in this case.

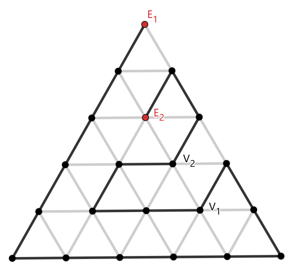

Figure 4.3.6: Case 2 In Lemma 4.3.2

Lemma 4.3.3. *Let T_6 be a triangular grid graph. Suppose both endpoints are in the outside zone. Then exists a solution.*

Case 1: Suppose the endpoints are located in the outside zone, and they are on the same edge. Observe that the inside zone is T_3. The path starts at E_2, and it must go through the corner vertex of the inside zone before it stops at V_2. Then we can follow the normal strategy to connect E_1 and V_1. Finally, we connect V_1 and V_2 then a solution is found in this case.

Case 2: Suppose both endpoints are in the outside zone, and they are on the same edge. Without loss of generality, suppose the two endpoints are located as shown in Figure 4.3.8. According to the strategy, the path comes out at E_1, then goes through the outside zone and

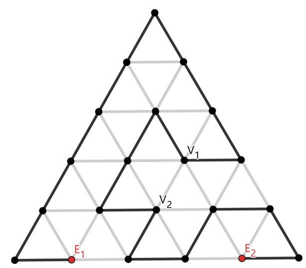

Figure 4.3.7: Case 1 In Lemma 4.3.3

hits V_1. Next, the path starts at E_2 and passes the vertices left in the outside zone and ends at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case.

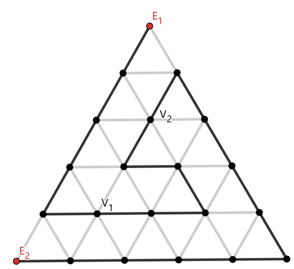

Figure 4.3.8: Case 2 In Lemma 4.3.3

Case 3: Suppose both endpoints are in the outside zone, and they are not on the same edge. In this case, we cannot follow the strategy directly, so we need to make adjustments. The path starts at E_2, then hits the corner vertex of the inside zone and stops at V_2. Next, we follow the normal strategy to connect E_1 and V_1. Finally, we connect V_1 and V_2, then a solution is found in this case.

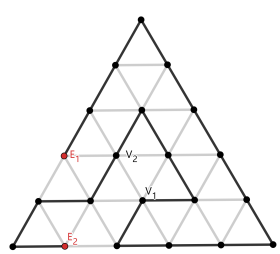

Figure 4.3.9: Case 3 In Lemma 4.3.3

Case 4: Suppose both endpoints are in the outside zone, and they are not on the same edge. Without loss of generality, suppose the two endpoints are located as shown in Figure 4.3.9. In this case, we cannot follow the strategy directly, so we need to make adjustments. The path starts at E_2, then hits the corner vertex of the inside zone and stops at V_2. Next, we follow the normal strategy to connect E_1 and V_1. Finally, we connect V_1 and V_2, then a solution is found in this case.

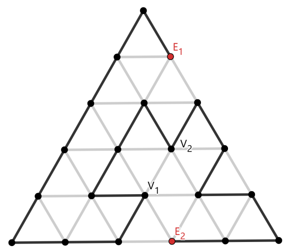

Figure 4.3.10: Case 4 In Lemma 4.3.3

Case 5: Suppose both endpoints are in the outside zone, and they are not on the same edge. Without loss of generality, suppose the two endpoints are located as shown in Figure 4.3.10. According to the strategy, the path comes out at E_1, then goes through the outside zone and hits V_1. Next, the path starts at E_2 and passes the vertices left in the outside zone, and ends at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case. □

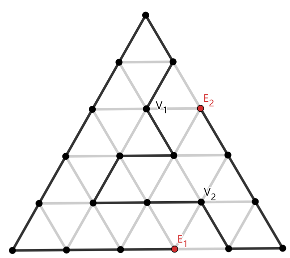

Figure 4.3.11: Case 5 In Lemma 4.3.3

By these lemmas, we have shown that in T_6 any two endpoints have at least one solution so long as they don't surround a corner.

5
Solutions In the T_7 Triangular Grid Graphs

In this chapter, we will show that the conjecture is true for $n = 5$, which is Theorem 5.2.1.

5.1 Cases

According to the position of the endpoints, we have the following separate cases:

(a) both endpoints are in the inside zone;

(b) one endpoint is in the inside zone, and the other is in the outside zone;

(c) both endpoints are in the outside zone;

  (i) endpoints are on the same edge;

  (ii) endpoints are on the different edges.

5.2 Theorem

Theorem 5.2.1. *Let $n \in \mathbb{N}$. Let T_n be a triangular grid graph. If $n = 7$, then any two endpoints have at least one solution so long as they don't surround a corner.*

To prove Theorem 5.2.1, each case is an individual lemma that accounts for every possible pair of endpoints in the triangular grid graph.

5.3 Lemmas

Lemma 5.3.1. *Let T_7 be a triangular grid graph. Suppose both endpoints are in the inside zone. Then there is a solution.*

Proof. Case 1: Suppose both endpoints are in the inside zone. In this case, the path starts at E_1 and hits the corner of T_4, which is the inside zone, and then it enters the outside zone and stops at V_1 in the inside zone. We have a similar strategy for E_2, that is, hit the corner vertex of the inside zone. The path comes out at E_2 and stops at V_2. We find a solution in this case when we connect V_1 and V_2.

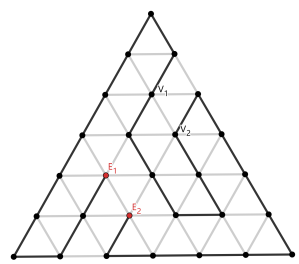

Figure 5.3.1: Case 1 in Lemma 5.3.1

Case 2: Suppose both endpoints are in the inside zone. In this case, the path starts at E_1, and it needs to hit the corner of the inside zone before it goes out, and then the path stops at V_1. Then we follow the standard strategy for E_2. Then we find a solution by connecting V_1 and V_2.

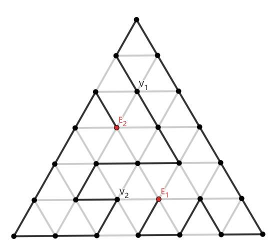

Figure 5.3.2: Case 2 in Lemma 5.3.1

Case 3: Suppose both endpoints are in the inside zone. We can follow the strategy directly. Then the path comes out at E_1, passes every vertex in the outside zone, and stops at V_1. Finally, we connect V_1 and V_2, then a solution is found in this case. □

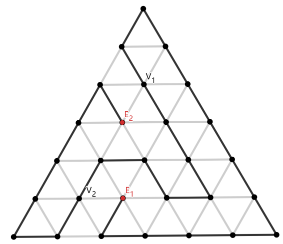

Figure 5.3.3: Case 3 in Lemma 5.3.1

Lemma 5.3.2. *Let T_7 be a triangular grid graph. Suppose one endpoint is in the inside zone, and the other is in the outside zone. Then there is a solution.*

Proof. Case 1: Suppose one endpoint is in the inside zone, and the other is in the outside zone. In this case, our strategy is to hit every vertex before the path enters the inside zone. So The path starts at E_1, goes up first, then passes all the other vertices in the outside zone and stops at V_1. Finally, we connect V_1 and E_2, then a solution is found in this case.

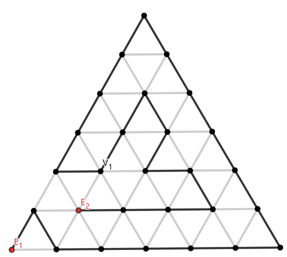

Figure 5.3.4: Case 1 in Lemma 5.3.2

Case 2: Suppose one endpoint is in the inside zone, and the other is in the outside zone. Without loss of generality, suppose they are in T_7 as shown in Figure 5.3.5. In this case, we can follow the strategy directly. Then the path comes out at E_1, passes every vertex in the outside zone, and stops at V_1. Finally, we connect V_1 and E_2, then a solution is found in this case. □

Theorem 5.3.3. *Let T_7 be a triangular grid graph. Suppose both endpoints are in the outside zone. Then there is a solution.*

Proof. Case 1: Suppose the endpoints are located in the outside zone and on the same edge. Observe that the inside zone is T_4. We use the same strategy as we used in T_5. The path starts at E_2 and must go through the corner vertex of the inside zone before stopping at V_2. Then we

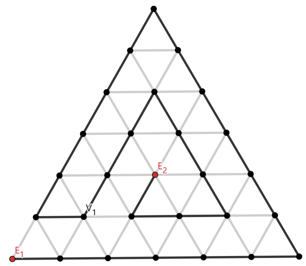

Figure 5.3.5: Case 2 in Lemma 5.3.2

can follow the standard strategy to connect E_1 and V_1. Finally, we connect V_1 and V_2 to find a solution in this case.

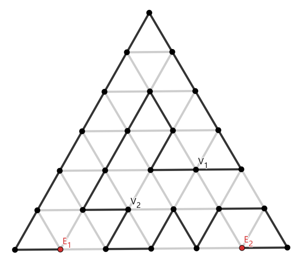

Figure 5.3.6: Case 1 in Lemma 5.3.3

Case 2: Without loss of generality, suppose the two endpoints are in T_7 as shown in Figure 5.3.7. According to the strategy, the path comes out at E_1, then goes through the outside zone and hits V_1. Next, the path starts at E_2, passes the vertices left in the outside zone, and ends at V_2. Finally, we join V_1 and V_2, then a solution is found in this case.

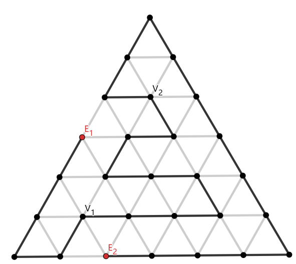

Figure 5.3.7: Case 2 in Lemma 5.3.3

Case 3: Suppose both endpoints are in the outside zone and not on the same edge. In this case, we cannot follow the strategy directly, so we need to make adjustments. The path starts at E_1, passes the left corner vertex, and stops at V_1. Next, we can follow our normal strategy to connect E_2 and V_2. Finally, we connect V_1 and V_2, then a solution is found in this case.

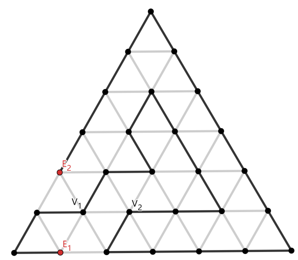

Figure 5.3.8: Case 3 in Lemma 5.3.3

Case 4: Suppose both endpoints are in the outside zone and not on the same edge. The path starts at E_1 and must go through the corner vertex of the inside zone before stopping at V_1. Then we can follow the standard strategy to connect E_1 and V_1. Finally, we connect V_1 and V_2 to find a solution in this case.

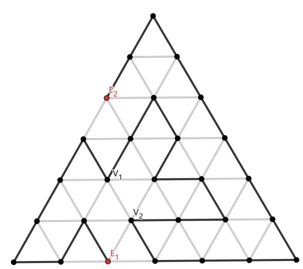

Figure 5.3.9: Case 4 in Lemma 5.3.3

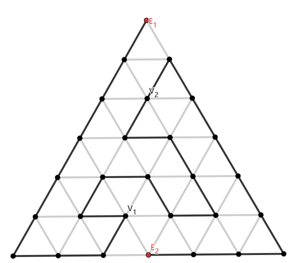

Figure 5.3.10: Case 5 in Lemma 5.3.3

Case 5: Suppose both endpoints are in the outside zone and on the same edge. Without loss of generality, suppose they are in the T_7 as shown in Figure 5.3.10. According to the strategy, the path comes out at E_1, then goes through the outside zone and hits V_1. Next, the path starts at E_2, passes the vertices left in the outside zone, and ends at V_2. Finally, we connect V_1 and V_2, then a solution is found in this case. □

By these lemmas, we have shown that in T_7, any two endpoints have at least one solution so long as they don't surround a corner.

6
The Final Proof for T_{n+3}

6.1 Proof

Lemma 6.1.1. *Let T_n be a triangular grid graph such that $n \geq 5$. If there is a solution in T_n and T_{n+1}, then there exists a solution in T_{n+3}.*

Proof. Through Chapters 3 to 5, we have shown that the conjecture is true for the base cases T_5, T_6, and T_7. Now We need to show that the conjecture is true for T_{n+1}. We have separate cases according to the positions of pairs of endpoints.

Case 1: Suppose the two endpoints are in the inside zone, not surrounding the corner vertex. Figure 6.1.1. shows that T_5 is the inside zone, and there is a solution in T_n. The corner vertex is neither of the two endpoints so that the path can go through all the other vertices in the outside zone and hit V to enter the inside zone. As we connect E_2 and V, we find a solution in T_n.

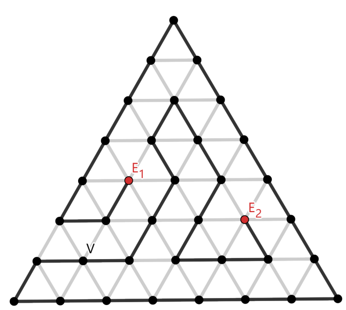

Figure 6.1.1: Case 1

Case 2: Suppose the two endpoints are in the inside zone and surrounding the corner vertex. Figure 6.1.2. shows that T_5 is the inside zone, and there is a solution in T_n. This case is special because the corner vertex is neither of the two endpoints, but they surround the corner. So, the path must go back and forth in the triangular graph in order to hit every vertex. Thus, we can find a solution in this case.

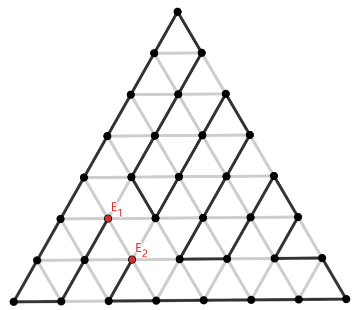

Figure 6.1.2: Case 2

Case 3: Suppose one of the endpoints is in the inside zone, and the other is in the outside zone. Figure 6.1.3. shows that T_5 is the inside zone, and there is a solution in T_n. The path starts at E_1 and goes around the outside zone, then hits V to enter the inside zone. Thus, we can get a solution by connecting V and E_2.

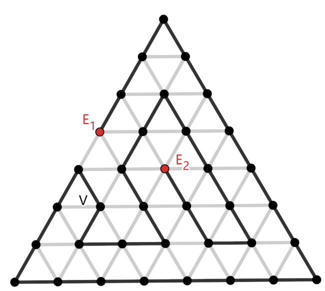

Figure 6.1.3: Case 3

Case 4: Suppose the two endpoints are in the outside zone and not on the same edge. Figure 6.1.4. shows that T_5 is the inside zone, and there is a solution in T_n. The path starts at E_1 and goes around the outside zone until one vertex before V_3, then enters the inside zone. The other side starts at E_2 and goes around until one vertex before E_1, and it enters the inside zone at V_2. Thus, we can get a solution by connecting V_2 and V_3.

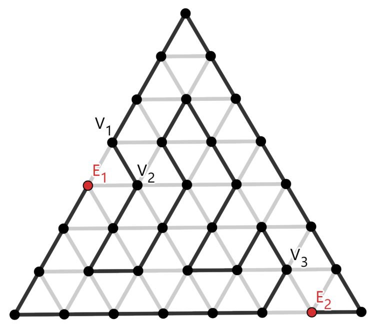

Figure 6.1.4: Case 4

Case 5: Suppose the two endpoints are in the outside zone and on the same edge. Figure 6.1.5. shows that T_5 is the inside zone, and there is a solution in T_n. We discuss this case because it shows that the conjecture is true for T_{n+1} is necessary. The path starts at E_2 and moves until right before E_1. Now we try to refill the left part of E_1 and outside T_{n+1}. It enters the corner on T_{n+1} that meets the lower part. The other side of the path starts at E_2 and enters the corner that meets the lower part. Then we can find a solution by connecting V_1 and V_2.

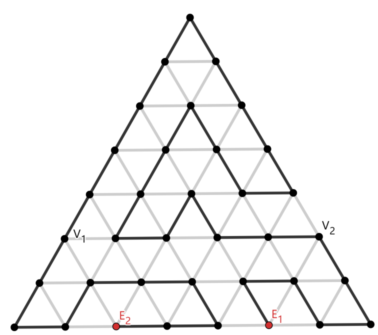

Figure 6.1.5: Case 5

Thus, if we know the conjecture is true for T_5 and T_6, then it is true for T_8; if the conjecture is true for T_6 and T_7, then it is true for T_9 and so on. Hence, I know this conjecture is true for all n in T_n.

Therefore, Let T_n be a triangular grid graph such that $n \geq 5$. If a solution exists in T_n and T_{n+1}, then a solution exists in T_{n+3}. □

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