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Polygonal Analogues to the Topological Tverberg and van Kampen-Flores Theorems

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Abstract

Tverberg's theorem states that any set of $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q disjoint subsets whose convex hulls intersect. Topologically, any continuous map $f : \partial\Delta^{(q-1)(d+1)} \to \mathbb{R}^d$ has q disjoint faces of the simplex $\partial\Delta^{(d+1)(q-1)}$ whose images intersect, given q is a prime power. The dimension of the simplex is tight, meaning if it is lower then f will admit a full Tverberg partition with probability zero. This paper discusses continuous functions $f : \partial\Delta^{2d(q-2)+(q-1)} \to \mathbb{C}^d$, where the simplicial dimension is lower than required for a full Tverberg partition. It results in having q disjoint faces of $\partial\Delta^{2d(q-2)+(q-1)}$ that, instead of intersecting, contain points that form a regular q-gon. The van Kampen-Flores theorem increases the dimension of the simplex in order to restrict the dimension of the q disjoint faces. Similarly, we found that if $2d(q-2)+1 < qk < 2d(q-1)$, then any continuous function $f : \partial\Delta^{2d(q-2)+2(q-1)} \to \mathbb{C}^d$ has a q-gon partition such that the q disjoint faces all come from the k-skeleton.

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Dedication

For my father, who taught me that dogs do indeed have eyebrows.

Acknowledgments

Thank you to Prof. Steven Simon, for your patience and guidance; I could not have asked for a better advisor. Thank you to the mathematics faculty for your mentorship and unwavering encouragement, for cultivating a strong mathematics community I am proud to be a part of, and for always putting your students first. Thank you to both red couches, despite the ants, for being there through all the late nights. Thank you to Apple Snatchers for being the closest friends I've had the pleasure of meeting. Thank you to my math buddies for being my stabilizers. I am forever grateful of the Me(a)gan's in my life for keeping me company in dilirium and holding me together through all the times we've fallen apart. Thank you to my family for all of the laughs, for your relentless support, and for your unlimited love.

1
Introduction

In 1921 Radon proved that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint subsets whose convex hulls intersect.[9] The problem was reformulated to looking at affine maps $f : \Delta^{d+1} \to \mathbb{R}^d$, where Δ^{d+1} is an $(d+1)$-dimensional simplex, having 2 disjoint faces of Δ^{d+1} whose images intersect. This naturally led to the question: if this is true for affine maps, is it also true for continuous maps? This was addressed with the Topological Radon Theorem, proven by Bajmóczy and Bárány in 1979.[1]

Tverberg's theorem is a generalization of Radon's theorem, stating that any set of $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q subsets whose convex hulls intersect. [13] In 1959, Birch proved Tverberg's Theorem for $d = 2$.[4] However, it was not until 1966 that Tverberg proved it for all d using algebraic topology.

Similarly to Radon's theorem, Tverberg's theorem is equivalent to saying that for any affine map $f : \Delta^{(q-1)(d+1)} \to \mathbb{R}^d$, there are q disjoint faces of $\Delta^{(q-1)(d+1)}$ whose images intersect, again shifting it from a question of convex geometry to a question of algebraic topology. The conjecture of whether we can replace our affine maps with non-affine continuous maps is known as the Topological Tverberg theorem. Unlike Radon's theorem, moving from affine maps to continuous maps gets messy with different values of q. Bárány, Shlosman, and Szűcs [2] proved it for prime q in 1981. In 1987, Özaydin [8] proved it for a prime power q, and in 2014 Frick [6] disproved it for all non-prime powers.

It was then asked whether we could restrict where the intersecting disjoint faces were coming from on the simplex, raising a question of restricting the dimension of our disjoint faces. Van Kampen and Flores found for Radon's theorem, if d is even, then we can find two disjoint faces of dimension at most $\frac{d}{2}$ that form a Radon partition.[5] From there Frick showed that if $k \geq \left\lceil \frac{q-1}{q} d \right\rceil$, then we can find a full Tverberg partition where the dimensions for all of the disjoint faces are less than or equal to k.

The number of points in Tverberg's theorem is incredibly tight, meaning that if we have fewer than $(q-1)(d+1)+1$ points in $\mathbb{R}^d$, there will not be a full Tverberg partition with probability one. However, there do exist collections of $n < (q-1)(d+1)+1$ points that admit a full q-fold Tverberg partition. This happens when the points are not in strong general position [3]; strong general position considers the affine hulls of all possible q partitions of our points, relating the codimension of their intersection to the sum of their codimension. As Perles and Sigron showed in [10], when our points are in strong general position, then any collection of points $n < (q-1)(d+1)$ will not admit a full Tverberg partition; in fact, not only will the convex hulls of our partitions not intersect, but neither will the affine hulls. A set of points is in strong general position with probability one, which demonstrates the optimality of Tverberg's theorem.[10]

Let $f : \Delta^N \to \mathbb{R}^d$ be continuous, like in the topological Tverberg Theorem. Our project considers when $N < (q-1)(d+1)$. Since the dimension for Tverberg's theorem is tight, we know, generically, we will not get an intersection of disjoint faces. However, using finite Fourier analysis [4], we were able to show that, if $N = d(q-2)+(q-1)$, then there will be q disjoint faces whose images contain points that form a regular q-gon [6]. In the affine case, this is equivalent to saying if we have $d(q-2)+(q-1)$ points in $\mathbb{R}^d$, then we can partition our points into q subsets whose convex hulls contain points that form a regular q-gon.

We then looked at where we can force these disjoint faces to come from on our simplex, similar to van Kampen-Flores. We show if $N = 2d(q-2) + 2(q-1)$ and $2d(q-2) \leq qk < 2d(q-1)$, then there exist q disjoint faces of dimension at most k whose images contain points that form a regular q-gon. Let $k = \left\lceil \frac{2(q-2)}{q} d \right\rceil = \frac{2d(q-2)+r}{q}$. If $r < 2d$, then we will always find a q-gon partition whose points come from the k-skeleton, meaning the dimension of the disjoint faces they come from has at most dimension $\left\lceil \frac{2(q-2)}{q} d \right\rceil$. For the affine case, this equivalently says if we have $N+1$ points in $\mathbb{C}^d$, and $2d(q-2) \leq qk < 2d(q-1)$, then there is some collection of at most $q(k+1)$ points that has a q-partition where the convex hulls the partitions contain points that form a regular q-gon.

2

Background

2.1 Preliminaries and Background Theorems

Definition 2.1.1. A set $X \subseteq \mathbb{R}^d$ is **convex** if for every two points $x, y \in X$, the line segment from x to y, defined by $l(x, y) = (1-t)x + ty$ for all $t \in [0, 1]$, is also contained X.

Let $Y = \{y_1, \ldots, y_q\}$ be a collection of points. A **convex combination** of the elements of Y is a sum

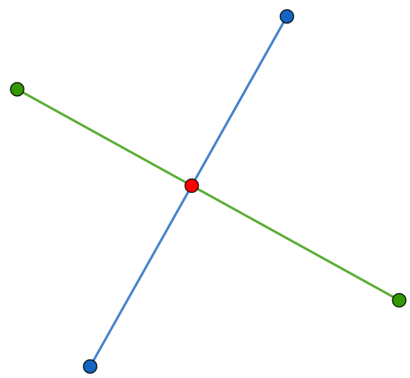
$$\alpha_1 y_1 + \cdots + \alpha_q y_q$$

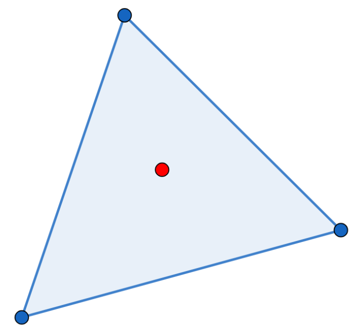
where $\alpha_i \in [0, 1]$ for all $i \in \{1, \ldots, q\}$ and $\alpha_1 + \cdots + \alpha_q = 1$.

The **convex hull** of Y, denoted conv Y, is the set of all convex combinations of Y, being the smallest convex set that contains Y.

Theorem 2.1.2 (Radon's Theorem, 1921)**.** [9] *Every set* $X = \{x_1, \ldots, x_{d+2}\}$ *of* $d + 2$ *points in* $\mathbb{R}^d$ *can be partitioned into two disjoint subsets whose convex hulls intersect.*

Example 2.1.3. Let $d = 2$. Then every set of $d + 2 = 4$ points $x_1, \ldots, x_4$ can be partitioned into two disjoint subsets whose convex hulls intersect. They come in two basic configurations:





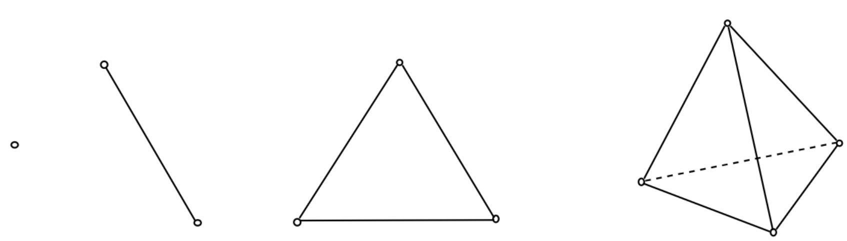
Definition 2.1.4. An **affine space** is like a shifted vector space, with no point acting as the origin. An **affine map** is a function $f : X \to Y$, where X and Y are affine spaces, such that $f((1-t)x+ty) = (1-t)f(x)+t\cdot f(y)$. Affine maps preserve points, lines, and parallel relationships.

Let A be an affine space. The points $\{x_1, \ldots, x_k\} \in A$ are **affinely independent** if the vectors $\{x_2 - x_1, \ldots, x_k - x_1\}$ are linearly independent.

Let B be a set. An **affine combination** of the elements of B is the sum $\sum_i \alpha_i b_i$ where $b_i \in B$ and $\alpha_i \in \mathbb{R}$ for all i such that $\sum_i \alpha_i = 1$.

The **affine hull** of B, denoted aff B, is the set of all affine combinations of elements of B. This is the smallest affine space that contains B. For example, the affine hull of a set of three affinely independent points is the plane going through them.

Definition 2.1.5. A **simplex** Δ is the convex hull of a set of affinely independent points. These affinely independent points are called **vertices**. A **face** of the simplex is the convex hull of an arbitrary set of vertices. The simplices $\Delta^0, \Delta^1, \Delta^2$, and Δ^3 are shown below:

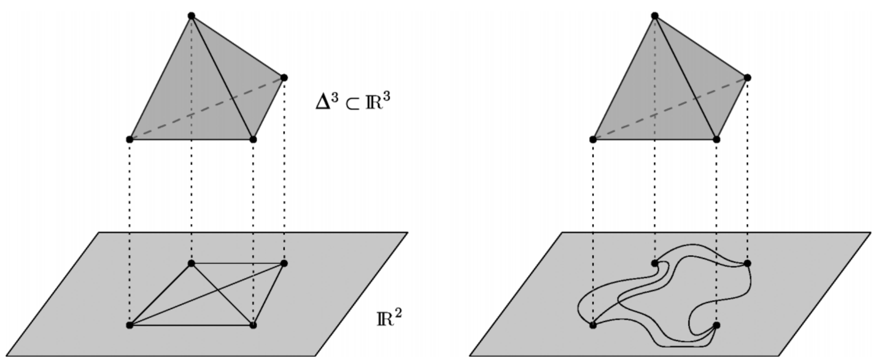


Although Radon's Theorem is a question of convex geometry, we can also view it from an algebraic topology perspective. We do so by considering affine maps from a $(d+1)$-dimensional simplex to the plane.[7]

Theorem 2.1.6 (Affine Radon Theorem)**.** [9] *For any affine map* $f : \Delta^{d+1} \to \mathbb{R}^d$*, where* Δ^{d+1} *is a* $(d+1)$*-dimensional simplex, there exist two disjoint faces* σ_1, σ_2 *of* Δ^{d+1} *such that* $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$*.*

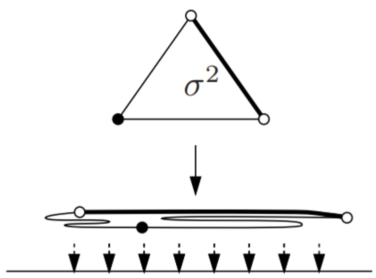
The Theorems 2.1.2 and 2.1.6 are equivalent, except in the degenerate case where the affine map sends multiple vertices to the same point. As described by Matoušek in [7], if the vertices of the simplex map down to our collection of points, then the face containing those vertices will map to the convex hull of our collection of points.

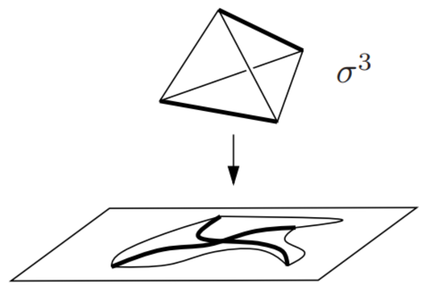
Theorem 2.1.6 naturally leads to considering all continuous maps, rather than just the affine case. Radon's theorm holds for all continuous functions and all simplicial dimensions. The distinction between the affine Radon theorem and topological Radon theorem is shown below, from [3] (p.464), where the affine map on the left and a non-affine continuous map on the right.



Theorem 2.1.7 (Topological Radon Theorem)**.** [9] *For any continuous function* $f : \Delta^{d+1} \to \mathbb{R}^d$*, where* Δ^{d+1} *is a* $(d+1)$*-dimensional simplex, there exist two disjoint faces* σ_1, σ_2 *of* Δ^{d+1} *such that* $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$*.*

Example 2.1.8. Below are two examples of the Topological Radon Theorem, from [7] (p. 89). The left image is when $d = 1$, so we are looking at the 2-dimensional simplex. The right image is where $d = 2$, so we are looking at the 3-dimensional simplex.



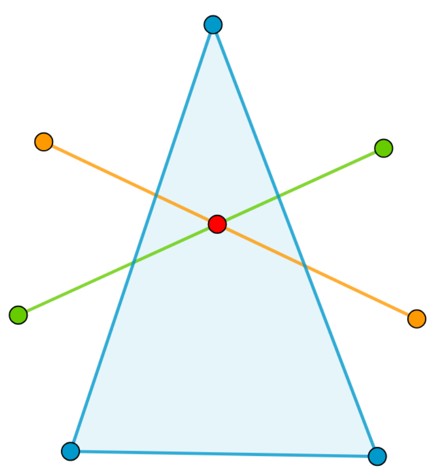


Radon's theorem induces the question of whether we can achieve something similar with more than two disjoint faces.

Theorem 2.1.9 (Tverberg's Theorem, 1966)**.** [13] *Any set of* $(d+1)(q-1)+1$ *points in* $\mathbb{R}^d$ *can be partitioned into* q *disjoint sets* $A_1, \ldots, A_q$ *such that conv* $A_1 \cap \cdots \cap$ *conv* $A_q \neq \emptyset$*.*

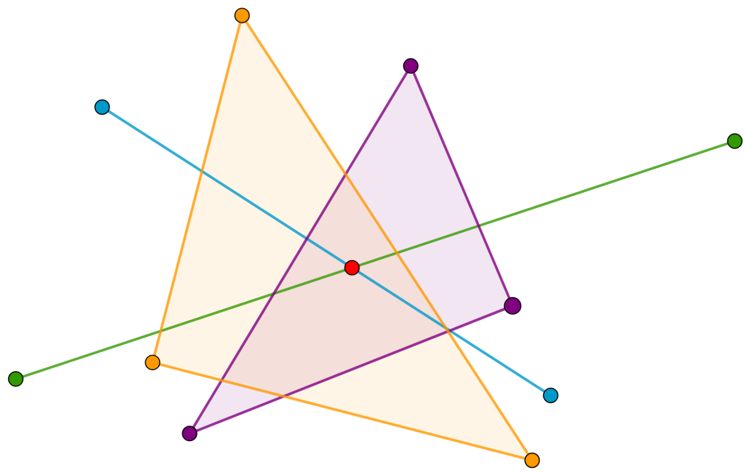
This is called a **q-fold Tverberg partition**. Note that Radon's Theorem is Tverberg's Theorem when $q = 2$.

Example 2.1.10. Below is a 3-fold Tverberg partition with $d = 2$. Observe that there are $(q-1)(d+1)+1 = (3-1)(2+1)+1 = 7$ points in $\mathbb{R}^2$. The different colors indicate the convex hulls of our 3 disjoint sets. Notice that they intersect at the red point, thus admitting a full Tverberg partition.



Example 2.1.11. Below is a 4-fold Tverberg partition with $d = 2$. Note that there are $(q-1)(d+1)+1 = (4-1)(2+1)+1 = 10$ points in $\mathbb{R}^2$. Here the convex hulls of our disjoint sets

are the two triangles and two line segments. Observe that they intersect at the red point, thus admitting a full Tverberg partition.



Similarly to Radon's theorem, we can shift this from a question of convex geometry to a question of algebraic topology by looking at affine maps from a $(q-1)(d+1)$-dimensional simplex to $\mathbb{R}^d$.

Theorem 2.1.12 (Affine Tverberg Theorem)**.** [13] *Let $d, q \in \mathbb{Z}$ such that $d \geq 1$ and $q \geq 2$. Let $N = (q-1)(d+1)$. Every affine map $f : \Delta^N \to \mathbb{R}^d$ has q pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of Δ^N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.*

Again, this naturally leads to the question of whether the affine Tverberg theorem can be extended to all continuous maps. While the topological Radon theorem is always true, the topological Tverberg theorem depends on the value of q.

Theorem 2.1.13 (Topological Tverberg Theorem)**.** [13] *Let $d, q \in \mathbb{Z}$ such that $d \geq 1$ and $q \geq 2$. Let $N = (q-1)(d+1)$. If q is a prime power, than every continuous map $f : \Delta^N \to \mathbb{R}^d$ has q pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of Δ^N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.*

Dissimilar to the affine Tverberg theorem, the topological Tverberg theorem requires the number of partitions q to be a prime power. In 1989, Özaydin proved in [8] that the topological Tverberg theorem is true for prime powers, and in 2014 Frick proved the inverse in [6].

3
Strong General Position

3.1 Strong General Position

Sometimes, even if we have fewer points than Tverberg's Theorem says is necessary for a full Tverberg partition, we still get a full Tverberg partition. As shown by Perles and Sigron in [10], this happens when the points are not in *strong general position*. Because this paper considers collections of points with fewer than $(q-1)(d+1)+1$ points, some of which will admit a full Tverberg partition, we must consider when our points are in strong general position and when they are not.

Definition 3.1.1. [10] A set S of points in $\mathbb{R}^d$ is in **general position** if every set of $d+1$ points of S is affinely independent.

This is a pretty weak condition. It means that no collection of $l \leq d+1$ points in $\mathbb{R}^d$ lie on the same $(l-2)$-flat. For example, no collection of 3 points in general position are colinear in $\mathbb{R}^2$, and no collection of 4 points in general position are coplanar in $\mathbb{R}^3$.

We define $\dim \emptyset := -1$. Let P be a set of points. We denote the dimension of the affine hull of P with $\dim \operatorname{aff} P$.

Proposition 3.1.2. *Let S be a set of $|S|$ points. The points in S are all affinely independent if and only if* $\dim \operatorname{aff} S = |S| - 1$.

Proof. Suppose all points in S are affinely independent. Let S be a set with one point in it. Note that the affine space of S has dimension zero, so $\dim \operatorname{aff} S = |S| - 1$. If we add an affinely independent point, then the dimension of the S goes up by one, and $\dim \operatorname{aff} S$ increases linearly at the same rate as $|S|$. Hence $\dim \operatorname{aff} S = |S| - 1$ for all collections of affinely independent points S.

Suppose $\dim \operatorname{aff} S = |S| - 1$. Suppose there exist distinct points $x_1, x_2, x_3 \in S$ such that x_1, x_2, x_3 are affinely dependent, while all other points are affinely independent. Let $T = S - \{x_1, x_2, x_3\}$. Then $\dim \operatorname{aff} S = \dim \operatorname{aff} T \cup \{x_1, x_2, x_3\} = \dim \operatorname{aff} T + \dim \operatorname{aff} \{x_1, x_2, x_3\} = [(|S| - 3) - 1] + 1 = |S| - 3 \neq |S| - 1$, which is a contradiction. Hence all points in S are affinely independent.

Therefore the points of S are affinely independent if and only if $\dim \operatorname{aff} S = |S| - 1$. $\square$

Remark 3.1.3. [10] A set of points $S \subset \mathbb{R}^d$ is in general position if and only if $\dim \operatorname{aff} F = \min(d, |F| - 1)$ for all $F \subseteq S$.

Proof. Suppose the set of points S is in general position. Let $F \subseteq S$. Then F is also in general position. Suppose $d + 1 \leq |F|$, meaning $d \leq |F| - 1 = \dim \operatorname{aff} F$. We know $\dim \operatorname{aff} F = d$, because $F \subseteq S \subseteq \mathbb{R}^d$, so the largest affine set F can be in is $\mathbb{R}^d$. Suppose $|F| < d + 1$. Hence by proposition 3.1.2, we know $\dim \operatorname{aff} F = |F| - 1 < d$. This tells us that $\dim \operatorname{aff} F = \min(d, |F| - 1)$ for all $F \subseteq S$.

Suppose $\dim \operatorname{aff} F = \min(d, |F| - 1)$. Let $F \subseteq S$ be such that $|F| \leq d + 1$. Suppose $|F| = d + 1$. Then $|F| - 1 = d$, so by 3.1.2, we know all points of F are affinely independent. Suppose $|F| = p + 1 < d + 1$. Then F can exist in at most p dimensions of $\mathbb{R}$, meaning $F \subset \mathbb{R}^p$. Then by 3.1.2, the points of F are affinely independent.

Thus a set of points $S \subset \mathbb{R}^d$ is in general position if and only if $\dim \operatorname{aff} F = \min(d, |F| - 1)$ for all $F \subseteq S$. $\square$

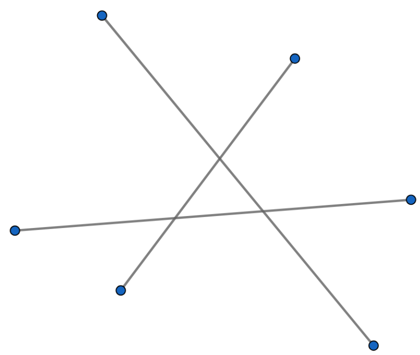
Definition 3.1.4. [10] A finite set $S \subset \mathbb{R}^d$ is in **strong general position** if:

(1) S is in general position.

(2) For any collection $\{F_1, \ldots, F_q\}$ of q pairwise disjoint subsets of S $(1 \leq q \leq |S|)$,

$$d - \dim \bigcap_{v=1}^{q} \text{aff } F_v = \min\left(d+1, \sum_{v=1}^{q}(d - \dim \text{aff } F_v)\right).$$

This means that for any collection of q subsets, the codimension of their intersection equals the sum of their codimensions. Below is an example, where $q = 3$ and $d = 2$:

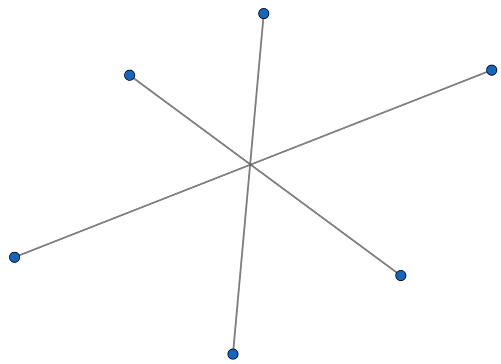


Since no three points are colinear, then our collection of points is in general position. Note that the intersection $\bigcap_{v=1}^{3} \text{aff } F_v = \emptyset$, as there is no total intersection of our affine hulls. Thus $\dim \bigcap_{v=1}^{3} \text{aff } F_v = -1$. Observe that each affine hull is a line segment, so $\dim \text{aff } F_v = 1$ for all v. Thus

$$\begin{aligned} d - \dim \bigcap_{v=1}^{3} \text{aff } F_v &= 2 - (-1) \\ &= \min(3, 3) \\ &= \min\left(2+1, \sum_{v=1}^{3}(2-1)\right) \\ &= \min\left(d+1, \sum_{v=1}^{3}(d - \dim \text{aff } F_v)\right) \end{aligned}$$

We would have to check for all 3-partitions of the points, but we would find that they are indeed in strong general position.

Below is an example of a collection of points that are not in strong general position:



Observe that these points have a Tverberg partition, even though they have one fewer points than required. It is because these points are not in strong general position. Notice how $\dim \bigcap_{v=1}^{3} \text{aff } F_v = 0$, so $d - \dim \bigcap_{v=1}^{3} \text{aff } F_v = 2$. However, each disjoint set F_v has dimension 1. Thus

$$\begin{aligned} d - \dim \bigcap_{v=1}^{3} \text{aff } F_v &= 2 - 0 \\ &\neq 3 \\ &= \min\left(2+1, \sum_{v=1}^{3}(2-1)\right) \\ &= \min\left(d+1, \sum_{v=1}^{3}(d - \dim \text{aff } F_v)\right) \end{aligned}$$

Because $d - \dim \bigcap_{v=1}^{q} \text{aff } F_v = 2 \neq 3 = \min\left(d+1, \sum_{v=1}^{q}(d - \dim \text{aff } F_v)\right)$, then the collection of points is not in strong general position.

Definition 3.1.5. [10] The **Tverberg number** $T(q,d)$ is defined by $T(q,d) = (d+1)(q-1)+1$ with $d, q \in \mathbb{N}$, as that is the number of points needed to ensure a q-fold Tverberg partition in $\mathbb{R}^d$.

This number $T(q,d)$ is extremely tight. If we have $N = T(q,d) - 1$ points in $\mathbb{R}^d$ that are in strong general position, there will not be a q-fold Tverberg partition.

Example 3.1.6. Let S be a set of $N = (q-1)(d+1) = T(q,d) - 1$ points in $\mathbb{R}^d$ that are in strong general position. For any $A_1, \ldots, A_q$ pairwise disjoint subsets of S, the intersection $\bigcap_{i=1}^{q} \text{aff } (A_i) = \emptyset$.

Proof. We want $d - \dim \bigcap_{i=1}^{q} \text{aff } A_i = d+1$, because it implies $\dim \bigcap_{i=1}^{q} \text{aff } A_i = -1$, meaning $\bigcap_{i=1}^{q} \text{aff } A_i = \emptyset$. Because S is in strong general position, we know that $d - \dim \bigcap_{v=1}^{q} \text{aff } F_v = \min\left(d+1, \sum_{v=1}^{q}(d - \dim \text{aff } F_v)\right)$. If $\min\left(d+1, \sum_{v=1}^{q}(d - \dim \text{aff } F_v)\right) = d+1$, then we are done.

Thus all we have to show is that $\sum_{i=1}^{q}(d - \dim \text{aff } A_i) \geq d+1$. Note that by remark 3.1.2, we know that $\dim \text{aff } A_i = |A_i| - 1$. Hence the sum $\sum_{i=1}^{q} \dim \text{aff } A_i = \sum_{i=1}^{q} |A_i| - 1 = \sum_{i=1}^{q} |A_i| - \sum_{i=1}^{q} 1 = N - q$. Therefore

$$\begin{aligned}
\sum_{i=1}^{q}(d - \dim \text{aff } A_i) &= \sum_{i=1}^{q} d - \sum_{i=1}^{q} \dim \text{aff } A_i \\
&= qd - (N - q) \\
&= q(d+1) - (q-1)(d+1) \\
&= (q - (q-1))(d+1) = d+1.
\end{aligned}$$

Hence $d - \bigcap_{i=1}^{q} \text{aff } (A_i) = d+1$, so $\dim \bigcap_{i=1}^{q} \text{aff } (A_i) = -1$. Thus $\bigcap_{i=1}^{q} \text{aff } (A_i) = \emptyset$. $\square$

We know from Tverberg's theorem that if we have at least $T(q,d)$ points in $\mathbb{R}^d$ we can ensure a q-fold Tverberg partition. What example 3.1.6 tells us is that a collection of points has a q-fold Tverberg partition if and only if we have at least $T(q,d)$ points in $\mathbb{R}^d$.

As shown by Perles and Sigron in [10], points are in strong general position with probability one. Thus we will get a Tverberg partition with fewer than $T(q,d)$ points with probability zero.

4

Fourier Analysis for Finite Cyclic Groups

4.1 $L^2(G)$

Let $G = \mathbb{Z}_m$ be a finite abelian cyclic group. Let $L^2(G)$ be defined as the set of all functions from G to the complex numbers:

$$L^2(G) := \{\text{all } f : G \to \mathbb{C}\}.$$

The set $L^2(G)$ is a complex vector space under function addition and scalar multiplication, and is equivalent to $\mathbb{C}[G]$.

It also comes equipped with a complex-valued inner product.

Definition 4.1.1. [12] For any $f_1, f_2 \in L^2(G)$, let

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)},$$

where $\overline{z}$ is the complex conjugate of z for all $z \in \mathbb{C}$. △

Remark 4.1.2. [12]The inner product satisfies the following properties:

(a) $\langle f_1, f_2 \rangle = \overline{\langle f_2, f_1 \rangle}$ for all $f_1, f_2 \in L^2(G)$,

(b) $\langle f, f \rangle \geq 0$ for all $f \in L^2(G)$, and $\langle f, f \rangle = 0$ if and only if f is the zero-function,

(c) $\langle f_1 + f_2, f_3 \rangle = \langle f_1, f_3 \rangle + \langle f_2, f_3 \rangle$ and $\langle f_1, f_2 + f_3 \rangle = \langle f_1, f_2 \rangle + \langle f_1, f_3 \rangle$ for all $f_1, f_2, f_3 \in L^2(G)$,

(d) $\langle \lambda f_1, f_2 \rangle = \lambda \langle f_1, f_2 \rangle$ and $\langle f_1, \lambda f_2 \rangle = \overline{\lambda} \langle f_1, f_2 \rangle$ for all $f_1, f_2 \in L^2(G)$ and any scalar $\lambda \in \mathbb{C}$.

Proof. (a) Observe $\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_q(g)\overline{f_2(g)} = \frac{1}{|G|}\sum_{g\in G}\overline{\overline{f_1(g)}}\,\overline{f_2(g)} = \frac{1}{|G|}\overline{\overline{f_1(g)}f_2(g)} = \overline{\langle f_2, f_1\rangle}$

(b) Let $f(g) = a+bi$ where $a, b \in \mathbb{R}$. Then $\langle f, f\rangle = \frac{1}{|G|}\sum_{g\in G}(a+bi)(a-bi) = \frac{1}{|G|}\sum_{g\in G} a^2+b^2$, which is always positive, unless $a = b = 0$ for all $g \in G$.

(c) Observe $\langle f_1 + f_2, f_3\rangle = \frac{1}{|G|}\sum_{g\in G}(f_1(g) + f_2(g))\overline{f_3(g)} = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_3(g)} + f_2(g)\overline{f_3(g)} = \langle f_1, f_3\rangle + \langle f_2, f_3\rangle$. Also observe $\langle f_1, f_2 + f_3\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{(f_2(g) + f_3(g))} = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} + f_1(g)\overline{f_3(g)} = \langle f_1, f_2\rangle + \langle f_1, f_3\rangle$.

(d) Observe $\langle \lambda f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G}\lambda f_1(g)\overline{f_2(g)} = \frac{\lambda}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} = \lambda\langle f_1, f_2\rangle$, and $\langle f_1, \lambda f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{\lambda f_2(g)} = \frac{\overline{\lambda}}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} = \overline{\lambda}\langle f_1, f_2\rangle$. □

The vector space $L^2(G)$ has a **norm**, called the L^2-norm, defined by $||f|| = \sqrt{\langle f, f\rangle}$.

4.2 Classification of Homomorphisms of G to $S^1 \subset \mathbb{C}$

We want to construct a particularly nice orthonormal basis for $L^2(G)$ so we can decompose any function $f \in L^2(G)$ into a linear combination of the basis elements. The nicer the basis elements, the easier the decompositions are to work with. It turns out that the set of homomorphisms $\chi : G \to S^1$ from G to the unit circle forms such a basis. [12]

Definition 4.2.1. Let $H^1(G)$ be defined by

$$H^1(G) := \{\text{homomorphisms } \chi : G \to \mathbb{C}^*\},$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$. △

Lemma 4.2.2. [12] *Let $G = \mathbb{Z}_m$ be a cyclic group, and let $\xi_m = e^{\frac{2\pi i}{m}}$ be the m^{th} primitive root of unity. Then*

$$H^1(G) = \{\chi_\epsilon \mid 0 \le \epsilon < m\},$$

where for each $0 \le \epsilon < m$, we define $\chi_\epsilon : G \to \mathbb{C}$ by raising ξ_m to successive powers of ϵ:

$$\chi_\epsilon(k) = [\xi_m^\epsilon]^k.$$

Proof. Let $\chi_\epsilon : G \to \mathbb{C}$ be defined by

$$\chi_\epsilon(k) = [\xi_m^\epsilon]^k$$

for all $0 \leq \epsilon < m$. Let $j, k \in G$. Then

$$\chi_\epsilon(k+j) = [\xi_m^\epsilon]^{k+j} = [\xi_m^\epsilon]^k[\xi_m^\epsilon]^j = \chi_\epsilon(k) \cdot \chi_\epsilon(j).$$

Hence χ_ϵ is a homomorphism, so $\{\chi_\epsilon \mid 0 \leq \epsilon < m\} \subseteq H^1(G)$.

Let $\phi \in H^1(G)$. Since ϕ is a homomorphism, then $1 = \phi(0) = \phi(m) = \phi(1+1+\cdots+1) = \phi(1)+\phi(1)+\cdots+\phi(1) = \phi(1)^m$ in G. Hence $\phi(1)$ is an m^{th} root of unity, meaning $\phi(1) = \xi_m^t$ for some $t \in \mathbb{Z}$. Since 1 is a generator for G, then $\phi(k) = \phi(1+\cdots+1) = \phi(1)+\cdots+\phi(1) = \phi(1)^k = (\xi_m^t)^k = \xi_m^{tk}$ will also be an m^{th} root of unity for all $k \in G$. Hence $\phi \in \{\chi_\epsilon \mid 0 \leq \epsilon < m\}$. Therefore $H^1(G) \subseteq \{\chi_\epsilon \mid 0 \leq \epsilon < m\}$. Thus $H^1(G) = \{\chi_\epsilon \mid 0 \leq \epsilon < m\}$. □

Note that $H^1(G)$ forms a group under multiplication. As shown above, there is a bijective correspondence between G and $H^1(G)$, defined by $\epsilon \mapsto \chi_\epsilon$. Let $L : G \to H^1(G)$ be a bijection defined by $L(\epsilon) = \chi_\epsilon$. Then $L(\epsilon+\delta) = \chi_{\epsilon+\delta} = \xi_m^{\epsilon+\delta} = \xi_m^\epsilon \xi_m^\delta = \chi_\epsilon \cdot \chi_\delta = L(\epsilon)L(\delta)$. Hence L is also a homomorphism. Thus G and $H^1(G)$ are isomorphic. [12]

4.3 $H^1(G)$ as an Orthogonal Basis for $L^2(G)$

To verify that $H^1(G)$ is indeed an orthonormal basis for $L^2(G)$, we must first confirm that $H^1(G)$ is an orthonormal set. We will then show that the elements both are linearly independent and span, thus forming a basis.

An orthonormal set is one where all of the elements are both orthogonal, meaning the inner product of any two distinct elements is 0, and normal, meaning their norm is 1. Hence we must show that the inner product of any two elements is 0 if they are distinct and 1 if they are not.

Lemma 4.3.1. [12]

$$\langle \chi_\epsilon, \chi_\delta \rangle = \begin{cases} 0 & \textit{if } \epsilon \neq \delta \\ 1 & \textit{if } \epsilon = \delta. \end{cases}$$

Proof. Let $\xi_m = e^{\frac{2\pi i}{m}}$, and let $0 \leq \epsilon < m$. We will first show that

$$\frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{k\epsilon} = \begin{cases} 0 & \text{if } \epsilon \neq 0 \\ 1 & \text{if } \epsilon = 0. \end{cases} \tag{4.3.1}$$

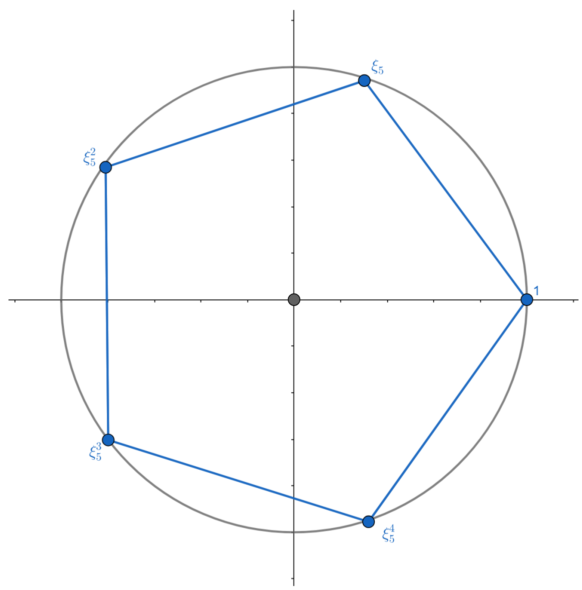
Case 1: Suppose $\epsilon \neq 0$. Then

$$\frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{k\epsilon} = \frac{1}{m}[1 + \xi_m^{\epsilon} + \xi_m^{2\epsilon} + \cdots + \xi_m^{(m-1)\epsilon}].$$

Observe

$$1 + \xi_m^{\epsilon} + \cdots + \xi_m^{(m-1)\epsilon} = \sum_{j=0}^{\infty} \xi_m^{j\epsilon} - \sum_{j=m}^{\infty} \xi_m^{j\epsilon} = \frac{1}{1-\xi_m^{\epsilon}} - \frac{\xi_m^{m\epsilon}}{1-\xi_m^{\epsilon}} = \frac{1-1}{1-\xi_m^{\epsilon}} = 0.$$

Thus $\frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{k\epsilon} = 0$. We can also tell that summing all m^{th} roots of unity together equals zero because they form a regular polygon centered at zero. Below is an example when $m = 5$:



Adding five vectors would get us back to the origin.

Case 2: Suppose $\epsilon = 0$. Then

$$\frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{k\epsilon} = \frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{0} = \frac{1}{m}\sum_{k=0}^{m-1} 1 = \frac{1}{m} \cdot m = 1.$$

Hence

$$\frac{1}{m}\sum_{k=0}^{m-1} \xi_m^{k\epsilon} = \begin{cases} 0 & \text{if } \epsilon \neq 0 \\ 1 & \text{if } \epsilon = 0. \end{cases}$$

Next we will show that $\langle \chi_\epsilon, \chi_\delta \rangle = \begin{cases} 0 & \text{if } \epsilon \neq \delta \\ 1 & \text{if } \epsilon = \delta. \end{cases}$

Let $G = \mathbb{Z}_m$. Let $\chi_\epsilon, \chi_\delta \in H^1(G)$ such that $\chi_\epsilon : 1 \mapsto \xi_m^\epsilon$ and $\chi_\delta : 1 \mapsto \xi_m^\delta$. Then by 4.3.1

$$\langle \chi_\epsilon, \chi_\delta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\epsilon(g) \chi_\delta^{-1}(g) = \frac{1}{|G|} \sum_{g \in G} \xi_m^{\epsilon g} \xi_m^{-\delta g} = \frac{1}{|G|} \sum_{g \in G} \xi_m^{g(\epsilon - \delta)} = \begin{cases} 0 & \text{if } \epsilon \neq \delta \\ 1 & \text{if } \epsilon = \delta. \end{cases}$$

Thus the set $H^1(G)$ is orthonormal. □

We wind up getting that the elements of $H^1(G)$ are linearly independent for free. Note that any orthogonal elements are linearly independent, and since $H^1(G)$ is an orthonormal set, all of its elements are orthogonal. Thus $H^1(G)$ is linearly independent.

Lastly, we must show that $H^1(G)$ spans. To show that it spans, we will show any $f \in L^2(G)$ can be expressed as a linear combination of $\chi_\epsilon \in H^1(G)$. Namely, we will show that any $f \in L^2(G)$ can be written as $f = \sum_\epsilon c_\epsilon \chi_\epsilon$. We will start by showing that if f can be written as a linear combination $f = \sum_\epsilon c_\epsilon \chi_\epsilon$, then c_ϵ must equal the inner product of f and χ_ϵ. We will then show that implies $f = \sum_\epsilon c_\epsilon \chi_\epsilon$. [12]

Lemma 4.3.2. [12] *(a) Let $f \in L^2(G)$. Suppose $f = \sum_\epsilon c_\epsilon \chi_\epsilon$ could be expressed as a linear combination of χ_ϵ. Then*

$$c_\epsilon = \langle f, \chi_\epsilon \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \chi_\epsilon^{-1}(g).$$

(b) Any $f \in L^2(G)$ can be decomposed as

$$f = \sum_\epsilon c_\epsilon \chi_\epsilon.$$

Proof. (a) Suppose $f = \sum_\epsilon c_\epsilon \chi_\epsilon = c_0 \chi_0 + \cdots + c_{q-1} \chi_{q-1}$ can be written as a linear combination of the χ_ϵ. Then

$$\begin{aligned} \langle f, \chi_i \rangle &= \left\langle \sum_{j \in G} c_j \chi_j, \chi_i \right\rangle \\ &= \sum_{j \in G} c_j \langle \chi_j, \chi_i \rangle \qquad 4.1.2 \\ &= c_0 \langle \chi_0, \chi_i \rangle + \cdots + c_i \langle \chi_i, \chi_i \rangle + \cdots + c_{m-1} \langle \chi_{m-1}, \chi_i \rangle \\ &= c_i. \qquad 4.3.1 \end{aligned}$$

Thus if $f = \sum_{\epsilon \in G} c_i \chi_i$, then $c_i = \langle f, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \chi_\epsilon^{-1}(g)$.

(b) Let $G = \mathbb{Z}_m$. Let $f \in L^2(G)$. Let $c_\epsilon = \langle f, \chi_\epsilon \rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\chi_\epsilon^{-1}(g)$. Then

$$\begin{aligned}
\sum_{\epsilon\in G} c_\epsilon \chi_\epsilon(a) &= \sum_{\epsilon=0}^{m-1}\left(\frac{1}{m}\sum_{k=0}^{m-1} f(k)\xi_m^{-k\epsilon}\right)\xi_m^{a\epsilon} \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\left(\sum_{\epsilon=0}^{m-1} f(k)\xi_m^{-k\epsilon}\right)\xi_m^{a\epsilon} \\
&= \frac{1}{m}\left(\sum_{k=0}^{m-1} f(k)\sum_{\epsilon=0}^{m-1}\xi_m^{-k\epsilon}\right)\xi_m^{a\epsilon} \\
&= \frac{1}{m}\sum_{k=0}^{m-1} f(k)\sum_{\epsilon=0}^{m-1}\xi_m^{\epsilon(a-k)} \\
&= \frac{1}{m}\left[f(0)\sum_{\epsilon=0}^{m-1}\xi_m^{\epsilon(a)} + \cdots + f(a)\sum_{\epsilon=0}^{m-1}\xi_m^{\epsilon(a-a)} + \cdots + f(m-1)\sum_{\epsilon=0}^{m-1}\xi_m^{\epsilon(a-(m-1))}\right] \\
&= \frac{1}{m}\left[f(0)\cdot 0 + \cdots + f(a)\cdot 1 + \cdots + f(m-1)\cdot 0\right] \qquad 4.3.1 \\
&= f(a).
\end{aligned}$$

Hence any element $f \in L^2(G)$ can be decomposed as $f = \sum_\epsilon c_\epsilon \chi_\epsilon$. $\square$

The coefficients c_ϵ are called the **Fourier Coefficients**, or **Fourier Transforms** of f with respect to χ_ϵ. The decomposition is called the **Fourier Inversion Formula**, as it completely determines the functions $f \in L^2(G)$ simply by knowing the Fourier Transforms.

Finite Fourier analysis will help us figure out whether the continuous function $f : \partial\Delta^n \to \mathbb{C}^d$ admits a Tverberg partition. We will do so by indexing the points on our simplex by a finite abelian group G. Then we can use the Fourier decomposition to see if there is a collection of points on our simplex that makes f constant, showing they must intersect in the image. We will know if the function f is constant based on how many Fourier coefficients c_ϵ are equal to zero. [11]

5
A Topological Description

5.1 Finite Fourier Approach to the Topological Tverberg Theorem

Suppose we are looking for a q-fold Tverberg partition from a simplex Δ^n. We can index all collections of q points $x_1 \in \sigma_q, \ldots, x_q \in \sigma_q$ from disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\partial\Delta^n$ by any group of order $|G| = q$.

Let $f = (f_1, \ldots, f_d) : \partial\Delta^n \to \mathbb{C}^d$ be a continuous map. From f, we get a collection of maps $F_1, \ldots, F_d : G \to \mathbb{C}$, one for each dimension of $\mathbb{C}^d$, defined by $F_i(g) = f_i(x_g)$.[11] If we let $G = \mathbb{Z}_q$ be a finite cyclic group, then our functions $F_1, \ldots, F_d$ have a nice Fourier decomposition:

$$F_i(g) = \sum_{\epsilon \in G} c_{(i,\epsilon)} \chi_\epsilon(g)$$

where

$$c_{(i,\epsilon)} = \frac{1}{|G|} \sum_{v \in G} f_i(x_v) \chi_\epsilon^{-1}(v) \in \mathbb{C}.$$

Recall $\chi_\epsilon(g) = \xi_q^{\epsilon g}$ for $\epsilon, g \in G$.

Observe $F_i = \sum_{\epsilon \in G} c_{(i,\epsilon)} \chi_\epsilon = c_{(i,0)} \chi_0 + \sum_{\epsilon \neq 0} c_{(i,\epsilon)} \chi_\epsilon = c_{(i,0)} + \sum_{\epsilon \neq 0} c_{(i,\epsilon)} \chi_\epsilon$; note χ_0 is trivial, as $\chi_0(t) = \xi_q^{0 \cdot t} = 1$ for all $t \in G$. Suppose F_i is a constant map for all $i \in \{1, \ldots, d\}$. Then f_i is constant for all $i \in \{1, \ldots, d\}$. Hence f is a constant function, so $f(x_g) = k$ for all $g \in G$, where $k \in \mathbb{C}^d$ is some constant. Hence all of our q points $x_0 \in \sigma_0, \ldots, x_{q-1} \in \sigma_{q-1}$ from disjoint faces $\sigma_0, \ldots, \sigma_{q-1}$ map to the single point k. That is a Tverberg partition.[11]

Lemma 5.1.1. [11] *Let $f = (f_1, \ldots, f_d) : \partial\Delta^n \to \mathbb{C}^d$ be a continuous function. Let $\{x_g\}_{g\in G}$ be a collection of q points $x_0 \in \sigma_0, \ldots, x_{q-1} \in \sigma_{q-1}$ from disjoint faces $\sigma_0, \ldots, \sigma_{q-1}$ of $\partial\Delta^n$, where $G = \mathbb{Z}_q$. Let $F_1, \ldots, F_d : G \to \mathbb{C}$ be defined by $F_i(g) = f_i(x_g)$, and let $F_i(g) = \sum_{\epsilon\in G} c_{(i,\epsilon)}\chi_\epsilon(g)$ be the Fourier decomposition of F_i for all $i \in \{1, \ldots, d\}$. If $c_{(i,\epsilon)} = 0$ for all $i \in \{1, \ldots, d\}$ and all $\epsilon \neq 0$, then f admits a full Tverberg partition.*

Proof. Suppose $c_{(i,\epsilon)} = 0$ for all $i \in \{1, \ldots, d\}$ and all $\epsilon \neq 0$. Then

$$F_i = c_{(i,0)} + \sum_{\epsilon\neq 0} c_{(i,\epsilon)}\chi_\epsilon = c_{(i,0)} + \sum_{\epsilon\neq 0} 0 \cdot \chi_\epsilon = c_{(i,0)},$$

which is a constant map. Thus, as explained above, f admits a Tverberg partition. $\square$

We now know that if all Fourier coefficients for the decomposition of F_i equal zero except $c_{(i,0)}$, then we get a full Tverberg partition. However, we cannot always ensure that there will be a collection of q points from disjoint faces of $\partial\Delta^n$ that will make the Fourier coefficients zero. We can test whether there exists a collection of points that make the Fourier coefficients zero with the following theorem by Simon from [11].

Theorem 5.1.2. [11] *Let $q \in \mathbb{N}$ and let $\epsilon_1, \ldots, \epsilon_m \in \mathbb{Z}_q$.*

*Let $n = 2dm + (q - 1)$. If the **vanishing polynomial***

$$h(y) = \Pi_{j=1}^{m}(\epsilon_j y)^d$$

is non-zero in $\mathbb{Z}[y]/q\mathbb{Z}$, then for any continuous map $f : \partial\Delta^n \to \mathbb{C}^d$, there exist q points of $\partial\Delta^n$ from pairwise disjoint faces such that $c_{(i,\epsilon_j)} = 0$ for all $j \in \{1, \ldots, m\}$ and all $i \in \{1, \ldots, d\}$ in the Fourier expansion.

Here m is the number of Fourier coefficients we want to set to zero. The vanishing polynomial of 5.1.2 will be pivotal in proving whether a function $f : \partial\Delta^n \to \mathbb{C}$ has a Tverberg partition.

Example 5.1.3. Let $f : \partial\Delta^n \to \mathbb{C}^d$ be a continuous function, where $n = 2dm + (q - 1)$ for q prime. Let $m = q - 1$, because if we can eliminate $q - 1$ coefficients $c_1, \ldots, c_{q-1}$, then f will admit a full Tverberg partition. Then the vanishing polynomial

$$h(y) = [y(2y)\cdots(q - 1)y]^d = [(q - 1)!]^d y^{d(q-1)}.$$

Note that by Wilson's Theorem, because q is prime then $(q-1)! \equiv -1 \pmod{q}$, so

$$h(y) = [(q-1)!]^d y^{d(q-1)} \equiv (-1)^d y^{d(q-1)}$$

is non-zero in $\mathbb{Z}[y]/q\mathbb{Z}$. Then by 5.1.1 and 5.1.2 there exist q points $x_1, \ldots, x_q$ of disjoint faces $\sigma_1, \ldots, \sigma_q$ such that $f(x_1) = \cdots = f(x_q)$.

Example 5.1.4. Let $f : \partial\Delta^n \to \mathbb{C}$ be a continuous function and let $q = 6$, meaning $n = 2(1)(6-1) + (6-1) = 15$. However, not every continuous function $f : \Delta^{15} \to \mathbb{C}$ can be guaranteed to admit a full 6-fold Tverberg partition [5], as the vanishing polynomial is

$$h(y) = \Pi_{j=1}^{q-1} \epsilon_j y = y(2y)(3y)(4y)(5y) = 120y^5 \equiv 0$$

in $\mathbb{Z}[y]/6\mathbb{Z}$. Thus Theorem 5.1.2 does not apply.

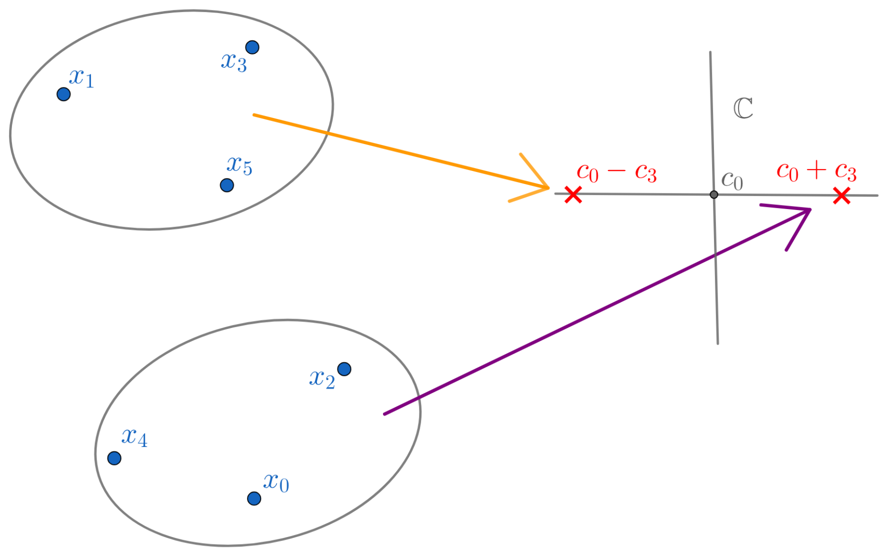
However, suppose all Fourier coefficients except c_0 and c_3 were zero. Then the vanishing polynomial is

$$h(y) = y(2y)(4y)(5y) = 40y^4 \equiv 4y^4$$

in $\mathbb{Z}[y]/6\mathbb{Z}$. Since the vanishing polynomial is not zero, then we can eliminate all coefficients except c_0 and c_3. Therefore

$$f(x_g) = c_0 + c_3 \chi_6^3(g) = c_0 + c_3 \xi_6^{3g} = c_0 + c_3(-1)^g.$$

Thus if g is odd then $f(x_g) = c_0 - c_3$ and if g is even then $f(x_g) = c_0 + c_3$. This means that $f(x_1), f(x_3), f(x_5)$ get mapped to one point, and $f(x_0), f(x_2), f(x_4)$ get mapped to another point.



Now we will show that Theorem 5.1.2 implies that we get a full Tverberg partition when q is an odd prime.

Example 5.1.5. Let $f : \partial\Delta^n \to \mathbb{R}^d$ be a continuous function, where $n = 2dm + (q-1)$ for q prime. Let c_ϵ be a Fourier coefficient. Note that

$$\overline{c_\epsilon} = \overline{\sum_{g\in G} f(x_g)\chi_\epsilon^{-1}(g)} = \sum_{g\in G} \overline{f(x_g)\chi_\epsilon^{-1}(g)} = \sum_{g\in G} \overline{f(x_g)}\,\overline{\chi_\epsilon^{-1}(g)}.$$

Because $f(x_g)$ is real-valued, then $\overline{f(x_g)} = f(x_g)$. Observe $\overline{\chi_\epsilon^{-1}(g)} = \xi_q^{-(-g\epsilon)} = [\xi_q^{-\epsilon\cdot g}]^{-1} = \chi_{-\epsilon}^{-1}(g)$. Then

$$\overline{c_\epsilon} = \sum_{g\in G} f(x_g)\chi_{-\epsilon}^{-1}(g) = c_{-\epsilon}.$$

If $c_\epsilon = 0$, then its conjugate is zero, meaning $c_{-\epsilon} = 0$ as well. Hence for a full Tverberg partition, we need only set $m = \frac{q-1}{2}$ Fourier coefficients to zero. Then $n = 2d\left(\frac{q-1}{2}\right) + (q-1) = d(q-1) + (q-1) = (d+1)(q-1)$. The vanishing polynomial

$$h(y) = \left[y(2y)\cdots\left(\frac{q-1}{2}\right)y\right]^d = \left[\left(\frac{q-1}{2}\right)!\right]^d y^{\frac{q-1}{2}\cdot d}.$$

is non-zero in $\mathbb{Z}[y]/q\mathbb{Z}$. Then by 5.1.2 there exist q points $x_0, \ldots, x_{q-1}$ from disjoint faces $\sigma_0, \ldots, \sigma_{q-1}$ of $\partial\Delta^n$ such that $f(x_0) = \cdots = f(x_{q-1})$, which we recall is the definition of a full q-fold Tverberg partition.

Example 5.1.6. Let $f : \partial\Delta^n \to \mathbb{C}^d$ be a continuous function, where $n = 2(1)m + (q-1)$ for q prime. Suppose we want to eliminate all Fourier coefficients except c_0 and c_1. Then the vanishing polynomial is

$$h(y) = [(2y)(3y)\cdots(q-1)y]^d = [(q-1)!]^d y^{(q-1)d} \equiv (-1)^d y^{(q-1)d}$$

in $\mathbb{Z}[y]/q\mathbb{Z}$. Since the vanishing polynomial is never zero, we know we can kill all coefficients but c_0 and c_1. Then

$$f(x_g) = c_0 + c_1\chi_q(g) = c_0 + c_1\xi_q^g.$$

By 4.3, we know the collection of points $\xi_q^0, \xi_q^1, \ldots, \xi_q^{q-1}$ form a regular q-gon, lying on the unit circle centered at zero. The coefficient c_1 does not change the shape, but rather makes the points form a regular q-gon centered at zero on a circle of radius c_1. By adding a c_0, it shifts the circle to be centered at c_0. Hence the points form a regular q-gon centered at c_0. Thus our collection of points $\{f(x_g)\}_{g=0}^{q-1}$ form a regular q-gon.

The proof of 5.1.2 follows a **configuration-space/test-map scheme**. First, we need a working understanding of group actions.

5.2 G-actions

Let G be a finite group, and let X be a topological space. The action of G on X is a function $\phi : G \times X \to X$ where $\phi(g, x) = g \cdot x$ for all $(g, x) \in G \times X$. A group action satisfies the following properties:

(1) if e is the identity element of G, then $\phi(e, x) = x$ for all $x \in X$, and

(2) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$.

A G-action is **free** if there is no $g \in G - \{e\}$ such that $\phi(g, x) = x$. If there is some $g \in G - \{e\}$ such that $\phi(g, x) = x$, then it is called a **fixed point**.

Example 5.2.1. Let $G = \mathbb{Z}_2$. We then get a $\mathbb{Z}_2$-action, which is a function $\phi : \mathbb{Z}_2 \times X \to X$ such that $\phi(g^2, x) = g^2 \cdot x = g(gx) = x$.

For example, let X be a topological space. Then an example of a $\mathbb{Z}_2$-action on $X\times X$ coordinate permutation, where $\phi : \mathbb{Z}_2 \times (X \times X) \to X \times X$ is defined by $1(x_1, x_2) = (x_{1+1}, x_{1+2}) = (x_2, x_1)$ for all $(x_1, x_2) \in X \times X$. The coordinate subsrcipts are evaluated in $\mathbb{Z}_2$. Note $0(x_1, x_2) = (x_{1+0}, x_{2+0}) = (x_1, x_2)$. Observe $1(1(x_1, x_2)) = 1(x_2, x_1) = (x_{2+1}, x_{1+1}) = (x_1, x_2) = (1 + 1)(x_1, x_2)$. Hence ϕ is a group action.

Note that this $\mathbb{Z}_2$ action is not free, because if $x_1 = x_2$, then $1(x_1, x_2) = (x_2, x_1) = (x_1, x_2)$ for all $x_1 \in X$. This gives rise to the notion of the **deleted product**. The deleted product is defined by

$$(X \times X)_{(2)} := \{(x, y) \mid x, y \in X \text{ and } x \neq y\}.$$

Observe that if we restrict our coordinate permutation $\mathbb{Z}_2$-action to the deleted product $(X \times X)_{(2)}$, then it does become a free action.

Example 5.2.2. Let $G = \mathbb{Z}_q$ be a restriction of the symmetric group action S_q. Let X be the set of all collections of q points $x = (x_1, \ldots, x_q)$ from the boundary of the simplex $\partial\Delta^n$. Then the $\mathbb{Z}_q$-action on x would permute the points of x:

$$g \cdot x = (x_{g+1}, x_{g+2}, \ldots, x_{g+q}) \text{ for all } g \in G,$$

where the subscript $g + n$ is evaluated in $\mathbb{Z}_q$ for all $n \in \{1, \ldots, q\}$. This action is not free. For example, let $x_1 = \cdots = x_q$. Then $g \cdot x = (x_{g+1}, x_{g+2}, \ldots, x_{g+q}) = x$ for all $g \in G$. However, it becomes free if the points $x \in \partial\Delta^n$ instead come from the q-fold deleted product $(\partial\Delta^n)^{\times q}_{(2)}$, where all of the points $(x_1, \ldots, x_q)$ come from pairwise disjoint faces.

Example 5.2.3. We can also look at the $\mathbb{Z}_2$ action from example 5.2.1 on $\mathbb{R}^d$ restricted to the $(d-1)$-sphere S^{d-1}. Let $x \in S^{d-1}$ be a point. Then the standard $\mathbb{Z}_2$-action is defined by $g \cdot x = -x$. Note that this is an antipodal action, meaning $g \cdot (-x) = -(g \cdot x)$ for all x.

This $\mathbb{Z}_2$ action is free. For example, consider the S^1 sphere in $\mathbb{R}^2$, being the unit circle. The only point in $\mathbb{R}^2$ such that $(x, y) = -(x, y) = (-x, -y)$ is the point $(0, 0)$, which does not lie on the unit circle. Hence this $\mathbb{Z}_2$-action has no fixed point.

Example 5.2.4. Let $G = \mathbb{Z}_p$ act on $\mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. Let $\chi_\epsilon : G \to \mathbb{C}^*$ be defined as usual. Then G acts on $\mathbb{C}^*$ with $\phi : G \times \mathbb{C}^* \to \mathbb{C}^*$ defined by left multiplication of χ_ϵ, meaning $g \cdot z = \chi_\epsilon(g) \cdot z$ for all $g \in G$ and $z \in \mathbb{C}^*$. Let $x \in \mathbb{C}^*$. Then $0 \cdot x = \chi_\epsilon(0)x = \xi_q^{\epsilon \cdot 0}x = x$. Let $h, g \in G$. Then $h \cdot (g \cdot x) = h \cdot (\chi_\epsilon(g)x) = \chi_\epsilon(h)\chi_\epsilon(g)x = \xi_q^{h\epsilon}\xi_q^{g\epsilon} = \xi_q^{(h+g)\epsilon} = \chi_\epsilon(h+g)x = (h+g)\cdot x$. Hence ϕ is a group action.

Given $\chi_{\epsilon_1}, \dots, \chi_{\epsilon_d} : G \to \mathbb{C}^*$, we have an action defined by componentwise left multiplication of χ_{ϵ_i} for all $i \in \{1, \dots, d\}$. If $z = (z_1, \dots, z_d) \in (\mathbb{C}^*)^d$, then $g \cdot z = (\chi_{\epsilon_1}(g)x, \dots, \chi_{\epsilon_d}(g)x)$.

Note that this $\mathbb{Z}_p$-action is free on the unit sphere $S^1 \subset \mathbb{C}^d$.

5.3 The Borsuk-Ulam Theorem

The Borsuk-Ulam theorem is a critical tool in algebraic topology. Moving forward we will use it analogously.

Definition 5.3.1. [7] Let X, Y be topological spaces with the standard $\mathbb{Z}_2$-action described in 5.2.3. An **antipodal map** $f : X \to Y$ is one where $f(-x) = -f(x)$ for all $x \in X$. △

Theorem 5.3.2 (Borsuk-Ulam)**.** [7] *Let $n \geq 0$. Then for every antipodal map $f : S^n \to \mathbb{R}^n$ there exists some point $x \in S^n$ such that $f(x) = 0$.*

Example 5.3.3. Let $h : \mathbb{R} \to S^1$ be defined by $h(t) = (\cos(t), \sin(t))$. Let $f : S^1 \to \mathbb{R}$ be continuous. Note that $f \circ h : \mathbb{R} \to \mathbb{R}$ is continuous. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = (f \circ h)(x) - (f \circ h)(-x)$; note that g is also continuous. Suppose there exists some $x \in \mathbb{R}$ such that $(f \circ h)(x) = (f \circ h)(-x)$. Then $g(x) = 0$. Suppose there is no point $x \in \mathbb{R}$ such that $(f \circ h)(x) = (f \circ h)(-x)$. Without loss of generality, suppose $(f \circ h)(x) > (f \circ h)(-x)$ for some $x \in S^1$. Then $g(x) \geq 0$. Then

$$g(-x) = (f \circ h)(-x) - (f \circ h)(-(-x)) = (f \circ h)(-x) - (f \circ h)(x) = -g(x) \leq 0.$$

Then by the Intermediate Value Theorem there exists some $c \in (-x, x)$ such that $g(c) = 0$. Hence $(f \circ h)(c) = (f \circ h)(-c)$.

In the case of the $\mathbb{Z}_2$-action, saying f is an antipodal map is equivalent to saying f is *equivariant.*

Definition 5.3.4. [7] Let X, Y be topological spaces and let G act on X and Y. Then $f : X \to Y$ is an **equivariant map** if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and all $x \in X$. $\triangle$

Example 5.3.5. Let $\mathbb{Z}_2$ be the standard action on S^1 and $\mathbb{R}^2$, as shown in 5.2.3. Let $f : S^1 \to \mathbb{R}^2$ be the identity map. Note that $f(-x) = -x = -(x) = -f(x)$ for all $x \in S^1$, so f is equivariant. However, since there is no $x \in S^1$ such that $x = \mathbf{0}$, then there is no $x \in S^1$ such that $f(x) = \mathbf{0}$. Thus f does not have a zero. This is because the dimensions of S^1 and $\mathbb{R}^2$ do not match.

Example 5.3.6. Let $\mathbb{Z}_2$ be the standard action on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map defined by $f(x) = 5$ for all $x \in \mathbb{R}$. Note that f is not equivariant, as $f(-x) = 5 \neq -5 = -f(x)$. Observe that f does not have a zero, as there does not exist some $x \in \mathbb{R}$ such that $f(x) = 0$. This is because f is not equivariant.

As shown in examples 5.3.5 and 5.3.6, both $f : X \to Y$ being equivariant and X and Y having matching dimensions is pivotal to f having zero. We will use this analogously later to ensure we can set our Fourier coefficients to zero.

5.4 Configuration Space/Test Map Scheme and Equivariance

Definition 5.4.1. The **configuration space** of a problem is the set of all possible outcomes, similar to the sample space in probability. In our case, it will be the set of all collections of q points from disjoint faces of $\partial\Delta^n$. The **test map** is the way we distinguish between elements in the configuration space, searching for a suitable solution. Our test map will determine what collection of q points will have a test map that sets all of the Fourier coefficients to zero. [11]

We can parameterize our collections of q points from disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\partial\Delta^n$ by the deleted product

$$X := (\partial\Delta^n)^{\times q}_{(2)} = \{x = (x_1, \ldots, x_q) \in \sigma_1 \times \cdots \times \sigma_q \mid \sigma_i \cap \sigma_j = \emptyset \text{ for all } i \neq j\}.$$

This is our *configuration space*. Let the $\mathbb{Z}_q$-action discussed in example 5.2.2 act on the configuration space X, after indexing subscripts $\{1, \dots, q\}$ by the abelian group $\mathbb{Z}_q$. This means our elements $x \in X$ are now indexed with $x = (x_0, \dots, x_{q-1})$. Note that because X is a deleted product, then this $\mathbb{Z}_q$-action is free. [11]

We consider a solution to be some $x \in X$ such that the Fourier transforms are zero. Hence we will define our *test map* $\mathcal{F} : X \to \mathbb{C}^{dm}$ by our Fourier transforms, because that is how we want to distinguish between elements of x. Recall that m is the number of Fourier coefficients we want to be zero. Thus the dimensions of X and $\mathbb{C}^{dm}$ match, as in example 5.3.5. The map $\mathcal{F} : X \to \mathbb{C}^{dm}$ is defined as follows:

$$\mathcal{F}(x) = \frac{1}{|G|} \sum_{g \in G} f_i(x_g) \chi_{\epsilon_j}^{-1}(g),$$

where $1 \leq i \leq d$ and $1 \leq j \leq m$. [11]

The group $G = \mathbb{Z}_q$ acts on the domain X via permutation, like in example 5.2.2. The group G acts linearly on the codomain $\mathbb{C}^{dm}$ with componentwise left multiplication, as described in example 5.2.4.

Note that $\mathcal{F}$ is equivariant with respect to the two actions, as shown below:

$$\begin{aligned}
\mathcal{F}(g \cdot x) &= \frac{1}{|G|} \sum_{h \in G} f(x_{h+g}) \chi_{\epsilon}^{-1}(h) \\
&= \frac{1}{|G|} \sum_{h \in G} f(x_{h+g}) \chi_{\epsilon}^{-1}(h) \chi_{\epsilon}^{-1}(g) \chi_{\epsilon}(g) \\
&= \chi_{\epsilon}(g) \frac{1}{|G|} \sum_{h \in G} f(x_{h+g}) \chi_{\epsilon}^{-1}(h+g) \\
&= \chi_{\epsilon}(g) \mathcal{F}(x) \\
&= g \cdot \mathcal{F}(x).
\end{aligned}$$

Similarly to the Borsuk-Ulam example 5.3.3, having $\mathcal{F}$ be equivariant and the dimensions of X and $\mathbb{C}^{dm}$ matching guarantees there is some $x \in X$ such that $\mathcal{F}(x) = 0$. [11]

Example 5.4.2. Let $f : \partial \Delta^n \to \mathbb{C}^d$, where $n = (2d+1)(q-1)$, meaning we aim to kill $m = q - 1$ Fourier coefficients. If the test map $\mathcal{F} : X \to \mathbb{C}^{d(q-1)}$ has a zero, then there is some

$x = (x_0, \ldots, x_{q-1}) \in X$ such that the $q-1$ Fourier coefficients $c_1, \ldots, c_{q-1}$ are zero. By 5.1.1, this tells us the points $x_0, \ldots, x_{q-1}$ form a Tverberg partition. If there does not exist some $x \in X$ such that $\mathcal{F}(x) = \mathbf{0}$, then there is no collection of points $x_0, \ldots, x_{q-1}$ that form a Tverberg partition.

6
Tverberg-type Results

6.1 The Theorem

Theorem 6.1.1. *Let $m = q-2$ and $n = 2dm+(q-1) = 2d(q-2)+(q-1) = (2d+1)(q-2)+1$. Let $f : \partial\Delta^n \to \mathbb{C}^d$ be a continuous function. Then there exists a collection of q points $x_1 \in \sigma_1, \dots, x_q \in \sigma_q$ from disjoint faces $\sigma_1, \dots, \sigma_q$ of $\partial\Delta^n$ such that either*

(1)$f(x_1) = \cdots = f(x_q)$ or

(2)$f(x_1), \dots, f(x_q)$ form a regular q-gon.

Remark 6.1.2. Note that if f is affine and maps the vertices of Δ^n to points that are in strong general position, then we need only consider (2). If f is affine, then this will be the case with probability one. [10]

Proof. If f is continuous, then it is possible to get a full Tverberg partition with $N < T(q, 2d)-1$, being case (1). However, if we do not get a full Tverberg partition, then our disjoint faces will contain points that, rather than intersect, form a regular q-gon, as in case (2).

Suppose there does not exist a collection of q points $x_1 \in \sigma_1, \dots, x_q \in \sigma_q$ from disjoint faces $\sigma_1, \dots, \sigma_q$ of $\partial\Delta^n$ such that $f(x_1) = \cdots = f(x_q)$. Let $\{x_g\}_{g\in G}$ be a collection of q points in from disjoint faces $\sigma_1, \dots, \sigma_q$ of $\partial\Delta^n$ indexed by $G = \mathbb{Z}_q$, with q prime. Let $f = (f_1, \dots, f_d) : \partial\Delta^n \to \mathbb{C}^d$ be a continuous function. This gives way to the collection of maps $F_1, \dots, F_d : G \to \mathbb{C}$ defined by $F_i(g) = f_i(x_g)$ for all $g \in G$. We also get the standard Fourier decomposition:

$$F_i(g) = \sum_{\epsilon \in G} c_{(i,\epsilon)} \chi_\epsilon(g)$$

where

$$c_{(i,\epsilon)} = \frac{1}{|G|} \sum_{v \in G} f_i(x_v) \chi_\epsilon^{-1}(v) \in \mathbb{C}.$$

For f to admit a full Tverberg partition, we needed to find some collection of points that made f was constant. We did that by having the number of Fouerier coefficients be $m = q - 1$. In order to have a collection of points that form a polygon, we will need one term that varies between points, so we will want to add another Fourier coefficient back in. We will always be setting all Fourier coefficients to zero except $c_{(i,0)}$ and $c_{(i,1)}$ for all $i \in \{1, \ldots, d\}$. Thus we want $m = q - 2$.

By example 5.1.6, we know that the vanishing polynomial will be non-zero in $\mathbb{Z}[y]/q\mathbb{Z}$. Hence by Theorem 5.1.2 there exists a collection of q points $(x_0, \ldots, x_{q-1})$ such that the Fourier coefficients $c_{(i,\epsilon)} = 0$ for all $\epsilon \in \{2, \ldots, q-1\}$ and all $i \in \{1, \ldots, d\}$.

Suppose we set all Fourier coefficients equal to zero except $c_{(i,0)}$ and $c_{(i,1)}$. Then

$$\begin{aligned} F_i(g) &= c_{(i,0)} + c_{(i,1)}\chi_1(g) + \cdots + c_{(i,q-1)}\chi_{q-1}(g) \\ &= c_{(i,0)} + c_{(i,1)}\chi_1(g) + 0 + \cdots + 0 \\ &= c_{(i,0)} + c_{(i,1)}\chi_1(g) \\ &= c_{(i,0)} + c_{(i,1)}\xi_q^g \end{aligned}$$

for all $i \in \{1, \ldots, d\}$. Observe that $\chi_1(g) = \xi_q^{1 \cdot g} = \xi_q^g$. Therefore $F_i(g) = c_{(i,0)} + c_{(i,1)}\chi_1(g) = c_{(i,0)} + c_{(i,1)}\xi_q^g$ for all $i \in \{1, \ldots, d\}$. Thus

$$\begin{aligned} F(g) &= (F_1(g), \ldots, F_d(g)) \\ &= (c_{(1,0)} + c_{(1,1)}\xi_q^g, \ldots, c_{(d,0)} + c_{(d,1)}\xi_q^g) \\ &= (c_{(1,0)}, \ldots, c_{(d,0)}) + \xi_q^g(c_{(1,1)}, \ldots, c_{(d,1)}) \\ &= c_0 + c_1\xi_q^g, \end{aligned}$$

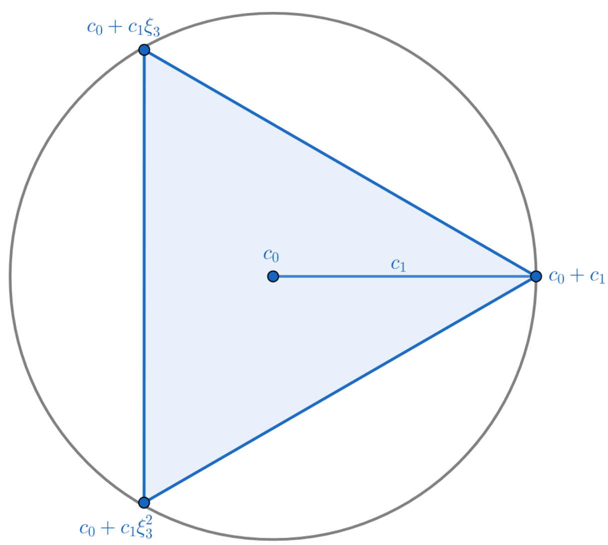
where $c_0 = (c_{(1,0)}, \dots, c_{(d,0)})$ and $c_1 = (c_{(1,1)}, \dots, c_{(d,1)})$ are constants in $\mathbb{C}^d$.

Then the average of F for all $g \in G$ is

$$\begin{aligned}\frac{F(0)+F(1)+\cdots+F(q-1)}{q} &= \frac{(c_0+c_1)+(c_0+\xi_q c_1)+\cdots+(c_0+\xi_q^{q-1}c_1)}{q}\\ &= \frac{qc_0+(1+\xi_q+\cdots+\xi_q^{q-1})c_1}{q}\\ &= \frac{qc_0}{q}\\ &= c_0.\end{aligned}$$

Hence the points $\{F(g)\}_{g\in G}$ are centered at c_0. Because each $F(g)$ is of the form $c_0 + c_1\xi_q^t$, we know that the distance from c_0 will always be the length of the vector c_1. Finally, the ξ_q^t changes the angle of the point with respect to c_0, but they are all powers of the same root of unity, so the points will be equidistantly distributed around c_0. Hence they will form a regular q-gon.

Example 6.1.3. Let $q = 3$ and $d = 1$. Then the points $F(0), F(1), F(2)$ are configured as such:



Notice how all three points lie on a circle centered at c_0 with radius c_1. Because the points lie on all of the third roots of unity (scaled by c_1), then they form an equilateral triangle.

Similarly, with q points the image will result in a regular q-gon. Thus the collection of points $\{f(x_g)\}_{g\in G}$ also form a regular q-gon. $\square$

Definition 6.1.4. Let $f : \partial\Delta^n \to \mathbb{C}^d$ is a continuous function with a collection of q points $x_1 \in \sigma_1, \ldots, x_q \in \sigma_q$ from disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\partial\Delta^n$ such that $f(x_1), \ldots, f(x_q)$ form a reqular q-gon. Then f admits a **q-gon partition**.

The following corollary is for the affine case.

Corollary 6.1.5. *If we have $n + 1 = (2d + 1)(q - 2) + 2$ points in $\mathbb{C}^d$, then we can partition them into q disjoint sets whose convex hulls have points that form a regular q-gon.*

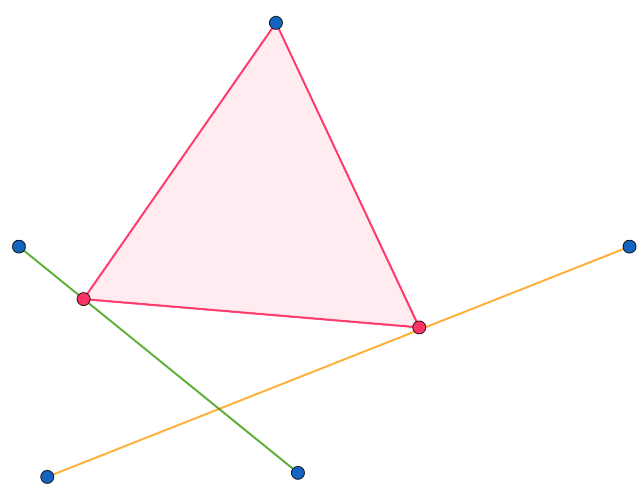
Proof. This relates to Theorem 6.1.1 similarly to how the Topological Tverberg theorem relates to Tverberg's theorem. □

6.2 Examples

Example 6.2.1. Let $d = 1$ and $q = 3$, meaning $G = \mathbb{Z}_3$. Then $n = 2d(q - 2) + (q - 1) = 2(3 - 2) + (3 - 1) = 2 + 2 = 4$. Let $f : \partial\Delta^4 \to \mathbb{C}$ be a continuous function. Because we have 3 Fourier coefficients, we need only set c_2 to zero. Observe the vanishing polynomial

$$h(y) = \Pi_{j=1}^{1}(\epsilon_j y)^d = 2(y) \equiv 2y$$

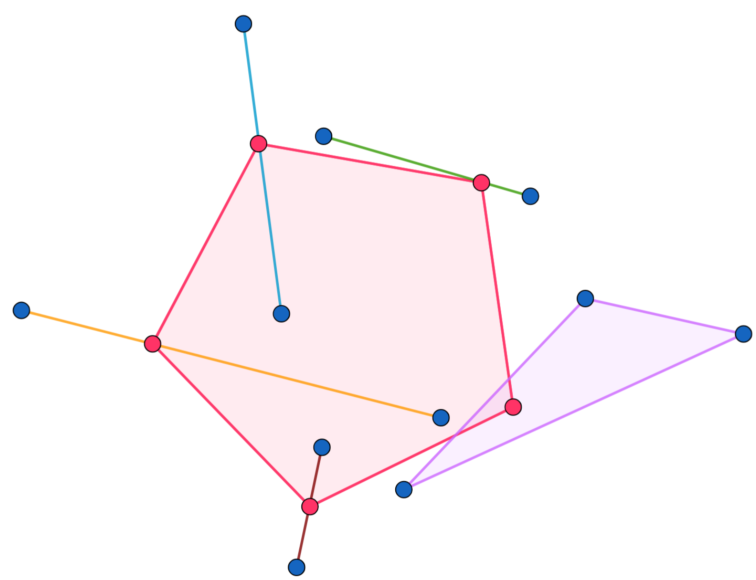
is non-zero in $\mathbb{Z}[y]/3\mathbb{Z}$. Then there exist 3 points x_1, x_2, x_3 from disjoint faces $\sigma_1, \sigma_2, \sigma_3$ of $\partial\Delta^4$ such that x_1, x_2, x_3 form an equilateral triangle. Thus for any $n+1 = 5$ points in $\mathbb{C} \cong \mathbb{R}^2$, we can partition them into three disjoint sets whose convex hulls contain points that form an equilateral triangle.



Example 6.2.2. Let $d = 1$ and $q = 5$, meaning $G = \mathbb{Z}_5$. Then $n = 2d(q-2) + (q-1) = 2(5-2) + (5-1) = 6 + 4 = 10$. Let $f : \partial\Delta^{10} \to \mathbb{C}$ be a continuous function. Note that we must set the Fourier coefficients c_2, c_3, c_4 to zero. Observe the vanishing polynomial

$$h(y) = \Pi_{j=1}^{3}(\epsilon_j y)^d = (2y)(3y)(4y) = 24y^3 \equiv 4y^3$$

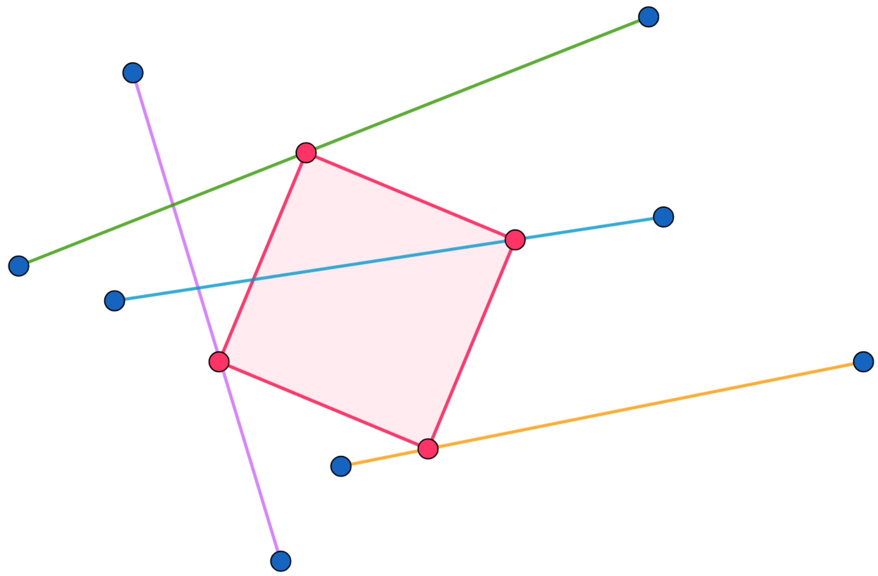
is non-zero in $\mathbb{Z}[y]/5\mathbb{Z}$. Then there exist 5 points $x_1, \ldots, x_5$ from disjoint faces $\sigma_1, \ldots, \sigma_5$ of $\partial\Delta^{10}$ such that $x_1, \ldots, x_5$ form a regular pentagon. Thus for any $n + 1 = 11$ points in $\mathbb{C} \cong \mathbb{R}^2$, we can partition them into five disjoint sets whose convex hulls contain points that form a regular pentagon.



Example 6.2.3. Let $d = 1$ and $q = 4$, meaning $G = \mathbb{Z}_4$. Then we will set c_2 and c_3 to zero. Observe that the vanishing polynomial

$$h(y) = 2y \cdot 3y = 6y^2 \equiv 2y^2$$

is nonzero in $\mathbb{Z}/4\mathbb{Z}$, despite 4 not being prime. This is a bit of a fluke, which happens because the polynomial is too short for the coefficient to reach a pair of zero divisors, so it does not vanish. Note that $n = 2(1)(4-2) + (4-1) = 7$. Let $f : \partial\Delta^7 \to \mathbb{C}$ be a continuous map. Then there exist 4 points $x_1, \ldots, x_4$ from disjoint faces $\sigma_1, \ldots, \sigma_4$ of $\partial\Delta^7$ such that $x_1, \ldots x_4$ form a square. Therefore for any $n + 1 = 8$ points in $\mathbb{C} \cong \mathbb{R}^2$, we can partition them into 4 disjoint sets whose convex hulls contain points that form a square.



7
Van Kapmen-Flores

7.1 Tverberg Unavoidable

Here we will be forcing our points that form our Tverberg partitions to come from certain faces on the simplex, namely the k-skeleton, which is the set of faces on the simplex of dimension at most k. We will do so by adding a constraint function to our continuous map, following a method developed by Frick in [5]. We can say that the continuous map $f : \partial\Delta^n \to \mathbb{C}^d$ admits a q-fold Tverberg partition if there are q points $\{x_g \in \sigma_g\}_{g\in G}$ from disjoint faces σ_g of the simplex $\partial\Delta^n$ such that $f(x_1) = \cdots = f(x_q)$. Similarly, when we add a distance constraint $g : \partial\Delta^n \to \mathbb{R}$, the collection of points will also have equal distance from the k-skeleton, meaning $g(x_1) = \cdots = g(x_q)$. Hence if that distance is zero for one point, it will be zero for all of them. Our goal is to find the smallest $k \in \mathbb{N}$ so that we can force our q points to come from the k-skeleton.

Lemma 7.1.1 (Frick)**.** [5] *Let $q \geq 2$ be a prime power, let $d \geq 1$ and $c \geq 0$. Let $N \geq N_c := (q-1)(d+1+c)$ and let $f : \Delta^N \to \mathbb{R}^d$, $g : \Delta^N \to \mathbb{R}^c$ be continuous. Then there exist q points $x_1, \ldots, x_q$ of disjoint faces $\sigma_1, \ldots, \sigma_q$, where $f(x_1) = \cdots = f(x_q)$ and $g(x_1) = \cdots = g(x_q)$.*

Proof. Since $f : \partial\Delta^N \to \mathbb{R}^d$ and $g : \partial\Delta^N \to \mathbb{R}^c$ both admit full Tverberg partitions, then the continuous function $h : \partial\Delta^N \to \mathbb{R}^d \oplus \mathbb{R}^c = \mathbb{R}^{c+d}$ defined by $h(x) = (f(x), g(x))$ also admits a full Tverberg partition. Then the dimension of the simplex is $N = (q-1)[(d+c)+1]$. □

Remark 7.1.2. For each constraint function applied, we increase the simplicial dimension N by $q-1$.

The dimension c is the number of constraints being applied to the Tverberg partition. Since we will be adding one distance constraint function, then $c = 1$. If we increase the dimension of our codomain, we must increase the dimension of our simplex so the dimension of our domain and codomain match, as in example 5.3.5.

Let $g(x) = \mathrm{dist}(x, \Delta^N_{(k)})$ be the distance function from a point x to the k-skeleton. Then Lemma 7.1.1 tells us that from our increased collection of points, not only will there be q points that intersect in the image, but these q points also have equal distance from the k-skeleton. Note that because the k-skeleton is closed and bounded, if $g(x) = \mathrm{dist}(x, \Delta^N_{(k)}) = 0$, then x lies on the k-skeleton. Hence if one point lies on the k-skeleton, we can ensure that all of them lie on the k-skeleton.[5]

Definition 7.1.3. [7] A **simplicial complex** is a nonempty collection of simplices Δ such that

(1) The face of any simplex in Δ is also in Δ

(2) The intersection of any two simplices σ_1, σ_2 in Δ is a face of both σ_1 and σ_2.

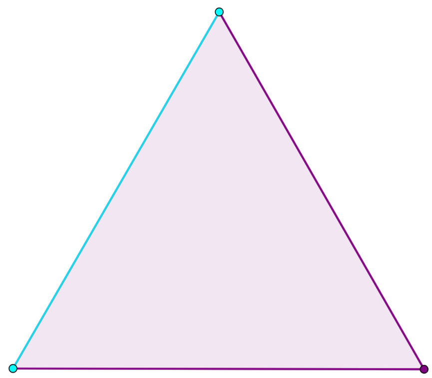
Given a simplex Δ^N, a simplicial complex of Δ^N is a collection of faces that are closed under intersection.

Definition 7.1.4. [7] A **subcomplex** Σ of a simplicial complex Δ is a subset of Δ that is itself a simplicial complex.

Definition 7.1.5. [5] Let $q, d, N \in \mathbb{Z}$ such that $q \geq 2$, $d \geq 1$, and $N \geq q-1$. A subcomplex $\Sigma \subseteq \Delta^N$ is **Tverberg unavoidable** if for every Tverberg partition $\{\sigma_1, \ldots, \sigma_q\}$ of Δ^N there exists some $j \in \{1, \ldots, q\}$ such that $\sigma_j \subseteq \Sigma$.

That is, a subcomplex $\Sigma \subseteq \Delta^N$ is Tverberg unavoidable if it is big enough in our simplex Δ^N so that every q-fold Tverberg partition has some disjoint face σ_i that lies in Σ.

Example 7.1.6. Let $q = 2$ and $N = 2$. Let Σ be the blue 1-simplex shown below.



We cannot pick the interior to be a disjoint face because faces are necessarily closed, while the interior is necessarily open. Then there are two types of faces to choose from: the vertices and the edges. Note that when we consider the edges, they contain the vertices at either end. Hence if we want to avoid the blue edge and vertices, we must pick the purple vertex across from it. However, now there are no other faces that are disjoint from the purple vertex and are not blue. Hence Σ is Tverberg unavoidable.

Lemma 7.1.7. [5] *Let $d, q, N \in \mathbb{Z}$ be such that $d \geq 1$, $q \geq 2$, and $N \geq q - 1$. Let $k \in \mathbb{N} \cup \{0\}$. If $q(k+2) > N + 1$, then the k-skeleton $\Delta^N_{(k)} \subseteq \Delta^N$ is Tverberg unavoidable.*

Proof. Suppose $q(k+2) > N + 1$. Suppose the k-skeleton $\Delta^N_{(k)}$ is not Tverberg unavoidable. Then every disjoint $\sigma_1, \ldots, \sigma_q$ has a dimension of at least $k+1$. Because every $k+1$-dimensional face has $k+2$ vertices, and all of the q faces are disjoint, then our simplex Δ^N would need at least $q(k+2)$ vertices. Then $N + 1 \geq q(k+2)$, which is a contradiction. Thus the k-skeleton is Tverberg unavoidable. $\square$

Lemma 7.1.8. [5] *Let $q, d, N \in \mathbb{Z}$ such that $q \geq 2$ is prime, $d \geq 1$, and $N \geq (q-1)(d+2)$. Let $f : \Delta^N \to \mathbb{R}^d$ be a continuous function that admits a q-fold Tverberg partition. Suppose* *that the subcomplex $\Sigma \subseteq \Delta^N$ is Tverberg unavoidable. Then there exist q pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of Δ^N, all contained in Σ, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.*

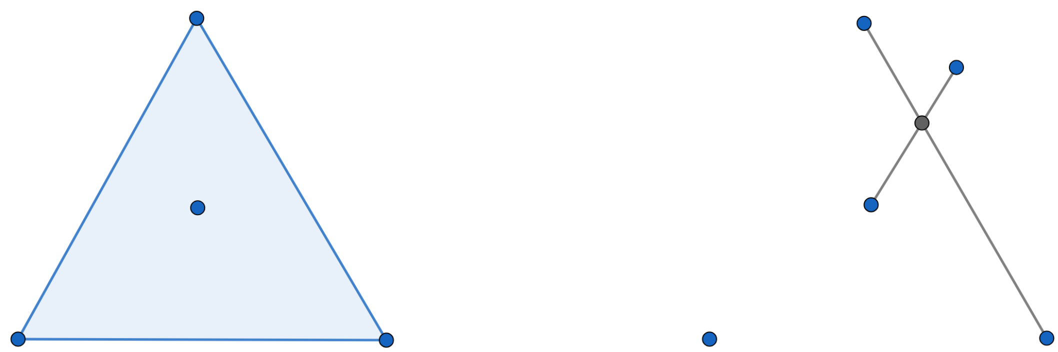
Proof. Note that if we can show this for $N = (q-1)(d+2)$, it will definitely work for larger values of N. Without loss of generality, let $N = (q-1)(d+2) = (q-1)(d+1+1)$. Suppose $\Sigma \subseteq \Delta^N$ is Tverberg unavoidable. Let $g : \Delta^N \to \mathbb{R}$ assign each point $x_i \in \Delta^N$ to its distance from the subcomplex Σ. Note that g is continuous. Then by Lemma 7.1.1 there exist q points $x_1, \ldots, x_q$ of disjoint faces $\sigma_1, \ldots, \sigma_q$ such that $f(x_1) = \cdots = f(x_q)$ and $g(x_1) = \cdots = g(x_q)$. Thus f admits a Tverberg partition, and the distance from x_i to Σ is equal for all $i \in \{1, \ldots, q\}$. Because Σ is Tverberg unavoidable, there exists some $i \in \{1, \ldots, q\}$ such that $g(x_i) = 0$. Hence $x_1, \ldots, x_q$ all lie in Σ. Thus the disjoint faces $\sigma_1, \ldots, \sigma_q$ are all in Σ, and $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$. $\square$

7.2 Van Kampen-Flores

Theorem 7.2.1 (van Kampen-Flores)**.** [5] *Let $d \in \mathbb{N}$ be even. Then for every continuous map $f : \Delta^{d+2} \to \mathbb{R}^d$ there are two disjoint faces σ_1, σ_2 of Δ^{d+2} such that σ_1, σ_2 have dimensions of at most $\frac{d}{2}$ and $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.*

Proof. Note that $q(k+2) = 2(\frac{d}{2}+2) = d+4 > d+3 = (d+2)+1 = N+1$. Hence by Lemma 7.1.7, the $\frac{d}{2}$-skeleton is Tverberg unavoidable. Then by Lemma 7.1.8 there exist σ_1, σ_2 contained in the $\frac{d}{2}$-skeleton $\Delta^{d+2}_{(\frac{d}{2})}$ such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$. Note that since $\sigma_1, \sigma_2 \subseteq \Delta^{d+2}_{(\frac{d}{2})}$, then the dimensions of σ_1 and σ_2 are at most $\frac{d}{2}$. $\square$

Example 7.2.2. Let $f : \Delta^4 \to \mathbb{R}^2$ be a continuous function. Then by Theorem 7.2.1, there exist two disjoint faces σ_1, σ_2 of Δ^4 such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$, where σ_1, σ_2 have at most dimension $\frac{2}{2} = 1$. Hence if we have $N+1 = 5$ points in $\mathbb{R}^2$, then 4 of them can be partitioned into 2 disjoint sets with at most $(\dim \sigma_1 + 1) = 2$ points whose convex hulls intersect. Originally, we needed $d+2 = 4$ points in $\mathbb{R}^2$ to ensure any intersection (left), but the additional point allows us to restrict the size of each partition (right).



Theorem 7.2.3 (Generalized van Kampen-Flores)**.** [5] *Let $q \geq 2$ be a prime power, let $d \geq 1$, let $N \geq (q-1)(d+2)$, and let $k \geq \left\lceil \frac{q-1}{q} d \right\rceil$. Then for every continuous map $f : \Delta^N \to \mathbb{R}^d$, there are q pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of Δ^N, where $\dim \sigma_i \leq k$ for all $i \in \{1, \ldots, q\}$, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.*

Proof. Note that if we show this for $N = (q-1)(d+2)$, then it will definitely be true for greater N. Without loss of generality, suppose $N = (q-1)(d+2)$.

Recall $k \geq \left\lceil \frac{q-1}{q} d \right\rceil$. This is because we want our collection of qk points to be in strong general position, so in order to ensure a full Tverberg partition we cannot have $k < \left\lceil \frac{q-1}{q} d \right\rceil$. Note that the codimension for each disjoint subset is $d-k$. Since there are q of them, the the sum of the codimensions of our subsets is $q(d-k)$. We want this to be equal to the codimension of our intersection, by the definition of strong general position. Because the intersection of our q disjoint faces will be nonempty, as it will admit a full Tverberg partition, then the codimension of the intersection of our faces will be less than or equal to d. Thus

$$\begin{aligned}
\sum_{i=1}^{q} (d - \dim \sigma_i) = d - \dim \bigcap_{i=1}^{q} \sigma_q &\leq d \\
q(d-k) &\leq d \\
d(q-1) &\leq qk \\
\frac{d(q-1)}{q} &\leq k.
\end{aligned}$$

Because k must be an interger, then $k \geq \left\lceil \frac{d(q-1)}{q} \right\rceil$.

Observe

$$\begin{aligned} N =&(q-1)(d+2) = (q-1)\left(\frac{d}{q}\cdot q+2\right) = \frac{d(q-1)}{q}\cdot q + 2(q-1) \\ &\leq \left\lceil \frac{d(q-1)}{q} \right\rceil \cdot q + 2(q-1) = kq + 2(q-1) < qk + 2q = q(k+2). \end{aligned}$$

Then by Lemma 7.1.7, we know the k-skeleton $\Delta^N_{(k)}$ is Tverberg unavoidable. Hence by Lemma 7.1.8, there exist $\sigma_1, \dots, \sigma_q$ of Δ^N, all contained in the k-skeleton $\Delta^N_{(k)}$, such that $f(\sigma_1) \cap \dots \cap f(\sigma_q) \neq \emptyset$. Note that since all disjoint σ_i are contained in the k-skeleton, they have at most dimension k. Hence $\dim \sigma_i \leq k$ for all $i \in \{1, \dots, q\}$. □

7.3 Results

When looking at Tverberg partitions, we increased the dimension of our simplex to add in more points, q of which formed a Tverberg partition. Similarly, when looking at q-gon partitions we need to increase the dimension of the simplex by $q-1$ for every constraint function appliex, so the dimensions match, as in remark 7.1.2. Since we will only be adding one distance constraint function, then $N = [2d(q-2) + (q-1)] + (q-1) = 2d(q-2) + 2(q-1)$.

Definition 7.3.1. Let $q, d, N \in \mathbb{Z}$ such that $q \geq 2, d \geq 1$, and $N \geq q-1$. Let $f : \partial\Delta^N \to \mathbb{R}^d$ be a continuous map with at least one q-gon partition. Then the subcomplex $\Sigma \subseteq \Delta^N$ is ***q*-gon unavoidable** if for every q-gon partition $\sigma_1, \dots, \sigma_q$ there exists some $j \in \{1, \dots, q\}$ such that $\sigma_j \subseteq \Sigma$.

Theorem 7.3.2. *Let $q, k, d, N \in \mathbb{Z}$ be such that $q \geq 2$ is prime, $k \geq 1$, $d \geq 1$, and $N = 2d(q-2)+2(q-1)$. Let $f : \partial\Delta^N \to \mathbb{C}^d$ be a continuous function. If $2d(q-2) \leq qk < 2d(q-1)$, then there exist q points $x_1, \dots, x_q$ from pairwise disjoint faces $\sigma_1, \dots, \sigma_q$ of the k-skeleton $\Delta^N_{(k)} \subseteq \Delta^N$ such that $f(x_1), \dots, f(x_q)$ form a regular q-gon.*

Proof. For this to be true, we not only need the k-skeleton to be q-gon unavoidable, but also need N to not be so large that it reaches the Tverberg number $T(q, 2d)$, or our $f(x_1), \dots, f(x_q)$ could intersect, rather than forming a regular q-gon.

Suppose $2d(q-2) \leq qk < 2d(q-1)$. Observe

$$N+1 = [2d(q-2)+2(q-1)]+1 \leq qk+2(q-1)+1 = q(k+2)-1 < q(k+2).$$

Then by Lemma 7.1.7, the k-skeleton of Δ^N is q-gon unavoidable.

We also want to make sure N+1 is still less than the Tverberg number $T(q, 2d)$. We need each point to come from disjoint faces of dimension at most $k+1$ to ensure they come from the k-skeleton. Hence the disjoint faces $\sigma_1, \ldots, \sigma_q$ have a total dimension of at most $q(k+1)$, with $x_1 \in \sigma_1, \ldots, x_q \in \sigma_q$. Because we want this to be less than the Tverberg number, then $q(k+1) < (2d+1)(q-1)+1$, which is true if and only if $qk < (2d+1)(q-1)+(1-q) = 2d(q-1)$.

Hence if $2d(q-2) \leq qk < 2d(q-1)$, then there are q points $x_1, \ldots, x_q$ from disjoint faces $\sigma_1, \ldots, \sigma_q$ of the k-skeleton $\Delta^N_{(k)}$ such that $f(x_1), \ldots, f(x_q)$ form a regular q-gon. $\square$

We want to minimize k, meaning we want to find the k that makes qk closest to $2d(q-1)$ while still being larger or equal to it. Note

$$2d(q-2) \leq qk$$
$$2d\left(\frac{q-2}{q}\right) \leq k.$$

Hence the smallest value of k that makes this inequality hold is $k = \left\lceil 2d\left(\frac{q-2}{q}\right)\right\rceil$. Since k is as small as it can be, it will mostly still hold for the inequality $qk < 2d(q-1)$.

Corollary 7.3.3. *Let $q, k, d, N \in \mathbb{Z}$ be such that $q > 2$ is prime, $k \geq 1$, $d \geq 1$, and $N = 2d(q-2)+2(q-1)$. Let $f : \partial\Delta^N \to \mathbb{C}^d$ be a continuous function, and let $k = \left\lceil \frac{2d(q-2)}{q} \right\rceil = \frac{2d(q-2)+r}{q}$. If $r < 2d$, then there exist q points $x_1, \ldots, x_q$ from pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of the k-skeleton $\Delta^N_{(k)} \subseteq \Delta^N$ such that $f(x_1), \ldots, f(x_q)$ form a regular q-gon.*

Proof. Suppose $r < 2d$. Observe

$$2d(q-1) \leq 2dq \leq \frac{2d(q-2)}{q}q \leq \left\lceil \frac{2d(q-2)}{q} \right\rceil q = qk$$

and

$$qk = \left\lceil \frac{2d(q-2)}{q} \right\rceil q = \left[\frac{2d(q-2)+r}{q} \right] q = 2d(q-2)+r < 2d(q-2)+2d = 2d(q-1).$$

Hence $2d(q-1) \leq qk \leq 2d(q-1)$, so by Theorem 7.3.2, we know there exist q points $x_1, \dots, x_q$ from pairwise disjoint faces $\sigma_1, \dots, \sigma_q$ of the k-skeleton $\Delta^N_{(k)} \subseteq \Delta^N$, where $k = \left\lceil \frac{2d(q-2)}{q} \right\rceil$, such that $f(x_1), \dots, f(x_q)$ form a regular q-gon. □

7.4 Examples

Corollary 7.4.1. *Given $q^2 - q + 1$ points in $\mathbb{C}^{\frac{d-1}{2}}$, there exist $q(q-1)$ points that can be partitioned into q disjoint sets of $q-1$ points such that the convex hulls of each disjoint set contains a point x_i where $x_1, \dots, x_q$ form a regular q-gon.*

Topologically, this says that if $f : \partial\Delta^N \to \mathbb{C}^{\frac{d-1}{2}}$ is a continuous function with $N = q^2 - q$, then there exist q disjoint faces $\sigma_1, \dots, \sigma_q$ of dimension at most $q-2$ containing points $x_1, \dots, x_q$ such that $f(x_1), \dots, f(x_q)$ form a regular q-gon.

Proof. Let $k = q - 2$ and $d = \frac{q-1}{2}$. Observe

$$
\begin{aligned}
N + 1 &= 2d(q-2) + 2(q-1) + 1 = (q-1)(q-2) + 2(q-1) + 1 = q(q-1) + 1 \\
&= (q-1+1)(q-1) + 1 = (2d+1)(q-1) + 1 = T(q, 2d).
\end{aligned}
$$

Note

$$
\begin{aligned}
2d(q-2) &= 2\left(\frac{q-1}{2}\right)(q-2) = (q-1)(q-2), \\
qk &= q(q-2) = q^2 - 2q, \text{ and} \\
2d(q-1) &= 2\left(\frac{q-1}{2}\right)(q-1) = (q-1)^2 = q^2 - 2q + 1.
\end{aligned}
$$

Observe $(q-1)(q-2) \leq q(q-2) = q^2 - 2q < q^2 - 2q + 1$. Then by Theorem 7.3.2, there exist $q(q-1)$ points of the original $q(q-1)+1$ points that form a q-gon partition, coming from q disjoint sets with at most $q-1$ points in them.

Thus any continuous function $f : \partial\Delta^N \to \mathbb{C}^{\frac{q-1}{2}}$ contains q disjoint faces $\sigma_1, \dots, \sigma_1$ from the $(q-1)-1 = q-2$-skeleton with points $x_1 \in \sigma_1, \dots, x_q \in \sigma_q$ such that $f(x_1), \dots, f(x_q)$ form a regular q-gon.

Observe that for the affine case the number of points $N+1$ is the Tverberg number $T(q, 2d)$. The number of points that are in our q disjoint partitions is $N = T(q, 2d) - 1$. $\square$

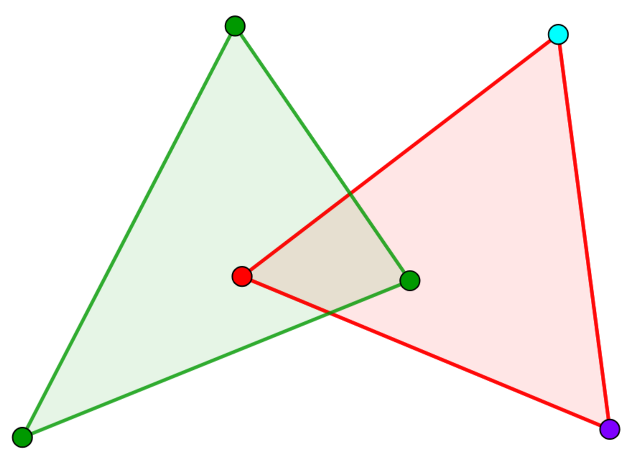
Example 7.4.2. Let $q = 3, d = 1$. Then $N = 2d(q-2) + 2(q-1) = 2 + 4 = 6$. We want some k such that $2 \leq 3k < 4$. Hence $k = 1$.

We could also use Corollary 7.3.3 to find k. Observe

$$k = \left\lceil \frac{2(q-2)}{q} d \right\rceil = \left\lceil \frac{2}{3} \cdot 1 \right\rceil = 1.$$

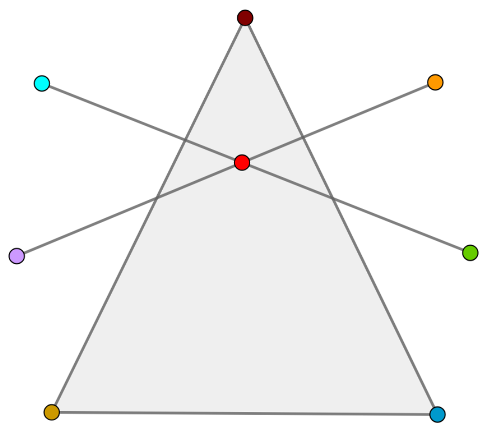
Thus for any continuous function $\partial\Delta^6 \to \mathbb{C}$, there exist 3 points x_1, x_2, x_3 from pairwise disjoint edges $\sigma_1, \sigma_2, \sigma_3$ such that $f(x_1), f(x_2), f(x_3)$ form an equilateral triangle.

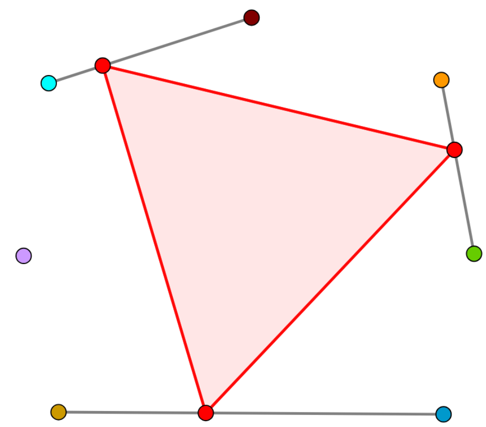
Normally, when $q = 3$ and $d = 1$, we have $N'+1 = 2d(q-2)+(q-1)+1 = 2+2+1 = 5$ points in the plane. Below is an example when we have 5 points in the plane, and one of our partitioned sets has more than $k+1 = 2$ points in it. Here the green, blue, and purple vertices are our 5 points, and the red triangle is equilateral. Note that the green partition has $3 > 2 = k+1$ points in it.



By Corollary 7.4.1, if we have $N+1 = 7$ points in the plane, we can find $q(k+1) = q(q-1) = 3 \cdot 2 = 6$ of them such that each of our 3 disjoint sets will have at most $k+1 = 2$ points in them, and there will be a point from each disjoint set that forms the vertices of an equilateral triangle.

Note that $N+1 = 7$ is the Tverberg number $T(3, 2)$. Below on the left we have a full Tverberg partition with 7 points, and on the right we have a q-gon partition from the same points, where our 3 disjoint sets union to a total of 6 points. Notice that all of the vertices of the equilateral triange lie on a line, meaning each disjoint set has at $k + 1 = 2$ points.





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