Early College Research Institute

Bard College at SIMON'S ROCK

Early College Folio

A Growth MindSTEM for Next Gen

Volume 3, Issue 1 | 2024

Early College Folio

Volume 3, Issue 1 | 2024

Writing in the Zones for the Reading of Proofs in the Mathematics Classroom: A How-To Guide

Miha E. Habič
Bard Academy and Bard College at Simon's Rock

Kaethe Minden
Bard Academy and Bard College at Simon's Rock

Abstract

We present a guide on how to use Writing in the Zones, a particular kind of Writing and Thinking activity, to support active student engagement with mathematical texts, and the reading of mathematical proofs specifically. Proofs in textbooks can often be opaque, and students commonly rely on direct teacher intervention in the classroom (i.e. lecturing) rather than engaging in a fruitful struggle with the text itself. We also include some specific examples of this approach and student outcomes that we observed.

1 Introduction to Writing in the Zones

The established college-level mathematics curriculum in the United States begins with several semesters of Calculus courses and then transitions, at a point which is difficult to pin down, to the core "pure math" courses such as Real Analysis, Abstract Algebra, etc. The transition is often facilitated by either a Linear Algebra class or a designated course designed to introduce students to mathematical proof and various proof methods.[1] While these courses place a lot of emphasis on writing proofs (by design), in our experience they reserve a lot less time for *reading* proofs, and mathematical texts in general.

The reading of a mathematical proof or, on a larger scale, a textbook or paper presents unique challenges precisely because mathematical writing is precise and often terse to a fault. Depending on the specific text, motivation may be sparse or completely omitted and the inexperienced reader is left with more questions than they had before they started reading. While in retrospect that is a

[1]At Bard College at Simon's Rock, the course is called *Introduction to Mathematical Proof*; at Bard College it is *Proofs and Fundamentals*; at UMass Amherst it is *Fundamental Concepts of Mathematics*. As the reader can see, there are infinitely many different names for the same idea.

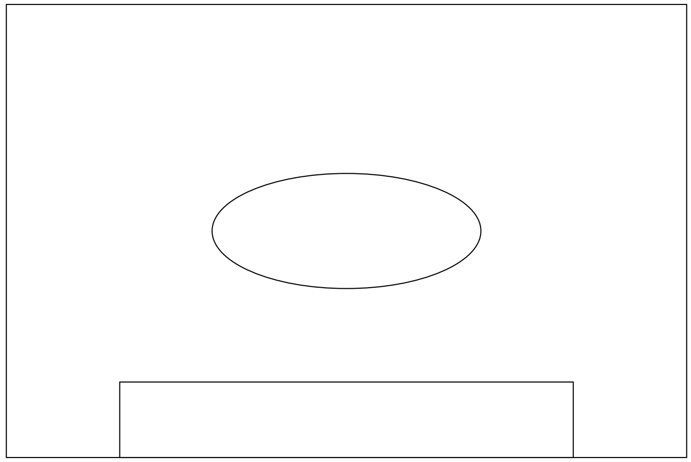

Figure 1: Layout before separating into zones.

great position to be in, it can be very frustrating in the moment, as a number of our students have reported.

In this paper we present our use of a method from Writing and Thinking pedagogy to address this issue and to help students persist in their struggle with mathematical texts and extract insight and ideas from the process.

2 Writing in the Zones: The Activity

Writing and Thinking pedagogy is based on teaching practices developed at the Institute for Writing and Thinking (IWT) at Bard College. The key idea is that students (or participants of any kind) engage with the material (either written, visual, audible, etc.) in writing. These methods are used throughout the Bard Network, and incoming students at Bard College at Simon's Rock spend the week before classes begin exploring this approach.

Writing in the Zones is a Writing and Thinking activity incorporating a number of writing prompts with a graphically organized structure. The organizations starts with an oval in the center and a rectangle at the bottom of the page (see fig. 1). Usually as instructors we hand out blank sheets of paper of large format for the students to use for the activity, so that it may be submitted after the class for some comments and to make sure the activity was completed by each student. An alternative conclusion is for students to hang their finished products on the walls of the classroom, and then inspect, consider, and comment on each other's work (a so-called *gallery walk*).

The remaining space of the page is subdivided into several zones. The default option is to use rays emanating from the central oval to the edges of the page, however students are free to divide in any way they prefer, as long as each zone takes up an approximately equal amount of space. The number of zones can change, depending on the kind of analysis a given proof or theorem calls for, or on how much time is available (although, in our experience, this activity works best when a whole hour class can be devoted to it). Students can use the back side of the page for their writing as well (although this can interfere with sharing later). They are also free to draw pictures and diagrams as needed.

The following is a generic list of prompts one can start from before adapting it to a specific theorem and proof to be read.[2]

Center Theorem name with reference (author, book, theorem number).

Zone 1 — First thoughts Write down your initial reaction to the statement of the theorem. Is there anything that sticks out to you? Include the definition of any technical words in the statement that seem important.

Zone 2 — Pointing Select a striking sentence, phrase, or image from any part of the text. Fill up the zone with writing about it or from it.

Zone 3 — Analysis/Close Reading Pick an excerpt from the proof that seems important or confusing. Why is this passage important? What is confusing about it? How might you approach explaining it to someone or attempt to understand it yourself?

Zone 4 — Believing and Doubting Go back to the statement of the theorem. What structure does the author use to prove the statement? Is this how you would have approached the proof? How might one disprove the statement?

Zone 5 — Making Inferences What question is this theorem answering? Why is it given at this point in the course or textbook?

Zone 6 — Summarizing What are the steps the proof goes through?

Zone 7 — Evidence What types of evidence does the author give in the proof? Does it rely on other theorems or lemmas? Are there any diagrams or examples? Are these helpful?

Zone 8 — Making Connections What other proofs or theorems does this one remind you of? Have you ever seen this type of reasoning used?

Rectangle Read what you wrote. What's the most important or central thing you're noticing or saying about this proof? What are you still curious about?

We give more concrete sample scripts in appendices A and B.

The amount of time devoted to each zone should be about equal (except for the Center and Rectangle zones, which can be relatively quick). This might mean that students struggle to fill the time for "easy" prompts (Zones 2 or 7, for example), and cannot quite fit everything they want to for more complex prompts (Zones 3 or 5). This tension can be frustrating at first, but becomes a source of focus if the activity is repeated several times throughout the semester.

[2] The zones are based on an internal handout from the IWT.

The overall goal of this structure, as we wrote above, is for students to engage with mathematical content directly through writing: examining a text, writing out their observations and questions on paper, and then engaging with those questions in the external forum. Our specific intent in using this activity in a math class, though, was to present the students a mathematical challenge and then remove the "panic button" of asking the professor as soon as something nontrivial pops up.

Researchers in mathematics (and other scholarly disciplines) are very familiar with the feeling of being stuck on something, and our hope was to introduce students to that feeling while also teaching them that a primary source (in this case their textbook) can have information that isn't apparent on first glance and that a closer, informed reading can yield new results. We have noticed a steady decline in students' willingness and ability to consult their textbooks or sources apart from their professors when studying and solving problems. Judging from students' responses, this comes from the fact that the explanation in a textbook or a paper is not necessarily uniquely tailored to the particular student. It seems that this is less of an issue in the humanities and the social sciences, where, it seems, the students do gain the experience of having to struggle with a particular text to gain deeper insight into the author's intended message. We hope that by including Writing and Thinking activities (such as Writing in the Zones) in mathematics classes (or STEM classes in general) we can show our students that those same skills are useful in the sciences as well, and help them be self-reliant.

3 Writing in the Zones: Reflections

3.1 Scaffolding the activity

This activity could fit into many math classrooms (for example, in a class dedicated to the writing of mathematical proofs, as we described above, or a course that already requires students to do some writing, in homework assignments or in class). Being a relatively complex Writing and Thinking task, this activity should not be the students' first encounter with writing in the mathematics classroom.

We have found it to be good practice to scaffold this type of engagement with mathematical text with simpler activities earlier in the semester (for example, group reading of a proof, with a *heat map*[3] activity after; or a *text expansion*[4].

Another long-term scaffolding strategy is to devote consistent time each class session (perhaps 10 minutes, starting with the first class) to "warm-up" writing

[3] Students underline words or phrases that seem important in the text. The leader guides the second reading, and students join in for the parts that they underlined.

[4] Students underline words or phrases which seem important or confusing. Afterwards, each student picks an underlined piece and writes a definition, question, or other thought addressing why they underlined that specific part. Finally, the leader guides the second reading, and once they get to a part a student chose, the reading stops and that student reads their writing; the reading of the original text then proceeds.

prompts given on the board. The questions can simply be about reciting definitions or theorems from the previous class, or trying to find mathematical examples. Students' products can sometimes be collected and given feedback on and other times briefly shared by the students. This gets the students used to the process of writing and sharing their writing in the course.

There is a variety of other introductory activities one can use to scaffold towards the more complex Writing in the Zones. Some examples can be found in [BCC13, SM06, Urq09], specifically including freewriting and journal writing. Although many available sources focus on K-12 education, the activities can be adapted to the college setting without too much difficulty. We would particularly like to point to [SM06] for their description of the time commitment necessary for incorporating writing into a math classroom, and the positive outcomes observed.

Students may have difficulty coming up with enough things to say to fill in the zones. Sometimes it is helpful to give examples (we give a sample of a completed activity in appendix C), but it helps if the students have some experience with Writing and Thinking instruction already. Repeating the activity several times over a number of weeks also helps students find strategies to express themselves.

For especially complicated proofs, it makes sense to ask the students to read the proof beforehand. One time this activity was led into by a homework assignment which asked the students to copy down, verbatim, the statement and proof directly from the textbook and bring it to class. Students were allowed to type the proof out as well (although this can cause some technical difficulties for students who aren't used to using equation editors or LaTeX).

3.2 Sharing and feedback

The sharing part of the activity at the end is important and should not be skipped. It can also be structured as pairs of students exchanging their products and summarizing each other's insight. This means the activity might spill into another class period, which may be longer than many of us would like to spend on one theorem. On the other hand, particularly involved theorems can take more than one lesson day anyway, at least for the students to feel its significance and truly attempt to understand it. So while it is rare that we can spend so much time on one theorem and its proof, the benefits are worthwhile, in our opinion.

It is also helpful to give feedback to the students on their activity, which is why it can be good to collect the students' work. One could also ask the students to produce a piece of writing (summarizing the proof more formally). Students can sometimes focus predominantly on teacher feedback and the activity's contribution to their grade. We recommend reducing the teacher feedback component over several repetitions of the activity, to nudge students towards relying on the feedback of their peers.

3.3 Student response and outcomes

A response we often receive after debuting this activity with a new class is that it is drawn-out and takes up too much time (especially if they are already confused about the content of the theorem). We have found it helpful to reserve some time to talk to the students about how this kind of activity is quite reflective of the process we actually go through as working mathematicians while reading a scholarly paper. We are convinced this is true for many fields in one way or another, and gives the students a template/framework for organizing their thoughts, which can be used (either explicitly or implicitly) in their life as a reader and learner.

Concrete outcomes for the use of Writing and Thinking activities, and Writing in the Zones in particular, are difficult to measure, and it is not always clear what, if any, impact an activity had on a student and their development. Nevertheless, students respond positively to a break in the routine of classes. Moreover, since we usually use this activity for more challenging theorems, it can be a good opportunity for students to review and connect a wide variety of material from earlier in the semester. For example, the activity described in appendix B was based on a reading of a proof of Urysohn's lemma. Students needed to know about dyadic rationals, apply lemmas from earlier in the book, and finish parts of the proof that the book did not finish. This was moderately successful. Students spent more time on the theorem than we would normally expect them to, and surely gained something in the process.

The activity is very open-ended, and intentionally so, which can lead to the students feeling stuck. Are they doing it right? Is this what I was looking for? This ties back to our earlier point about teacher feedback (which ideally the students would be slowly weaned off of over time) and about scaffolding (Writing in the Zones should not be the first Writing and Thinking activity the students are exposed to). We do give students some possible responses if they ask (see also appendix C), but find that it is great to encourage as much freedom and diversity in approach as possible. The guiding principle, which should be spelled out in the beginning, is that the goal of the activity is for students to be able to articulate their questions and the points they are stuck on, and then, hopefully, to be able to resolve those questions, either individually or by working together.

MIHA HABIČ is a faculty member in Mathematics at Bard College at Simon's Rock. His research involves set theory and other areas of mathematical logic. He is particularly interested in how infinite objects behave differently from finite ones, and what analogies can be drawn between infinite objects of different sizes.

KAETHE MINDEN started teaching mathematics at Bard College at Simon's Rock as an assistant professor in 2019. Before Simon's Rock, Kaethe taught at Marlboro College as a mathematics fellow and also taught mathematics courses while a graduate student at The Graduate Center of the City University of New York (CUNY). She received her PhD in Mathematics from CUNY in 2017 and her BS in

Mathematics from the University of California, Los Angeles in 2010. Her research lies in the fields of logic and set theory—particularly large cardinals and forcing. Some recent work with undergraduate students connects to combinatorial group theory, involving infinite latin squares.

A Sample Script: Introduction to Proofs

The following is a script for a Writing in the Zones activity to read the proof of the irrationality of $\sqrt{2}$.[5] This activity was done very early in the semester and the students weren't prompted to do any reading in advance.

Setup [3 min] Draw an oval in the center of the page and a rectangle at the bottom edge of the page. Then separate the remaining empty space into five zones, all about the same size. Number the zones.

Center [2 mins] Copy the theorem statement into the center zone.

Zone 1 [7 mins] What is your reaction to the statement of the theorem? Is it familiar? Include definitions of any terms and symbols that seem important.

Zone 2 [7 mins] Pick one or two sentences from the proof that seem important or confusing. What makes them so? How would you go about resolving the confusion?

Zone 3 [7 mins] What is the overall structure of the proof? Do you see any connections between the way the theorem is phrased and that structure?

Zone 4 [7 mins] How sensitive is the proof to the exact theorem? Could you use (essentially) the same proof to prove something else?

Zone 5 [7 mins] Are there any unspoken assumptions in the proof? Is the author relying on the reader to fill in any steps?

Rectangle [5 mins] Read over what you wrote. What is the central thing you are saying about this proof? What are you still curious about?

Share In turn, each student shares one of their zones with the rest of the class. After the share, take a minute or two to note if any of your questions that you had written down have now been answered or transformed.

B Sample Script: Topology

Urysohn's lemma is a fundamental result in point-set topology about separating closed subsets by functions.

[5] The textbook in the class was [Ham20]. The theorem in question is Theorem 1 of the book.

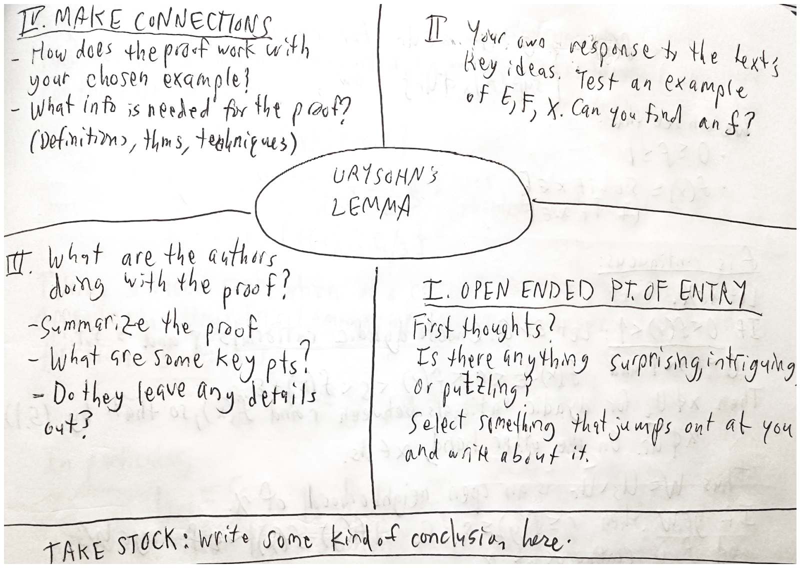

Figure 2: Activity for Urysohn's Lemma.

Urysohn's Lemma Let E and F be disjoint, closed subsets of a normal topological space X. There is a continuous function $f : X \to [0, 1]$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ 1 & \text{if } x \in F \end{cases}.$$

Figure 2 shows a diagram of the zones and prompts that was drawn on the board to guide students through the activity of reading the proof.[6]

Student responses were collected and looked over by the instructor, then handed back the next class. The topic for that class was Tietze's Extension theorem (which is a natural next step after Urysohn's lemma), and to debrief and connect the two topics, the following prompts were designed for the first half of the class:

1. Do you have any questions remaining about Urysohn's Lemma, after looking over your page from the last class?
2. Break up into groups and attempt to answer your questions. Determine which parts of the proof are missing, fill in the details.
3. Look over Tietze's Extension Theorem. How is it different from Urysohn's Lemma? Does the proof use Urysohn's Lemma?

[6]The textbook in the class was [GG99]. Urysohn's Lemma is Theorem 5.3 of the text.

C Sample Response: Real Analysis

One of the authors happened to be sitting in on the other's Real Analysis II[7] course on a day when the Writing in the Zones Activity was being implemented for a first introduction to the proof of the Inverse Function Theorem. We reproduce a brief version of it in fig. 3.

The Inverse Function Theorem Let $E \subseteq \mathbb{R}^n$ be open. Let $f : E \to \mathbb{R}^n$ be a function which is continuously differentiable on E. Suppose $x_0 \in E$ is such that the linear transformation $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there is an open set U in E containing x_0 and an open set V in $\mathbb{R}^n$ containing $f(x_0)$ such that f is a bijection from U to V. In particular, there is an inverse map $f^{-1} : V \to U$. Furthermore, this inverse map is differentiable at $f(x_0)$ and

$$(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}.$$

References

[BCC13] Ali Bicer, Robert M Capraro, and Mary M Capraro, *Integrating writing into mathematics classroom to increase students' problem solving skills*, International Online Journal of Educational Sciences **5** (2013), no. 2, 361–369.

[GG99] Theodore W. Gamelin and Robert Everist Greene, *Introduction to topology*, 2nd ed., Dover Publications, 1999.

[Ham20] Joel David Hamkins, *Proof and the art of mathematics*, MIT Press, 2020.

[SM06] Bethany Seto and David E Meel, *Writing in mathematics: Making it work*, PRIMUS: problems, resources, and issues in mathematics undergraduate studies (2006), no. 3, pp. 204–232.

[Tao16] Terence Tao, *Analysis II*, 3 ed., Texts and Readings in Mathematics, Hindustan Book Agency, 2016.

[Urq09] Vicki Urquhart, *Using writing in mathematics to deepen student learning.*, Mid-continent Research for Education and Learning, 2009.

[7] The textbook used for the class was [Tao16]. The Inverse Function Theorem is Theorem 17.7.2 of the text.

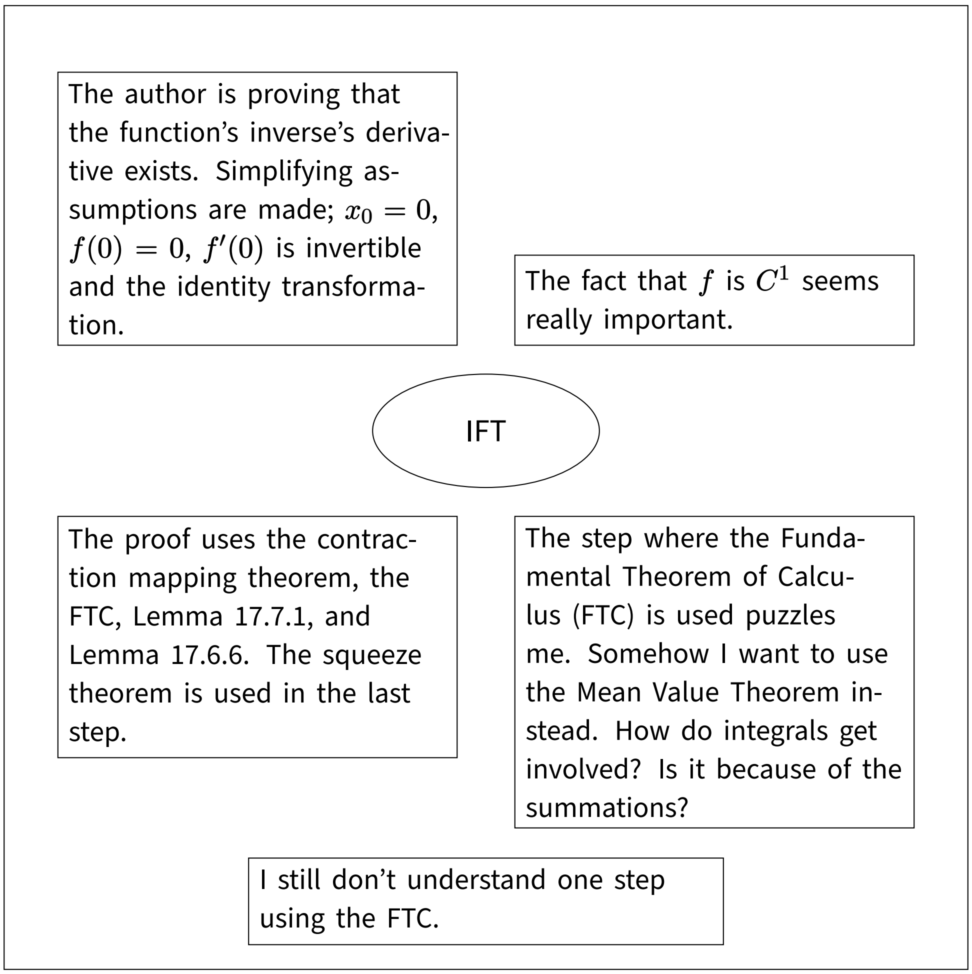

Figure 3: An initial response to the proof of the Inverse Function Theorem using the activity.

