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The Logic of Internal Relations

Jake S. Cardillo
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The Logic of Internal Relations

Senior Project submitted to
The Division of Social Studies
of Bard College

by

Jake S. Cardillo

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I hold that logic is what is fundamental in philosophy,
and that schools should be characterized rather by their logic
than by their metaphysic.
—Bertrand Russell, "Logical Atomism," 1924

Introduction

Bertrand Russell initiated one of the formative battles of analytic philosophy at the turn of the 20th century with a wholesale assault on the headquarters of British monist philosophy: "the doctrine of internal relations."[1] In Russell's words, the doctrine says that "the fact that two objects have a certain relation implies….something in the 'natures' of the two objects, in virtue of which they have the relation in question."[2] From this doctrine alone, he alleged, every other doctrine of metaphysical monism follows—thus did he sometimes call it "the axiom of internal relations"[3]—including the doctrine that there is one and only one really existing entity, the Absolute, and the corollary that any proposition about any entity is ultimately a proposition about the Absolute. Of course, what it means for *relata* to stand in relations 'in virtue of their natures' is not at all obvious; Russell understood it to mean that every relation is reducible, in some sense, to properties of *relata*. One of the most important consequences of this doctrine, so understood, is that relations do not really exist in themselves (hence are 'internal' to *relata*), such that the doctrine of internal relations is a variety of anti-realism about relations. An 'external' relation, by contrast, is a relation which does exist in itself, and in which *relata* stand independent of their 'natures.'

In the main, Russell directed his criticisms against the monism of F.H. Bradley, probably the most influential of the so-called 'British Hegelians,' whose 1893 master-

[1] Russell, "The Basis of Realism," 160.
[2] Russell, "The Monistic Theory of Truth," 160-1.
[3] Russell, "The Monistic Theory of Truth," 160.

work *Appearance and Reality* was the impetus for Russell's anti-monist crusade. Russell's criticisms are also self-directed, since early in his philosophical career, he adhered to a Bradley-inspired form of monism. But there is more at stake in the dispute than whether or not relations exist, or whether or not there are many entities as opposed to just one. Russell believed that metaphysical doctrines are held on the basis of logical doctrines, and specifically on the basis of doctrines about *logical form*. Already before Gottlob Frege introduced the modern predicate calculus, Augustus De Morgan had observed that Aristotelian term logic is not equipped for an adequate treatment of relational propositions and their implications.[4] Russell argued that the source of the inadequacy is that the analysis of propositions into subject-predicate form is fundamentally mistaken, and reveals itself as such especially in the case of relational propositions. However, subject-predicate analysis, according to Russell, is not only logically inadequate, it is metaphysically noxious, because the inadequacy of the subject-predicate form when applied to relational propositions leads the traditional logician to attempt a reduction of propositions about relations to propositions about monadic properties, such that relations themselves become superfluous. This may, Russell allows, work for some relational propositions, but when applied to propositions about ordering relations like '*x* is less than *y*' or '*x* is a part of *y*,' the subject-predicate analysis fails to capture the asymmetry property. In the face of defeat, the monist's only options are to admit to the irreducibility of external relations, or to commit to untenable metaphysical consequences. Russell's insight, then, is that the

[4] Tarski, "On the Calculus of Relations," 73. Tarski refers to "[De Morgan's] famous aphorism, that all the logic of Aristotle does not permit us, from the fact that a horse is an animal, to conclude that the head of a horse is the head of an animal," and credits "his effort to break the bonds of traditional logic and to expand the limits of logical inquiry" by "his attention to the general concept of relations."

metaphysical doctrine about relations one accepts depends on the logical analysis of relational propositions one accepts: the metaphysical dispute between monists and anti-monists is really a dispute over the correct logical form of propositions, and propositional analysis is the correct method of resolving the metaphysical dispute.

I will argue, though, that Russell's insight does not convey the full import of the disagreement between Russell and Bradley, which originates not in the difference between two types of propositional analysis, but in a deeper methodological conflict over what logic is and how it can and should be applied in metaphysical reasoning. To that end, I will present four distinct logical methods of metaphysical reasoning, each employed by a distinct metaphysical philosopher. In part 1, I will lay out in detail the procedures of Bradley's method, a dialectical approach to conceptual analysis that I call 'intelligibility testing.' In particular, I will examine Bradley's method as applied to the concepts 'object,' 'property,' and 'relation' to show that Bradley did not accept anything like the doctrine of internal relations in the form depicted by Russell. Moreover, I will show that intelligibility testing permits Bradley to reject the existence of both internal *and* external relations, and thereby show that Bradley reaches his anti-realism about relations on the basis of the intelligibility failure of the concept 'relation,' and not on the basis of a subject-predicate analysis of relational propositions.

In part 2, I will examine a critique of Russell's logic by one of Bradley's fellow-travelers, Bernard Bosanquet, which will help characterize how the British monists understood the role of logic in metaphysics, and especially what logic needs to be capable of in order to occupy such a role. Bosanquet criticizes Russell's formal system for its insensitivity to propositional content, advancing two arguments: first, that the material con-

ditional, which features prominently in the system, licenses ridiculous premises and conclusions in conditional reasoning, and second, that the axiomatization of mathematical theories in the system obscures distinctions in subject matter between distinct theories. Bosanquet prefers to think in terms of propositional systems that can be ranked according to their 'logical stability.' A propositional system is logically stable when it meets a pair of rationality conditions which make it eligible for metaphysical reasoning about some domain of actual content in the world. Bradley's intelligibility tests can be incorporated into Bosanquet's propositional systems to give a general picture of the British monist conception of logic and its application in metaphysics.

In part 3, I will present Russell's arguments against the doctrine of internal relations, and demonstrate how and in what sense he thinks this doctrine follows from a subject-predicate analysis of relational propositions. Then I will demonstrate how and in what sense Russell's "doctrine of external relations,"[5] which says that some relations are external, follows from his own analysis of relational propositions, the 'propositional function' analysis. Additionally, it will become clear that Russell's position on internal relations is not as far from Bradley's as he would have us believe.

In part 4, I will show that Russell's ally in the anti-monist struggle, G.E. Moore, who agreed with Russell about the existence of external relations, in fact only agreed with Russell in a superficial sense, because he defined the terms 'internal' and 'external' in a radically different way than Russell did. Moore provided a substantive critique of the metaphysical limitations of Russell's logic; in particular, he claimed that the material conditional was incapable of properly formulating the doctrine of internal relations, and

[5] Russell, "The Basis of Realism," 160.

as an alternative introduced an 'entailment' connective into Russell's system, which is logically indistinguishable from necessary implication. This lets the system express modal versions of the principles of the Indiscernibility of Identicals and the Identity of Indiscernibles, which Moore used to formulate the doctrine of internal relations such as to define 'internal' to mean 'necessary' and 'external' to mean 'contingent.' I point to some comments of Russell's which indicate that he would not have agreed with Moore's definitions, and also note that mathematical ordering relations, for Russell an exemplar of external relations, come out internal on Moore's definition. Moreover, I look at some arguments Russell made against modal formulations of the doctrine of internal relations and confirm Moore's claim that material implication hampered Russell's ability to argue effectively against these forms of the doctrine. In sum, Russell and Moore are neither arguing against the same doctrine of internal relations, nor arguing for the same doctrine of external relations, and thus any agreement between them is essentially hollow.

Finally, I will conclude that Moore's modal method is the right one for reasoning about internal and external relations, especially when outfitted with a 'possible worlds' interpretation. I offer a modal definition according to which ordering relations are external, and which does not succumb to any of the difficulties that Russell identified. While this definition does not justify the doctrine of internal relations, it does, in my opinion, justify the concept 'internal relation' for the purpose of making informative statements about the 'natures' of entities. The focus on logical methods of metaphysical reasoning in this paper is meant to illustrate that the metaphysical disagreement over internal and external relations is, at root, a conflict in logical methodology, thereby generalizing Russell's insight beyond the case of propositional analysis.

1. Intelligibility Tests: Bradley

Bradley's logical method in *Appearance and Reality* is to propose a formulation of some intuitive metaphysical concept which we apply in our everyday experience to understand reality, and then submit it to a test of its 'intelligibility.' The test procedure begins with a conceptual formulation F_C of a concept C, and F_C passes the test so long as it does not lead to contradiction, circularity, regress, or nonsense (any logical deficiency will do). If F_C fails, the following step is to test the intelligibility of its negation F^*_C, which passes or fails under the same conditions as F_C. Since F_C and F^*_C are mutually exclusive, if F_C passes, then F^*_C fails, and vice versa; however, the failure of one does not guarantee that the other passes. If either F_C or F^*_C passes, then C passes, but if both F_C and F^*_C fail, then C fails. If C fails, modify F_C to an alternative formulation F'_C and start again. If C passes, it would, presumably, be permissible to use C in a metaphysical system; it could, for instance, occur in the definitions of other concepts. The objective of an intelligibility test is to determine "how without error we may think of reality,"[6] and even though Bradley admits that some erroneous concepts may be "necessary in practice," they are nonetheless "theoretically unintelligible."[7] That is, we may not be able to get along in everyday experience without a concept C, but Bradley's intelligibility test for C determines whether or not we are justified employing it in metaphysical reasoning.

Bradley begins by testing our most intuitive metaphysical concept, the 'property-bearing object.' Since "[w]e find the world's contents grouped into things and their qualities,"[8] we might naturally think of the property-bearing object O in one of two mutually

[6] Bradley, *Appearance and Reality*, 20.
[7] Bradley, *Appearance and Reality*, 21.
[8] Bradley, *Appearance and Reality*, 16.

exclusive ways: as *just* a set of properties (F_O), or *not* just a set of properties (F^*_O). F_O says that O is the set of properties jointly sufficient to constitute O; F^*_O says it is an additional unit to which the properties belong. Beginning with F_O, Bradley observes that the object is not a set of properties "if you take them each severally,"[9] since if we try to predicate a property of such an object, we either predicate a property of itself, or we predicate a property of another property, and "[w]e certainly do not predicate one of the other; for, if we attempt to identify them, they at once resist."[10] Take the object constituted by the properties P_1 (black), P_2 (white), and P_3 (red). Predicating P_1 of P_1 is trivial (black is black); meanwhile, predicating P_1 of P_2 is false, since they are different properties (white is not black). However, if we try to predicate a property of the *set* of properties, then we either predicate that property of a set which already contains it, in which case we say nothing about the object—"if the predicate makes no difference, it is idle"[11]––or we add another property to the set, in which case we enlarge the object, but we say nothing about the object we started with. That is, predicating P_1 of the set $\{P_1, P_2, P_3\}$ is either trivial, because it simply lists an element of the set (black is a member of {black, white, red}), or false, because P_1 is a member of the set and is not identical with it (black is not {black, white, red}). And predicating a property P_4 (read all over) of $\{P_1, P_2, P_3\}$ generates $\{P_1, P_2, P_3\}$; P_4 ({black, white, red, read all over}), in which case it predicates P_4 of $\{P_1, P_2, P_3, P_4\}$ and not of $\{P_1, P_2, P_3\}$, the original object of predication. And then P_4 has the problem P_1 had: it is either trivially or falsely predicated of $\{P_1, P_2, P_3, P_4\}$.

[9] Bradley, *Appearance and Reality*, 16.
[10] Bradley, *Appearance and Reality*, 17.
[11] Bradley, *Appearance and Reality*, 17.

Predication, then, seems to require an additional unit which we can predicate properties of, so that we are not just predicating properties of properties or of sets of properties. The intelligibility failure of F_O transitions immediately to the test of F^*_O, a conceptual formulation positing something *more* than just a set of properties. But F^*_O fails as well, because "if….we inquire what there can be in the thing beside its several qualities, we are baffled once more."[12] We can identify no bare object without properties to serve as the additional unit, as without its properties an object is unidentifiable—thus "[w]e can discover no real unity existing outside these qualities, or, again, existing within them."[13] Since both F_O and F^*_O fail their intelligibility tests, O is unintelligible. The only way to preserve the concept of the 'property-bearing object' is to modify the formulation and continue the process of intelligibility testing, "to set [it] down as mere appearance"[14] and search for conceptions which we can apply intelligibly to reality.

The modification F'_O says that an object O is a *relation* of properties which "co-exist in a certain way"[15] so as to constitute O. The introduction of the concept 'relation' through modification is justified only if the concept passes its own intelligibility test—it must be legitimately introduced into the metaphysical system. There are, as before, two mutually exclusive conceptual formulations of 'relation': a relation R is either dependent on some property in virtue of which its *relata* stand in R (F_R), or it is independent of its *relata* and they stand in a pre-existing R (F^*_R). In Russell's terms, relations are either 'internal' or 'external.'

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[12] Bradley, *Appearance and Reality*, 16.
[13] Bradley, *Appearance and Reality*, 16.
[14] Bradley, *Appearance and Reality*, 21.
[15] Bradley, *Appearance and Reality*, 16.

Speaking of a relation $R(a, b)$, Bradley puzzles over what it might mean to say that *relata* stand in a relation in virtue of some property they have: "if you mean that [a] and [b], taken each severally, even 'have' this relation, you are asserting what is false. But if you mean that [a] and [b] in such a relation are so related, you appear to mean nothing."[16] In other words, if a and b *each* have a property in virtue of which they stand in R to each other, then a has that property and b has that property, so they stand in R independently of each other. And if a and b taken *together* have a property in virtue of which they stand in R to each other, that amounts to a and b having the property of standing in the relation R, which is the fact we are trying to understand in the first place. Take the relation 'a dances with b.' Presumably, a cannot dance with b, nor b with a, unless a and b dance together; but 'a and b dance together' is another way of expressing the relation 'a dances with b,' not a property in virtue of which they stand in that relation (or move around in that relation, as it were). Think of the two possible formulations of 'internal' as sub-formulations of F_R, so that F_R passes its intelligibility test just in case one of its sub-formulations passes. Then, because neither sub-formulation can meaningfully define what sort of property something has in virtue of which it stands in a relation, the formulation F_R fails its intelligibility test. This argument, whatever its merits, is Bradley's explicit rejection of the concept 'internal relation.'

It is in the process of testing F^*_R that Bradley first invokes his famous regress argument against the concept 'external relation.' I have tried to describe the procedures of Bradley's intelligibility tests in part because I want to show that his argument *against* 'external' relations does not imply an argument *for* 'internal' relations. Intelligibility tests

[16] Bradley, *Appearance and Reality*, 17.

allow Bradley to reject any conceptual formulation *and* its negation, since both are submitted to the same test and pass or fail under the same conditions. The tests can be continued through modification, so there is no need to settle in favor of a formulation or its negation; they are mutually exclusive but not mutually exhaustive.

The test of F^*_R assumes the negative result of the test of F_R, that "the relation [R] has been admitted different from [a] and [b], and no longer is predicated of them."[17] Still, "[s]omething….seems to be said of this relation [R], and said, again, of [a] and [b]. And this something is not to be the ascription of one to the other."[18] Even though R is now independent of any property of a or b, and thus, according to Bradley, is not *predicated* of the *relata*, something is still *said* about the relation R and the *relata* a and b. Because no property is predicated of a or b, and because a and b are genuinely related by R, Bradley concludes that there "would appear to be another relation, [R'], in which [R], on one side, and, on the other side, [a] and [b] stand."[19] What is said of R is that it relates a and b, and what is said of a and b is that they are related by R—so R stands in the relation of 'relating' (R') to a and b. This is the source of the regress: R' likewise must stand in yet another relation R'' to R and (a, b), and so on infinitely. F^*_R fails because the independently existing relation R is unable to relate a to b, as the relation $R(a, b)$ presupposes $R'(R, (a, b))$, and $R'(R, (a, b))$ presupposes $R''(R', (R, (a, b)))$, &c. And given the failure of both F_R and F^*_R, R is unintelligible.

[17] Bradley, *Appearance and Reality*, 18.
[18] Bradley, *Appearance and Reality*, 18.
[19] Bradley, *Appearance and Reality*, 18.

But Bradley doesn't dispense with the concept 'relation' immediately, concluding that "the problem….really turns on the respective natures of quality and relation."[20] Given that, we might expect Bradley to run tests on two separate and self-sufficient concepts, the concept 'property' and the concept 'relation,' respectively, but instead he wants to show that "[r]elation presupposes quality, and quality relation. Each can be something neither together with, nor apart from, the other."[21] A conceptual formulation F_P for the concept 'property' (P) will say that "[q]ualities are nothing without relations,"[22] that is, no property P exists without standing in some relation R. The test of F_P is a test of the *dependence* of P on R, and the test of F^*_P, which says that P indeed may exist without standing in any relation R, is a test of the *independence* of P from R.

Start with F_P, which says that a property P is dependent on some relation R, so that P would not exist if it did not stand in R. At this point, we might expect that Bradley is never going to let *any* conceptual formulation pass its intelligibility test, but here the strategy is a bit different: Bradley is going to argue that both F_P *and* F^*_P pass their intelligibility tests, so that the conceptual formulation of P *and* its negation both pass their intelligibility tests, resulting in the intelligibility failure of P. To show that F_P passes, Bradley argues that because properties are different from each other, and because difference is a relation, the very existence of properties depends on their standing in the relation of difference to each other:

> For consider, the qualities [F] and [G] are to be different from each other; and if so, that difference must fall somewhere. If it falls, in any degree or to any extent, outside [F] or [G], we have relation at once. But, on the other hand, how can difference and otherness fall inside? If we have in [F] any such otherness, then inside [F] we must

[20] Bradley, *Appearance and Reality*, 21.
[21] Bradley, *Appearance and Reality*, 21.
[22] Bradley, *Appearance and Reality*, 21.

> distinguish its own quality and its otherness. And, if so, then the unsolved problem breaks out inside each quality, and separates each into two qualities in relation.[23]

Note, first, that if properties were *not* different from each other, then they could not differentiate objects from each other—and consequently "all must fall into one."[24] It is because Bradley considers such a monistic consequence *undesirable* that he assumes that properties are different from each other, trivial though it is (we will see Russell accuse him of accepting precisely this monistic consequence). Assuming, then, that properties are different from each other, we can ask what differentiates them from each other: is it a relation of difference between properties, or a property of difference in the properties themselves? If the former, then difference is, of course, a relation, and F_P passes; if the latter, then difference is a property different from the property it differentiates, so "the unsolved problem" returns: what differentiates the property of difference from the property it differentiates? We are forced to conclude that what differentiates the property of difference from the property it differentiates is a relation of difference, so that F_P passes. If difference is a relation, and properties are differentiated by standing in a relation of difference, then properties are dependent on relations, and F_P passes; if difference is a property, then properties have a property which differentiates them and stand in a relation of difference to that property, so again properties are dependent on relations, and F_P passes; therefore, F_P passes.

The fact that F_P has passed, according to the intelligibility test procedure, guarantees the failure of F^*_P by default. We know that the failure of a conceptual formulation does not immediately result in its negation passing, because the failure of a concept lets us

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[23] Bradley, *Appearance and Reality*, 24.
[24] Bradley, *Appearance and Reality*, 25.

modify it in the hope that it will eventually pass and become usable in the system; a formulation and its negation are not mutually exhaustive. But we also know that if a conceptual formulation passes, there is no need to test its negation, as the concept passes if one and only one formulation passes. So why test F^*_P at all? Notice that in a couple of the foregoing tests, specifically those of F^*_O and F^*_R, we assumed the *negative* results of the previous tests as premises. But since F_P has passed, we have acquired a *positive* result, and therefore if we assume the positive result in a test of F^*_P and derive some logical deficiency, F^*_P would pass—a *reductio* on F^*_P would be sufficient to show that F^*_P passes and, thus, that F_P fails. But since we saw that F_P passed its own test, and thus that F^*_P failed even prior to testing, F_P and F^*_P will have each both passed and failed, resulting in the intelligibility failure of *P*.

Assuming, then, the positive result of the test of F_P, that properties are dependent on relations, Bradley points out that this does not imply that properties are "resolved into the relations,"[25] since relations presuppose *relata*—"relations must depend upon terms, just as much as terms upon relations."[26] We saw that the relation on which a property depends is its relation of difference to other properties, and in this it is like anything which is different from any other thing: different things depend on the relation of difference. Now, if a property *P* both depends on and is depended on by a relation, *P* splits into something *a* on which the relation depends and something *a′* which depends on that relation. *P* is no longer one thing, but two different things, and since two different things stand in the relation of difference, *a* and *a′* stand in the relation of difference to each other. What troubles

[25] Bradley, *Appearance and Reality*, 25.
[26] Bradley, *Appearance and Reality*, 26.

Bradley is that "this diversity is fatal to the internal unity of each"[27]: if a and a' stand in the relation of difference to each other, then they are no better off than P. For instance, a is both something a'' on which its relation of difference depends, and it is yet another different thing a''' which depends on standing in that relation. Then, both a'' and a''' are subject to the same split, so they are "dissipated in an endless process of distinction."[28] Because the property P splits into a and a', and because a splits into a'' and a''', and so on indefinitely, we have to conclude that the assumption that properties depend on relations involves us in another regress. Thus, by assuming the positive result of the test of F_P, we assume the failure of F^*_P and thereby derive a regress, showing by *reductio* that the failure of F^*_P has absurd consequences, i.e., that F^*_P passes. So because F_P and F^*_P are mutually exclusive conceptual formulations of P, and because each passes its intelligibility test, P must be an unintelligible concept.

One might wonder in what sense Bradley's method is a 'logic.' It is, undoubtedly, a procedure for evaluating conceptual formulations and their permissibility in systems of metaphysical reasoning, but such systems are unlike the formal systems in which modern logicians operate. What, exactly, is the difference?

2. Logical Stability: Bosanquet

One of Bradley's contemporaries, Bernard Bosanquet, attempted to pinpoint the difference in his textbook *Logic, or the Morphology of Knowledge*. Between the publication of the first edition in 1888 and the second in 1911, Russell had published his *Principles of Mathematics*, as well as an article on type theory and the first volume of *Principia*

[27] Bradley, *Appearance and Reality*, 27.
[28] Bradley, *Appearance and Reality*, 26.

Mathematica (with Alfred North Whitehead). Bosanquet, apparently, felt compelled to respond to Russell's work in an appendix to the second edition of his *Logic* called "On the Relation of Symbolic Logic to the Theory of the Present Work," in which he counterposes Russell's "Formal Logic, construed as including pure Mathematics" to his own "Philosophical Logic" in "the tradition….of Aristotle and Plato."[29] Though Bosanquet admits that he cannot pretend to challenge Russell's formal logic on its own terms, he nonetheless advances a couple objections which help to characterize his own discomfort with "pure logical implication"[30] detached from "an underlying real system."[31]

His first objection concerns "Mr. Russell's extreme use of the hypothetical proposition in illustrating the meaning of implications,"[32] i.e., Russell's definition of the conditional as 'material implication.' Bosanquet is especially distraught by the validity of an implication like 'If a donkey is Plato, he is a great philosopher,' which in Russell's analysis becomes 'for any x, if x is a donkey and x = Plato, then x is a great philosopher.' If 'x is a donkey' is false when 'x = Plato' is true, the antecedent is a false conjunction guaranteeing the truth of the "hypothetical proposition," which admits an "illegitimate hypothesis"[33] and thereby "scatters….underlying reality to the winds."[34]

Bosanquet, of course, is far from the only logician in the history of philosophy to express reservations about material implication; e.g., the 'paradoxes of material implication' inspired the subsequent development of modal logic. But where that was an attempt to expand and reform symbolic logic to account for its alleged shortcomings, Bosanquet's

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[29] Bosanquet, *Logic*, 40.
[30] Bosanquet, *Logic*, 43.
[31] Bosanquet, *Logic*, 41.
[32] Bosanquet, *Logic*, 41n.
[33] Bosanquet, *Logic*, 44.
[34] Bosanquet, *Logic*, 41n.

"Philosophical Logic" is a radically different project which, as his first objection indicates, intends to capture the 'real' content 'underlying' logical implications. Unlike formal logic, which is occupied "not with facts, but with the mutual implications of propositions,"[35] philosophical logic should be able to represent reasoning about metaphysical reality and provide criteria for evaluating the 'real systems' in which such reasoning takes place. Because formal logic does not bother with the metaphysical content of its propositions, it can tolerate unrealistic assumptions and absurd conclusions.

Bosanquet's second objection is directed against Russell's view that "pure Mathematics is to consist….in logical deductions from logical principles,"[36] as again a system of reasoning which Bosanquet takes to be 'about' a particular class of actual entities has been replaced by a formal system which can be said to be 'about' whatever satisfies its abstract logical relations. Thus, "very great and serious differences between two sets of terms may be neglected (such as differences between points in a Euclidean plane and complex numbers) if only a common set of formal deductions can be found which apply to them."[37] This objection evokes historical anxieties about David Hilbert's axiomatization of Euclidean geometry, which interpreted the terms 'point' and 'line' to refer to whichever entities satisfy the axioms, and not, essentially, to points and lines. "[A]ctual space,"[38] according to Bosanquet, is the "underlying reality" grounding any geometrical theorems and deductions—geometry is *true* because it is the theory of empirically real space. The formalization of pure implications, like the axiomatization of pure geometries, threatens to deprive logic and mathematics of their essential subject matters, and thus of

[35] Bosanquet, *Logic*, 42.
[36] Bosanquet, *Logic*, 43.
[37] Bosanquet, *Logic*, 43.
[38] Bosanquet, *Logic*, 42.

any metaphysical salience. Indeed, if Russell appreciated this point, he would not be able to reduce mathematics to logic, as their respective contents would resist conflation.

Philosophical logic, dissatisfied as it is with 'pure implication,' is instead "interested in the conditions of logical stability," which Bosanquet defines as "that characteristic of a system of propositions which makes it free from self-contradiction and from contradiction with the rest of experience."[39] A propositional system is not just a set of propositions in an implication relation, but must observe, ideally, the conditions of formal and empirical consistency: the propositions are consistent with each other and with any possible evidence. "Its degree of non-liability to contradiction, internal or external, is its degree of logical stability,"[40] so that we can imagine a hierarchy of propositional systems, with those of the lowest degree of stability on the bottom and those of the highest degree on the top; the *most* logically stable system, even if only a regulative ideal, would be "the whole system of judgments which represents our organized experience."[41] If a proposition is introduced into the system which lowers its degree of stability, then an inference is justified on the grounds that it is "necessary to the avoidance of contradiction in the system."[42] Implications between propositions only serve to balance them against each other such as to maintain or improve the overall stability of the system. Propositions do not even have truth-values independently of their role in balancing the system; truth-values are capable of degrees, as "a proposition could be more or less true because of being more or less supported by other propositions," whereas in formal systems the truth-value

[39] Bosanquet, *Logic*, 45.
[40] Bosanquet, *Logic*, 46.
[41] Bosanquet, *Logic*, 46.
[42] Bosanquet, *Logic*, 46.

of a proposition is "a private affair of its own, not lying in its coherence or incoherence with a system."[43]

Such remarks, at least considered outside their context in Bosanquet's wider project, are exceedingly vague, but the idea seems to be that one begins with extremely simple propositions about "minima of experience" or "isolated data,"[44] such as observation reports, and then, when one such proposition contradicts another—since they are just observation reports, formal and empirical consistency will amount to the same thing—the system is balanced out by propositions which restore consistency. A propositional system consisting only of observation reports will be highly unstable, because it is easy for empirical observations to contradict each other: a system including the proposition 'it is cold' will become inconsistent upon the introduction of the proposition 'it is warm,' and this may lead to the introduction of propositions about, say, continuous changes in temperature. Of course, this does not much resemble a typical 'inference' or 'implication,' but its purpose is to enable the construction of increasingly comprehensive systems by progressively smoothing out logical instabilities. It is rather more like certain kinds of 'defeasible' implication, in which a proposition p defeasibly implies q just in case p offers support for q (in some sense), even if q is not a logical consequence of p. And though Bosanquet's logic has many of the standard features of a 'coherence theory,' the situation is complicated by the requirement of empirical adequacy. A propositional system attempts to systematize our experiential data, so a formally consistent system which conflicts with empirical facts will still be 'incoherent' in the sense Bosanquet intends.

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[43] Bosanquet, *Logic*, 47.
[44] Bosanquet, *Logic*, 47.

I have sketched Bosanquet's notion of 'logical stability' in the hope that it might serve as a context in which to understand Bradley's intelligibility testing. Where Bradley's logic tests the intelligibility of concepts and hence their permissibility in conceptual systems, Bosanquet's logic describes the conditions and degrees of logical stability for propositional systems, and how highly stable systems can be constructed from highly unstable systems. The objects of study are not, as Bosanquet stresses, formal proof systems, but metaphysical systems of the sort which a philosopher might construct, and the rationality conditions on such systems. Bradley's tests insure that the concepts a metaphysical system employs do not contradict themselves, and Bosanquet's degrees of stability measure the overall tendency to contradiction in the system. If a formal logic is inconsistent, it loses value as an object of study, but for the 'British Hegelians,' perhaps unsurprisingly, contradiction has a generative function. When a contradiction is detected in a conceptual formulation, a modified formulation is proposed to account for it so that progress can be made in conceptual analysis; when a proposition is introduced into a system to reconcile a contradiction, the overall degree of stability increases. This 'dialectical' aspect of philosophical logic is what makes it suitable for the study of metaphysical systems, the consistency or stability of which cannot be assumed if their concepts and propositions are meant to apply to empirical reality. Philosophical logic deals in the *content* of concepts and propositions because it is their content which superimposes, or fails to superimpose, on the world itself. A coherent propositional system employing only intelligible concepts is appropriate for metaphysical reasoning in the way that a formal system of implications is not: if a logical system ignores content, it will be of no use in defining metaphysical concepts and evaluating metaphysical propositions.

3. Propositional Analysis: Russell

Russell's earliest discussion of internal and external relations occurs during his investigation, in the *Principles of Mathematics*, into ordering relations, especially "transitive asymmetrical relations,"[45] or strict orders like the less-than relation **<**. Addressing "the philosophic dislike of relations," which induces philosophers to "hold that no relations can possess absolute and metaphysical validity,"[46] he places the blame squarely on an antiquated subject-predicate analysis of propositions, which, when confronted by a relational proposition, attempts to reduce the relation to properties of the *relata*, such that the relation depends on the *internal* 'natures' of the *relata* and is not an independently existing entity *external* to them. Russell is, here as elsewhere, a bit elliptical when tracing the origins of this metaphysical view to subject-predicate logic, but the intuition of the subject-predicate logician seems to be that, assuming that to each subject corresponds a substance, and to each predicate a property inhering in a substance, a relation is anomalous in that it seems to connect substances rather than inhere in them. So if a relational predicate is to correspond to something capable of inhering in a substance, a relation must be ontically dependent on properties of *relata*, not a medium existing in its own right.

There are, accordingly, two strategies of reducing relations to the properties of *relata*, the first of which Russell calls 'monadistic,' the second 'monistic.' The monadistic reduction analyzes the relational proposition $R(a, b)$ "into two propositions, which we may call ar_1 and br_2, which give to a and b respectively adjectives supposed to be to-

[45] Russell, *Principles*, 220.
[46] Russell, *Principles*, 221.

gether equivalent to R."[47] In other words, a and b each receive their own predicate, r_1 and r_2 respectively, each corresponding to a property which, when taken together with the other, conditions the relation R between a and b. The monistic reduction "regards the relation as a property of the whole composed of a and b, and as thus equivalent to a proposition which we may denote by $(ab)r$,"[48] where r is predicated of both a and b taken together. If these two definitions of 'internal' sound familiar, it is because they very closely resemble the two sub-formulations of Bradley's conceptual formulation F_R, both of which Bradley rejects: the monadistic view on the grounds that reducing a relation to properties of the separate *relata* does not account for the fact that they are *connected* by the relation, and the monist view on the grounds that reducing a relation to a property of the *relata* taken together accounts *only* for the trivial fact that they *are* connected by the relation. So it might be something of a surprise when Russell attributes the monistic view to "Spinoza and Mr. Bradley"![49]

Russell wants to show that an internalist reduction stumbles especially on the analysis of asymmetrical relations. Plausibly, certain symmetrical relations are reducible to the properties of *relata*. Take the symmetrical relation '*a* differs in color from *b*,' and say that *a* is green and *b* is yellow. The monadist reduces the relational proposition with a single relational predicate into two propositions with monadic predicates: '*a*(green)' and '*b*(yellow).' The predicate 'green,' predicated of *a*, does not refer to *b*, and the predicate 'yellow,' predicated of *b*, does not refer to *a*, so the predicate 'differs in color' does not

[47] Russell, *Principles*, 221.
[48] Russell, *Principles*, 221.
[49] Russell, *Principles*, 221.

seem to correspond to anything which spans more than one substance, and the relation can be said to depend on properties of the individual *relata*.

However, when it comes to an asymmetrical relation, like '$a < b$,' the relational predicate '$<$' cannot be analyzed into '*a*(less than)' and '*b*(not less than),' as clearly the proposition says that *a* is not only less than *something*, it is less than *b* (and we also cannot rule it out that *b* is less than something, if not *a*). So the proposition must resolve into the two propositions '*a*(less than *b*)' and '*b*(not less than *a*),' in which case the predicate 'less than *b*,' predicated of *a*, refers to *b*, and the predicate 'not less than *a*,' predicated of *b*, refers to *a*. However, "if [*a*] has an adjective corresponding to the fact that it is [less than *b*], this adjective is logically subsequent to, and is merely derived from, the direct relation of [*a*] to [*b*],"[50] so the allegedly monadic predicates are tacitly non-monadic, and cannot correspond to properties each predicated of only one *relatum*. Russell concludes that "the attempted analysis of the relation fails, and we are forced to admit what the theory was designed to avoid, a so-called 'external' relation, i.e. one implying no complexity in either of the related terms."[51] When the monadist attempts to resolve an asymmetrical relational predicate into two monadic predicates, each predicated solely of an individual *relatum*, each predicate inevitably refers to the other *relatum* of which it is not predicated, thereby presupposing an irreducible asymmetrical relation.

The monistic reduction does not fare much better. '$a < b$' reduces into a single proposition '*ab*(less than),' but it may equally well reduce into the proposition '*ba*(less than),' since the monist cannot presuppose the asymmetrical relation between *a* and *b* in the whole *ab* if he wants to reduce it to a property thereof. Moreover, the monistic reduc-

[50] Russell, *Principles*, 222.
[51] Russell, *Principles*, 223-4.

tion should apply as well to the parthood relation on which it relies, and the proposition '*a* is a part of *b*' should reduce to a proposition about the whole composed of *a* and *b*, or '*ab*(is a part of).' But the parthood relation is itself asymmetrical, so we do not know if *a* is a part of *b* or if *b* is a part of *a*, as '*ba*(is a part of)' is an equally good analysis if *ab* and *ba* are the same whole. If they are not the same whole, the monist presupposes the relation he is trying to reduce; if they are the same whole, the parthood relation becomes symmetrical, and in that case everything is a part of everything else, because if *a* is a part of *b* and *b* is a part of *a*, every part of *a* is a part of *b* and every part of *b* is a part of *a*, and likewise for whatever else *a* and *b* are parts of. If my arm is a part of my body, and my hand is a part of my arm, then my hand is a part of my body. But if my body is also a part of my arm, and if my leg is a part of my body, then my leg is also a part of my arm, and then I might as well have a foot in place of my hand. And if everything is a part of everything else and thus a part of one big thing, so that there really are no parts and wholes, we are led to accept the especially pernicious consequence that "the only true whole, the Absolute, has no parts at all."[52]

Russell has thus teased out how metaphysical monism follows from a subject-predicate analysis of relational propositions, or more broadly how assumptions about logical form have metaphysical consequences. Analyzing a relational proposition into subject-predicate form reduces a relational predicate to a monadic predicate which, according to the monist, applies to a whole composed of the *relata*. The monistic analysis of mereological propositions reduces the diadic predicate 'is a part of' to a monadic predicate applied to a whole composed of the part and the whole, thus rendering the

[52] Russell, *Principles*, 226.

parthood relation symmetrical and, as a metaphysical consequence, wholes and parts indistinguishable. The only whole there is, the whole composed of everything there is ('the Absolute'), is partless—otherwise the symmetrical parthood relation would require that it was a part of its parts, in which case it would be one of the things there is and, thus, not the whole composed of everything there is. In reality, there are no things which are parts of the Absolute, which is, paradoxically, a partless whole, the one and only thing there is. The absurdity of monism, as Russell sees it, is a result of errors in the logical analysis of propositions; and if we want to avoid the conclusions that relations are unreal and that the Absolute is the only thing there is, we must abandon the subject-predicate analysis which yields them.

Similar arguments against internal relations, however, can be made in purely metaphysical terms. For instance, Russell argues from metaphysical assumptions about objects and properties that relations cannot be ontically dependent on properties of *relata*, and must therefore exist independently, much as Bradley does. In fact, Russell recycles an argument of Bradley's, the test of conceptual formulation F_P, which says that properties are ontically dependent on relations, and which Russell takes as an argument in favor of external relations. Considering relations which "are reducible to internal states of the apparently related objects," Russell observes that "this class of relations presupposes a plurality of objects, and hence involves the relation of diversity….before we can distinguish the internal states of [*a*] from those of [*b*], we must first distinguish [*a*] from [*b*]….i.e., [*a*] and [*b*] must *be* different, before they can have different states."[53] Bradley's argument was meant to convey that there could not be multiple properties unless they were mutu-

[53] Russell, *Principles*, 447.

ally differentiated from each other by standing in the relation of difference, thus assuming the independent existence of at least one relation. Notably, Bradley thought the alternative was a vulgar monism in which no properties were different from each other, and accordingly no objects were differentiable from each other by their properties.

Russell appropriates this argument in what looks like an attempt to show that there are *no* internal relations whatsoever—any such relation presupposes different *relata* and hence cannot be reduced to different internal states thereof without presupposing the relation of difference. In his arguments against the subject-predicate analysis of relational propositions, he was only prepared to conclude that *some* relations (e.g. asymmetrical relations) must be external, whereas others (e.g. symmetrical relations) might be amenable to internalist reduction. In this case, he concludes that admitting the existence of *any* internal relations is metaphysically undesirable, since in order to avoid presupposing the external relation of difference, one must appeal to "the notion that the apparent relations of two things consist in the internal states of one thing," namely the Absolute, a return to "rigid monism."[54] Russell, for his part, does not mention Bradley's argument in connection with his own, though it seems obvious that he would have encountered it first in *Appearance and Reality*.

There are a couple important differences between the arguments, one minor and one major. The minor difference is that, in Russell's argument, the relation of difference obtains between different *relata* in order that it may obtain between their internal states; in Bradley's argument, it is the converse, as the relation of difference obtains between different properties of *relata* in order that it may obtain between the *relata*. The major dif-

[54] Russell, *Principles*, 447.

ference is one of logical method, and is deeper than any difference of logical formalism: Russell is more or less correct that Bradley does not accept the existence of relations and is, ultimately, a metaphysical monist, but the subject-predicate diagnosis does not accurately trace the origins of Bradley's position, because it neglects how Bradley's dialectical method of intelligibility testing leads him there. Bradley does not so much reject the existence of relations as reject the intelligibility of the concept of 'relation,' and so concludes that the application of the concept to empirical reality is unjustified. Neither the 'internal' nor the 'external' formulations of the concept pass their intelligibility tests, so there is no reason to accept relations in either form. Russell does remark that an internalist view of relations "is, in fact, a theory that there are no relations," as "has been recognized by the most logical adherents of the dogma—e.g. Spinoza and Mr. Bradley—who have asserted that there is only one thing, God or the Absolute, and only one type of proposition, namely that ascribing predicates to the Absolute."[55] But he provides no compelling argument to the effect that Bradley accepts these conclusions *on the grounds* that relations are reducible to properties of *relata*, a view which Bradley evidently does *not* endorse. While I agree with Russell that metaphysical doctrines boil down to matters of logic, he seems to misstep when he derives Bradley's monism from erroneous assumptions about logical *form*. My contention is, rather, that Bradley's monism is a consequence of his logical *method*, and that the divergence in logical method between Russell and Bradley, as partly described by Bosanquet, is at the root of their disagreement over the metaphysics of relations, more so than any dispute over the correct logical analysis of propositions.

[55] Russell, *Principles*, 447-8.

In his later essays "The Monistic Theory of Truth," from 1906/7, and "The Basis of Realism," from 1911, Russell begins to defend what he calls "the doctrine of external relations,"[56] the core of a 'pluralist' metaphysics compatible with his own analysis of relational propositions. "The doctrine may be expressed by saying that 1) relatedness does not imply any corresponding complexity in the *relata*; 2) any given entity is a constituent of many different complexes."[57] In saying that a relation does not 'imply complexity' in the *relata*, Russell means that *relata* stand in relations irrespective of any non-relational facts about their internal structure, contrary to what "the axiom of internal relations"[58] would require. If we were to accept that axiom, along with a pair of plausible metaphysical premises—i). that there are infinitely many entities, and ii). that every entity stands in some relation or other to every other entity—then every *relatum* "would have to be strictly infinitely complex."[59] If we think of an entity, as before, as one part of the whole of everything that exists, then "each part will have a nature which exhibits its relation to every other part and to the whole,"[60] thus reflecting the entire system of the universe within itself, "like….Leibniz's monads."[61] An entity *a* standing in a relation *R* to *b* will bear a tacitly non-monadic property which in a literal sense 'contains' *b*, and *a* will thus bear a non-relational, though tacitly non-monadic, property corresponding to each relation in which it stands and containing each *relatum* to which it relates. And because *a* stands in some relation or other to every entity in the universe, *a* will be a container for everything else. Any proposition we assert about *a* will at the same time be a proposition

[56] Russell, "The Basis of Realism," 160.
[57] Russell, "The Basis of Realism," 158.
[58] Russell, "The Monistic Theory of Truth," 160.
[59] Russell, "The Basis of Realism," 159.
[60] Russell, "The Monistic Theory of Truth," 161.
[61] Russell, "The Monistic Theory of Truth," 162.

about every entity contained in *a*, and as such "would involve referring to everything else in the universe."[62] So the proposition *R*(*a*, *b*) refers not only to *a* and *b*, but also to everything contained in *a* and everything contained in *b*. In that case, how do we know that the proposition is really *about a* and *b* and not about any of the infinitely many entities contained in *a* and *b*? We might as well be asserting that *R*(*c*, *d*), since if *a* contains everything, it contains *c*, and if *b* contains everything, it contains *d*, in which case the assertion that *R*(*a*, *b*) is an assertion about *c* and *d* as much as *a* and *b*. Indeed, since *a* and *b* each contain everything, we might as well be asserting that everything stands in *R* to everything, so that the proposition *R*(*a*, *b*) is really a proposition about the relation of the Absolute to itself. "Thus," Russell says, "nothing quite true can be said about [*a*] without taking account of the whole universe, and then what is said about [*a*] will be the same as what would be said about anything else, since the natures of different things must….all express the same system of relations."[63] However, it seems clear that when we assert that *R*(*a*, *b*), we assert *only* that *a* stands in *R* to *b*: we refer only to *a* and *b* and we assert only that *they* stand in *R* to each other. Thus does the axiom of internal relations inflate the content of any proposition well beyond what it actually asserts, conflating every entity with every other entity, and thus any proposition about any entity with a proposition about every entity, so that whenever we seem to be making an assertion about something in particular, we are in fact making an assertion about everything in general. Monadism and monism, then, follow *compatibly* from the axiom of internal relations, as the infinite complexity of each *relatum* opens the backdoor to the Absolute.

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[62] Russell, "The Monistic Theory of Truth," 162.
[63] Russell, "The Monistic Theory of Truth," 162.

Just as the axiom of internal relations is the metaphysical complement of the subject-predicate analysis of relational propositions, the doctrine of external relations complements Russell's 'propositional function' analysis. A propositional function $\phi(x_1,\ldots x_n)$ takes an *n*-tuple of entities $a_1,\ldots a_n$ as an argument and outputs either a true or false proposition depending on whether $a_1,\ldots a_n$ stand in the relation ϕ or not. So the true proposition $\prec(7, 9)$ is the output of the propositional function $\prec(x, y)$ when it takes the ordered pair (7, 9) as an argument, and the false proposition $\prec(3062, 9)$ is the output when it takes (3062, 9) as an argument. When '$7 \prec 9$' is analyzed such that '7' is the subject which 'less-than-9' is predicated of, the predicate 'less-than-9' corresponds to a property inhering in the substance corresponding to the subject '7,' and then 7 and 9 become metaphysically inextricable, the one contained in the other.

But if, on the other hand, we analyze the relational proposition $\prec(7, 9)$ into the propositional function $\prec(x, y)$ and the ordered pair (7, 9), we end up with two distinct constituents: a relation detached from any *relata* in particular and a set of entities detached from any relation in particular. The propositional function corresponds to a relation external from its *relata* and therefore metaphysically real in itself, and the ordered pair to a set of self-sufficient entities which enter into relations without conceding their self-identity; when combined, the propositional function and the ordered pair form a relational proposition, which corresponds to an external relation between independent *relata*. Arguments and functions, unlike the substances and properties corresponding to subjects and predicates, are detachable, so that relations and *relata* maintain their metaphysical independence. 7 can be the argument for x in the propositional function $\prec(x, y)$ while any other number is the argument for y, or vice versa, so that any number whatever is inter-

changeable in either argument place in the function: the same 7 is a constituent of 7 < 9 as of 7 < 3062 or of 2 < 7, and as such is a constituent of infinitely many relational propositions without itself containing infinitely many numbers.

4. Modal Entailment: Moore

Russell's friend and colleague G.E. Moore made a late contribution to the debate with his 1919 paper "External and Internal Relations," in which he agrees with Russell that not all relations are internal, but does so under a very different interpretation of the meaning of 'internal.' To begin with, Moore thinks that what he calls "the dogma of internal relations"[64]—refusing to dignify it as an axiom—is not really about relations *per se*, but about 'relational properties.' A relational property is the property of standing in a relation to something, so that if *a* stands in the relation *R* to *b*, *a* bears the relational property of standing-in-*R*-to-*b*. Moore does not entirely justify the distinction, except to say that it helps us clarify that, according to the dogma of internal relations, if *a* stands in *R* to *b*, it is not just standing in *R* which is internal to *a*, it is standing in *R to b*. Moore does not consider the possibility of which Russell made so much, that such properties might be tacitly non-monadic and hence 'imply complexity' in the *relata*; relational properties are, on Moore's view, perfectly straightforward and metaphysically unproblematic.

Speaking in terms of this distinction, Moore claims that "one thing which is always implied by the dogma….is that, in the case of every relational property, it can always be truly asserted of any term *x* which has that property, that any term which had not had it

[64] Moore, "External and Internal Relations," 8.

would necessarily be different from *x*."[65] This modal interpretation of the dogma of internal relations, or at least of an important consequence of the dogma, is ambiguous between two readings: i). it is necessary that for any *x* and any *y*, if *x* bears the relational property ϕ, then if $y = x$, then *y* bears ϕ; or ii). for any *x* and any *y*, if *x* bears the relational property ϕ, then it is necessary that if $y = x$, then *y* bears ϕ. In other words, i). says that *necessarily* if anything bears a relational property, then it bears that relational property, and ii). says that if anything bears a relational property, then it *necessarily* bears that relational property. Moore thinks that i). is obviously true of all relational properties, since it is a consequence of the logical necessity of the Indiscernibility of Identicals, the principle that identical entities bear all the same properties. On the other hand, though ii). might be true of *some* relational properties, it would be absurd to think that it was true of *all* relational properties, since that would amount to the view that all relations are necessary, which is obviously false. If the dogma of internal relations amounts to the view that i). is true of all relational properties, then it is a weak and uncontroversial dogma; but Moore maintains that those who adhere to the dogma adhere to the strong and controversial dogma that ii). is true of all relational properties. On this reading, what it means to say that a relational property is 'internal' is that it is a *necessary relational property*.

Moore observes that Russell's logic—and specifically the connective '⊃', interpreted as material implication—cannot represent the distinction between i). and ii). Russell's system *can* express the relevant consequence of the Indiscernibility of Identicals: in contemporary notation, '$\forall x \forall y(\phi x \supset (y = x \supset \phi y))$'. This only says that it is *actually* the case that if anything bears the relational property ϕ, it is not the case that anything identical to

[65] Moore, "External and Internal Relations," 5.

it fails to bear ϕ; but it says nothing about what is *necessarily* the case. Thus neither i). nor ii). can be represented in Russell's system, and to improve its expressive power, Moore introduces the *entailment* connective 'ent'. Now i). can be represented as '$\forall x \forall y(\phi x \text{ ent } (y = x \supset \phi y))$', and ii). as '$\forall x \forall y(\phi x \supset (y = x \text{ ent } \phi y))$'. But how should we understand 'ent'? Moore makes the intriguing comment that "undoubtedly what Professor Lewis means by '[*p*] *strictly* implies [*q*]' is what I mean by [*p*] ent [*q*],"[66] i.e., Moore's 'entailment' corresponds to C.I. Lewis' 'strict implication,' or, in contemporary terms, to 'necessary implication.' This is perhaps unsurprising, since the distinction between the two readings depends on the scope of the modal operator 'it is necessary that.' In contemporary notation, we can use the box operator '$\Box$' for logical necessity to represent i). as '$\forall x \forall y \Box(\phi x \supset (y = x \supset \phi y))$' and ii). as '$\forall x \forall y(\phi x \supset \Box(y = x \supset \phi y))$'. The necessity operator has wide scope in i)., so that the implication on the left is a necessary implication, and narrow scope in ii)., so that the implication on the right is a necessary implication. If ii). is true for all relational properties, then the fact that something bears a relational property necessarily implies that it bears that property, in which case it bears all of its relational properties necessarily, and therefore every relation in which it stands is an internal relation. Moore, on the other hand, holds that if ii). is true for any relational properties, it is probably only true for some of them, in which case some entities bear some relational properties only contingently, and therefore some relations in which they stand are external relations. Though Russell too holds this position, his logic does not have the formal resources to express it, and must be able to incorporate necessary implication—whether in the form of Moore's entailment connective, Lewis' strict implication

[66] Moore, "External and Internal Relations," 12.

connective, or the box operator for logical necessity—if it is to do any serious metaphysical reasoning about internal and external relations.

Russell never addressed Moore's interpretation directly, but he did reject the following modal argument for the axiom of internal relations: "if two terms have a certain relation, they cannot but have it, and if they did not have it they would be different; which seems to show that there is something in the terms themselves which leads to their being related as they are."[67] It is significant that Russell understands this as an *argument for* the axiom, and not as a formulation of it: on the assumption that entities which stand in a relation to each other would not be those entities if they did not stand in that relation, we can conclude that there must be something in their 'natures' in virtue of which they stand in that relation. Unfortunately, Russell does *not* understand this argument modally at all, and hence his counter-arguments lack the force of Moore's.

Russell entertains two non-modal readings of the argument. The first takes "if they did not have it they would be different" to mean that "if they were not so related they would be other than they are," to which Russell objects that "if two terms are related in a certain way, it follows that, if they were not so related, every imaginable consequence would ensue."[68] In other words, if *a* and *b* stand in the relation *R*, then if they do not stand in the relation *R*, we have a contradiction, and from a contradiction anything follows. Of course, Russell reads "if they were not so related they would be other than they are" as a material rather than a counterfactual conditional, in which case Russell's objection is legitimate, but it is unlikely that anyone arguing for the axiom of internal relations would assent to that reading. Because Russell mistakes the *counterfactual* assumption that *a* and

[67] Russell, "The Monistic Theory of Truth," 165.
[68] Russell, "The Monistic Theory of Truth," 166.

b do not stand in *R* for a merely *false* assumption, the conclusions he draws are irrelevant to the modal argument for the axiom, illustrating yet another disadvantage of material implication in metaphysical reasoning.

In any case, Russell considers another reading of "if they did not have it they would be different" which takes it to mean that "if [*a*] and [*b*] are related in a certain way, then anything not so related must be other than [*a*] and [*b*],"[69] a simple consequence of the non-modal form of the Indiscernibility of Identicals. Nonetheless, Russell claims that it "only proves that what is not related as [*a*] and [*b*] are must be *numerically* diverse from [*a*] or [*b*]; it will not prove difference of adjectives, unless we assume the axiom of internal relations."[70] The Indiscernibility of Identicals implies that, if *a* and *b* stand in *R* to each other, then anything *x* and *y* which do not stand in *R* to each other are non-identical with *a* and *b*. But it does not imply that *x* and *y*, respectively, fail to bear any of the non-relational properties of *a* and *b*, i.e., the Indiscernibility of Identicals does not imply the Discernibility of Non-Identicals, the contrapositive of the principle of the Identity of Indiscernibles, which says that entities which bear all the same properties are identical. The axiom of internal relations, which holds that every relation is reducible to non-relational properties of *relata*, requires the reduction of *R* to some non-relational properties of *a* or *b*, but if $x \neq a$ and $y \neq b$ does not imply that *x* and *y* fail to bear any of those non-relational properties, then we have not shown that some non-relational properties of *a* or *b* are those properties in virtue of which they stand in *R*, since *x* and *y* may well bear those properties without standing in *R*. If we were to *assume* the axiom of internal relations, then we could immediately conclude that *x* and *y*, respectively, do not bear any of the non-relational

[69] Russell, "The Monistic Theory of Truth," 166.
[70] Russell, "The Monistic Theory of Truth," 166.

properties of *a* and *b*, since they do not stand in *R* and, according to the axiom, to stand in *R* is to bear the requisite non-relational properties. But clearly the intention was to argue for the axiom without assuming it. Russell's counter-argument, then, depends on the claim that the Indiscernibility of Identicals does not imply the Identity of Indiscernibles, and thus on the possibility that some entities are non-identical yet indiscernible. A version of Leibniz' Law, however, states that the two principles imply each other, and Russell offers nothing in the way of a refutation. Metaphysical tradition does nothing to dissuade Moore, either, from announcing that "it is obvious that the principle of Identity of Indiscernibles is not true,"[71] without further justification. Dogmatism aside, both philosophers at least recognize that these principles can help clarify the problem.

Moore's innovation in this regard is to employ a modal form of the principle of the Indiscernibility of Identicals to reason about the 'natures' of *relata*. Because that principle, in its non-modal form, expresses a logical relation between property-bearing and identity, it is particularly suitable for reasoning about the 'natures' of entities, if by the 'nature' of an entity we mean that which distinguishes it from other entities. But by employing its modal form, which states that the principle is logically necessary, Moore broadens the meaning of 'nature' from how an entity *is* to how it *must be*, insofar as it exists. Given the logical necessity of the Indiscernibility of Identicals, the relevant consequence for the metaphysics of relations is that, necessarily, if anything bears a certain *relational* property, then anything identical to it bears that property; and it is a confusion about the scope of 'necessarily' which leads to the dogma of internal relations, since it is plainly false that the *fact* that something bears a relational property implies that it *must*

[71] Moore, "External and Internal Relations," 14.

bear it. The distinction between the modal and non-modal forms of the Indiscernibility of Identicals is essential to Moore's point: though it is true that anything which happens to bear a relational property will, as a necessary *condition* of its self-identity, bear that property, it is not true that anything which happens to bear a relational property will, as a necessary *feature* of its self-identity, bear that property.

Russell was generally skeptical that we could reason about the 'natures' of entities at all. Throughout his discussion of internal relations, he assumes that for *relata* to stand in a relation in virtue of their 'natures' means that the relation is reducible to non-relational properties of the *relata*, which identifies the 'nature' of an entity with the set of its properties. But Russell was explicitly opposed to this conception of 'nature,' since it would allow "[a]ny casual collection of [properties]….to compose an [entity], if [entities] are not other than the system of their own [properties]."[72] Moore, however, is comfortable enough with this conception to declare that "by a term's nature is meant….all its [properties] as distinguished from its relational properties,"[73] or the set of its non-relational properties. This conception of 'nature' figures in a formulation of the dogma of internal relations which might be more agreeable to Russell: "if you take *all* the [non-relational properties] which [a] term has, it will again follow in the case of each relational property, from the proposition that the term has *all* those [non-relational properties], that it has the relational property in question."[74] If the 'nature' of the entity *a* is the set of non-relational properties θ, and if *a* bears the relational property ϕ, then the proposition that *a* bears θ *necessarily implies* that *a* bears ϕ.

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[72] Russell, "The Monistic Theory of Truth," 167.
[73] Moore, "External and Internal Relations," 14.
[74] Moore, "External and Internal Relations," 14.

In Russell's view, what it means for a relation to be 'reducible to' non-relational properties, or for *relata* to stand in a relation 'in virtue of' their non-relational properties, is that any relational proposition, when properly analyzed, is equivalent to a non-relational proposition in which only monadic predicates occur. Moore's modal method of reasoning about relations, however, works *within* a logical system, at a level 'above' propositional analysis. His notion of 'entailment,' or necessary implication, is capable of giving a much more precise meaning to 'in virtue of,' such that an entity bears a relational property in virtue of its non-relational properties just in case it bears that relational property, necessarily, whenever it bears those non-relational properties. We might better identify the 'nature' of the entity *a* with the set of its *necessary* non-relational properties θ, and then we can easily see that, if *a* bears θ, and if *a* bears θ necessarily implies that *a* bears the relational property ϕ, then ϕ is a necessary relational property of *a*. The dogma of internal relations then goes: every relational property of a *relatum* is necessarily implied by the set of its necessary non-relational properties, i.e., is a necessary relational property. This, Moore thinks, is patently unacceptable, so he and Russell agree that there must be some external relations.

Unfortunately, they seem to disagree over the definitions of 'internal' and 'external.' Speaking of his doctrine of external relations, Russell cryptically remarks that it "is not correctly expressed by saying that two terms which have a certain relation might not have had that relation. Such a statement introduces the notion of possibility and thus raises irrelevant difficulties."[75] Russell does not elaborate on what these difficulties might be, or why they are irrelevant, but remember that Russell's paradigm case of an external rela-

[75] Russell, "The Basis of Realism," 158.

tion is a transitive asymmetrical relation like **<**. The relation 7 **<** 9 is external to 7 because it does not confer any tacitly non-monadic properties on 7 such that 7 becomes a container for infinitely many numbers; 7 stands in the less-than relation to infinitely many numbers irrespective of any non-relational facts about its 'nature,' and the less-than relation is external to any of the numbers which stand in it. However, 7 **<** 9 is a *necessary* relation, as all mathematical relations are, such that 7 bears the necessary relational property less-than-9, which is thus, on Moore's account, *internal* to 7. Russell may have recognized that an external relation is not necessarily a contingent relation, and that an internal relation is not necessarily a necessary relation, and sought on such grounds to preclude any modal interpretation of the axiom of internal relations. Whatever his reasons, if he and Moore are not on the same page with respect to the *examples* of internal and external relations, then we should not expect them to be on the same page with respect to the *definitions* of 'internal' and 'external.'

Russell demonstrated, to his satisfaction, that the axiom of internal relations is consistent with a subject-predicate analysis of relational propositions, and that the doctrine of external relations is consistent with a function-argument analysis; 'internal' and 'external' were defined in terms of the reducibility or irreducibility of relational predicates. Methodologically speaking, Russell thought that, so long as we get the analysis of propositions right, metaphysics will sort itself out: correct (incorrect) metaphysical doctrines are derived from correct (incorrect) propositional analyses, and a proper understanding of logical form will resolve our metaphysical quandaries. Moore, instead, meets Bosanquet's challenge head on, enriching Russell's logic such as to make it capable of robust metaphysical reasoning; the definitions of 'internal' and 'external' follow from proposi-

tions in the expanded system. Because he builds on Russell's system, Moore must assume a propositional function analysis, but that does not, on its own, dictate which metaphysical doctrines he must accept. Once again, it seems that Russell has overestimated what one can conclude on the basis of propositional analysis alone, as when he charged Bradley with a vulgar monism based on subject-predicate analysis, though Bradley's monism in fact derives from the inapplicability of any metaphysical concept to empirical reality, a result which Bradley obtains through his dialectical method of intelligibility testing. All this to say that when Moore agrees with Russell's doctrine of external relations, he does so according to a definition of 'external' with which Russell would not have agreed, and which he derives from propositions expressed in a formal system, not from the form of expression itself. The agreement between Russell and Moore, then, is largely superficial, since there is a deeper disagreement over the definitions of 'internal' and 'external,' itself due to a methodological conflict over the metaphysical implications of logic, over which aspects of logic have metaphysical implications, and over which logical methods are adequate and appropriate for metaphysical reasoning.

Conclusion

If Moore is right that an internal relation is just a necessary relation and an external relation is just a contingent relation, and that not every relation is necessary and some relations are contingent, then it might be reasonable to dismiss the categories 'internal' and 'external' from the metaphysics of relations altogether. Especially given subsequent developments in modal metaphysics, we seem to have a fairly clear notion of what it is for a relation to be necessary and what it is for a relation to be contingent: a relation $R(a, b)$ is

necessary if and only if *a* stands in *R* to *b* in every possible world in which *a* and *b* exist, and *R*(*a*, *b*) is *contingent* if and only if *a* does not stand in *R* to *b* in every possible world in which *a* and *b* exist. Applying this language to Moore's definition of 'internal,' we can say that, if θ is the set of non-relational properties which *a* bears in every possible world in which it exists, then a relational property ϕ is *internal* to *a* if and only if *a* bears ϕ in every possible world in which *a* bears θ. In that case ϕ is a necessary relational property of *a*, because *a* bears ϕ in every possible world in which *a* exists.

This 'possible worlds' interpretation of modal concepts has become standard in metaphysical reasoning about the 'natures' of entities, where an entity is said to bear a property 'by nature' just in case it bears that property in every possible world in which it exists. It is tricky to say, though, what it means for an *n*-tuple to stand in a relation 'by nature,' because what is the 'nature' of an *n*-tuple? Here we can see the advantage that Moore gains by referring to 'relational properties,' as it allows him to reason about the 'natures' of the respective *relata*, rather than about the 'nature' of an ordered set, whatever that might be. In the foregoing, when we said that a relation is 'internal' or 'external,' we meant either that it is internal or external to an individual *relatum* (the 'monadistic' definition), or that it is internal or external to an *n*-tuple of *relata* (the 'monistic' definition). But a relation is necessary or contingent only to an *n*-tuple of *relata*, while a relational *property* is necessary or contingent only to an individual *relatum*, since it is, if we are to believe Moore over Russell, a monadic property. This, however, seems to put Moore at something of a disadvantage as well, because he can neither say of a *relation* that it is internal or external to an individual *relatum*, nor of a *relational property* that it is internal or external to an *n*-tuple of *relata*, and this puts talk of relations and relational

properties at cross-purposes when it comes to distinguishing ‘monadistic’ from ‘monistic’ definitions. In any case, this is another aspect of the issue on which Moore and Russell do not confront each other directly, and I will continue to speak in terms of relations and not relational properties.

Nevertheless, we might sensibly conclude that, because the concepts ‘internal relation’ and ‘external relation’ are, even after dozens of pages of exposition, still hopelessly ambiguous, we should expend with them entirely, and stick to the much more intelligible concepts ‘necessary relation’ and ‘contingent relation,’ spelled out in terms of possible worlds. Russell, furthermore, seems to have demonstrated that there is no sense in trying to reduce relational propositions to non-relational propositions: one always ends up presupposing the relations one is trying to eliminate. Finally, the monists themselves, upon close inspection of their logical methods, look like they may not have even adhered to the doctrine of internal relations or accepted the existence of internal relations, and that, coupled with the decisive argumentation of Russell and Moore, might be incentive enough to retire this whole messy business about internal and external relations to the ancient history of misbegotten ideas.

That, I think, would be unfortunate. One of the most appealing characteristics of monism is its view of the world, or of Nature, as a rationally organized system, a complex totality of entities standing in exclusive and inflexible relations. Russell seems to have had little sympathy for such a world-view, and empiricist philosophers tend to think of Nature, or at least of natural science, as much less cohesive and unified than it might seem. But I have little doubt that cosmologists are thinking along these lines when they speak of ‘the universe,’ and that physicists pursue, at least ideally, a theory which covers

everything from the smallest fundamental particles to the largest galaxies. More significantly, because we have granted Moore the point that it is unlikely that every relation is necessary to its *relata*, and consequently that the relational properties of an entity do not generally 'follow from' the set of necessary non-relational properties thereof, we might want to wonder how far natural science would get if, in the course of investigating the relations of entities, it sought out only necessary relations. Even if natural science investigates only the 'natures' of entities in the modal sense, such entities will not necessarily stand in relations 'in virtue of their natures' (if we take Moore's point), which indicates that a good many of the relations between them will be contingent relations.

And yet, Nature seems to arrange its entities in tight structures which do not break up and rearrange at random. Aren't contingent relations too casual and flexible to tell natural scientists much about the organization of Nature? Even worse, where necessary relations at least tell us something informative and ampliative with respect to the 'natures' of entities, contingent relations are irrelevant to modal identity. This is where we might want to agree with Russell, that an internal relation is not always a necessary relation, and that an external relation is not always a contingent relation. Russell, indeed, has already given us an example of a necessary external relation: the less-than relation $<$ on the natural numbers. If we can find an example of a contingent internal relation, then we might have found a relation which tells us something informative and ampliative about the 'nature' of an entity, but which is not a relation in which that entity stands in every possible world in which it exists. Such a relation would be exclusive and inflexible, thus not nearly as casual and flexible as a contingent external relation, and as such would be able to figure in scientific investigations into the rational organization of Nature. In that case, the modal

identity of an entity would not be the only relevant benchmark for understanding the 'nature' of an entity, at least where natural science is concerned.

But Russell did not provide anything as precise as Moore's definitions; and when updated with a 'possible worlds' interpretation, Moore's modal method is much more conducive to reasoning about internal and external relations than Russell's method of propositional analysis, even if Moore's definitions do not help us make sense of the monist world-view. It would be difficult to come up with an example of a contingent internal relation so long as we do not have a working definition of 'internal relation' at our disposal, so I propose employing an updated Moore-style modal method to give definitions of 'internal' and 'external' which account for Russell's examples. Here is my proposal: a relation R is *internal* to an element x_i of an n-tuple $\langle x_1,\ldots x_n\rangle \in R$ $(1 \leq i \leq n)$ if and only if, for every possible world in which $\langle x_1,\ldots x_n\rangle \in R$, there is no $\langle y_1,\ldots y_n\rangle \in R$ such that $y_i = x_i$ and $y_j \neq x_j$ for every $j \neq i$ $(1 \leq j \leq n)$. And a relation R is *external* to an element x_i of an n-tuple $\langle x_1,\ldots x_n\rangle \in R$ $(1 \leq i \leq n)$ if and only if it is not internal to x_i. Informally, in the case of a binary relation $R(a, b)$, R is internal to a if and only if, for every possible world in which a stands in R to b, a does not stand in R to anything *except for* b. $R(a, b)$ is external to a if and only if there are possible worlds in which a stands in R to b and a stands in R to something *other than* b.

These definitions insure that, whenever a stands in an internal relation R to b, it stands in R exclusively and inflexibly to b. But it is not thereby necessary that a stands in R to b in every possible world in which a exists, nor that it stands in R to anything in every such possible world. E.g., the relation expressed by 'I was born in the United States' is internal to me, because in every possible world in which I was born in the

United States, I was not born in any other country. (It is not, however, internal to the United States, inasmuch as there are many possible worlds in which I am born in the United States along with many others.) Of course, my mother might have gone into labor during a vacation in Mexico, or I might never have been born at all, so there is nothing *necessary* about my standing in this relation to the United States; but it is nonetheless informative and ampliative with respect to my 'nature,' broadly construed, since I *am* an American, and that is not a trivial or insignificant relational property. According to this definition of 'internal,' there are contingent internal relations, and they are informative and ampliative, as well as exclusive and inflexible.

According to the above definition of 'external,' the less-than relation **<** is, in addition to being a necessary relation between any natural numbers, external to each natural number. E.g., 7 **<** 9 is necessary to <7, 9>, in that 7 stands in the less-than relation **<** to 9 in every possible world in which 7 and 9 exist, and 7 **<** 9 is external to 7, in that 7 stands in **<** to infinitely many other numbers in every possible world in which it stands in **<** to 9. Interestingly, the successor function **S** on the natural numbers is, in addition to being necessary to any pair which stands in it, internal to each natural number, because, e.g., 1 is the successor of 0 in every possible world in which 1 and 0 exist, and in every possible world in which 1 is the successor of 0, 1 is not the successor of any other number. This means, perhaps to Russell's dismay, that not every asymmetrical relation comes out external according to this definition, since **S** is asymmetrical, e.g., 0 is not the successor of 1. But at least **S** is not an ordering relation, since it does not obey the transitivity or trichotomy properties, so we might content ourselves by saying that, so long as ordering relations all come out external on this definition, Russell wouldn't be too upset about cer-

tain asymmetrical relations coming out internal. I have not done an exhaustive search for examples of internal ordering relations, but these working definitions are merely proposals, meant primarily to locate and sort examples, and could certainly be improved upon.[76]

To return to the case of natural science, if contingent internal relations fill the role they need to, in that they are informative and ampliative with respect to the 'natures' of entities, as well as exclusive and inflexible enough to tell us something about how these entities fit into the well-organized system of Nature, then we should be able to find examples in the natural sciences of contingent internal relations. One example might be the β decay of a free neutron, whereby the neutron, which becomes unstable outside the nucleus, decays into the triple {proton, electron, anti-neutrino}. There is no means of confirming, as there is in the case of mathematical relations, that this is a contingent internal relation. But it seems at least plausible that there are possible worlds in which the neutron decays into some other set of particles, or perhaps that there are possible worlds in which the neutron is stable outside the nucleus and does not decay at all. However, there does not seem to be any possible world so casually and flexibly arranged that the neutron could decay sometimes into this set of particles, and sometimes into some other set of particles. Or, at least, if the neutron could, in some possible world, decay into any number of sets of particles, that would indicate that statements about the β decay of the free neutron are neither informative nor ampliative with respect to its 'nature.' Contingent external relations are unlikely to tell us anything about the 'natures' of the *relata* which stand

[76] Robert Martin points out that any time-indexed relation will come out internal on this definition. E.g., if I am swimming in Lake Taghkanic at time *t*, then in every possible world in which I am swimming in Lake Taghkanic at *t*, I am not swimming in any other body of water at *t*. I am not sure what to say about this, except that it seems like yet another instance of the inconsiderate interference of time with logic.

in them, since they tend to have the character of brute facts. But if the natural sciences are replete with statements about contingent internal relations, then the world-view most appropriate to it would be one for which Nature could have been otherwise, but insofar as it is how it is, its most fundamental entities stand in relations to each other exclusively and inflexibly, and thereby does Nature maintain its regular and predictable course of events.

Would Russell recognize these as 'internal relations'? Internal relations, by his lights, are supposed to be reducible to non-relational monadic properties of *relata*. But this confers a 'corresponding complexity' on the *relata* in the form of tacitly non-monadic properties which undermine the reducibility of the relation. If an entity fits into the system of the universe, it stands in some relation or other to every other entity in the universe, thus rendering it infinitely complex. But the above definition of 'internal' is undisturbed by any of these difficulties: it does not require the reducibility of internal relations to non-relational monadic properties of *relata*, and thus it does not 'imply complexity' in the *relatum* to which the relation is internal. And even if it did, the definition of 'internal' precludes the possibility that a *relatum* standing in an internal relation could stand therein to infinitely many other entities—to stand in an internal relation is to stand in that relation to a restricted range of entities. If these entities *were* to confer complexity upon the *relatum*, it would thereby at most assume a finite complexity, and if this took the form of tacitly non-monadic properties, these would not threaten the reducibility of the internal relation to monadic properties, because no such reduction is required by the definition.

So in what sense is it a definition of 'internal'? Why not call these relations something else, like 'exclusive' or 'committed' relations, and oppose them to 'open' or 'cas-

ual' relations (to use a romantic metaphor)? I call these relations 'internal' because their definitions are designed to avoid the difficulties Russell describes, by i). conceding the irreducible existence of relations, and ii). letting only those entities which stand in external relations relate to infinitely many other entities, so that even if complexity is conferred by internal relations, it is not infinite complexity, and Nature does not impute its total system of relations to each entity which stands in an internal relation. I call them 'internal' also because they are designed to occupy a role in the natural sciences in which they are informative and ampliative with respect to the 'natures' of entities, as well as exclusive and inflexible enough to embody the tight arrangements of Nature, a role which I associate with a monist world-view. Though Russell and Moore may have discredited the doctrine of internal relations, they have not, and arguably did not intend to, establish that all relations are external. This leaves room for a viable definition of 'internal relation' to redeem what is redeemable in metaphysical monism.

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