Gibbs Phenomenon for Jacobi Approximations

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Abstract

The classical Gibbs phenomenon is a peculiarity that arises when approximating functions near a jump discontinuity with the Fourier series. Namely, the Fourier series "overshoots" (and "undershoots") the discontinuity by approximately 9% of the total jump. This same phenomenon, with the same value of the overshoot, has been shown to occur when approximating jump-discontinuous functions using specific families of orthogonal polynomials. In this paper, we extend these results and prove that the Gibbs phenomenon exists for approximations of functions with interior jump discontinuities with the two-parameter family of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In particular we show that for all α, β the approximation overshoots and undershoots the function by the same value as in the classical case – approximately 9% of the jump.

Contents

Dedication

To my mom, who has worked tirelessly to ensure both her daughters got every opportunity she did not and to my sister, who made fun of me every time I said SPROJ.

Acknowledgments

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1
Introduction

Function approximations are at the heart of mathematics and in broad terms is the process by which we translate certain functions into other functions that closely match our original function. This procedure proves particularly useful in areas of applied mathematics, physics, and engineering. The Gibbs phenomenon is a peculiarity of this process that occurs when approximating a function with a simple jump discontinuity with a family of continuous functions. Recall that a function with a jump discontinuity is a function, f on an interval $[a, b]$ such that for some $x_0 \in [a, b]$ we have

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x).$$

Definition 1.0.1. Let f be a function defined on $[a, b]$. If f is discontinuous at x_0 and if $\lim_{x \to x_0^-} f(x_0)$ and $\lim_{x \to x_0^+} f(x_0)$ exist, then f is said to have a discontinuity of the *first kind* or a *simple discontinuity* at x_0.

There are many functions that satisfy this condition, but the classic example that we use in this paper is the square wave on the interval $[-\pi, \pi]$ defined by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0. \\ 0 & x = 0 \end{cases}$$

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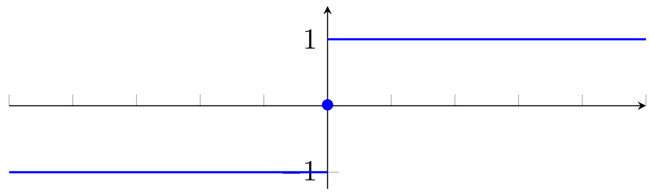


Figure 1.0.1. Square wave on the interval $[-\pi, \pi]$ with amplitude 1.

Notice that at $x = 0$, $\lim_{x\to x_0^+} f(0) = 1 \neq \lim_{x\to x_0^-} f(0)$ because, $\lim_{x\to x_0^+} f(0) = 1$, whereas $\lim_{x\to x_0^+} f(0) = -1$.

1.1 Fourier Series

Classically, square waves are approximated by the Fourier series. A Fourier series is a weighted combination of sines and cosines used to represent or expand functions. A Fourier series approximation of the square wave is a linear combination of frequencies and amplitudes that sound or look like a square wave, an example of this is available **here**. In general, they are useful when solving partial differential equations and ordinary differential equations with periodic boundary conditions [6].

Recall, a function f is said to be *integrable* if and only if it is bounded and continuous almost everywhere, that is continuous except at a finite number of points. A proof of this can be found in [7, (7.6.5)].

Definition 1.1.1. The Fourier series representation of a function $f(x)$ that is integrable on an interval $[a, b]$ is:

$$\mathscr{F}(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1.1.1}$$

where

$$a_0 = \frac{1}{b-a}\int_a^b f(x)dx \tag{1.1.2}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos(nx)dx \tag{1.1.3}$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin(nx)dx \tag{1.1.4}$$

for $n = 0, 1, 2, \ldots$.

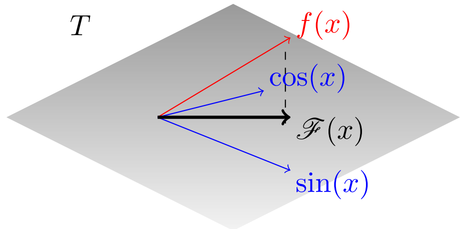


Figure 1.1.1. The orthogonal projection of a function, $f(x)$, in the vector space of integrable functions onto the subspace T spanned by $\{\sin(nx), \cos(nx)\}_{n=1}^{\infty}$.

A more intuitive approach to approximations by the Fourier series is to view it as an orthogonal projection of a function in the vector space of integrable functions onto the subspace T spanned by the functions $\{\sin(nx), \cos(nx)\}_{n=1}^{\infty}$ shown in Figure 1.1.1. The coefficients in the definition above are analogous for function spaces of the usual projection coefficients in linear algebra using dot products.

It is unclear whether or not the square wave defined above belongs to the space T, thus we use the Fourier series representation to project it onto the subspace, represented by $\mathscr{F}(x)$ in Figure 1.1.1 and check whether $\mathscr{F}(x) = f(x)$. In the same vein, we can choose other similar families of functions to project our square wave onto that need not be spanned by sines and cosines.

When working with continuous functions, Fourier series approximations are incredibly accurate in matching the original function (it gives a least-squares approximation); however, when working with jump-discontinuous functions, the Fourier series compensates by having an overshoot or undershoot at the point of discontinuity. This shows immediately that $\mathscr{F}(x) \neq f(x)$ and so $f(x) \notin T$. Interestingly however, the overshoot and undershoot for functions with simple jump discontinuities is proportional to the size of the jump, and is approximately 9% of the jump and this is known as the Gibbs phenomenon shown in Figure 2.1.1. The exact value of the Gibbs constant is expressed as the following integral

$$\gamma = h\frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \sim 1.18h \qquad (1.1.5)$$

where h is the amplitude of the “jump”.

One can then ask whether this happens for different, non-trigonometric approximations of the square wave (and similar functions with simple discontinuities). There has been recent work that explores the Gibbs phenomenon for various families of orthogonal polynomials, such as Legendre, Laguerre, Chebyshev, Hermite [15] as well as for more general families such as the Gegenbauer polynomials [14]. In every case, the authors found that the Gibbs constant overshoot that arises for each class of polynomials appears to be the same as that for Fourier series expansions, that is roughly 9% of the jump. In this paper, we attempt to encompass all these results by showing that the Gibbs constant arises for the Jacobi polynomials, which subsume the polynomials mentioned above. More specifically, we show both computationally and theoretically, that the Gibbs constant for the Jacobi polynomials is, in fact, exactly the same as in the case of Fourier series.

We begin by introducing the classical Gibbs phenomenon and the corresponding computations to show the exact value of the Gibbs constant at a jump discontinuity for the square wave. Chapter 3 reviews mathematical preliminaries needed to work with and generate orthogonal polynomials. Chapter 4 introduces the concept of orthogonal polynomials and provides an overview of their universal properties. It also provides a semi-detailed description of common polynomials for which the Gibbs phenomenon has been proved. Finally, in Chapter 5 we first find the Jacobi expansion of the sgn function and then focus on calculating the critical points closest to the jump discontinuity. We then use the critical points to discuss the behavior of the overshoot and undershoot around the jump-discontinuity as we consider higher order Jacobi expansions. We ultimately express the Gibbs constant associated with the Jacobi expansion in the integral formula (1.1.5). Finally, we show that the Gibbs phenomenon not only exists for Jacobi approximations, but is the same for every value of α and β.

1.2 Computations and Links

Note we used PARI [16] to perform computations for this paper. Additionally, all of the figures in this paper were made in Desmos [3]. We include interactive graphs for polynomials, which are hyperlinked within the text and identified in **bold**. For the convenience of readers we provide the direct links to all interactive materials below:

- square wave sound (https://www.youtube.com/watch?v=1W_uV-p-7_kab_channel=ResolaQQ)
- Gegenbauer polynomials (https://www.desmos.com/calculator/9twtoi1aeg)
- Jacobi polynomial (https://www.desmos.com/calculator/jc1q0zqvty)
- Jacobi Sgn approximation (https://www.desmos.com/calculator/ds0deufq0c).

2
Classical Gibbs Phenomenon

We begin by showing the theoretical approach to proving the classical Gibbs phenomenon, approximating a square wave with the Fourier Series. The Gibbs phenomenon was first discovered by Henry Wilbraham in 1848 and rediscovered by J. Willard Gibbs in 1899 [12]. The Gibbs phenomenon highlights the difficulty of approximating a discontinuous function by continuous functions and similarly, the difficulty of approximating continuous functions by discontinuous functions such as wavelets. The Gibbs phenomenon is common to any approximation of jump discontinuous functions to smooth functions, including trigonometric functions or polynomials. Its discovery has resulted in the creation of smoother methods of approximating discontinuous functions such as wavelet approximations [10].

2.1 Fourier Series Approximation of the Square Wave

In this section, we use notation from [6] to find the Fourier series approximation of the Square Wave. We use a square wave on $[-\pi, \pi]$ defined as follows:

$$f(x) = \begin{cases} \frac{h}{2} & 0 < x < \pi \\ -\frac{h}{2} & -\pi < x < 0 \\ 0 & x = 0. \end{cases}$$

Then, the corresponding Fourier series $\mathscr{F}(x)$ is found using the relations defined by Definition 1.1.1.

$$\begin{aligned}
a_0 &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{h}{2}dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{h}{2}dx \\
&= -\frac{hx}{2\pi}\bigg|_{-\pi}^{0} + \frac{hx}{2\pi}\bigg|_{0}^{\pi} = 0 \\
a_n &= \frac{1}{\pi}\int_{a}^{b} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{h}{2}\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{h}{2}\cos(nx)dx \\
&= -\frac{h}{2\pi}\frac{\sin(nx)}{n}\bigg|_{-\pi}^{0} + \frac{h}{2\pi}\frac{\sin(nx)}{n}\bigg|_{0}^{\pi} = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{h}{2}\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{h}{2}\sin(nx)dx \\
&= \frac{h}{2\pi}\frac{\cos(nx)}{n}\bigg|_{-\pi}^{0} - \frac{h}{2\pi}\frac{\cos(nx)}{n}\bigg|_{0}^{\pi} = \frac{h}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]
\end{aligned}$$

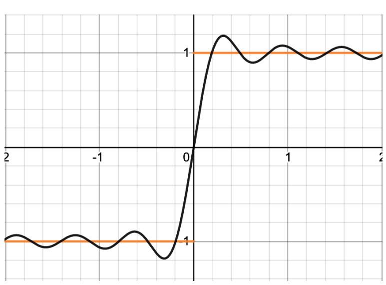
For b_n, we have the following two solutions depending on n.

$$b_n = \begin{cases} \frac{2h}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}.$$

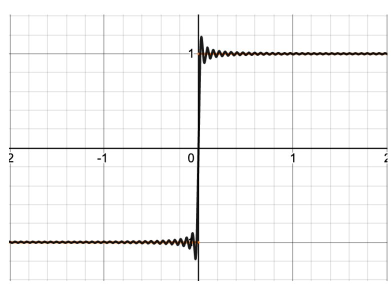
It follows that the Fourier series is:

$$\mathscr{F}(x) = \frac{2h}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right). \tag{2.1.1}$$

At this point, we can plot the Fourier series approximation with the original square wave. For simplicity, we assume $h = 1$, to retrieve the square wave defined in the Introduction, however any nonzero $h \in \mathbb{R}$ results in a similar plot.



(a)



(b)

Figure 2.1.1. Fourier Series Approximation of the square wave for (a) $n = 10$ and (b) $n = 100$.

Notice that as we approximate for larger n, we get closer to $f(x)$; however, the overshoot and undershoot remains and gets shifted closer to the point of discontinuity. Moreover, we notice pictorially that the overshoot and undershoot are the maximum and minimum of $\mathscr{F}_n(x)$ respectively. It suffices then, to verify the Gibbs Phenomenon, to calculate the value of the maximum and minimum for $\mathscr{F}_n(x)$.

2.2 Calculation of the Gibbs Phenomenon

2.2.1 *Summation of the Series*

We begin by fixing a positive integer r and considering the rth approximation:

$$\mathscr{F}_r(x) = \frac{2h}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(rx)}{r}\right).$$

Ideally, to calculate the Gibbs Phenomenon, we would want to find the maximum value, M, for $\mathscr{F}_r(x)$ by finding the first critical point, x_0 such that $\mathscr{F}_r(x_0) = M$, and $\frac{d}{dx}\mathscr{F}_r(x) = 0$. However, if we were to consider the classical approach for finding when $\frac{d}{dx}\mathscr{F}_r(x) = 0$ on $[-\pi, \pi]$ we would have to take the derivative of a sum and set equal to 0, which would be incredibly difficult. Instead, we represent the sum of the finite series by a single function.

Substituting (1.1.2), (1.1.3) and (1.1.4) into (1.1.1) we get:

$$\begin{aligned}\mathscr{F}_r(x) &= a_0 + \sum_{n=0}^{r} a_n \cos(nx) + \sum_{n=0}^{r} b_n \sin(nx)\\ &= \frac{2}{b-a}\int_a^b f(t)dt + \sum_{n=0}^{r}\frac{2}{b-a}\int_a^b f(t)\cos(nt)dt\cos(nx)\\ &+ \sum_{n=0}^{r}\frac{2}{b-a}\int_a^b f(t)\sin(nt)dt\sin(nx)\end{aligned}$$

Since $a = -\pi$ and $b = \pi$, our expression becomes:

$$\begin{aligned}\mathscr{F}_r(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{n=0}^{r}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt\right)\cos(nx)\\ &+ \sum_{n=0}^{r}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt\right)\sin(nx)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=0}^{r}\int_{-\pi}^{\pi}\left(\cos(nt)\cos(nx) + \sin(nt)\sin(nx)\right)dt.\end{aligned}$$

Note, $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$. It follows:

$$\mathscr{F}_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=0}^{r}\int_{-\pi}^{\pi} f(t)\cos n(t-x)dt. \tag{2.2.1}$$

If we set $\mathscr{F}(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx)$, then

$$\mathscr{F}(x) = \frac{a_0}{2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} f(t)\cos n(t-x)dt \tag{2.2.2}$$

and $\mathscr{F}_r(x)$ is the rth partial sum. Combining all of this into one expression

$$\begin{aligned}\mathscr{F}_r(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{n=0}^{r}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)dt\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=0}^{r}\cos n(t-x)\right]dt.\end{aligned}$$

We will now sum this series by passing to complex numbers. Recall that the sum of a finite geometric series is given by the following:

$$1 + s + s^2 + \ldots + s^n = \frac{1-s^{n+1}}{1-s},$$

which is valid for all $s \in \mathbb{C} - \{1\}$.

Let $s = e^{i\theta} = \cos\theta + i\sin\theta$. It follows that,

$$1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}.$$

where $\Re(1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta}) = 1 + \cos(\theta) + \cos(2\theta) + \ldots + \cos(n\theta)$ and,

$\mathrm{Im}(1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta}) = \sin(\theta) + \sin(2\theta) + \ldots + \sin(n\theta)$. We have,

$$\frac{1}{2} + \sum_{n=0}^{r}\cos n(t-x) = \Re\left[\frac{1}{2} + \sum_{n=0}^{r} e^{in(t-x)}\right] \tag{2.2.3}$$

and our equation for $\mathscr{F}_r(x)$ becomes:

$$\mathscr{F}_r(x) = \Re\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=0}^{r} e^{in(t-x)}\right]dt\right]. \tag{2.2.4}$$

Notice, the finite sum of exponentials is a geometric series and thus we have

$$\begin{aligned}\Re\left[\frac{1}{2} + \sum_{n=0}^{r} e^{in(t-x)}\right] &= \Re\left[\frac{1}{2} + e^{i(t-x)}\frac{1-e^{i(r+1)(t-x)}}{1-e^{i(t-x)}}\right]\\ &= \Re\left[\frac{1}{2} + e^{i(t-x)}\frac{e^{i(r+1)(t-x)}-1}{e^{i(t-x)}-1}\right].\end{aligned}$$

Since $\sin\left(\frac{\theta}{2}\right) = \frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}$, $2ie^{i\theta/2}\sin\left(\frac{\theta}{2}\right) = e^{i\theta}-1$. Setting $\theta = t-x$, we have:

$$\begin{aligned}
\Re\left[\frac{1}{2}+\sum_{n=0}^{r} e^{in(t-x)}\right] &= \Re\left[\frac{1}{2}+e^{i(t-x)}\frac{2ie^{i(r+1)(t-x)/2}\sin\left(\frac{(t-x)(r+1)}{2}\right)}{2ie^{i(t-x)/2}\sin\left(\frac{t-x}{2}\right)}\right] \\
&= \Re\left[\frac{1}{2}+\frac{\sin\left(\frac{(r+1)(t-x)}{2}\right)}{\sin\left(\frac{t-x}{2}\right)}e^{ir(t-x)/2+i(t-x)}\right] \\
&= \frac{1}{2}+\frac{\sin\left(\frac{(r+1)(t-x)}{2}\right)}{\sin\left(\frac{t-x}{2}\right)}\cos\left(\frac{(r+1)(t-x)}{2}\right).
\end{aligned}$$

Exploiting the identity $\sin(a+b) = \sin(a)\cos(b)+\sin(b)\cos(a)$ in the above equation gives the desired result

$$\mathscr{F}_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt. \tag{2.2.5}$$

In order for this integral to make sense, we need to show it converges for all t, in particular where $t = x$. We can show this by letting $\theta = t-x$ and considering the following limit:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\sin[(r+\frac{1}{2})(\theta)]}{\sin(\frac{1}{2}\theta)} &= \lim_{\theta\to 0}\left[\frac{(r+\frac{1}{2})\sin[(r+\frac{1}{2})(\theta)]}{(r+\frac{1}{2})(\theta)}\cdot\frac{\frac{1}{2}\theta}{\frac{1}{2}\sin(\frac{1}{2}\theta)}\right] \\
&= \lim_{\theta\to 0}\left[\frac{(r+\frac{1}{2})\sin[(r+\frac{1}{2})(\theta)]}{(r+\frac{1}{2})(\theta)}\right]\cdot\left[\lim_{\theta\to 0}\frac{\frac{1}{2}\theta}{\frac{1}{2}\sin(\frac{1}{2}\theta)}\right] \\
&= 2\left(r+\frac{1}{2}\right) \quad \text{by L'Hopital's rule.}
\end{aligned}$$

2.2.2 Inputting the Square Wave

Returning to our square wave described in Section 2.1, we substitute directly for $f(t)$. We break the integral into two integrals corresponding to the different values of $f(x)$. Then,

$$\mathscr{F}_r(x) = \frac{1}{2\pi}\int_{-\pi}^{0}\left(\frac{-h}{2}\right)\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt + \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{h}{2}\right)\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt \tag{2.2.6}$$

$$\begin{aligned}
&= \frac{-h}{4\pi}\int_{-\pi}^{0}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt + \frac{h}{4\pi}\int_{0}^{\pi}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt \\
&= \frac{h}{4\pi}\int_{0}^{\pi}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt - \frac{h}{4\pi}\int_{0}^{\pi}\frac{\sin[(r+\frac{1}{2})(t+x)]}{\sin\frac{1}{2}(t+x)}dt.
\end{aligned}$$

The second integral is a product of doing a u-substitution where $u = -t$ and since it is merely a dummy variable, we can substitute t again. We substitute once again and this time let $s = t-x$

and $s = t + x$, respectively. It follows:

$$\mathscr{F}_r(x) = \frac{h}{4\pi}\int_{-x}^{\pi-x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds - \frac{h}{4\pi}\int_{x}^{\pi+x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds.$$

The two integrals have integrands of the same mathematical form for values between x and $\pi - x$ but of opposite signs and thus cancel out. In particular,

$$\frac{h}{4\pi}\int_{-x}^{\pi-x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds + \frac{h}{4\pi}\int_{x}^{\pi} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds$$

and

$$\frac{h}{4\pi}\int_{x}^{\pi+x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds = \frac{h}{4\pi}\int_{x}^{\pi} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds + \frac{h}{4\pi}\int_{\pi}^{\pi+x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds.$$

We can visualize this using the following figure:

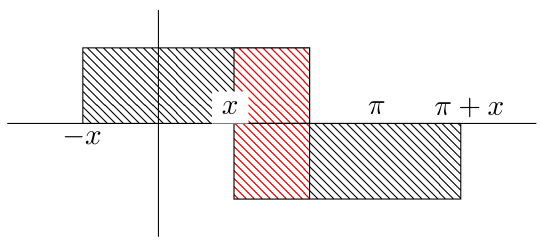


Figure 2.2.1. Cancelling out integrands of the same form over the same interval.

It follows that we have the following integrals:

$$\mathscr{F}_r(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds - \frac{h}{4\pi}\int_{\pi-x}^{\pi+x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds. \tag{2.2.7}$$

2.2.3 Calculation of the overshoot

We now have a formula for $\mathscr{F}_r : [-\pi, \pi] \to \mathbb{R}$, given by 2.2.7. We want to now explore how $\mathscr{F}_r$ behaves around our discontinuity, that is around $x = 0$. Let us begin by looking at the second integral. Notice, as $x \to 0$:

$$\lim_{x\to 0} \frac{h}{4\pi}\int_{\pi-x}^{\pi+x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds = 0$$

because the integrand is continuous and centered around π. Then, our function becomes:

$$\mathscr{F}_r(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)}ds.$$

We cannot conclude this integral is 0 because the integrand is discontinuous at $x = 0$, so we proceed with our calculations. Since the integrand is an even function, we have:

$$\begin{aligned}\mathscr{F}_r(x) &= 2 \cdot \frac{h}{4\pi} \int_0^x \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)} ds \\ &= \frac{h}{2\pi} \int_0^x \frac{\sin[(r+\frac{1}{2})(s)]}{\sin\frac{1}{2}(s)} ds.\end{aligned}$$

Now we perform two substitutions. First, we consider $(r + \frac{1}{2}) = p$. Then,

$$\mathscr{F}_r(x) = 2 \cdot \frac{h}{4\pi} \int_0^x \frac{\sin(ps)}{\sin\frac{1}{2}(s)} ds.$$

Next let $\xi = ps$. Then, $ds = \frac{d\xi}{p}$ and $\frac{s}{2} = \frac{\xi}{2p}$. For the lower bound we have when $s = 0$, $\xi = 0$ and for the upper bound we have when $s = x$, $\xi = px$. On substituting we have,

$$\mathscr{F}_r(x) = \frac{h}{2\pi} \int_0^{px} \frac{\sin(\xi)}{\sin(\frac{\xi}{2p})} \frac{d\xi}{p}.$$

Recall, we were looking for an analytic formula for $\mathscr{F}_r$ so that we could conveniently find the the maximum, M, on the interval $[-\pi, \pi]$. Now that we have the analytical formula, we can proceed by taking the derivative and finding the first positive critical point.

$$\frac{d}{dx}\mathscr{F}_r(x) = \frac{d}{dx}\frac{h}{2\pi} \int_0^{px} \frac{\sin(\xi)}{\sin(\frac{\xi}{2p})} \frac{d\xi}{p}.$$

By the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Our upper bound is px, so we can use the chain rule in the following manner. Let $h(x) = \int_a^x f(t)dt$. Then, $\int_a^p x f(t)dt = h(px)$. By the Fundamental Theorem of Calculus we also have $h'(x) = f(x)$ and thus the chain rule gives us $h'(px) = pf(px)$. Returning to the formula for the derivative of $\mathscr{F}_r(x)$, we have:

$$\frac{d}{dx}\mathscr{F}_r(x) = \frac{\sin(px)}{\sin(\frac{x}{2})}. \tag{2.2.8}$$

We want to know for what value of x on $[-\pi, \pi]$ does (2.2.8) equal 0. In order for that to be true, we would require $\sin(px) = 0$, which is true when $px = m\pi$ for $m \in \mathbb{Z}$. Then $x = \frac{m\pi}{p}$. Here, $m = 1$. This is because our overshoot happens near the point of discontinuity, where $x = 0$, apparent in Figure 2.1.1. While all $m > 1$ are critical points, $m = 1$, is the point at which we will get our maximum, M, since it is our first positive critical point.

Now we can return to our to function $\mathscr{F}_r$ and substitute for $x = \frac{\pi}{p}$. It follows:

$$\mathscr{F}_r\left(\frac{\pi}{p}\right) = \frac{h}{2\pi}\int_0^{p(\frac{\pi}{p})} \frac{\sin(\xi)}{p\sin(\frac{\xi}{2p})}d\xi \tag{2.2.9}$$

$$= \frac{h}{2\pi}\int_0^{\pi} \frac{\sin(\xi)}{p\sin(\frac{\xi}{2p})}d\xi. \tag{2.2.10}$$

Recall that $p = \left(r + \frac{1}{2}\right)$. This implies that as r approaches infinity (that is, we consider more and more terms in (2.2.2)) p also approaches infinity by construction. By the Taylor series expansion of $\sin(x)$, $p\sin(\frac{\xi}{2p})$ behaves in the following manner:

$$p\sin\left(\frac{\xi}{2p}\right) = p\left[\frac{\xi}{2p} - \frac{\left(\frac{\xi}{2p}\right)^3}{3!} + \frac{\left(\frac{\xi}{2p}\right)^5}{5!} - \dots\right]$$
$$= \frac{\xi}{2} - \frac{\xi^3}{4p^2\cdot 3!} + \frac{\xi}{32p^4\cdot 5!} - \dots.$$

Then, as $p\to\infty$, $p\sin(\frac{\xi}{2p}) \to \frac{\xi}{2}$. It follows that as $r\to\infty$

$$\mathscr{F}_r\left(\frac{\pi}{p}\right) = \frac{h}{2\pi}\int_0^{\pi} \frac{\sin(\xi)}{\frac{\xi}{2}}d\xi$$
$$= h\cdot\frac{1}{\pi}\int_0^{\pi}\frac{\sin(\xi)}{\xi}d\xi.$$

The exact calculation of this integral requires Complex Analysis since $\frac{\sin x}{x}$ has no elementary derivative. The calculation is done in [8]. However, we can represent this integral as a Riemann sum

$$\frac{2h}{\pi}\int_0^{\pi}\frac{\sin x}{x}\,\mathrm{d}x = 2h\sum_{j=0}^{\infty}\frac{(-1)^j\pi^{2j}}{(2j+1)(2j+1)!} \sim 1.18h.$$

Pictorially, this is the area underneath the curve between $[0,\pi]$ in Figure 2.2.2.

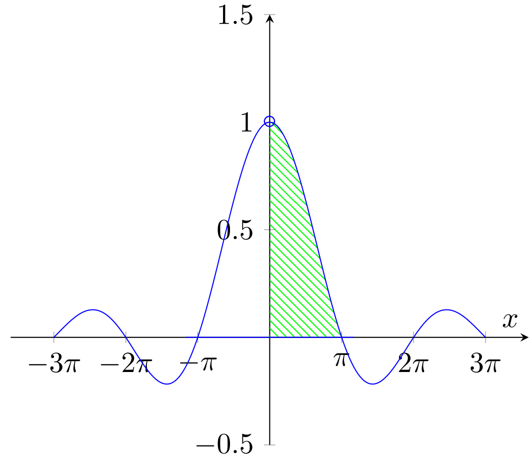


Figure 2.2.2. Pictorial representation of γ defined in the Introduction.

3
Background

In this chapter we review definitions and theorems needed in the subsequent chapters. We discuss definitions surrounding abstract vector spaces, which allows us to work with functions (and consequently polynomials) using the same set up as the traditional vectors in $\mathbb{R}^n$. Further details regarding the definitions and proofs can be found in [6].

3.1 Inner Product

The inner product is a generalization of the dot product of vectors in $\mathbb{R}^n$. That is, in a vector space, the inner product is a way of multiplying two vectors and outputting a scalar.

Definition 3.1.1. For a real vector space V, an **inner product** is a function

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R}$$

with the following properties.

- For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and a, $b \in \mathbb{R}$ it holds $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w}\rangle = a\langle \mathbf{u}, \mathbf{w}\rangle$ + b$\langle \mathbf{v}, \mathbf{w}\rangle$,
- For all $\mathbf{u}, \mathbf{v} \in V$ it holds $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$,
- For all $u \in V$, $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$ and,

- $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$.

V together with $\langle \cdot, \cdot \rangle$ is called an inner product space.

Examples of inner product spaces include the real numbers, $\mathbb{R}$ where the inner product is given by multiplication. That is, $\langle x, y \rangle = xy$ for all $x, y \in \mathbb{R}$. Another common example of an inner product space is the Euclidean space $\mathbb{R}^n$ where the inner product is given by the dot product. That is, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$ where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$.

The inner product space we focus on in this paper is the vector space of real-valued integrable functions whose domain is a closed interval $[a, b]$ with inner product defined as follows:

$$\int_a^b f(x)g(x)dx$$

where $f(x)$ and $g(x)$ are real functions.

First we prove that real-valued integrable functions form a vector space and then we show that the integral of the product of two real-valued integrable functions over an interval $[a, b]$ satisfies the inner product axioms.

Let C be the set of all real-valued integrable functions, let $c, d \in \mathbb{R}$ and let $f(x), g(x), h(x) \in C$. It is easy to check that

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

satisfies the properties of inner product spaces for all elements $\in C$. Namely,

1. $$\begin{aligned} \langle cf + dg, h \rangle &= \int_a^b (cf(x) + dg(x))h(x)dx \\ &= \int_a^b [cf(x)h(x) + df(x)h(x)]dx \\ &= c\int_a^b f(x)h(x)dx + d\int_a^b g(x)h(x)dx \\ &= c\langle f, h \rangle + d\langle g, h \rangle. \end{aligned}$$

2. $\langle f, g \rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle g, f \rangle.$

3. $\langle f, f \rangle = \int_a^b f(x)f(x)dx = \int_a^b |f(x)|^2 dx \geq 0 \quad \text{since } f \text{ is real-valued.}$

4. Suppose $\langle f, f\rangle = \int_a^b f(x)f(x)dx = \int_a^b |f(x)|^2dx = 0$. Since f is real valued, this is true if and only if $f = 0$.

When talking about an inner product space, we often discuss a *norm*, that is, the length of a vector.

Definition 3.1.2. Let V be a vector space and $\langle\cdot,\cdot\rangle$ be a inner product on V. The **norm** function, or length, is a function

$$\|\ \| : V \to \mathbb{R}$$

and defined as

$$\|u\| = \sqrt{\langle u, u\rangle}$$

for $u \in V$.

It follows that a norm in the space of continuous functions is given by:

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b [f(x)]^2dx}.$$

3.2 Orthogonality

In elementary geometry, we often think of orthogonality as perpendicularity. However, a more general definition of orthogonality takes into consideration an inner product.

Definition 3.2.1. Let V be a vector space and $\langle\cdot,\cdot\rangle$ be an inner product on V. Two vectors $\mathbf{u}, \mathbf{v} \in V$ are **orthogonal**, or perpendicular, if and only if

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0.$$

For vectors in $\mathbb{R}^n$, the inner product is the same as the dot product ($\mathbf{w}\cdot\mathbf{v} = |w||v|\cos\theta$), and thus it is straight forward that two vectors that are perpendicular are orthogonal. We can extend this to integrable functions on $[a, b]$ with inner product $\langle f, g\rangle = \int_a^b f(x)g(x)dx$, that is, two integrable functions are orthogonal over an interval $[a, b]$ if

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx = 0 \tag{3.2.1}$$

Example 3.2.2. The functions $\cos x$ and $\sin(x)$ are orthogonal on the interval $[0, 2\pi]$ because

$$\begin{aligned}\langle \cos(x), \sin(x) \rangle &= \int_0^{2\pi} \sin(x)\cos(x)dx \\ &= \frac{1}{2}\int_0^{2\pi} \sin(2x)dx \\ &= -\frac{1}{4}\cos(2x)\Big]_0^{2\pi} = 0.\end{aligned}$$

3.2.1 Gram-Schmidt Orthogonalization

Orthogonal bases, like cosines and sines, are incredibly convenient to carry out computations. In 1900, Jorgen Gram and Erhard Schmidt made a standard process to construct an orthogonal set of vectors or functions over any interval and with respect to an arbitrary weight from a nonorthogonal set of linearly independent vectors or functions [6]. The Gram-Schmidt procedure is often used to generate orthogonal polynomials as we will see in the future chapters.

Theorem 3.2.3. *Let $V, \langle \cdot, \cdot \rangle$ be an inner product space and let $\{u_1, \ldots, u_n\}$ be a basis of V. Then an **orthogonal basis** of V is given by the vectors $\{v_1, \ldots, v_n\}$ where*

$$\begin{aligned}v_1 &= u_1 \\ v_2 &= u_2 - \frac{\langle u_1, v_1 \rangle}{\|v_1\|} v_1 \\ v_3 &= u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|} v_2 \\ \vdots &= \vdots \\ v_n &= u_n - \sum_{i=1}^{n-1} \frac{\langle u_n, v_i \rangle}{\|v_i\|} v_i\end{aligned}$$

Proof. A proof of this theorem can be found in [5]. □

3.3 The Gamma Function

Suppose that $s > 0$ and define

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt.$$

$\Gamma(s)$ is known as the Gamma function and is a generalization of the factorial function for non-integer values developed by Leonhard Euler in the 18th century. The function is particularly useful in the field of special functions. We use two key propositions associated with the Gamma function in this paper.

Proposition 3.3.1. *If $s > 0$, then $\Gamma(s+1) = s\Gamma(s)$.*

Proof. We prove this using integration by parts. We have

$$\Gamma(s+1) = \int_0^\infty e^{-t}t^{s+1-1}dt = \int_0^\infty e^{-t}t^s dt = e^{-t} - t^s\Big|_0^\infty + \int_0^\infty se^{-t}t^{s-1}dt = 0 + s\Gamma(s).$$

To do integration by parts we let $u = t^s$, $dw = st^{s-1}$, $dv = e^{-t}$, $v = -e^{-t}$ and recall that $\int udv = uv - \int vdu$. □

Corollary 3.3.2. *If n is a positive integer, then $\Gamma(n) = (n-1)!$.*

Proof. Using the previous proposition, we have that

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-2)(n-1)\Gamma(n-2) = \ldots = (n-1)(n-2)\cdot\ldots\cdot\Gamma(1).$$

However by the definition of the Gamma function, we know that

$$\Gamma(1) = \int_0^\infty e^{-t}t^0 dt = -e^{-t}\Big|_0^\infty = 1.$$

Thus,

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-2)(n-1)\Gamma(n-2) = \ldots = (n-1)(n-2)\cdot\ldots\cdot 1 = (n-1)!.$$

□

3.4 Pochhammer Symbol

The Pochhammer symbol or rising factorial, defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{3.4.1}$$

$$= x(x+1)\ldots(x+n-1) \tag{3.4.2}$$

is notation often used in the theory of special functions and was introduced by Leo August Pochhammer (1841-1920) [11]. There are many forms of the Pochhammer symbol used throughout the different fields of mathematics, but in this paper we focus strictly on the Pochhammer symbol associated with the Gamma function.

4
Orthogonal Polynomials

Orthogonal polynomials were studied intensively in the 19th-century through the study of continued fractions and the moment problem. [17] They have been widely used in mathematics, science and engineering often times as basis functions to help understand more complicated functions. Moreover, since they are polynomials, they are often nicer to work with. This section introduces the basic properties of orthogonal polynomials, which are later used in our analysis of the Gibbs phenomenon for Jacobi polynomials.

Definition 4.0.1. A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ where $p_n(x)$ is of exact degree n, is orthogonal on the interval $[a, b]$ with respect to a weight function $w(x) > 0$, which is continuous and positive on $[a, b]$ if

$$\int_a^b w(x)p_n(x)p_m(x)dx = c_n\delta_{nm},$$

where

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

A "weight function" is often used when performing a sum, integral, or average to give some elements more weight or influence on the final result than other elements found in the set. [11] Classically, the weight function is equal to 1 in which case we recover the formula for $\langle f, g\rangle$.

The above definition relies on the fact that $\mathbb{R}$ together with $\int_a^b w(x)f(x)g(x)dx$ form an inner product space, hence we proceed by checking this is true.

Proof. We check that

$$\langle f, g\rangle_w := \int_a^b w(x)f(x)g(x)dx$$

satisfies the axioms of inner product.

1. Let $x, y \in \mathbb{R}$. $\langle xf + yg, h\rangle_w = \int_a^b w(x)(xf(x) + yg(x))h(x)dx$
$$= \int_a^b [xw(x)f(x)h(x) + yw(x)g(x)h(x)]dx$$
$$= x\int_a^b w(x)f(x)h(x)dx + y\int_a^b w(x)g(x)h(x)dx$$
$$= x\langle f, h\rangle_w + y\langle g, h\rangle_w.$$
2. $\langle f, g\rangle_w = \int_a^b w(x)f(x)g(x)dx = \int_a^b w(x)g(x)f(x)dx = \langle g, f\rangle_w$.
3. $\langle f, f\rangle_w = \int_a^b w(x)f(x)f(x)dx = \int_a^b w(x)|f(x)|^2 dx \geq 0 \quad \text{since } w(x) > 0$.
4. Suppose $\langle f, f\rangle_w = \int_a^b w(x)f(x)f(x)dx = \int_a^b w(x)|f(x)|^2 dx = 0$. Since $w(x)|f(x)|^2$ is a continuous positive function of x, $w(x)|f(x)|^2 = 0$ for all $x \in [a, b]$ and so $|f(t)|^2 = 0$ and $f(t) = 0$ for all $x \in [a, b]$.

□

Remark 4.0.2. Note, we distinguish between $\langle f, g\rangle$ and $\langle f, g\rangle_w$ to identify the difference between a weighted inner product and unweighted inner product ($w(x) = 1$) introduced in Chapter 2. However, going forward, we drop the subscript and assume weighted inner products.

Before proceeding to the properties of orthogonal polynomials, we explore an example of generating them using the Gram-Schmidt.

Example 4.0.3. We wish to obtain a set of orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle = \int_{-1}^1 w(x)f(x)g(x)dx$$

with $w(x) = 1$. To do this, we apply Gram-Schmidt orthogonalization to the set $\{1, x, x^2, x^3, \ldots\}$. Then, by Definition 3.2.3 the $n$the element of our orthogonal basis $\{v_k\}$ can be represented by

$$v_n = u_n - \sum_{i=0}^{n-1} \frac{\langle u_n, v_i \rangle}{\|v_i\|} v_i.$$

Setting $u_n(x) = x^n$ for $n = 0, 1, 2, \ldots$ and setting $v_0(x) = 1$ our orthogonal set $\{v_k\}$, $k = 1, 2, \ldots$ is obtained as follows:

$$\begin{aligned}
v_0(x) &= 1. \\
v_1(x) &= u_1 - \frac{\langle u_1, v_0 \rangle}{\|v_0\|} v_0 \\
&= x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = x - \frac{\frac{1}{2} x^2|_{-1}^{1}}{x|_{-1}^{1}} = x - \frac{\frac{1}{2}(1-1)}{2} = x. \\
v_2(x) &= u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|} v_1 - \frac{\langle u_2, v_0 \rangle}{\|v_0\|} v_0 \\
&= x^2 - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} x - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} = x^2 - \frac{\frac{1}{4} x^4|_{-1}^{1}}{\frac{1}{3} x^3|_{-1}^{1}} x - \frac{\frac{1}{3} x^3|_{-1}^{1}}{x|_{-1}^{1}} \\
&= x^2 - \frac{1}{3}. \\
v_3(x) &= u_3 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|} v_2 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|} v_1 - \frac{\langle u_3, v_0 \rangle}{\|v_0\|} v_0 \\
&= x^3 - \frac{\int_{-1}^{1} x^3 (x^2 - \frac{1}{3}) dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx} \left(x^2 - \frac{1}{3} \right) - \frac{\int_{-1}^{1} x^3 (x) dx}{\int_{-1}^{1} x^2 dx} x - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} dx} \\
&= x^3 - \frac{\frac{1}{6} x^6 - \frac{1}{12} x^4|_{-1}^{1}}{\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{x}{9}|_{-1}^{1}} \left(x^2 - \frac{1}{3} \right) - \frac{\frac{1}{5} x^5|_{-1}^{1}}{\frac{1}{3} x^3|_{-1}^{1}} x - \frac{\frac{1}{4} x^4|_{-1}^{1}}{x|_{-1}^{1}} \\
&= x^3 - \frac{\frac{2}{5}}{\frac{2}{3}} x = x^3 - \frac{3}{5} x.
\end{aligned}$$

Continuing this process and normalizing so that $v_n(1) = 1$ gives us the Legendre polynomials, shown in Figure 4.2.1. Different values of $w(x)$ yields different families of orthogonal polynomials.

4.1 Properties of Orthogonal Polynomials

In this section, we introduce three key properties of orthogonal polynomials that hold across different families. These properties prove important in the computations associated with orthogonal polynomial approximations. We begin by introducing the three term recurrence relationship, which provides the necessary set up for the Christoffel-Darboux sum and finally introduce the

Rodrigues formula. Both the Christoffel-Darboux sum and the Rodrigues formula are used in the calculation of the derivative of the Jacobi expansion of the sgn function. Proofs and consequences of the following theorems can also be found in [11] and [5].

Theorem 4.1.1. *A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *on* $[a,b]$ *satisfies a three-term recurrence relation*

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

for $n = 1, 2, 3, \dots$ *where* A_n*,* B_n *and* C_n *are real constants and*

$$A_n = \frac{k_{n+1}}{k_n},$$
$$C_{n+1} = \frac{A_{n+1}}{A_n}\frac{c_{n+1}}{c_n}$$

where $k_n > 0$ *is the leading coefficient of* p_n *and* c_n *is found using Definition 4.0.1.*

Proof. Since both p_{n+1} and $xp_n(x)$ have degree $(n+1)$, we can determine A_n such that

$$p_{n+1}(x) - A_n x p_n(x)$$

is a polynomial of at most degree n by choosing A_n to cancel out leading coefficients. It follows that there exist constants $b_k, k = 0, \dots, n$, such that

$$p_{n+1}(x) - A_n x p_n(x) = \sum_{k=0}^{n} b_k p_k(x). \tag{4.1.1}$$

If $Q(x)$ is any polynomial of degree $m < n$, we know by Definition 4.0.1 that $\int_a^b P_n(x)Q(x)w(x)dx = 0$ since $m \neq n$.

Multiplying both sides of (4.1.1) by $p_m(x)w(x)$ where $m \in \{0, 1, \dots, n-2\}$ we get

$$p_{n+1}(x)p_m(x)w(x) - A_n x p_n(x)p_m(x)w(x) = \sum_{k=0}^{n} b_k p_k(x)p_m(x)w(x).$$

Integrating both sides on the interval $[a,b]$, we get

$$\int_a^b p_{n+1}(x)p_m(x)w(x) - \int_a^b A_n x p_n(x)p_m(x)w(x) = \sum_{k=0}^{n}\int_a^b b_k p_k(x)p_m(x)w(x).$$

By Definition 4.0.1, the first integral on the left side is zero for every $m \in \{0, 1, \ldots, n-2\}$ and since $xp_m(x)$ is a polynomial of degree $(m+1) \leq (n-1)$ by definition of m, we conclude the entire left hand side of the equation is zero.

Similarly, the right hand side of the equation is only non-zero for $k = m$, once again by Definition 4.0.1. Hence, $b_m c_m = 0$ for all $m \in \{0, 1, \ldots, n-2\}$, and since $c_m \neq 0$, we have that $b_m = 0$.

Hence, $p_{n+1}(x) - A_n x p_n(x) = b_{n-1}p_{n-1}(x) + b_n p_n(x)$ and rearranging this gives us $p_{n+1}(x) = (A_n x + b_n)p_n(x) + b_{n-1}p_{n-1}(x)$. Let $b_n = B_n$ and $b_{n-1} = -C_n$. It follows:

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) + C_n p_{n-1}(x) \tag{4.1.2}$$

which is the required three-term recurrence relation.

From (4.1.2), it is clear that $A_n = \frac{k_{n+1}}{k_n}$. To prove the C_{n+1} relation, we begin by multiplying $p_{n-1}(x)w(x)$ and integrate both sides. It follows:

$$0 = A_n \int_a^b x p_n(x) p_{n-1}(x) w(x) dx - C_n \int_a^b p_{n-1}^2(x) w(x) dx. \tag{4.1.3}$$

Now, $p_{n-1} = k_{n-1}x^{n-1}$ + a polynomial of degree $\leq n-2$.

Consequently, $p_n = k_n x^n$ + a polynomial of degree $\leq n-1$. Then,

$$x p_{n-1}(x) = \frac{k_{n-1}}{k_n} p_n(x) + \sum_{k=0}^{n-1} d_k p_k(x). \tag{4.1.4}$$

We see that from (2.2.3),

$$0 = A_n \frac{k_{n-1}}{k_n} c_n - C_n c_n.$$

Since $A_n = \frac{k_{n+1}}{k_n}$, we have

$$C_{n+1} = \frac{A_{n+1}}{A_n} \frac{c_n + 1}{c_n}$$

which concludes our proof. This recurrence relation is particularly useful when calculating p_{n+1}, given you already have p_n and p_{n-1}. □

A consequence of the three-term recurrence relation is the Christoffel-Darboux sum, which is the polynomial analog of the sum in (2.2.3). The Christoffel-Darboux sum allows us to change the expression for $\pi_n^{(\alpha,\beta)}$ in Chapter 6 from a finite sum to a single function, which is easier to analyze.

Theorem 4.1.2. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials with respect to some weight function,* $w(x)$ *on an interval* $[a,b]$ *for* $a, b \in \mathbb{R}$*. Then, given* k_n *is the leading coefficient of* $p_n(x)$ *and that*

$$\int_a^b p_n^2(x)w(x)dx = c_n \neq 0$$

then $\{p_n\}_{n=0}^{\infty}$ *satisfies*

$$\sum_{m=0}^{n} \frac{p_m(x)p_m(y)}{c_m} = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)c_n}. \tag{4.1.5}$$

Proof. From 4.0.1, we have

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x),$$

where $A_n = \frac{k_{n+1}}{k_n}$ and $C_n = \frac{A_{n+1}}{A_n}\frac{c_{n+1}}{c_n}$.

Multiplying through by $p_n(y)$ gives us

$$p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) - C_n p_{n-1}(x)p_n(y). \tag{4.1.6}$$

Similarly, we can consider $p_{n+1}(y)$ and multiply through by $p_n(x)$ to get

$$p_{n+1}(y)p_n(x) = (A_n x + B_n)p_n(y)p_n(x) - C_n p_{n-1}(y)p_n(x). \tag{4.1.7}$$

Subtracting (4.1.7) from (4.1.6) we get

$$p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) = A_n(x-y)p_n(x)p_n(y) - C_n[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)].$$

Diving both sides by $A_n c_n(x-y)$ we get

$$\frac{1}{A_n}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{c_n(x-y)} = \frac{p_n(x)p_n(y)}{c_n} - \frac{1}{A_{n-1}}\left[\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{c_{n-1}(x-y)}\right]$$

Recall, $A_n = \frac{k_{n+1}}{k_n}$ and consequently, $\frac{1}{A_n} = \frac{k_n}{k_{n+1}}$. Then, repeated application of this gives us (4.1.5). □

In general, any orthogonal polynomial can be represented by the Rodrigues formula given in the next definition, which provides information on the interval of orthogonality, the weight function for that family of orthogonal polynomials and the range of parameters for which the polynomials are orthogonal.

Definition 4.1.3. Let $p_n(x)$ denote the nth degree orthogonal polynomial. Then

$$p_n(x) = \frac{1}{a_n w(x)} \frac{d^n}{dx^n} \left(w(x)[Q(x)]^n \right) \tag{4.1.8}$$

where $w(x)$ is a positive weight function and $Q(x)$ is a polynomial in x with coefficients that do not depend on n. There exists a constant a_n, $n \in \mathbb{N}$ associated with the weight function, $w(x)$, which retrieves various families of classical orthogonal polynomials.

4.2 Families of Orthogonal Polynomials

In this section, we introduce some families of orthogonal polynomials which have been shown to have a Gibbs phenomenon when approximating functions with a simple discontinuity, with the same value as that of the classical Gibbs phenomenon.

4.2.1 Legendre Polynomials

Legendre polynomials appear when solving the ordinary differential equation referred to as Legendre's differential equation which comes up when solving Laplace's equation in spherical coordinates. More specifically, when performing a separation of variables in spherical coordinates to solve the Laplace's equation (and assuming additional physical symmetry) the angular equation is the Legendre equation and the solutions to the angular equation are the Legendre polynomials [2]. Adrien-Marie Legendre began using, what are now referred to as Legendre polynomials in 1784 while studying the attraction of spheroids and ellipsoids [9].

Legendre Polynomials are orthogonal with respect to the weight function $w(x) = 1$ over the interval $[-1, 1]$, which is the content of the following theorem whose proof can be found in [13].

Theorem 4.2.1. *The orthogonality property of the Legendre polynomials is given below*

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}$$

where m and n are non-negative integers and and δ_{mn} is the Kronecker function.

Theorem 4.2.2. *The Rodrigues formula for Legendre polynomials is given by*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Definition 4.2.3. The general form of a Legendre polynomial of order n is given by the sum:

$$P_n(x) = \sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

where $M = \frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd.

The first few Legendre polynomials are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

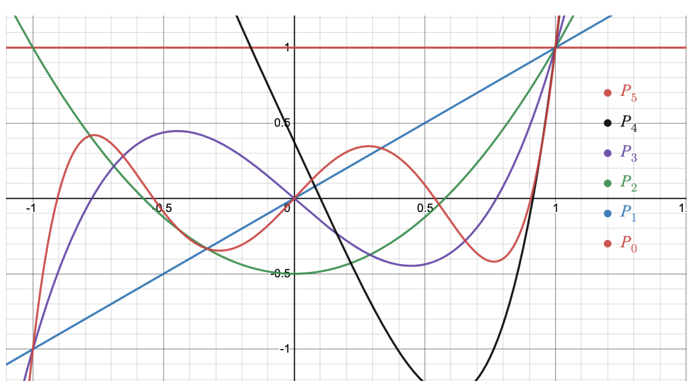


Figure 4.2.1. First five Legendre polynomials.

4.2.2 Hermite Polynomials

Hermite polynomials were first discovered by Pierre-Simon Laplace in 1810 and later studied by Pafnuty Chebyshev in 1859, but ultimately credited by Charles Hermite, who wrote on the polynomials in 1864, describing them as new [11]. Hermite polynomials have applications in signal processing, probability in connection with Brownian motion and in physics, when describing the eigenstates of the quantum harmonic oscillator [11]. There are two commonly discussed Hermite polynomials, the "physicist's Hermite polynomials" and the "probabilist's Hermite polynomials." Here we only discuss the physicist's Hermite polynomials.

The physicist's Hermite polynomials are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on $[-\infty, \infty]$.

Theorem 4.2.4. *The orthogonality property of the Hermite polynomials is given below*

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{mn}$$

where m and n are non-negative, e^{-x^2} is a positive function, and δ_{mn} is the Kronecker function.

Theorem 4.2.5. *The Rodrigues formula for Hermite Polynomials is given by*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Definition 4.2.6. The general form of a Hermite polynomial of order n is given by the sum:

$$H_n(x) = \begin{cases} n! \sum_{l=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-l}}{(2l)!\left(\frac{n}{2}-l\right)!}(2x)^{2l} & \text{for even } n, \\ n! \sum_{l=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-l}}{(2l+1)!\left(\frac{n-1}{2}-l\right)!}(2x)^{2l+1} & \text{for odd } n. \end{cases}$$

The first few Hermite polynomials are:

$$H_0(x) = 1,$$

$$H_1(x) = 2x,$$

$$H_2(x) = 4x^2 - 2,$$

$$H_3(x) = 8x^3 - 12x,$$

$$H_4(x) = 16x^4 - 48x^2 + 12,$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

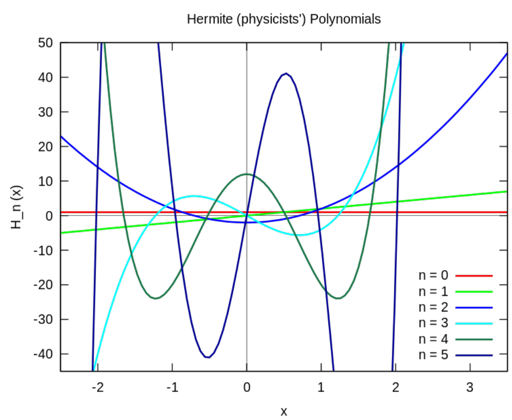


Figure 4.2.2. First five Hermite polynomials. Note, we use Gnuplot with original code by [1] as Desmos does not allow for independent axis increments, and for Hermite polynomials, the range is more variant than the domain.

4.2.3 Gegenbauer Polynomials

The Gegenbauer polynomials, also known as Ultraspherical polynomials, were discovered by Leopold Bernhard Gegenbauer in 1874 and were solutions to the Gegenbauer differential equation and are generalizations of the associated Legendre polynomials [4]. They appear naturally as extensions of Legendre polynomials in the context of potential theory and harmonic analysis. [14] provides an explicit framework for the proof of the Gibbs phenomenon for Gegenbauer polynomials, which we extend to the Jacobi polynomials in the subsequent chapters. The Gegenbauer polynomials are a one-parameter family of orthogonal polynomials that are orthogonal with respect to the weight function $w(x) = (1 - x^2)^{\alpha - 1/2}$ on $[-1, 1]$.

Theorem 4.2.7. *The orthogonality property of the Gegenbauer polynomials is given below for a fixed* α

$$\int_{-1}^{1} C_m^{(\alpha)}(x) C_n^{(\alpha)}(x)(1-x^2)^{\alpha-\frac{1}{2}}\, dx = \begin{cases} \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}\delta_{mn} & \text{for } \alpha \neq 0 \\ \frac{2\pi}{n^2} & \text{for } \alpha = 0 \end{cases}$$

where m *and* n *are non-negative,* $(1-x^2)^{\alpha-\frac{1}{2}}$ *is a positive function, and* δ_{mn} *is the Kronecker function.*

Note, setting $\alpha = \frac{1}{2}$, we get the Legendre polynomials as $w(x)$ becomes $(1-x^2)^0 = 1$.

Theorem 4.2.8. *The Rodrigues formula for Gegenbauer Polynomials is given by*

$$C_n^{(\alpha)}(x) = \frac{(-1)^n}{2^n n!} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}(1-x^2)^{-\alpha+1/2}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-1/2}\right].$$

Definition 4.2.9. The general form of a Gegenbauer polynomial of order n for a fixed α is given by the sum:

$$C_n^{(\alpha)}(x) = \begin{cases} \displaystyle\sum_{k=0}^{\frac{n}{2}} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!((\frac{n}{2}-k)!}(2z)^{2k} & \text{for even } n, \\ \displaystyle\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!\left(\frac{n-1}{2}-k\right)!}(2z)^{2k+1} & \text{for odd } n. \end{cases}$$

Remark 4.2.10. If α is a positive integer, we can write the Γ functions as factorials.

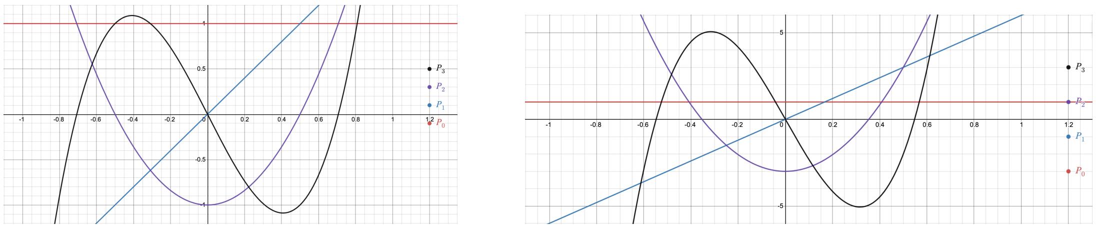
The first few Gegenbauer polynomials are:

$$C_0(x) = 1,$$

$$C_1(x) = 2\alpha x,$$

$$C_2(x) = -\alpha + 2\alpha(1+\alpha)x^2,$$

$$C_3(x) = -2\alpha(1+\alpha)x + \frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3.$$



(a) (b)

Figure 4.2.3. First five Gegenbauer polynomials for (a) $\alpha = 1$ and (b) $\alpha = 3$. An interactive plot can be found **here**.

5

Jacobi Polynomials

Jacobi polynomials were discovered by Gustav Jacob Jacobi, who was influential in many different areas of mathematics, particularly in mathematical physics. The Jacobi polynomials possess all the previously mentioned properties of orthogonal polynomials but are useful because they subsume other families of orthogonal polynomials including the Gegenbauer polynomials, Legendre polynomials, Hermite polynomials and many others. For example, Legendre polynomials are equivalent to $P_n^{(0,0)}(x)$ and Gegenbauer polynomials are equivalent to $P_n^{(\alpha,)}(x)$. Because it has been shown that the Gegenbauer, and therefore the Legendre polynomials have a universal Gibbs constant, the natural next step is to ask if any Jacobi polynomial has a universal Gibbs constant. In this chapter, we provide definitions and theorems related to the Jacobi Polynomials that are needed when working with them. Extensive proofs and analysis of all theorems and definitions can be found in [5]; in particular see Chapter 4.

The Jacobi Polynomials are a two-parameter family that are orthogonal on the interval $[-1,1]$ with respect to the weight function, $w^{(\alpha,\beta)} : [-1,1] \to \mathbb{R}$ defined by

$$w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}. \tag{5.0.1}$$

In order for $w^{(\alpha,\beta)}(x)$ to be integrable we require $\alpha > -1$ and $\beta > -1$ [5].

Recall that any family of orthogonal polynomials can be generated using the Gram-Schmidt procedure on the set $\{1, x, x^2, \ldots\}$ and integrating with respect to a specific weight function [5]. For Jacobi polynomials, we set the weight function to be $w^{(\alpha,\beta)}(x)$ defined above and define the inner product by

$$\langle f, g\rangle_{\alpha,\beta} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)g(x)dx.$$

5.1 Properties of Jacobi Polynomials

In this section, we introduce theorems necessary to get the binomial form of the nth Jacobi polynomial, $P_n^{(\alpha,\beta)}$ which is easy to work with when expanding the sgn function and also encoding into computer algebra systems, such as PARI. In particular, without an explicit representation we are left with a recursive definition such as the Rodrigues formula, which is not as amenable for computation. We begin by introducing the second-order differential equation that the Jacobi polynomials satisfy.

Theorem 5.1.1. *The Jacobi Polynomials $y = P_n^{(\alpha,\beta)}(x)$ are a solution to the linear homogeneous differential equation of second order:*

$$(1-x)^2 y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0 \tag{5.1.1}$$

or

$$\frac{d}{dx}\left(w^{(\alpha+1,\beta+1)}y'\right) + n(n+\alpha+\beta+1)w^{(\alpha,\beta)}y = 0. \tag{5.1.2}$$

Proof. [5, Theorem 4.2.1] □

A consequence of this theorem is the useful formula

$$\frac{d}{dx}\{P_n^{(\alpha,\beta)}(x)\} = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha,\beta)}(x). \tag{5.1.3}$$

We now move onto the Rodrigues formula for Jacobi polynomials, which proves essential in a multitude of calculations including getting the binomial form and consequently the orthogonality condition.

Theorem 5.1.2. *The Jacobi polynomials of degree n are defined by:*

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)} = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \tag{5.1.4}$$

where $\alpha, \beta > -1$ and $x \in [-1, 1]$.

Proof. [5, Theorem 4.3.1] □

Note, when $\alpha = \beta = 0$ we have the Legendre polynomial and when $\alpha = \beta$, we have the Gegenbauer polynomial.

Szego obtains the following explicit formula for $P_n^{(\alpha,\beta)}$ by working inductively with the Rodrigues formula above. This induction step requires the product rule, chain rule, and a combinational argument to collect everything into one sum. For details, see [5, p. 68].

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n}\binom{n+a}{n-j}\binom{n+\beta}{j}\left(\frac{x-1}{2}\right)^j\left(\frac{x+1}{2}\right)^{n-j}. \tag{5.1.5}$$

Remark 5.1.3. The binomial form given in (5.1.5) highlights the varied impact of α and β on Jacobi polynomials. The values of α influence the right-half the interval, whereas the values of β influence the left-half of the interval. We can show this by computing the explicit expressions for $P_n^{(\alpha,\beta)}(1)$ and $P_n^{(\alpha,\beta)}(-1)$:

$$\begin{aligned}
P_n^{(\alpha,\beta)}(1) &= \sum_{j=0}^{n}\binom{n+a}{n-j}\binom{n+\beta}{j}\left(\frac{1-1}{2}\right)^j\left(\frac{1+1}{2}\right)^{n-j} \\
&= \sum_{j=0}^{n}\binom{n+a}{n-j}\binom{n+\beta}{j}. \\
P_n^{(\alpha,\beta)}(-1) &= \sum_{j=0}^{n}\binom{n+a}{n-j}\binom{n+\beta}{j}(\frac{-1-1}{2})^j\left(\frac{-1+1}{2}\right)^{n-j} \\
&= \sum_{j=0}^{n}\binom{n+a}{n-j}\binom{n+\beta}{j}(-1)^j.
\end{aligned}$$

It follows then, that if an $\alpha \neq \beta$ then our polynomial is not symmetric. To visualize this, we look at the 5th Jacobi Polynomial for $(\alpha, \beta) = (2, 3)$ and $(\alpha, \beta) = (3, 2)$ in Figure 5.1.1. An interactive plot of Jacobi Polynomials can be found **here**.

In anticipation of calculations in Chapter 6, we use the binomial form to calculate the leading coefficient, k_n for the nth Jacobi polynomial.

Theorem 5.1.4. *Let $P_n^{\alpha,\beta)}(x)$ be the nth Jacobi polynomial. Then, the leading coefficient, k_n of $P_n^{\alpha,\beta)}(x)$ is defined by*

$$\frac{(n+\alpha+\beta+1)_n}{2^n n!}. \tag{5.1.6}$$

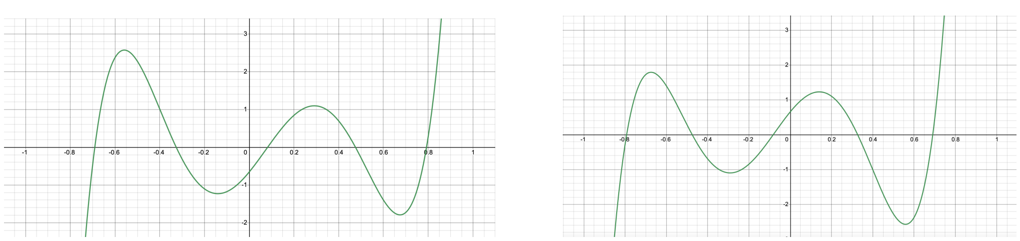
Proof. Gauss showed that there exists a hypergeometric representation of $P_n^{(\alpha,\beta)}$. In this paper, we will not go into the detail regarding the hypergeometric equation of Gauss and corresponding proofs; however, one consequence of the hypergeometric representation is

$$P_n^{(\alpha,\beta)} = \frac{1}{n!}\sum_{v=0}^{n}\binom{n}{v}(n+\alpha+\beta+1)\dots(n+\alpha+\beta+v)\cdot(\alpha+v+1)\dots(\alpha+n)\left(\frac{x-1}{2}\right)^v.$$

and we can replace the general coefficient by $(n+\alpha+\beta+1)(n+\alpha+\beta+2)\dots(2n+\alpha+\beta)$ for $v=n$. Combining this with $\left(\frac{x-1}{2}\right)^n = \frac{(x-1)^n}{2^n}$, we have for $v=n$, the leading coefficient is defined by

$$\frac{(n+\alpha+\beta+1)_n}{2^n n!}.$$

□

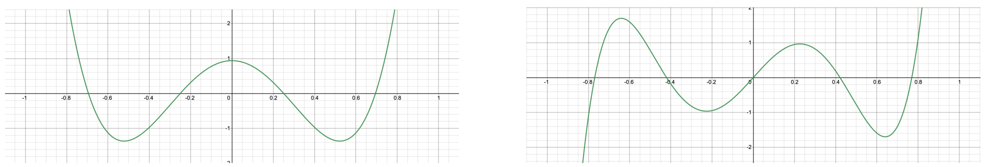


(a) (b)

Figure 5.1.1. Fifth Jacobi Polynomial for (a) $(\alpha,\beta)=(2,3)$ and (b) $(\alpha,\beta)=(3,2)$.

Recall that a function is called an even function if $f(-x)=f(x)$ and an odd function if $f(x)=-f(x)$. For Jacobi polynomials, we have that $P_n^{(\alpha,\beta)}(-x) = -P_n^{(\beta,\alpha)}(x)$. It follows that the Jacobi polynomials are neither odd nor even functions. However, for $\alpha=\beta$ (which gives us the Gegenbauer polynomials) we get a function that is symmetric about the origin or about the

y-axis. In particular, for $P_n^{(\alpha,\beta)}$ is an odd function if n is odd and an even function if n is even. This is shown in Figure 5.1.2. This distinction is important in the calculation of Gibbs constant for Jacobi approximations, namely in the calculation of the critical points.



(a) (b)

Figure 5.1.2. (a) Fourth and (b) fifth Jacobi Polynomial for $(\alpha, \beta) = (2, 2)$

One of the consequences of the binomial form is a simplfied proof of the orthogonality condition for Jacobi polynomials. [11]'s proof is especially accessible, but a proof can also be found in [5].

Theorem 5.1.5. *The orthogonality property of the Jacobi polynomials for a fixed α, β is given by*

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)k!} \delta_{mn}$$

δ_{mn} is Kronecker function and $\Gamma(x)$ is the gamma function.

Proof. [11, Theorem 3.3.1]. □

The orthogonality condition is used in the calculation of the norm of Jacobi polynomials. Specifically we have:

$$\|P_n^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)k!}. \tag{5.1.7}$$

If in addition, $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, then the Gamma functions can be expressed conveniently using a factorial:

$$\frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{(\alpha+k)!(\beta+k)!}{(\alpha+\beta+k)!k!}.$$

6

Gibbs Phenomenon for Jacobi Polynomials

In this chapter we use the properties described in previous chapters to explore whether the Gibbs phenomenon exists for the Jacobi polynomials in the approximation of a function with one interior jump discontinuity. To make our computations easier, we use the sgn function

$$\operatorname{sgn}(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

which is discontinuous at $x = 0$. Note, the sgn function is defined on all $\mathbb{R} - \{0\}$ but Jacobi polynomials are orthogonal only on the interval $[-1, 1]$ which is why we restrict the domain of sgn.

6.1 Jacobi Expansion

In order to approximate the sgn function with Jacobi polynomials, we first define the Jacobi Series approximation of an arbitrary function u as follows:

$$\pi^{(\alpha,\beta)}(u)(x) = \sum_{k=0}^{\infty} \hat{u}_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) \tag{6.1.1}$$

where $P_k^{(\alpha,\beta)}(x)$ is the k-th Jacobi polynomial evaluated at x and $\hat{u}_k^{(\alpha,\beta)}$ are the Jacobi coefficients defined by orthogonal projection

$$\hat{u}_k^{(\alpha,\beta)} = \frac{\left\langle u, P_k^{(\alpha,\beta)} \right\rangle_{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|^2}. \tag{6.1.2}$$

Remark 6.1.1. Going forward we remove the subscript (α, β), but $\left\langle u, P_k^{(\alpha,\beta)} \right\rangle$ implies an inner product with respect to (α, β). Namely,

$$\left\langle u, P_k^{(\alpha,\beta)} \right\rangle := \int_{-1}^{1} u P_k^{(\alpha,\beta)} w^{(\alpha,\beta)}(x) dx.$$

Note, once again we can visualize this approximation as an orthogonal projection of a function u in the vector space of integrable functions onto the subspace spanned by the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ shown in Figure 6.1.1.

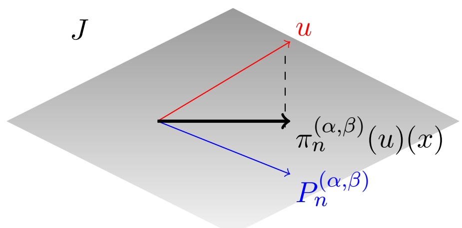


Figure 6.1.1. The orthogonal projection of a function, u, in the vector space of integrable functions onto the subspace J spanned by $\{P_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$.

We can approximate the infinite series by an N-truncated one:

$$\pi_n^{(\alpha,\beta)}(u)(x) = \sum_{k=0}^{n} \hat{u}_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x). \tag{6.1.3}$$

Now, we can set $u = \operatorname{sgn}(x)$ and begin setting up our approximation for the square wave.

$$\pi_n^{(\alpha,\beta)}(\operatorname{sgn})(x) = \sum_{k=0}^{n} \widehat{\operatorname{sgn}} P_k^{(\alpha,\beta)}(x). \tag{6.1.4}$$

By (6.1.2),

$$\widehat{\operatorname{sgn}}_k^{(\alpha,\beta)} = \frac{\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)} \right\rangle}{\|P_k^{(\alpha,\beta)}\|^2} \tag{6.1.5}$$

We begin by calculating the norm using Definition 3.1.2

$$\begin{aligned}
\|P_k^{(\alpha,\beta)}\|^2 &= \sqrt{\left\langle P_k^{(\alpha,\beta)}, P_k^{(\alpha,\beta)} \right\rangle^2} \\
&= \left\langle P_k^{(\alpha,\beta)}, P_k^{(\alpha,\beta)} \right\rangle \\
&= \int_{-1}^{1} w^{(\alpha,\beta)}(x) (P_k^{(\alpha,\beta)})^2 dx
\end{aligned}$$

where $w^{(\alpha,\beta)}(x)$ is the weight function for Jacobi polynomials and thus $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1-x)^{\beta}$. Substituting we have,

$$\|P_k^{(\alpha,\beta)}\|^2 = \int_{-1}^{1} (1-x)^{\alpha}(1-x)^{\beta}(P_k^{(\alpha,\beta)})^2 dx$$

Note, this is the same integral in Theorem 5.1.5, and by (5.1.7) we have

$$\|P_k^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\frac{(k+\alpha)!(k+\beta)!}{(k+\alpha+\beta)!k!}$$

where both $\alpha, \beta \in \mathbb{Z}_{>-1}$.

Remark 6.1.2. If $a, b \neq \mathbb{Z}$ we replace the factorials with Γ functions defined in Chapter 3, but we still require $\alpha, \beta \in \mathbb{Z}_{>-1}$.

Then, we calculate the inner product $\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)} \right\rangle$:

$$\begin{aligned}\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)} \right\rangle &= \int_{-1}^{1} \operatorname{sgn}(x) P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx \\ &= \int_{-1}^{0} (-1) P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx + \int_{0}^{1} (1) P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx\end{aligned}$$

In order to calculate this integral, we use the differential equation defined in Theorem 5.1.1 which states that:

$$\frac{d}{dx}\left[w^{(\alpha+1,\beta+1)}(x)\frac{d}{dx}\left(P_k^{(\alpha,\beta)}(x)\right)\right] + k(k+\alpha+\beta+1)w^{(\alpha,\beta)}P_k^{(\alpha,\beta)} = 0. \tag{6.1.6}$$

It follows, by the Fundamental Theorem of Calculus integrating through gives us:

$$\begin{aligned} w^{(\alpha+1,\beta+1)}(x)\frac{d}{dx}\left(P_k^{(\alpha,\beta)}(x)\right) + k(k+\alpha+\beta+1)\int w^{(\alpha,\beta)}P_k^{(\alpha,\beta)}dx &= \int 0 \\ k(k+\alpha+\beta+1)\int w^{(\alpha,\beta)}P_k^{(\alpha,\beta)}dx &= -w^{(\alpha+1,\beta+1)}(x)\frac{d}{dx}\left(P_k^{(\alpha,\beta)}(x)\right) + C \\ \int w^{(\alpha,\beta)}P_k^{(\alpha,\beta)}dx &= -\frac{w^{(\alpha+1,\beta+1)}(x)}{k(k+\alpha+\beta+1)}\frac{d}{dx}\left(P_k^{(\alpha,\beta)}(x)\right) + C.\end{aligned}$$

From (5.1.3) we know

$$\frac{d}{dx}\left(P_k^{(\alpha,\beta)}(x)\right) = \frac{\alpha+\beta+k+1}{2}P_{k-1}^{(\alpha+1,\beta+1)}(x) \tag{6.1.7}$$

and thus we have:

$$\int (1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)dx = -\frac{w^{(\alpha+1,\beta+1)}(x)}{k(k+\alpha+\beta+1)}\frac{\alpha+\beta+k+1}{2}P_{k-1}^{(\alpha+1,\beta+1)}(x)+C \quad (6.1.8)$$

$$= -\frac{w^{(\alpha+1,\beta+1)}(x)}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(x)+C. \quad (6.1.9)$$

Returning to our set up for $\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)}\right\rangle$ we have

$$\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)}\right\rangle = -\int_{-1}^{0} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx + \int_{0}^{1} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$

Using (6.1.8), we evaluate each integral.

$$\begin{aligned}\int_{-1}^{0} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx &= -\frac{w^{(\alpha+1,\beta+1)}(0)}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(0) - \left(-\frac{w^{(\alpha+1,\beta+1)}(-1)}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(-1)\right)\\ &= -\frac{1}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(0) - \left(-\frac{0}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(-1)\right)\\ &= -\frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{2k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{0}^{1} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx &= -\frac{w^{(\alpha+1,\beta+1)}(1)}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(1) - \left(-\frac{w^{(\alpha+1,\beta+1)}(0)}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(0)\right)\\ &= -\frac{0}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(1) - \left(-\frac{1}{2k}P_{k-1}^{(\alpha+1,\beta+1)}(0)\right)\\ &= \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{2k}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle \operatorname{sgn}(x), P_k^{(\alpha,\beta)}\right\rangle dx &= -\left(\frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{2k}\right) + \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{2k}\\ &= \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k}.\end{aligned}$$

Using (6.1.5), we get an explicit expression for the Jacobi Coefficients of our function,

$$\begin{aligned}\widehat{\operatorname{sgn}}_k^{(\alpha,\beta)} &= \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k}\cdot\frac{(2k+\alpha+\beta+1)(k+\alpha+\beta)!k!}{2^{\alpha+\beta+1}(k+\alpha)!(k+\beta)!}\\ &= \frac{(2k+\alpha+\beta+1)(k+\alpha+\beta)!(k-1)!}{2^{\alpha+\beta+1}(k+\alpha)!(k+\beta)!}P_{k-1}^{(\alpha+1,\beta+1)}(0). \quad (6.1.10)\end{aligned}$$

Substituting back into 6.1.4

$$\pi_n^{(\alpha,\beta)}(\text{sgn})(x) = \sum_{k=0}^{N} \frac{(2k+\alpha+\beta+1)(k+\alpha+\beta)!(k-1)!}{2^{\alpha+\beta+1}(k+\alpha)!(k+\beta)!} P_{k-1}^{(\alpha+1,\beta+1)}(0) P_k^{(\alpha,\beta)}(x).$$

Notice when $k = 0$ we have $P_{0-1}(0) = 0$ trivially. Thus, our sum begins at $k = 1$

$$\pi_n^{(\alpha,\beta)}(\text{sgn})(x) = \sum_{k=1}^{N} \frac{(2k+\alpha+\beta+1)(k+\alpha+\beta)!(k-1)!}{2^{\alpha+\beta+1}(k+\alpha)!(k+\beta)!} P_{k-1}^{(\alpha+1,\beta+1)}(0) P_k^{(\alpha,\beta)}(x). \tag{6.1.11}$$

At this point, we can compute this sum for any n and any (α, β). Using Desmos, we show two plots below.

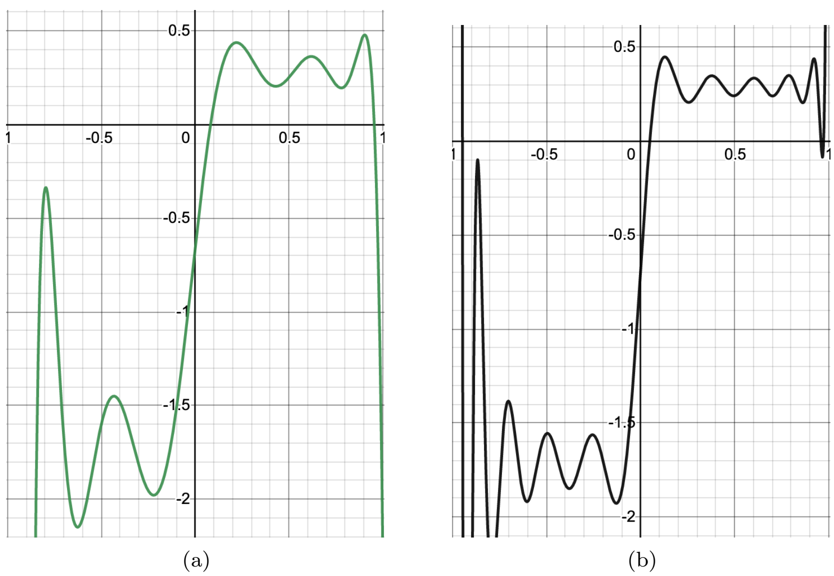
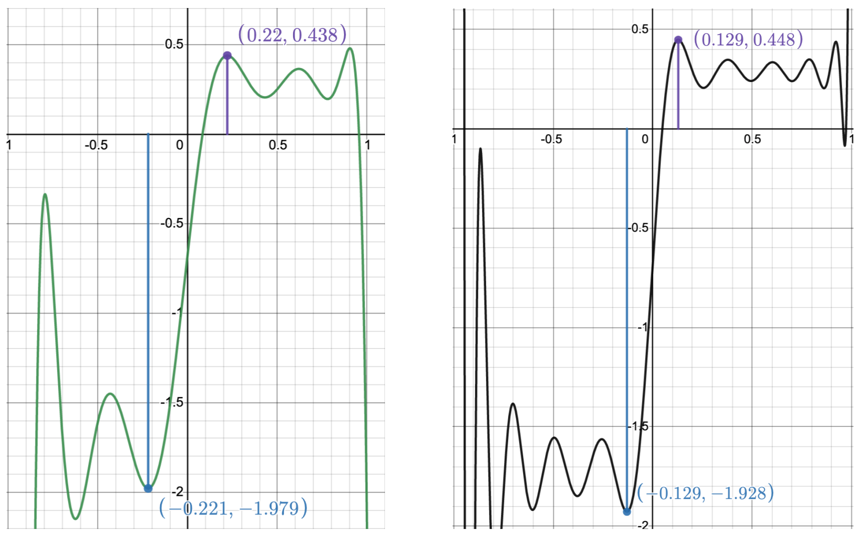


Figure 6.1.2. Jacobi polynomial approximation for square wave for $(\alpha, \beta) = (2, 5)$ and (a) $n = 10$ and (b) $n = 20$.

Notice, when $\alpha \neq \beta$ (which would yield a Gegenbauer polynomial if $\alpha = \beta$), the approximating function is no longer passes through the origin. However, there is still an overshoot and an undershoot but since the approximating function is not odd, the x-coordinate of the undershoot and overshoot are not centered around 0. Also, as n increases, both the overshoot and undershoot approach the point of discontinuity, which is in line with the classical Gibbs phenomenon. When looking at the difference between the smallest positive critical point and the biggest negative critical point from the y-intercept, we find that we do in fact have a Gibbs constant, and in particular we get closer to γ as n increases. This is shown in Figure 6.1.3.

Remark 6.1.3. Interestingly enough, when $|\alpha - \beta| = 1$ the x-coordinate of the overshoot and undershoot are the same in magnitude for all n. On the other hand, for $|\alpha - \beta| > 1$, where $\alpha, \beta \in \mathbb{Z} > -1$ the magnitude of x-coordinate of the overshoot and undershoot approaches the same value as n gets bigger. Both of these phenomena can be visualized using the Desmos graph **here**.



(a) (b)

Figure 6.1.3. Difference between the smallest positive critical point and the biggest negative critical point divided by two for $(\alpha, \beta) = (2, 5)$ and (a) $n = 10$ is ≈ 1.209 and (b) $n = 20$ is ≈ 0.188.

Remark 6.1.4. We also find an interesting result when checking where $\pi_n^{(\alpha,\beta)}$ crosses the y-axis is for specific α, β. In particular, as $n_{even} \to \infty$, the y-intercepts decrease monotonically, and as $n_{odd} \to \infty$, the y-intercepts increase monotonically. Since both sequences are bounded and monotone, we expect they converge to the same value as $n \to \infty$. The following PARI code can be used to observe this behavior up to $n = 300$:

```
P(n,a,b,x) = sum(j=0,n,binomial(n+a,n-j)*binomial(n+b,j)*((x-1)/2)^j
*((x+1)/2)^(n-j));
S(k,a,b) = P(k-1,a+1,b+1,0)*(2*k+a+b+1)*(k+a+b)!*(k-1)!/((2^(1+b+a))
*(k+a)!*(k+b)!);
PI(n,a,b,x) = sum(k=1,n,S(k,a,b)*P(k,a,b,x));
```

```
yINT(n,a,b) = 1.0*PI(n,a,b,0); #multiplying by 1.0 for decimal notation.
for(n-1,300, print(n, "   ", yINT(n,a,b))); #user must input a and b.
```

6.2 Calculation of the Gibbs Constant

In this section, we work out a more direct formula to calculate the critical points of $\pi_n^{(\alpha,\beta)}(\text{sgn})(x)$; specifically, the smallest positive critical point and the largest negative. To obtain them, we first work out an alternative expression for $\pi_n^{(\alpha,\beta)}(\text{sgn})(x)$ that is more amenable to computation, using the Christoffel-Darboux identity defined by Theorem 4.1.5.

Lemma 6.2.1. *With all notation as described in the previous section, we have*

$$\frac{d}{dx}\left(\pi_n^{(\alpha,\beta)}(\text{sgn})(x)\right) = d_n^{(\alpha,\beta)}\left(\frac{P_n^{(\alpha+1,\beta+1)}(0)P_{n+1}^{(\alpha+1,\beta+1)}(x) - P_{n+1}^{(\alpha+1,\beta+1)}(0)P_n^{(\alpha+1,\beta+1)}(x)}{x}\right), \tag{6.2.1}$$

where

$$d_n^{(\alpha,\beta)} = \frac{4(n+1)(n+\alpha+\beta+3)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)\|P_n^{(\alpha+1,\beta+1)}\|^2}.$$

Proof. Recall that $\pi_n^{(\alpha,\beta)}(\text{sgn})(x) = \sum_{k=1}^n \widehat{\text{sgn}}_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x)$, where

$$\widehat{\text{sgn}}_k^{(\alpha,\beta)} = \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{\|P_k^{(\alpha,\beta)}\|^2 k}.$$

We begin the sum at $k=1$ because for $k=0$, $\widehat{\text{sgn}}_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) = 0$. Also, by definition we have

$$\begin{aligned}\pi_n^{(\alpha,\beta)}(\text{sgn})(x) &= \sum_{k=0}^n \widehat{\text{sgn}}(x) P_k^{(\alpha,\beta)}(x)\\ &= \sum_{k=0}^n \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k}P_k^{(\alpha,\beta)}(x).\end{aligned}$$

Now, taking the derivative term-by-term we get

$$\begin{aligned}\frac{d}{dx}(\pi_n^{(\alpha,\beta)}(\text{sgn})(x)) &= \frac{d}{dx}\sum_{k=0}^n \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k}P_k^{(\alpha,\beta)}(x)\\ &= \sum_{k=0}^n \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k}\frac{d}{dx}P_k^{(\alpha,\beta)}(x).\end{aligned}$$

By (5.1.3),

$$\begin{aligned} &= \sum_{k=0}^{n} \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2} \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{k} \frac{\alpha+\beta+k+1}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x) \\ &= \sum_{k=0}^{n} \frac{\alpha+\beta+k+1}{2k\|P_k^{(\alpha,\beta)}(x)\|^2} P_{k-1}^{(\alpha+1,\beta+1)}(0) P_{k-1}^{(\alpha+1,\beta+1)}(x). \end{aligned}$$

Recall that

$$\|P_k^{(\alpha,\beta)}(x)\|^2 = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{(k+\alpha)!(k+\beta)!}{(k+\alpha+\beta)!k!}$$

and thus

$$\begin{aligned} \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2} \frac{1}{k} \frac{\alpha+\beta+k+1}{2} &= \frac{2k+\alpha+\beta+1}{2^{\alpha+\beta+1}} \frac{(k+\alpha+\beta)!k!}{(k+\alpha)!(k+\beta)!} \frac{1}{k} \frac{\alpha+\beta+k+1}{2} \\ &= \frac{(2k+\alpha+\beta+1)(\alpha+\beta+k+1)!(k-1)!}{(2^{\alpha+\beta+2})(k+\alpha)!(k+\beta)!}. \end{aligned}$$

We notice that this expression is almost equivalent to the norm of the $k-1$st polynomial.

$$\begin{aligned} \frac{1}{\|P_{k-1}^{(\alpha+1,\beta+1)}(x)\|^2} &= \frac{(2(k-1)+\alpha+1+\beta+1+1)(k-1+\alpha+1+\beta+1)!(k-1)!}{(2^{\alpha+1+\beta+1+1})(k-1+\alpha+1)!(k-1+\beta+1)!} \\ &= \frac{(2k-2+\alpha+\beta+3)(k+\alpha+\beta+1)!(k-1)!}{(2^{\alpha+\beta+3})(k+\alpha)!(k+\beta)!} \\ &= \frac{(2k+\alpha+\beta+1)(k+\alpha+\beta+1)!(k-1)!}{(2^{\alpha+\beta+3})(k+\alpha)!(k+\beta)!}. \end{aligned}$$

In particular, we establish

$$\frac{1}{\|P_{k-1}^{(\alpha+1,\beta+1)}(x)\|^2} = \left(\frac{\alpha+\beta+k+1}{4k}\right) \frac{1}{\|P_k^{(\alpha,\beta)}(x)\|^2}. \tag{6.2.2}$$

Then substituting back into our derivative equation we have

$$\begin{aligned} \frac{d}{dx}(\pi_n^{(\alpha,\beta)}(\text{sgn})(x)) &= \sum_{k=1}^{n} \frac{P_{k-1}^{(\alpha+1,\beta+1)}(0)}{\frac{1}{2}\|P_{k-1}^{(\alpha+1,\beta+1)}\|^2 k} P_{k-1}^{(\alpha+1,\beta+1)}(x) \\ &= 2\sum_{k=1}^{n} \frac{P_{k-1}^{(\alpha+1,\beta+1)}(x) P_{k-1}^{(\alpha+1,\beta+1)}(0)}{\|P_{k-1}^{(\alpha+1,\beta+1)}\|^2} \end{aligned}$$

At this point, we can use the Christoffel-Darboux dentity defined by Theorem 4.1.5. Then, our sum becomes

$$\sum_{k=0}^{n} 2\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x) P_{k-1}^{(\alpha+1,\beta+1)}(0)}{\|P_{k-1}^{(\alpha+1,\beta+1)}\|\|P_{k-1}^{(\alpha+1,\beta+1)}\|} = 2\frac{k_n}{c_n k_{n+1}} \frac{P_N^{(\alpha+1,\beta+1)}(x) P_{N-1}^{(\alpha+1,\beta+1)}(0) - P_N^{(\alpha+1,\beta+1)}(0) P_{N-1}^{(\alpha+1,\beta+1)}(x)}{x}. \tag{6.2.3}$$

Recall that k_n is the leading coefficient of $p_n(x)$ defined by Theorem 5.1.4:

$$k_n = \frac{(n+\alpha+\beta+1)_n}{2^n n!}$$

thus k_n's associated with $Q_n(x)$ are

$$k_n = \frac{(n+\alpha+1+\beta+1+1)_n}{2^n n!} \tag{6.2.4}$$

$$k_{n+1} = \frac{(n+1+\alpha+1+\beta+1+1)_{n+1}}{2^{n+1}(n+1)!}. \tag{6.2.5}$$

For ease of computation we drop the norms and introduce them at the end again. Then $\frac{2k_n}{k_{n+1}}$ becomes

$$\begin{aligned}\frac{2k_n}{k_{n+1}} &= \frac{(n+\alpha+\beta+3)_n}{2^n n!} \cdot \frac{2^{n+1}(n+1)!}{(n+\alpha+\beta+4)_{n+1}} \\ &= \frac{4(n+\alpha+\beta+3)_n(n+1)}{(n+\alpha+\beta+4)_{n+1}}.\end{aligned}$$

By definition of the Pochhamer symbol,

$$\begin{aligned}\frac{(n+\alpha+\beta+3)_n}{(n+\alpha+\beta+4)_{n+1}} &= \frac{(n+\alpha+\beta+3)(n+\alpha+\beta+4)\dots(n+\alpha+\beta+3+n-1)}{(n+\alpha+\beta+4)(n+\alpha+\beta+5)\dots(n+\alpha+\beta+4+n)} \\ &= \frac{(n+\alpha+\beta+3)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)}.\end{aligned}$$

c_n is the orthogonality constant defined by (5.1.7):

$$c_n = \|P_n^{(\alpha+1,\beta+1)}\|^2$$

and consequently,

$$\begin{aligned}\frac{1}{\|P_n^{(\alpha+1,\beta+1)}\|^2} &= \frac{(2+n+\alpha+1+\beta+1+1)}{2^{\alpha+1+\beta+1+1}} \frac{(n+\alpha+1+\beta+1)!}{(n+\alpha+1)!(n+\beta+1)!} \\ &= \frac{(2n+\alpha+\beta+3)}{2^{\alpha+\beta+3}} \cdot \frac{(n+\alpha+\beta+2)!}{(n+\alpha+1)!(n+\beta+1)!}.\end{aligned}$$

However, for the purposes of our calculations we leave $c_n = \|P_n^{(\alpha+1,\beta+1)}\|^2$ and thus we have

$$\frac{2k_n}{c_n k_{n+1}} = \frac{4(n+1)(n+\alpha+\beta+3)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)} \cdot \frac{1}{\|P_n^{(\alpha+1,\beta+1)}\|^2}.$$

Returning to our sum in (6.2.3) we have

$$2\sum_{k=0}^{n} \frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)P_{k-1}^{(\alpha+1,\beta+1)}(0)}{\|P_{k-1}^{(\alpha+1,\beta+1)}\|\|P_{k-1}^{(\alpha+1,\beta+1)}\|} = \frac{2k_N}{c_N k_{N+1}} \left(\frac{P_N^{(\alpha+1,\beta+1)}(x)P_{N-1}^{(\alpha+1,\beta+1)}(0) - P_N^{(\alpha+1,\beta+1)}(0)P_{N-1}^{(\alpha+1,\beta+1)}(x)}{x} \right).$$

We conclude that

$$\frac{d}{dx}(\pi_n^{(\alpha,\beta)}(\mathrm{sgn})(x)) = d_n^{(\alpha,\beta)}\left(\frac{P_n^{(\alpha+1,\beta+1)}(0)P_{n+1}^{(\alpha+1,\beta+1)}(x) - P_{n+1}^{(\alpha+1,\beta+1)}(0)P_n^{(\alpha+1,\beta+1)}(x)}{x}\right), \tag{6.2.6}$$

where

$$d_n^{(\alpha,\beta)} = \frac{4(n+1)(n+\alpha+\beta+3)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)\|P_n^{(\alpha+1,\beta+1)}\|^2}. \tag{6.2.7}$$

□

Remark 6.2.2. Observe that

$$P_n^{(\alpha+1,\beta+1)}(0)P_{n+1}^{(\alpha+1,\beta+1)}(x) - P_{n+1}^{(\alpha+1,\beta+1)}(0)P_n^{(\alpha+1,\beta+1)}(x)$$

vanishes when $x = 0$, hence is divisible by x. We will need this observation when computing an integral expression for $\pi_N^{(\alpha,\beta)}(\mathrm{sgn})(x)$.

6.2.1 Limit Formula

Because it will be useful to us in our proofs below, we work in this subsection to study the value of $\pi_n^{(\alpha,\beta)}(\mathrm{sgn})(0)$ and its behavior as $n \to \infty$. Since $\pi_n^{(\alpha,\beta)}(\mathrm{sgn})(x)$ is a linear combination of polynomials, and thus a polynomial itself, it is clearly defined at $x = 0$.

Let

$$F_n^{(\alpha,\beta)}(x) = P_n^{(\alpha+1,\beta+1)}(0)P_{n+1}^{(\alpha+1,\beta+1)}(x) - P_{n+1}^{(\alpha+1,\beta+1)}(0)P_n^{(\alpha+1,\beta+1)}(x). \tag{6.2.8}$$

Observe that F_n has degree n, so that after division by x and integration, the right hand side is a polynomial of degree n. (6.2.6) gives us that

$$\pi_n^{(\alpha,\beta)}(\mathrm{sgn})(z) = \int_0^z d_n^{(\alpha,\beta)} \frac{F_n^{(\alpha,\beta)}(x)}{x} dx, \tag{6.2.9}$$

where z is the smallest positive zero of $\frac{d}{dx}(\pi_n^{(\alpha,\beta)})$.

Lemma 6.2.3. *With all notation as above, the limit* $\lim_{n\to\infty} \pi_n^{(\alpha,\beta)}(\mathrm{sgn})(0)$ *exists. We denote it by* $\lambda^{(\alpha,\beta)}$.

Proof Overview Substituting directly into (6.2.9) we have

$$\pi_n^{(\alpha,\beta)}(\text{sgn})(0) = \lim_{x\to 0^+} \int_0^x d_n^{(\alpha,\beta)} \frac{F_n^{(\alpha,\beta)}(x)}{x} dx.$$

However, when $\alpha \neq \beta$ there exists no simple equation for $P_n^{(\alpha,\beta)}(0)$ for the Jacobi polynomials, and consequently we end up with an integrand we do not understand. However, from Remark 6.1.4 we know that as $n \to \infty$, by the boundedness and monotonicity of even and odd n's we have a convergence.

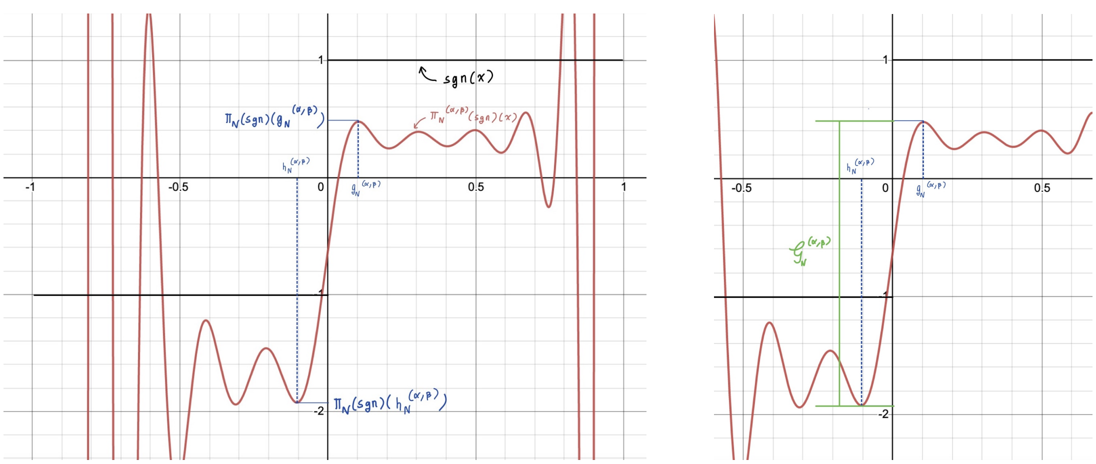
Definition 6.2.4. For a fixed N, we define $g_N^{(\alpha,\beta)}$ to be the function that associates to (α,β) to be the smallest positive zero of $\frac{d}{dx}\left(\pi_N^{(\alpha,\beta)}(\text{sgn})(x)\right)$ and define $h_N^{(\alpha,\beta)}$ that associates to (α,β) to be the biggest negative zero of $\frac{d}{dx}\left(\pi_N^{(\alpha,\beta)}(\text{sgn})(x)\right)$.

Using this definition, we define

$$\mathscr{G}_N^{(\alpha,\beta)} = \pi_N^{(\alpha,\beta)}(\text{sgn})(g_N^{(\alpha,\beta)}) - \pi_N^{(\alpha,\beta)}(\text{sgn})(h_N^{(\alpha,\beta)}), \tag{6.2.10}$$

and the Gibbs constant $\mathscr{G}^{(\alpha,\beta)}$ is defined as the limit of $\mathscr{G}_N^{(\alpha,\beta)}$ as $N \to \infty$:

$$\mathscr{G}^{(\alpha,\beta)} = \lim_{N\to\infty} \mathscr{G}_N^{(\alpha,\beta)}. \tag{6.2.11}$$



(a) (b)

Figure 6.2.1. The smallest positive zero $g_N^{(\alpha,\beta)}$ and biggest negative zero $h_N^{(\alpha,\beta)}$ and $\mathscr{G}_N^{(\alpha,\beta)}$ as defined in d=Definition 6.2.10.

Theorem 6.2.5. *For all $\alpha, \beta > -1$,*

$$\mathscr{G}^{(\alpha,\beta)} = \mathscr{G}^{(0,0)} = 2\int_0^{\pi} \frac{\sin t}{t} dt = \gamma. \tag{6.2.12}$$

Theorem 6.2.5 tells us two things. First, that the Gibbs Constant exists for all $\alpha, \beta > -1$, and second that it is the same for all values of α, β. In order to prove this, we require two lemmas defined in [14] for which the proofs are identical to those provided for the Gegenbauer polynomials, except every instance of λ is replaced by (α, β).

Lemma 6.2.6. *For n even, $n \to +\infty$ and for all $(\alpha, \beta) > -1$*

$$\lim_{n\ even \to +\infty} n\left(\frac{\pi}{2} - \theta^{(\alpha,\beta)}_{[n/2],n}\right) = \frac{\pi}{2}.$$

For n odd, $n \to +\infty$ and for all $(\alpha, \beta) > -1$

$$\lim_{n\ odd \to +\infty} n\left(\frac{\pi}{2} - \theta^{(\alpha,\beta)}_{[n/2],n}\right) = \pi.$$

Note that $x^{(\alpha,\beta)}_{[n/2],n} = \cos\theta^{(\alpha,\beta)}_{[n/2],n}$ is the smallest positive zero of $P_n^{(\alpha,\beta)}$.

Lemma 6.2.7. *For $(\alpha, \beta) > -1$ and $u \in \mathbb{R}$,*

$$\lim_{n \to +\infty} (-1)^n n^{1-(\alpha,\beta)} P^{(\alpha,\beta)}_{2n+1}\left(\sin\frac{u}{2n}\right) = \frac{1}{\Gamma(\alpha,\beta)} \sin u$$

Remark 6.2.8. *Comments on Lemma 6.2.6.* The purpose of this lemma is to provide a translation of the critical points of $\pi_N^{(\alpha,\beta)}(\text{sgn})(x)$ into values in the set $\{\cos(\theta_i^{(\alpha,\beta)})\}$. By [11, Theorem 4.0.1], we know that $\pi_N^{(\alpha,\beta)}(\text{sgn})(x)$ has N real critical points, all between -1 and 1. Define $\xi_i^{(\alpha,\beta)}$ as the ith critical point for a fixed α, β

$$-1 < \xi_1^{(\alpha,\beta)}, \xi_2^{(\alpha,\beta)}, \ldots, \xi_N^{(\alpha,\beta)} < 1.$$

Then, each $\xi_i^{(\alpha,\beta)} = \cos(\theta_i^{(\alpha,\beta)})$ where $\theta_i^{(\alpha,\beta)} \in (0, \pi)$. It follows that the smallest positive zero $g_N^{(\alpha,\beta)}$ is associated with the value of $\cos(\theta_i^{(\alpha,\beta)})$ less than $\frac{\pi}{2}$ that corresponds to a critical point. Likewise, the largest negative zero, $h_N^{(\alpha,\beta)}$ is associated with the value of $\cos(\theta_i^{(\alpha,\beta)})$ greater than $\frac{\pi}{2}$ that corresponds to a critical point, as shown in Figure 6.2.2.

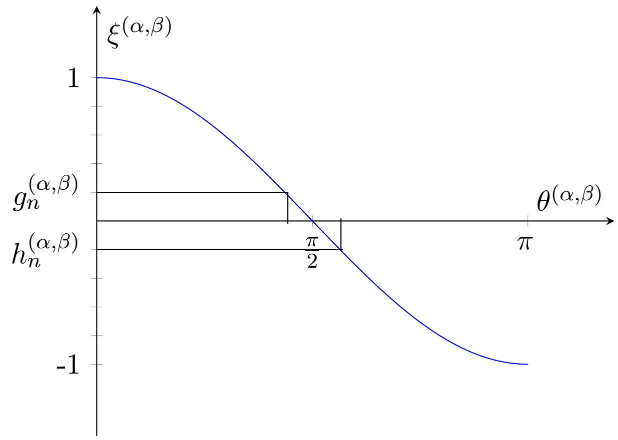


Figure 6.2.2. Translation of critical points, $\xi_i^{(\alpha,\beta)}$ into angles, $\cos(\theta_i^{(\alpha,\beta)})$, where $g_n^{(\alpha,\beta)}$ is the smallest positive zero and $h_n^{(\alpha,\beta)}$ is the largest negative zero.

We now return to the proof of Theorem 6.2.5.

Proof. From (6.2.6) and the fact that $g_N^{(\alpha,\beta)}$ is the smallest positive zero we deduce

$$\pi_n^{(\alpha,\beta)}(\text{sgn})(g_N^{(\alpha,\beta)}(\alpha,\beta)) = \int_0^{g_N^{(\alpha,\beta)}(\alpha,\beta)} d_n^{(\alpha,\beta)} F_n^{(\alpha,\beta)}(x)dx.$$

For $(\alpha,\beta) = (0,0)$, the Jacobi polynomials collapse into the Legendre Polynomials, which is a special case worked out in [14, p. 12].

We complete our proof by checking general case, $(\alpha,\beta) \in [-1,1]$, we define $g_N^{(\alpha,\beta)} = \cos[\theta_{[N/2],N}^{(\alpha+1,\beta+1)}]$ and let $x = \sin\theta$ then

$$\int_0^{g_N^{(\alpha,\beta)}(\alpha,\beta)} \frac{F_n^{(\alpha,\beta)}(x)}{x}dx = \int_0^{\frac{\pi}{2}-\theta_{[N/2],N}^{(\alpha+1,\beta+1)}} \frac{F_n^{(\alpha,\beta)}(\sin\theta)}{\sin\theta}\cos\theta d\theta.$$

Note the upper and lower bound change accordingly with the change of variables. In particular when $\sin\theta = 0$ then $\theta = 0$ and when $\sin\theta = g_N^{(\alpha,\beta)}$, $\theta = \frac{\pi}{2} - \theta_{[N/2],N}^{(\alpha+1,\beta+1)}$ by Lemma 6.2.6. We perform another change of variables by letting $\theta = \frac{u}{2(N+1)}$. Then, $d\theta = \frac{du}{2(N+1)}$ and we have

$$\int_0^{g_N^{(\alpha,\beta)}} \frac{F_n^{(\alpha,\beta)}(x)}{x}dx = \int_0^{\phi_N^{(\alpha,\beta)}} \frac{F_n^{(\alpha,\beta)}(\sin\left(\frac{u}{2(N+1)}\right))}{2(N+1)\sin\left(\frac{u}{2(N+1)}\right)}\cos\left(\frac{u}{2(N+1)}\right)du. \tag{6.2.13}$$

The lower bound remains the same by definition. However, by Lemma 6.2.6 the upper bound:

$$\phi_N^{(\alpha,\beta)} = 2(N+1)\left(\frac{\pi}{2} - \theta_{[N/2],N}^{(\alpha+1,\beta+1)}\right) \tag{6.2.14}$$

goes to π as $N \to \infty$. The cosine term goes to 1 as $N \to \infty$, since $\frac{u}{2(N+1)} \to 0$. As $N \to \infty$, the term in the denominator goes to u because

$$2(N+1)\sin\left(\frac{u}{2(N+1)}\right) = \frac{\sin\left(\frac{u}{2(N+1)}\right)}{\frac{1}{2(N+1)}}$$
$$= \frac{\sin\left(\frac{u}{2(N+1)}\right)}{\frac{u}{u2(N+1)}}$$
$$= u\left(\frac{\sin\left(\frac{u}{2(N+1)}\right)}{\frac{u}{2(N+1)}}\right).$$

Recall $\lim_{x\to\infty} \frac{\sin x}{x} = 1$. Thus,

$$\lim_{N\to\infty} u\left(\frac{\sin\left(\frac{u}{2(N+1)}\right)}{\frac{u}{2(N+1)}}\right) = u \lim_{N\to\infty} \frac{\sin\left(\frac{u}{2(N+1)}\right)}{\frac{u}{2(N+1)}} = u.$$

Refer to [5, Section 7.32] for identities of Jacobi polynomials with trigonometric arguments. An argument analogous to [14] finishes the argument. That is, we combine (6.2.7) and Lemma 6.2.7 and get

$$\lim_{n\to\infty} d_n^{(\alpha,\beta)} F_n^{(\alpha,\beta)}\left(\sin\left(\frac{u}{2(N+1)}\right)\right) = \frac{2}{\pi}\sin u. \tag{6.2.15}$$

Substituting this back into (6.2.13) in we have:

$$\lim_{n\to\infty} \pi_n^{(\alpha,\beta)}(\text{sgn})(z) = \frac{2}{\pi}\int_0^\pi \frac{\sin u}{u} du \tag{6.2.16}$$

which is precisely our constant γ defined in the Introduction. This concludes our proof.

□

7
Conclusion and Future Work

We have computationally shown that all Jacobi expansions with interior jump discontinuities have a Gibbs constant and we have provided a framework for analytically proving this. It remains to prove the behaviour of the x-coordinate of the overshoot and undershoot as n gets bigger, described in Remark 6.1.3. It also remains to prove that the limit $\lim_{n\to+\infty} \pi_n^{(\alpha,\beta)}(\text{sgn})(0)$ exists, described in Lemma 6.2.3. Both of these claims have been shown computationally.

One family of Jacobi polynomials we have yet to fully understand is the Laguerre family which is a Jacobi polynomial for $\alpha = \infty$ and $\beta = \infty$. In particular, Laguerre polynomials are orthogonal over $[0, \infty)$ with respect to the weight function, $w^{\alpha}(x) = x^{\alpha}e^{-x}$. We predict the Gibbs phenomenon will also hold for these polynomials, but this has yet to be shown analytically.

As we consider other types of approximations, a question that arises is whether similar phenomenon exist for approximating continuous functions with discontinuous functions. An example of this would be approximating sine and cosine functions with wavelets. Unlike considering a Riemann sum under a curve, we would select a specific basis and take linear combinations to minimize the discrepancies between the two curves. It is possible that the sum of the overshoots and undershoots in this approximation sum up to γ. Ultimately, our findings and discussions demonstrate that there are still many questions surrounding the Gibbs phenomenon suitable for research projects and further investigations.

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