Upper Bounds for the Number of Lattice Edges Needed to Represent 4-Regular Graphs as Lattice Graphs

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Upper Bounds for the Number of Lattice Edges Needed to Represent 4-Regular Graphs as Lattice Graphs

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
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Annandale-on-Hudson, New York
May, 2019
Abstract

A lattice graph is a graph whose drawing, embedded in Euclidean space $\mathbb{R}^2$, has vertices that are the points with integer coefficients, and has edges that are unit length and are parallel to the coordinate axes. A 4-regular graph is a graph where each vertex has four edges containing it; a loop containing a vertex counts as two edges. The goal for my senior project is to find upper bounds for the number of lattice edges needed to represent 4-regular graphs as lattice graphs.
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Dedication

To my family.
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1

Background

1.1 Introduction to Graph Theory

The following definition is from [1].

Definition 1.1.1. A graph $G = (V, E)$ is a mathematical structure consisting of two finite sets $V$ and $E$. The elements of $V$ are called vertices (or nodes), and the elements of $E$ are called edges. Each edge has a set of one or two vertices associated to it, which are called its endpoints.

Definition 1.1.2. Degree is the number of edges incident to the vertex, with loops counted twice.

Definition 1.1.3. A loop is an edge that connects a vertex to itself.

Definition 1.1.4. Double edges are two edges that join the same two vertices.

Figure 1.1.1 is a graph that each vertices has degree 2. And the graph has 5 vertices and 5 edges.

Definition 1.1.5. A simple graph has neither loops nor double edges.

Definition 1.1.6. A regular graph is a graph where all vertices have the same degree. A regular graph with vertices of degree $k$ is called a $k$-regular graph.
Definition 1.1.7. A 4-regular graph is a graph where each vertex has degree 4. A 4-regular graph can contain loops and double edges.

Figure 1.1.2 is a picture of a 4-regular graph that contains double edges.

Figure 1.1.3 is a picture of a 4-regular graph that contains loops.

The following two definitions are from [1]. $V_p, V_c$ are the number of vertices of the simple graph. $E_p, E_c$ are the number of edges of the simple graph.

Definition 1.1.8. A path graph $P$ is a simple graph with $|V_p| = |E_p| + 1$ that can be drawn so that all of its vertices and edges lie on a single straight line.

For example, Figure 1.2.7 is a path graph which has 4 vertices and 3 edges.
Definition 1.1.9. A cycle graph\ is a single vertex with a self-loop or a simple graph $C$ with $|V_c| = |E_c|$ that can be drawn so that all of its vertices and edges lie on a single circle. △

For example, Figure 3.1.1 is a cycle graph which has 6 vertices and 6 edges.

1.2 $D$ and $C$ Notation in 4-regular graph

Definition 1.2.1. Let $D_1$ be the graph in Figure 1.2.2, let $D_2$ be the graph in Figure 1.2.3 and let $D_n$ be the graph in Figure 1.2.4, where the graph has $n$ ellipses. A $D$-graph is a graph $D_i$ for some $i \in \mathbb{N}$. Let $C$ be the graph in Figure 1.2.1. △

For example, we can use $D$ and $C$ notation to express Figure 1.2.5, we will get $CD_1D_2C$. The loops at the two ending vertices in Figure 1.2.5 are not being expressed in $D$ and $C$ notation, because we will always assume them to be there in this kind of graph.
Definition 1.2.2. A 4-regular path graph is a 4-regular graph that comes from removing all the edges of a path graph and replacing each edges removed either with a $C$ graph or with a $D_n$ graph that takes $n + 2$ adjacent vertices, and adding a loop at each end vertex.

For example, Figure 1.2.6 is a 4-regular path graph that comes from path graph Figure 1.2.7. The three edges of the path graph is removed and replaced by a $D_1$ graph and a $C$ graph with and a loop at each end.

Definition 1.2.3. A 4-regular cycle graph is a 4-regular graph that comes from removing all the edges of a cycle graph and replacing each edges removed either with a $C$ graph or with a $D_n$ graph that takes $n + 2$ adjacent vertices.
1.3 4-REGULAR GRAPHS IN LATTICE GRAPHS

For example, Figure 1.2.8 is a 4-regular cycle graph that comes from cycle graph Figure 1.2.9. The six edges of the cycle graph is removed and replaced by six $C$ graph.

1.3 4-Regular Graphs in Lattice Graphs

**Definition 1.3.1.** A **Lattice graph** is a graph whose drawing, embedded in Euclidean space $\mathbb{R}^2$ where the vertices are the points with integer coefficients, and the edges are unions of unit length edges that are parallel to the coordinate axes.

Figure 1.3.1 is a picture of a lattice.

We want to represent 4-regular graphs in lattice graphs.

For example, Figure 1.3.2 is a picture of a 4-regular graph with 2 vertices. When we represent Figure 1.3.2 in lattice graph, Figure 1.3.3 is one possible lattice graph.

Figure 1.3.3 is a picture of a lattice graph that has 2 vertices and 16 lattice edges.
Question 1: Is Figure 1.3.3 is the most efficient way to represent any 4-regular graphs with 2 vertices?

In question 1, we know that Figure 1.3.3 is the lattice graph that represents Figure 1.3.2. But to find out if Figure 1.3.3 is the most efficient way to represent any 4-regular graphs with 2 vertices, we have to find out if there are any possible drawings of 4-regular graphs with 2 vertices that has less lattice edges. Figure 1.3.4 is another possible drawing of 4-regular graph with 2 vertices. And Figure 1.3.5 would be Figure 1.3.4 represented in lattice graph, which has 2 vertices and 12 lattice edges. Another one consists of two copies of Figure 1.1.3, which lattice graph has 16 lattice edges. Thus, Figure 1.3.3 is not the most efficient way to represent any 4-regular graphs with 2 vertices since Figure 1.3.4 has less lattice edges.
1.3. 4-REGULAR GRAPHS IN LATTICE GRAPHS

Figure 1.3.3.

Figure 1.3.4.

Figure 1.3.5.
# 2
Counting 4-Regular Path Graphs

## 2.1 Symmetry in Counting 4-Regular Path Graphs

When counting 4-regular path graphs, there will be graphs that are symmetric to each other. Symmetric graphs only count as one graph.

For example, take $D_1$, the addition of a single $C$ can change $D_1$ to $D_1C$ or $CD_1$, as seen in Figure 2.1.2 and Figure 2.1.1, respectively, but since they are symmetric, they are the same graph, it only count as one way to arrange $D_1C$. But the addition of 2 $C$’s can change $D_1C$ to $D_1CC$ or $CD_1C$, which has 2 possible outcome without symmetry.

![Figure 2.1.1.](image)

![Figure 2.1.2.](image)

## 2.2 Counting 4-Regular Path Graphs with Single $D_n$

In this section, we count the number of 4-regular path graphs that only have one $D_n$. The smallest size of $D_n$ is $D_1$ which takes up 3 vertices. Let $V$ be the number of vertices.
When $V = 3$, there is $D_1$.

When $V = 4$, there are $D_1C$ and $D_2$.

When $V = 5$, there are $D_1CC$, $CD_1C$, $D_2C$ and $D_3$.

When $V = 6$, there are $D_1CCC$, $CD_1CC$, $CD_1C$, $D_2CC$, $CD_2C$, $D_3C$, $CD_3C$, $D_4C$ and $D_5$.

**Theorem 2.2.1.** Let $V \in \mathbb{N}$, and suppose $V \geq 3$. Let $k \in \mathbb{N}$ be some number such that $k \leq V - 2$. The total number of 4-regular path graphs that have a single $D_k$ graph is

$$\left\lfloor \frac{V - k}{2} \right\rfloor.$$ 

**Proof.** Suppose we have a 4-regular path graph that has a single $D_k$ graph. Then there are $V - (k + 2)$ copies of $C$ graphs in the 4-regular path graph. Let $i$ be the number of $C$ graphs on the left side of $D_k$, and let $j$ be the number of $C$ graphs on the right side of $D_k$. Then $i, j \in \{0, 1, 2, 3, \ldots, V - (k + 2)\}$ and $i + j = V - (k + 2)$, and therefore $i = V - (k + 2) - j$. By symmetry, we assume $i \leq j$. Hence, $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{V - (k - 2)}{2} \right\rfloor\}$. Then there are $\left\lfloor \frac{V - (k - 2)}{2} \right\rfloor + 1 = \left\lfloor \frac{V - (k - 2)}{2} \right\rfloor \left\lfloor \frac{V - k}{2} \right\rfloor$ possibilities to arrange $D_k$ up to symmetry for a fix number $k$. Hence, there are $\left\lfloor \frac{V - k}{2} \right\rfloor$ 4-regular path graphs. 

2.3 Counting 4-Regular Path Graphs with Two $D_n$

2.3.1 Counting 4-Regular Path Graphs with Two Same $D_n$

In this subsection, we count the number of 4-regular path graphs that have two $D_n$ that are the same. When there are two $D_n$ that are the same, it creates a lot more symmetry graphs than the 4-regular path graphs with two different $D_n$.

**Theorem 2.3.1.** Let $V \in \mathbb{N}$, and suppose $V \geq 5$. Let $k \in \mathbb{N}$ be some number such that $k \leq \left\lfloor \frac{V - 3}{2} \right\rfloor$. The total number of 4-regular path graphs that have two same $D_k$ graphs is

$$\sum_{j=0}^{V-(2k+3)} \left\lfloor \frac{V - 2k - 1 - j}{2} \right\rfloor.$$
2.3. COUNTING 4-REGULAR PATH GRAPHS WITH TWO $D_N$

Proof. Suppose we have a 4-regular path graph that has two same $D_k$ graphs. Then there are $V - (2k + 3)$ copies of $C$ graphs in the 4-regular path graph. Let $i$ be the number of $C$ graphs on the left side of both $D_k$, let $j$ be the number of $C$ graphs in between both $D_k$, and let $m$ be the number of $C$ graphs on the right side of both $D_k$. Then $i, j, m \in \{0, 1, 2, 3, \ldots, V - (2k + 3)\}$ and $i + j + m = V - (2k + 3)$, and therefore $i = V - (2k + 3 + j) - m$. By symmetry, we assume $i \leq m$. Let $j \in \{0, 1, 2, \ldots, V - (2k + 3)\}$. Let $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{V - (2k + 3 + j)}{2} \right\rfloor \}$. Then, there are $\left\lfloor \frac{V - (2k + 3 + j)}{2} \right\rfloor + 1 = \left\lfloor \frac{V - (2k + 3 + j) + 1}{2} \right\rfloor = \left\lfloor \frac{V - 2k - 1 - j}{2} \right\rfloor$ possibilities to arrange two same $D_k$ up to symmetry for a fix number $k$. Hence, there are $\sum_{j=0}^{V-(2k+3)} \left\lfloor \frac{V - 2k - 1 - j}{2} \right\rfloor$ 4-regular path graphs.

2.3.2 Counting 4-Regular Path Graphs with Two Different $D_n$

In this subsection, we count the number of 4 regular path graphs that have two $D_n$ that are different. When there are two different $D_n$, it creates a lot less symmetry graphs than the 4-regular path graphs with two same $D_n$.

Theorem 2.3.2. Let $V \in \mathbb{N}$, and suppose $V \geq 5$. Let $k, h \in \mathbb{N}$ be some number such that $k + h \leq V - 3$ and $k \neq h$. The total number of 4-regular path graphs that have $D_k$ and $D_h$ graph is

$$\sum_{j=0}^{V-k-h-3} \frac{(V - k - h - 1 - j)(V - k - h - 2 - j)}{2}.$$ 

Proof. Suppose we have a 4-regular path graph that have a $D_k$ and a $D_h$ graph. Then there are $V - (k+h+3)$ copies of $C$ graphs in the 4-regular path graph. Let $i$ be the number of $C$ graphs on the left side of $D_k$ and $D_h$, let $j$ be the number of $C$ graphs in between of $D_k$ and $D_h$, let $m$ be the number of $C$ graphs on the right side of $D_k$ and $D_h$. Then $i, j, m \in \{0, 1, 2, 3, \ldots, V - (k+h+3)\}$ and $i + j + m = V - (k+h+3)$, and therefore $i = V - (k+h+3+j) - m$. There is no symmetry between $D_k$ and $D_h$, since they have different size of $D_n$. Let $j \in \{0, 1, 2, 3, \ldots, V - (k+h+3)\}$. Let $i \in \{0, 1, 2, 3, \ldots, V - (k+h+3+j)\}$. Then there are $V - (k+h+3+j) + 1$ possibilities to arrange a $D_k$ and a $D_h$ for a fix number $k$ and $h$. Hence, there are $1 + 2 + 3 + \cdots + [V - (k+h+2+j)] = \sum_{j=0}^{V-k-h-3} \left\lfloor \frac{V - 2k - 1 - j}{2} \right\rfloor$ 4-regular path graphs.
2. COUNTING 4-REGULAR PATH GRAPHS

\[ \frac{(V-k-h-1-j)(V-k-h-2-j)}{2} \] 4-regular path graphs for the given value of \( j \). Summing over \( j \) yields

\[ \sum_{j=0}^{V-k-h-3} \frac{(V-k-h-1-j)(V-k-h-2-j)}{2}. \]
3
Counting 4-Regular Cycle Graphs

3.1 Definition of 4-Regular Cycle Graphs

In this section of counting 4-regular cycle graphs, we only consider the 4-regular graphs where the vertices of the 4-regular graph are in a cycle. For example, Figure 3.1.1 is a 4-regular cycle graph with 6 vertices.

![Figure 3.1.1.](image)

3.2 Counting 4-Regular Cycle Graphs with Single $D_n$

There needs to be at least 2 vertices to form a 4-regular cycle graph, and there needs to be at least 3 vertices to form a $D$-graph. Thus, there needs to be at least 3 vertices to form a 4-regular cycle graph with a single $D_n$. 
Figure 3.2.1 is a 2 vertices 4-regular cycle graph and Figure 3.2.2 is a 4-regular cycle graph with a single $D_1$. We take the two loops at both side of the 4-regular path graph with a single $D_1$ and connect them to make a 4-regular cycle graph with a single $D_1$. Since a single $D_1$ has 3 vertices, there needs to be at least 3 vertices to form a 4-regular cycle graph with $D_n$.

![Figure 3.2.1.](image1)

![Figure 3.2.2.](image2)

**Theorem 3.2.1.** Let $V \in \mathbb{N}$, and suppose $V \geq 3$. Let $k \in \mathbb{N}$ be some number such that $k \leq V - 2$. The total number of 4-regular cycle graph that contain a single $D_k$-graph is 1.

**Proof.** Since there is only a single $D$-graph in the 4-regular cycle graphs, they are all identical and counted as one. Thus, The total number of 4-regular cycle graph that contain a single $D_k$ for a fixed number $k$ is 1. 

3.3 Counting 4-Regular Cycle Graphs with Two $D_n$

There needs to be at least 4 vertices to form a 4-regular cycle graph with two $D_n$. 
3.3. COUNTING 4-REGULAR CYCLE GRAPHS WITH TWO $D_N$

3.3.1 Counting 4-Regular Cycle Graphs with Two Same $D_n$

**Theorem 3.3.1.** Let $V \in \mathbb{N}$, and suppose $V \geq 4$. Let $k \in \mathbb{N}$ be some number such that $k \leq \lfloor \frac{V-2}{2} \rfloor$. The total number of 4-regular cycle graphs that have two same $D_k$ graphs is

$$\lfloor \frac{V-2k}{2} \rfloor.$$

**Proof.** Suppose we have a 4-regular cycle graph that has two same $D_k$ graph. Then there are $V-(2k+2)$ copies of $C$ graphs in the 4-regular cycle graph. Let $i, j$ be the number of $C$ graphs in between both $D_k$. Then $i, j \in \{0, 1, 2, 3, \ldots, V-(2k+2)\}$ and $i + j = V-(2k+2)$, and therefore $i = V-(2k+2) - j$. By symmetry, we assume $i \leq j$. Hence, $i \in \{0, 1, 2, 3, \ldots, \lfloor \frac{V-(2k-2)}{2} \rfloor\}$. Then there are $\lfloor \frac{V-(2k-2)}{2} \rfloor + 1 = \lfloor \frac{V-(2k-2)}{2} + 1 \rfloor = \lfloor \frac{V-2k}{2} \rfloor$ possibilities to arrange $D_k$ up to symmetry for a fix number $k$. Hence, there are $\lfloor \frac{V-2k}{2} \rfloor$ 4-regular graphs that have two same $D_k$ graphs. \hfill $\square$

3.3.2 Counting 4-Regular Cycle Graphs with Two Different $D_n$

**Theorem 3.3.2.** Let $V \in \mathbb{N}$, and suppose $V \geq 4$. Let $k, h \in \mathbb{N}$ be some number such that $k + h \leq V - 2$ and $k \neq h$. The total number of 4-regular cycle graphs that have a $D_k$ and a $D_h$ graph is

$$\frac{(V-k-h)(V-k-h-1)}{2}.$$

**Proof.** Suppose we have a 4-regular cycle graph that have a $D_k$ and a $D_h$ graph. Then there are $V-(k+h+2)$ copies of $C$ graphs in the 4-regular cycle graph. Let $i, j$ be the number of $C$ graphs in between $D_k$ and $D_h$. Then $i, j \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$ and $i + j = V-(k+h+2)$, and therefore $i = V-(k+h+2) - j$. Since there is no symmetry with $D_k$ and $D_h$. For each $j \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$, we know that $i \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$. Then there are $V-(k+h+2) + 1$ possibilities to arrange a $D_k$ and a $D_h$ for a fix number $k$ and $h$. Hence, there are $1 + 2 + 3 + \cdots + V-(k+h+1)$ 4-regular cycle graphs that have a $D_k$ and a $D_h$ graph for the given values of $k$ and $h$, which can be written as $\frac{(V-k-h)(V-k-h-1)}{2}$. \hfill $\square$
4

Counting Lattice Edges in Lattice Graphs from 4-Regular Path Graphs

4.1 Finding the Lattice Graph that Requires the Minimum Number of Lattice Edges

**Definition 4.1.1.** $\hat{D}_n$ a 4-regular path graph with single $D_n$ and no $C$. △

Figure 4.1.1 is a 4-regular path graph with single $D_1$. Figure 4.1.2 is a lattice graph from Figure 4.1.1. Figure 4.1.2 is not the lattice graph that requires the minimum number of lattice edges since we can flip the lattice graph and get Figure 4.1.3, which has 16 lattice edges. Since Figure 4.1.3 is consist of the smallest lattice squares, Figure 4.1.3 is the lattice graph that requires the minimum number of lattice edges.
4. COUNTING LATTICE EDGES IN LATTICE GRAPHS FROM 4-REGULAR PATH GRAPHS

Figure 4.1.2.

Figure 4.1.3.

4.2 Special Cases in Counting Lattice Edges in Lattice Graphs with Single \( D_n \)

Figure 4.2.1 is a 4-regular path graph with single \( D_2 \) and no \( C \) graphs. Figure 4.2.2 is a lattice graph from Figure 4.2.2. The square in the center can only attach to 4 other squares in the lattice graph since it only has 4 vertices. Two squares are coming from the loops at each end of the 4-regular path graph. So \( D_2 \) is the largest size of \( D_n \) that can be express this way. So \( \hat{D}_1 \) and \( \hat{D}_2 \) are two specially cases when generalizing the lattice graph.
4.3 Counting Lattice Edges in Lattice Graphs with Single \( D_n \) Without \( C \)

Figure 4.3.1 is a lattice graph from a 4-regular path graph with single \( D_3 \). Since it has 5 lattice squares to attach, it cannot be attached by a single lattice square in the center. We know that a lattice square can only be attached if there is a vertex, so the goal is to create the most vertices for lattice square. The most efficient way to create vertices is to create a zigzag shape lattice graph for the more squares to attach.

For example, Figure 4.3.1 represents \( \hat{D}_3 \) which has 30 lattice edges. Figure 4.3.2 represents \( \hat{D}_4 \) which has 36 lattice edges. Figure 4.3.3 represents \( \hat{D}_5 \) which has 44 lattice edges. Figure 4.3.4 represents \( \hat{D}_6 \) which has 50 lattice edges. Figure 4.3.5 represents \( \hat{D}_7 \) which has 58 lattice edges.
From the sequence, we know that the pattern starts at 4-regular graph with single $D_3$ and no $C$. For lattice graph with a single $D_n$ and no $C$. Let $n \in \mathbb{N}$, and suppose $n \geq 3$. With odd numbers of $n$, the function is $7n + 9$. With even numbers of $n$, the function is $7n + 8$. And both function can be express by $7n + 8 + \frac{1 - (-1)^n}{2}$ as in Lemma 4.3.1.

**Lemma 4.3.1.** Let $n, k \in \mathbb{N}$. Suppose $n \geq 3$. Then

$$7n + 8 + \frac{1 - (-1)^n}{2} = \begin{cases} 7n + 9 & \text{if } n \text{ is odd} \\ 7n + 8 & \text{if } n \text{ is even} \end{cases}$$

**Proof.** Suppose $n$ is an odd number. Then $n = 2k + 1$ for some $k \in \mathbb{N}$. Then $7n + 8 + \frac{1 - (-1)^n}{2} = 7(2k + 1) + 8 + 1 = 14k + 16 = 7(2k + 1) + 9 = 7n + 9$. Suppose $n$ is an even number. Then $n = 2k$ for some $k \in \mathbb{N}$. Then $7n + 8 + \frac{1 - (-1)^n}{2} = 7(2k) + 8 + 0 = 7n + 8$. Hence,

$$7n + 8 + \frac{1 - (-1)^n}{2} = \begin{cases} 7n + 9 & \text{if } n \text{ is odd} \\ 7n + 8 & \text{if } n \text{ is even} \end{cases}.$$
4.4. COUNTING LATTICE EDGES IN LATTICE GRAPHS WITH SINGLE $D_n$ WITH $C$21

![Image](image.png)

Figure 4.3.5.

**Theorem 4.3.2.** Let $n \in \mathbb{N}$. Then,

1. $\hat{D}_1$ needs at worst 16 lattice edges to represent.
2. $\hat{D}_2$ needs at worst 20 lattice edges to represent.
3. If $n \in \mathbb{N}$ and $n \geq 2$, then $\hat{D}_n$ needs at worst $7n + 8 + \frac{1-(-1)^n}{2}$ lattice edges to represent.

4.4 Counting Lattice Edges in Lattice Graphs with Single $D_n$ With $C$

Figure 4.4.1 is a lattice graph from a 4-regular path graph with single $D_1$ and $C$ graphs. Let $k \in \mathbb{N}$. Suppose there are $k$ number of $C$ graphs. Noted that $C$ graphs can be attached to both each of the 4-regular graphs, so Figure 4.4.1 is not the only way to attach $C$ graphs. Since each $C$ graph has 4 lattice edges, then the number of lattice edges needed can be calculated by just adding $4k$ to the number of lattice edges needed in $\hat{D}_1$.

Hence, a 4-regular path graph with single $D_1$ and $k$ number of $C$ graphs needs at worst $16 + 4k$ lattice edges to represent.

Figure 4.4.2 is a lattice graph from a 4-regular path graph with single $D_2$ and $C$ graphs. Then by similar calculation, a 4-regular path graph with single $D_2$ and $k$ number of $C$ graphs needs at worst $20 + 4k$ lattice edges to represent.

The sequence starts at $n \geq 3$. For example, Figure 4.4.3 is a lattice graph from a 4-regular path graph with single $D_7$ and $C$ graphs. Then by similar calculation, a 4-regular path graph with single $D_1$ and $k$ number of $C$ graphs needs at worst $58 + 4k$ lattice edges to represent.
Let \( n \in \mathbb{N} \), and suppose \( n \geq 3 \). By similar calculation, a lattice graph from a 4-regular path graph with single \( D_n \) and \( C \) graphs needs at worst \( 7n + 8 + \frac{1-(-1)^n}{2} + 4k \) to represent.

**Theorem 4.4.1.** Let \( n, k \in \mathbb{N} \). Then,

1. A lattice graph from a 4-regular path graph with single \( D_1 \) and \( k \) number of \( C \) needs at worst \( 16 + 4k \) lattice edges to represent.

2. A lattice graph from a 4-regular path graph with single \( D_2 \) with \( k \) number of \( C \) needs at worst \( 20 + 4k \) lattice edges to represent.

3. If \( n \in \mathbb{N} \) and \( n \geq 3 \), then a lattice graph from a 4-regular path graph with single \( D_n \) with \( k \) number of \( C \) needs at worst \( 7n + 8 + \frac{1-(-1)^n}{2} + 4k \) lattice edges to represent.
4.5 Counting Lattice Edges in Lattice Graphs with $D_i$ and $D_j$ Without $C$

Figure 4.5.1 is a lattice graph from a 4-regular path graph with two $D_1$ and no $C$. Figure 4.5.1 has 24 lattice edges.

Figure 4.5.2 is a lattice graph from a 4-regular path graph with single $D_1$, $D_2$ and no $C$. Figure 4.5.2 has 34 lattice edges.

Figure 4.5.3 is a lattice graph from a 4-regular path graph with two $D_2$ and no $C$. Figure 4.5.3 has 40 lattice edges.

The sequence starts at $n \geq 3$. Figure 4.5.4 is a lattice graph from a 4-regular path graph with two $D_3$ and no $C$. Figure 4.5.4 has 52 lattice edges. The number of lattice edges is coming from
adding the number of lattice edges of two $\hat{D}_3$ which has four ending loops in total, and subtract the number of lattice edges of two ending loops, since there can only be two ending loops.

Let $i, j \in \mathbb{N}$, suppose $n \geq 3$. Then a lattice graph from a 4-regular path graph with single $D_i$, $D_j$ and no $C$ needs at worst $7(i + j) + 8 + \frac{1-(-1)^i}{2} + \frac{1-(-1)^j}{2}$.

**Theorem 4.5.1.** Let $i, j \in \mathbb{N}$. Then,

1. A lattice graph from a 4-regular path graph with two $D_1$ without $C$ needs at worst 24 lattice edges to represent.

2. A lattice graph from a 4-regular path graph with single $D_1$ and $D_2$ without $C$ needs at worst 36 lattice edges to represent.

3. A lattice graph from a 4-regular path graph with two $D_2$ without $C$ needs at worst 40 lattice edges to represent.
4.6 Counting Lattice Edges in Lattice Graphs with $D_i$ and $D_j$ With $C$

Let $k \in \mathbb{N}$. From previous section, we know that the addition of $k$ number of $C$ graphs can be calculated by just adding $4k$ to the number of lattice edges needed.

For example, Figure 4.6.1 is a lattice graph from a 4-regular path graph with two $D_3$ and three $C$. It has 64 lattice edges. Note that $C$ graphs can be placed in between two $D_n$ or at the end of each $D_n$ graphs.

Let $k \in \mathbb{N}$. Figure 4.6.2 is a lattice graph from a 4-regular path graph with two $D_1$ and $k$ number of $C$. The locations of $C$ graphs are not going to change the lattice edges needed to represent. It needs at worst $24 + 4k$ lattice edges to represent.

Let $k \in \mathbb{N}$. Figure 4.6.3 is a lattice graph from a 4-regular path graph with single $D_1$, $D_2$ and $k$ number of $C$. In this case, there are $C$ graphs in between $D_1$ and $D_2$. It needs at worst $28 + 4k$ lattice edges to represent.
If there is no $C$ graphs in between $D_1$ and $D_2$, it need at worst $30 + 4k$ lattice edges to represent.

Let $k \in \mathbb{N}$. Figure 4.6.4 is a lattice graph from a 4-regular path graph with two $D_2$ and $k$ number of $C$. In this case, there are $C$ graphs in between two $D_2$. It needs at worst $32 + 4k$ lattice edges to represent.

If there is no $C$ graphs in between two $D_2$, it need at worst $36 + 4k$ lattice edges to represent.

**Theorem 4.6.1.** Let $i, j, k \in \mathbb{N}$. Then,

1. A lattice graph from a 4-regular path graph with two $D_1$ with $k$ number of $C$ needs at worst $24 + 4k$ lattice edges to represent.

2. A lattice graph from a 4-regular path graph with single $D_1$, $D_2$ with $k$ number of $C$, and have $C$ graphs in between $D_1$, $D_2$ needs at worst $28 + 4k$ lattice edges to represent.
4.6. COUNTING LATTICE EDGES IN LATTICE GRAPHS WITH $D_1$ AND $D_2$ WITH $C^{27}$

3. A lattice graph from a 4-regular path graph with single $D_1$, $D_2$ with $k$ number of $C$, and no $C$ graphs in between $D_1$, $D_2$ needs at worst $30 + 4k$ lattice edges to represent.

4. A lattice graph from a 4-regular path graph with two $D_2$ with $k$ number of $C$, and have $C$ graphs in between two $D_2$ needs at worst $32 + 4k$ lattice edges to represent.

5. A lattice graph from a 4-regular path graph with two $D_2$ with $k$ number of $C$, and no $C$ graphs in between two $D_2$ needs at worst $36 + 4k$ lattice edges to represent.

6. If $i, j \in \mathbb{N}$, $i, j \geq 3$, then a lattice graph from a 4-regular path graph with single $D_i$, $D_j$ with $k$ number of $C$ needs at worst $7(i + j) + 8 + \frac{1 - (-1)^i}{2} + \frac{1 - (-1)^j}{2} + 4k$ lattice edges to represent.
5

Counting Lattice Edges in Lattice Graph from 4-Regular Cycle Graphs

5.1 Counting Lattice Edges in Lattice Graphs with $C$

To make a 4-regular cycle graph, there needs to be at least two $C$ graphs. Figure 5.1.1 is a lattice graph that represents a 4-regular cycle graph with two $C$ graphs. Figure 5.1.1 has 16 lattice edges.

![Figure 5.1.1](image)

Figure 5.1.1.

Figure 5.1.2 is a lattice graph that represents a 4-regular cycle graph with three $C$ graphs. Figure 5.1.2 has 16 lattice edges.

Figure 5.1.3 is a lattice graph that represents a 4-regular cycle graph with four $C$ graphs. When the number of $C$ graphs in the 4-regular graph is divisible by 4, the lattice graph can be represented by using only 4 lattice edges squares. Figure 5.1.3 has 16 lattice edges.

Figure 5.1.4 is a lattice graph that represents a 4-regular cycle graph with five $C$ graphs. Figure 5.1.4 has 28 lattice edges.
Figure 5.1.2.

Figure 5.1.3.

Figure 5.1.5 is a lattice graph that represents a 4-regular cycle graph with six $C$ graphs. Figure 5.1.5 has 28 lattice edges.

Figure 5.1.6 is a lattice graph that represents a 4-regular cycle graph with seven $C$ graphs. Figure 5.1.6 has 32 lattice edges.

Figure 5.1.7 is a lattice graph that represents a 4-regular cycle graph with eight $C$ graphs. This lattice graph can be represented by using only 4 lattice edges squares since it has eight $C$ graphs. Figure 5.1.7 has 32 lattice edges.

Figure 5.1.8 is a lattice graph that represents a 4-regular cycle graph with nine $C$ graphs. Figure 5.1.8 has 40 lattice edges.

Figure 5.1.9 is a lattice graph that represents a 4-regular cycle graph with ten $C$ graphs. Figure 5.1.9 has 44 lattice edges.

Figure 5.1.10 is a lattice graph that represents a 4-regular cycle graph with eleven $C$ graphs. Figure 5.1.10 has 48 lattice edges.

Figure 5.1.11 is a lattice graph that represents a 4-regular cycle graph with twelve $C$ graphs. This lattice graph can be represented by using only 4 lattice edges squares since it has twelve $C$ graphs. Figure 5.1.11 has 48 lattice edges.
5.1. COUNTING LATTICE EDGES IN LATTICE GRAPHS WITH $C$

Let $p \in \mathbb{N}$ be the number of $C$ graphs in a lattice cycle graph with no $D_k$. Let $C^p$ be the number of lattice edges in a lattice cycle graph with no $D_k$. By observation, the sequence is as following.

1. Let $n \in \mathbb{N}$ and suppose $p = 4n$. Then, $C^p = C^{4n} = 16n$

2. Let $n \in \mathbb{N}$ and suppose $p = 4n + 1$. Then, $C^p = C^{4n+1} = 16n + 12$

3. Let $n \in \mathbb{N}$ and suppose $p = 4n + 2$. Then, $C^p = C^{4n+2} = 4(4n + 1) + 8 = 16n + 12$

4. Let $n \in \mathbb{N}$ and suppose $p = 4n + 3$. Then, $C^p = C^{4n+3} = 4(4n + 2) + 8 = 16n + 16$

**Theorem 5.1.1.** Let $\bar{f} : \mathbb{Z} \to \mathbb{Z}$ be define by

$$f(p) = \begin{cases} 
0 & \text{if } p = 4n \text{ for some } n \in \mathbb{Z} \\
12 & \text{if } p = 4n + 1 \text{ for some } n \in \mathbb{Z} \\
12 & \text{if } p = 4n + 2 \text{ for some } n \in \mathbb{Z} \\
16 & \text{if } p = 4n + 3 \text{ for some } n \in \mathbb{Z}
\end{cases}$$

Then $C^p = 16\left[\frac{p}{4}\right] + \bar{f}(p)$
5.2 Counting Lattice Edges in Lattice Graphs with Single $D_n$ and $C$

Figure 5.2.1 is a lattice graph that represents a 4-regular cycle graph with single $D_3$ and eight $C$ graphs. This lattice graph is created by attaching a $D_3$ to a 4-regular cycle graph with nine $C$ graphs. Then one of the $C$ graphs is removed and attach $D_3$. Figure 5.2.1 has 58 lattice edges.

Figure 5.2.2 is a lattice graph that represents a 4-regular cycle graph with single $D_3$ and eight $C$ graphs. This lattice graph is created by attaching eight $C$ graphs to a $D_3$ to from a cycle graph. Figure 5.2.2 has 58 lattice edges.

Figure 5.2.1 and Figure 5.2.2 have the same number of lattice edges, different way of attachment will not change the lattice edges needed to represent at worst. We know that $\hat{D}_n$ needs at worst $7n + 8 + \frac{1-(-1)^n}{2}$ lattice edges to represent. To attach the $D_n$ graph, the end loops at each end are removed and one more $C$ graph needs to be added to the 4-regular cycle graph.
5.2. **COUNTING LATTICE EDGES IN LATTICE GRAPHS WITH SINGLE $D_N$ AND $C_3$**

![Figure 5.1.8.](image1)

![Figure 5.1.9.](image2)

represented by lattice graph. So a lattice graph from a 4-regular cycle graph with single $D_n$ with $k$ number of $C$ will need $7n + 8 + \frac{1-(-1)^n}{2} - 8 + C^{p+1} = 7n + \frac{1-(-1)^n}{2} + C^{p+1}$ lattice edges to represent.

**Theorem 5.2.1.** Let $n, k \in \mathbb{N}$. Then a lattice graph from a 4-regular cycle graph with single $D_n$ with $k$ number of $C$ needs at worst $7n + \frac{1-(-1)^n}{2} + C^{p+1}$ lattice edges to represent.
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Figure 5.1.10.

Figure 5.1.11.
5.2. C\textsc{ounting} Lattice edges in lattice graphs with single $D_N$ and $C_{35}$

Figure 5.2.1.

Figure 5.2.2.
Bibliography