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Upper Bounds for the Number of Lattice Edges Needed to Represent 4-Regular Graphs as Lattice Graphs

Shenze Li
Bard College

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Upper Bounds for the Number of Lattice Edges Needed to Represent 4-Regular Graphs as Lattice Graphs

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Shenze Li

Annandale-on-Hudson, New York
May, 2019

Abstract

A lattice graph is a graph whose drawing, embedded in Euclidean space $\mathbb{R}^2$, has vertices that are the points with integer coefficients, and has edges that are unit length and are parallel to the coordinate axes. A 4-regular graph is a graph where each vertex has four edges containing it; a loop containing a vertex counts as two edges. The goal for my senior project is to find upper bounds for the number of lattice edges needed to represent 4-regular graphs as lattice graphs.

Contents

Dedication

To my family.

Acknowledgments

First of all, I would like to thank my academic advisor and senior project advisor, Professor Ethan Bloch, for his guidance, help and valuable advise throughout this senior project. I would also like to thank Professor Stefan Mendez-Diez and Professor Japheth Wood for their suggestions to improve this senior project. I want to thank all my family members for their support and trust throughout my life, providing me with hidden but influential love in all their unique ways.

1
Background

1.1 Introduction to Graph Theory

The following definition is from [1].

Definition 1.1.1. A **graph** $G = (V, E)$ is a mathematical structure consisting of two finite sets V and E. The elements of V are called **vertices** (or **nodes**), and the elements of E are called **edges**. Each edge has a set of one or two vertices associated to it, which are called its **endpoints**. △

Definition 1.1.2. **Degree** is the number of edges incident to the vertex, with loops counted twice. △

Definition 1.1.3. A **loop** is an edge that connects a vertex to itself. △

Definition 1.1.4. **Double edges** are two edges that join the same two vertices. △

Figure 1.1.1 is a graph that each vertices has degree 2. And the graph has 5 vertices and 5 edges.

Definition 1.1.5. A **simple graph** has neither loops nor double edges. △

Definition 1.1.6. A **regular graph** is a graph where all vertices have the same degree. A regular graph with vertices of degree k is called a k-regular graph. △

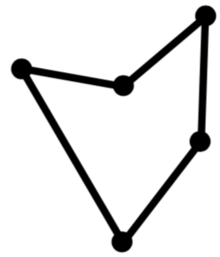

Figure 1.1.1.

Definition 1.1.7. A **4-regular graph** is a graph where each vertex has degree 4. A 4-regular graph can contain loops and double edges. △

Figure 1.1.2 is a picture of a 4-regular graph that contains double edges.

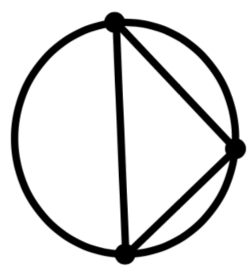

Figure 1.1.2.

Figure 1.1.3 is a picture of a 4-regular graph that contains loops.

Figure 1.1.3.

The following two definitions are from [1]. V_p, V_c are the number of vertices of the simple graph. E_p, E_c are the number of edges of the simple graph.

Definition 1.1.8. A **path graph** P is a simple graph with $|V_p| = |E_p| + 1$ that can be drawn so that all of its vertices and edges lie on a single straight line. △

For example, Figure 1.2.7 is a path graph which has 4 vertices and 3 edges.

Figure 1.1.4.

Definition 1.1.9. A **cycle graph** is a single vertex with a self-loop or a simple graph C with $|V_c| = |E_c|$ that can be drawn so that all of its vertices and edges lie on a single circle. △

For example, Figure 3.1.1 is a cycle graph which has 6 vertices and 6 edges.

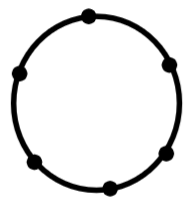

Figure 1.1.5.

1.2 D and C Notation in 4-regular graph

Definition 1.2.1. Let D_1 be the graph in Figure 1.2.2 , let D_2 be the graph in Figure 1.2.3 and let D_n be the graph in Figure 1.2.4, where the graph has n ellipses. A D**-graph** is a graph D_i for some $i \in \mathbb{N}$. Let C be the graph in Figure 1.2.1. △

Figure 1.2.1.

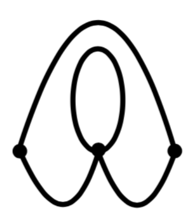

Figure 1.2.2.

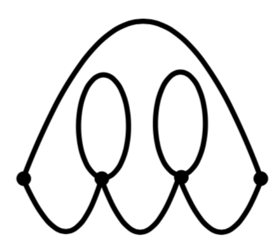

Figure 1.2.3.

For example, we can use D and C notation to express Figure 1.2.5, we will get CD_1D_2C. The loops at the two ending vertices in Figure 1.2.5 are not being expressed in D and C notation, because we will always assume them to be there in this kind of graph.

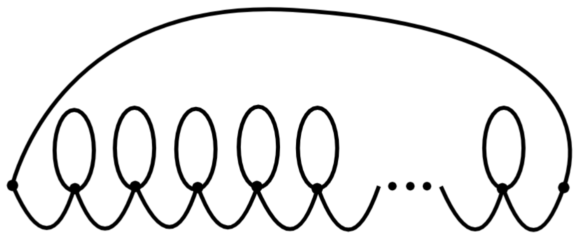

Figure 1.2.4.

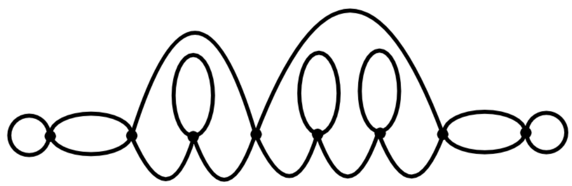

Figure 1.2.5.

Definition 1.2.2. A 4**-regular path graph** is a 4-regular graph that comes from removing all the edges of a path graph and replacing each edges removed either with a C graph or with a D_n graph that takes $n+2$ adjacent vertices, and adding a loop at each end vertex. △

For example, Figure 1.2.6 is a 4-regular path graph that comes from path graph Figure 1.2.7. The three edges of the path graph is removed and replaced by a D_1 graph and a C graph with and a loop at each end.

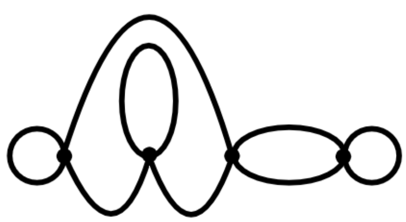

Figure 1.2.6.

Definition 1.2.3. A 4**-regular cycle graph** is a 4-regular graph that comes from removing all the edges of a cycle graph and replacing each edges removed either with a C graph or with a D_n graph that takes $n+2$ adjacent vertices. △

Figure 1.2.7.

For example, Figure 1.2.8 is a 4-regular cycle graph that comes from cycle graph Figure 1.2.9. The six edges of the cycle graph is removed and replaced by six C graph.

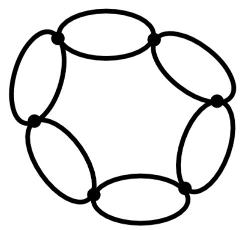

Figure 1.2.8.

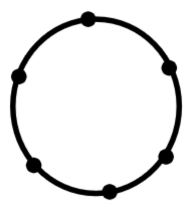

Figure 1.2.9.

1.3 4-Regular Graphs in Lattice Graphs

Definition 1.3.1. A **Lattice graph** is a graph whose drawing, embedded in Euclidean space $\mathbb{R}^2$ where the vertices are the points with integer coefficients, and the edges are unions of unit length edges that are parallel to the coordinate axes. △

Figure 1.3.1 is a picture of a lattice.

We want to represent 4-regular graphs in lattice graphs.

For example, Figure 1.3.2 is a picture of a 4-regular graph with 2 vertices. When we represent Figure 1.3.2 in lattice graph, Figure 1.3.3 is one possible lattice graph.

Figure 1.3.3 is a picture of a lattice graph that has 2 vertices and 16 lattice edges.

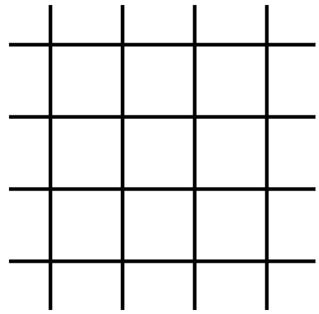

Figure 1.3.1.

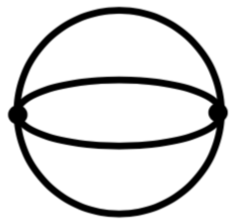

Figure 1.3.2.

Question 1: *Is Figure* 1.3.3 *is the most efficient way to represent any* 4*-regular graphs with* 2 *vertices?*

In question 1, we know that Figure 1.3.3 is the lattice graph that represents Figure 1.3.2. But to find out if Figure 1.3.3 is the most efficient way to represent any 4-regular graphs with 2 vertices, we have to find out if there are any possible drawings of 4-regular graphs with 2 vertices that has less lattice edges. Figure 1.3.4 is another possible drawing of 4-regular graph with 2 vertices. And Figure 1.3.5 would be Figure 1.3.4 represented in lattice graph, which has 2 vertices and 12 lattice edges. Another one consists of two copies of Figure 1.1.3, which lattice graph has 16 lattice edges. Thus, Figure 1.3.3 is not the most efficient way to represent any 4-regular graphs with 2 vertices since Figure 1.3.4 has less lattice edges.

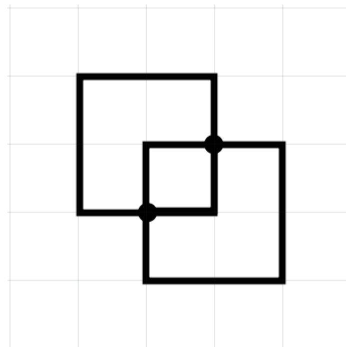

Figure 1.3.3.

Figure 1.3.4.

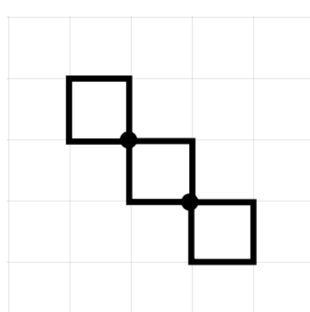

Figure 1.3.5.

2

Counting 4-Regular Path Graphs

2.1 Symmetry in Counting 4-Regular Path Graphs

When counting 4-regular path graphs, there will be graphs that are symmetric to each other. Symmetric graphs only count as one graph.

For example, take D_1, the addition of a single C can change D_1 to D_1C or CD_1, as seen in Figure 2.1.2 and Figure 2.1.1, respectively, but since they are symmetric, they are the same graph, it only count as one way to arrange D_1C. But the addition of 2 C's can change D_1C to D_1CC or CD_1C, which has 2 possible outcome without symmetry.

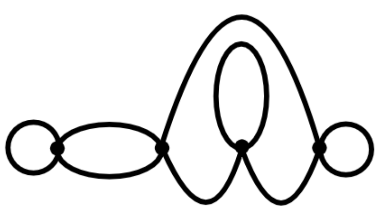

Figure 2.1.1.

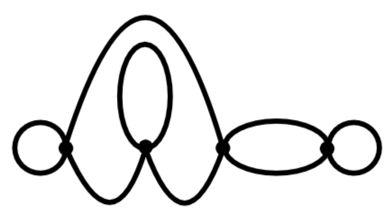

Figure 2.1.2.

2.2 Counting 4-Regular Path Graphs with Single D_n

In this section, we count the number of 4-regular path graphs that only have one D_n. The smallest size of D_n is D_1 which takes up 3 vertices. Let V be the number of vertices.

When $V=3$, there is D_1.

When $V=4$, there are D_1C and D_2.

When $V=5$, there are D_1CC, CD_1C, D_2C and D_3.

When $V=6$, there are D_1CCC, CD_1CC, D_2CC, CD_2C, D_3C and D_4.

When $V=7$, there are D_1CCCC, CD_1CCC, CCD_1CC, D_2CCC, CD_2CC, D_3CC, CD_3C, D_4C and D_5.

Theorem 2.2.1. *Let $V \in \mathbb{N}$, and suppose $V \geq 3$. Let $k \in \mathbb{N}$ be some number such that $k \leq V-2$. The total number of 4-regular path graphs that have a single D_k graph is*

$$\lfloor \frac{V-k}{2} \rfloor.$$

Proof. Suppose we have a 4-regular path graph that has a single D_k graph. Then there are $V-(k+2)$ copies of C graphs in the 4-regular path graph. Let i be the number of C graphs on the left side of D_k, and let j be the number of C graphs on the right side of D_k. Then $i, j \in \{0,1,2,3,\ldots,V-(k+2)\}$ and $i+j=V-(k+2)$, and therefore $i=V-(k+2)-j$. By symmetry, we assume $i \leq j$. Hence, $i \in \{0,1,2,\ldots,\lfloor \frac{V-(k-2)}{2} \rfloor\}$. Then there are $\lfloor \frac{V-(k-2)}{2} \rfloor + 1 = \lfloor \frac{V-(k-2)}{2} + 1 \rfloor = \lfloor \frac{V-k}{2} \rfloor$ possibilities to arrange D_k up to symmetry for a fix number k. Hence, there are $\lfloor \frac{V-k}{2} \rfloor$ 4-regular path graphs. □

2.3 Counting 4-Regular Path Graphs with Two D_n

2.3.1 Counting 4-Regular Path Graphs with Two Same D_n

In this subsection, we count the number of 4-regular path graphs that have two D_n that are the same. When there are two D_n that are the same, it creates a lot more symmetry graphs than the 4-regular path graphs with two different D_n.

Theorem 2.3.1. *Let $V \in \mathbb{N}$, and suppose $V \geq 5$. Let $k \in \mathbb{N}$ be some number such that $k \leq \lfloor \frac{V-3}{2} \rfloor$. The total number of 4-regular path graphs that have two same D_k graphs is*

$$\sum_{j=0}^{V-(2k+3)} \lfloor \frac{V-2k-1-j}{2} \rfloor.$$

Proof. Suppose we have a 4-regular path graph that has two same D_k graphs. Then there are $V-(2k+3)$ copies of C graphs in the 4-regular path graph. Let i be the number of C graphs on the left side of both D_k, let j be the number of C graphs in between both D_k, and let m be the number of C graphs on the right side of both D_k. Then $i, j, m \in \{0, 1, 2, 3, \ldots, V-(2k+3)\}$ and $i+j+m = V-(2k+3)$, and therefore $i = V-(2k+3+j)-m$. By symmetry, we assume $i \leq m$. Let $j \in \{0, 1, 2, \ldots, V-(2k+3)\}$. Let $i \in \{0, 1, 2, \ldots, \lfloor \frac{V-(2k+3+j)}{2} \rfloor\}$. Then, there are $\lfloor \frac{V-(2k+3+j)}{2} \rfloor + 1 = \lfloor \frac{V-(2k+3+j)}{2} + 1 \rfloor = \lfloor \frac{V-2k-1-j}{2} \rfloor$ possibilities to arrange two same D_k up to symmetry for a fix number k. Hence, there are $\sum_{j=0}^{V-(2k+3)} \lfloor \frac{V-2k-1-j}{2} \rfloor$ 4-regular path graphs. □

2.3.2 Counting 4-Regular Path Graphs with Two Different D_n

In this subsection, we count the number of 4 regular path graphs that have two D_n that are different. When there are two different D_n, it creates a lot less symmetry graphs than the 4-regular path graphs with two same D_n.

Theorem 2.3.2. *Let $V \in \mathbb{N}$, and suppose $V \geq 5$. Let $k, h \in \mathbb{N}$ be some number such that $k+h \leq V-3$ and $k \neq h$. The total number of 4-regular path graphs that have D_k and D_h graph is*

$$\sum_{j=0}^{V-k-h-3} \frac{(V-k-h-1-j)(V-k-h-2-j)}{2}.$$

Proof. Suppose we have a 4-regular path graph that have a D_k and a D_h graph. Then there are $V-(k+h+3)$ copies of C graphs in the 4-regular path graph. Let i be the number of C graphs on the left side of D_k and D_h, let j be the number of C graphs in between of D_k and D_h, let m be the number of C graphs on the right side of D_k and D_h. Then $i, j, m \in \{0, 1, 2, 3, \ldots, V-(k+h+3)\}$ and $i+j+m = V-(k+h+3)$, and therefore $i = V-(k+h+3+j)-m$. There is no symmetry between D_k and D_h, since they have different size of D_n. Let $j \in \{0, 1, 2, 3, \ldots, V-(k+h+3)\}$. Let $i \in \{0, 1, 2, 3, \ldots, V-(k+h+3+j)\}$. Then there are $V-(k+h+3+j)+1$ possibilities to arrange a D_k and a D_h for a fix number k and h. Hence, there are $1+2+3+\cdots+[V-(k+h+2+j)] =$

$\frac{(V-k-h-1-j)(V-k-h-2-j)}{2}$ 4-regular path graphs for the given value of j. Summing over j yields $\sum_{j=0}^{V-k-h-3} \frac{(V-k-h-1-j)(V-k-h-2-j)}{2}$. □

3
Counting 4-Regular Cycle Graphs

3.1 Definition of 4-Regular Cycle Graphs

In this section of counting 4-regular cycle graphs, we only consider the 4-regular graphs where the vertices of the 4-regular graph are in a cycle. For example, Figure 3.1.1 is a 4-regular cycle graph with 6 vertices.

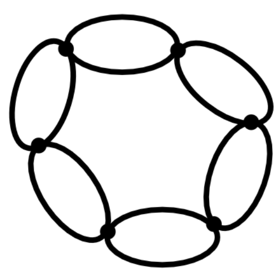

Figure 3.1.1.

3.2 Counting 4-Regular Cycle Graphs with Single D_n

There needs to be at least 2 vertices to form a 4-regular cycle graph, and there needs to be at least 3 vertices to form a D-graph. Thus, there needs to be at least 3 vertices to form a 4-regular cycle graph with a single D_n.

Figure 3.2.1 is a 2 vertices 4-regular cycle graph and Figure 3.2.2 is a 4-regular cycle graph with a single D_1. We take the two loops at both side of the 4-regular path graph with a single D_1 and connect them to make a 4-regular cycle graph with a single D_1. Since a single D_1 has 3 vertices, there needs to be at least 3 vertices to form a 4-regular cycle graph with D_n.

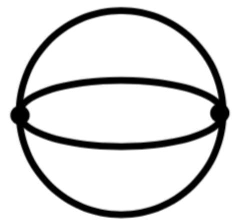

Figure 3.2.1.

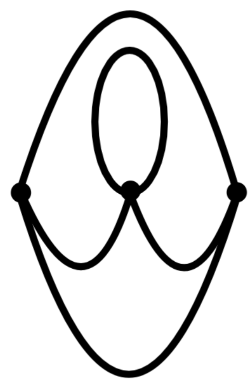

Figure 3.2.2.

Theorem 3.2.1. *Let $V \in \mathbb{N}$, and suppose $V \geq 3$. Let $k \in \mathbb{N}$ be some number such that $k \leq V-2$. The total number of 4-regular cycle graph that contain a single D_k-graph is 1.*

Proof. Since there is only a single D-graph in the 4-regular cycle graphs, they are all identical and counted as one. Thus, The total number of 4-regular cycle graph that contain a single D_k for a fixed number k is 1. □

3.3 Counting 4-Regular Cycle Graphs with Two D_n

There needs to be at least 4 vertices to form a 4-regular cycle graph with two D_n.

3.3.1 Counting 4-Regular Cycle Graphs with Two Same D_n

Theorem 3.3.1. *Let $V \in \mathbb{N}$, and suppose $V \geq 4$. Let $k \in \mathbb{N}$ be some number such that $k \leq \lfloor \frac{V-2}{2} \rfloor$. The total number of 4-regular cycle graphs that have two same D_k graphs is*

$$\lfloor \frac{V-2k}{2} \rfloor.$$

Proof. Suppose we have a 4-regular cycle graph that has two same D_k graph. Then there are $V-(2k+2)$ copies of C graphs in the 4-regular cycle graph. Let i, j be the number of C graphs in between both D_k. Then $i, j \in \{0, 1, 2, 3, \ldots, V-(2k+2)\}$ and $i+j = V-(2k+2)$, and therefore $i = V-(2k+2)-j$. By symmetry, we assume $i \leq j$. Hence, $i \in \{0, 1, 2, 3, \ldots, \lfloor \frac{V-(2k-2)}{2} \rfloor\}$. Then there are $\lfloor \frac{V-(2k-2)}{2} \rfloor + 1 = \lfloor \frac{V-(2k-2)}{2} + 1 \rfloor = \lfloor \frac{V-2k}{2} \rfloor$ possiblities to arrange D_k up to symmetry for a fix number k. Hence, there are $\lfloor \frac{V-2k}{2} \rfloor$ 4-regular graphs that have two same D_k graphs. □

3.3.2 Counting 4-Regular Cycle Graphs with Two Different D_n

Theorem 3.3.2. *Let $V \in \mathbb{N}$, and suppose $V \geq 4$. Let $k, h \in \mathbb{N}$ be some number such that $k + h \leq V - 2$ and $k \neq h$. The total number of 4-regular cycle graphs that have a D_k and a D_h graph is*

$$\frac{(V-k-h)(V-k-h-1)}{2}.$$

Proof. Suppose we have a 4-regular cycle graph that have a D_k and a D_h graph. Then there are $V-(k+h+2)$ copies of C graphs in the 4-regular cycle graph. Let i, j be the number of C graphs in between D_k and D_h. Then $i, j \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$ and $i+j = V-(k+h+2)$, and therefore $i = V-(k+h+2)-j$. Since there is no symmetry with D_k and D_h. For each $j \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$, we know that $i \in \{0, 1, 2, 3, \ldots, V-(k+h+2)\}$. Then there are $V-(k+h+2)+1$ possibilities to arrange a D_k and a D_h for a fix number k and h. Hence, there are $1+2+3+\cdots+V-(k+h+1)$ 4-regular cycle graphs that have a D_k and a D_h graph for the given values of k and h, which can be written as $\frac{(V-k-h)(V-k-h-1)}{2}$. □

4

Counting Lattice Edges in Lattice Graphs from 4-Regular Path Graphs

4.1 Finding the Lattice Graph that Requires the Minimum Number of Lattice Edges

Definition 4.1.1. $\hat{D}_n$ a 4-regular path graph with single D_n and no C. △

Figure 4.1.1 is a 4-regular path graph with single D_1. Figure 4.1.2 is a lattice graph from Figure 4.1.1. Figure 4.1.2 is not the lattice graph that requires the minimum number of lattice edges since we can flip the lattice graph and get Figure 4.1.3, which has 16 lattice edges. Since Figure 4.1.3 is consist of the smallest lattice squares, Figure 4.1.3 is the lattice graph that requires the minimum number of lattice edges.

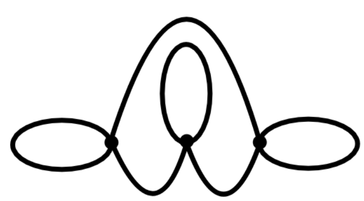

Figure 4.1.1.

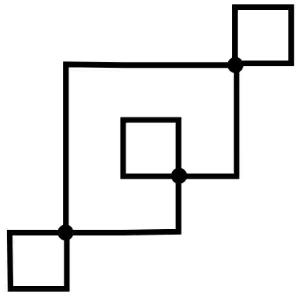

Figure 4.1.2.

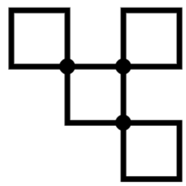

Figure 4.1.3.

4.2 Special Cases in Counting Lattice Edges in Lattice Graphs with Single D_n

Figure 4.2.1 is a 4-regular path graph with single D_2 and no C graphs. Figure 4.2.2 is a lattice graph from Figure 4.2.2. The square in the center can only attach to 4 other squares in the lattice graph since it only has 4 vertices. Two squares are coming from the loops at each end of the 4-regular path graph. So D_2 is the largest size of D_n that can be express this way. So $\hat{D}_1$ and $\hat{D}_2$ are two specially cases when generalizing the lattice graph.

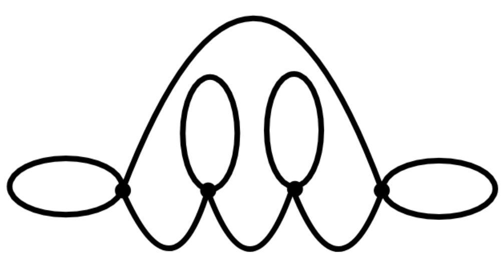

Figure 4.2.1.

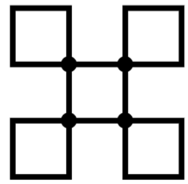

Figure 4.2.2.

4.3 Counting Lattice Edges in Lattice Graphs with Single D_n Without C

Figure 4.3.1 is a lattice graph from a 4-regular path graph with single D_3. Since it has 5 lattice squares to attach, it cannot be attached by a single lattice square in the center. We know that a lattice square can only be attached if there is a vertex, so the goal is to create the most vertices for lattice square. The most efficient way to create vertices is to create a zigzag shape lattice graph for the more squares to attach.

For example, Figure 4.3.1 represents $\hat{D}_3$ which has 30 lattice edges. Figure 4.3.2 represents $\hat{D}_4$ which has 36 lattice edges. Figure 4.3.3 represents $\hat{D}_5$ which has 44 lattice edges. Figure 4.3.4 represents $\hat{D}_6$ which has 50 lattice edges. Figure 4.3.5 represents $\hat{D}_7$ which has 58 lattice edges.

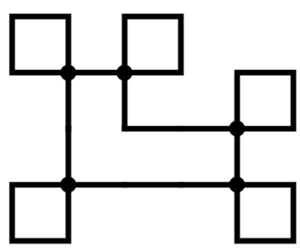

Figure 4.3.1.

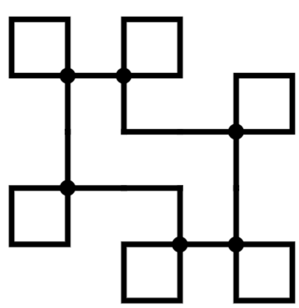

Figure 4.3.2.

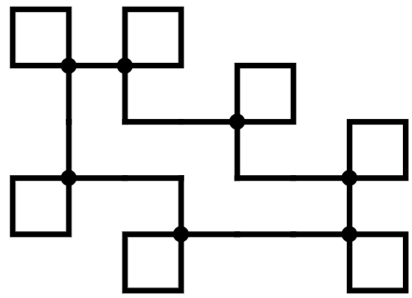

Figure 4.3.3.

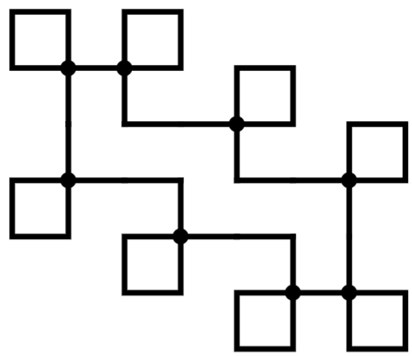

Figure 4.3.4.

From the sequence, we know that the pattern starts at 4-regular graph with single D_3 and no C. For lattice graph with a single D_n and no C. Let $n \in \mathbb{N}$, and suppose $n \geq 3$. With odd numbers of n, the function is $7n + 9$. With even numbers of n, the function is $7n + 8$. And both function can be express by $7n + 8 + \frac{1-(-1)^n}{2}$ as in Lemma 4.3.1.

Lemma 4.3.1. *Let $n, k \in \mathbb{N}$. Suppose $n \geq 3$. Then*

$$7n + 8 + \frac{1-(-1)^n}{2} = \begin{cases} 7n + 9 & \textit{if } n \textit{ is odd} \\ 7n + 8 & \textit{if } n \textit{ is even} \end{cases}$$

Proof. Suppose n is an odd number. Then $n = 2k + 1$ for some $k \in \mathbb{N}$. Then $7n + 8 + \frac{1-(-1)^n}{2} = 7(2k + 1) + 8 + 1 = 14k + 16 = 7(2k + 1) + 9 = 7n + 9$. Suppose n is an even number. Then $n = 2k$ for some $k \in \mathbb{N}$. Then $7n + 8 + \frac{1-(-1)^n}{2} = 7(2k) + 8 + 0 = 7n + 8$. Hence,

$$7n + 8 + \frac{1-(-1)^n}{2} = \begin{cases} 7n + 9 & \text{if } n \text{ is odd} \\ 7n + 8 & \text{if } n \text{ is even} \end{cases}.$$

$\square$

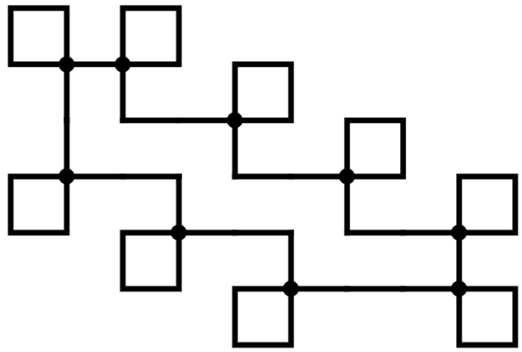

Figure 4.3.5.

Theorem 4.3.2. *Let $n \in \mathbb{N}$. Then,*

1. *$\hat{D}_1$ needs at worst 16 lattice edges to represent.*
2. *$\hat{D}_2$ needs at worst 20 lattice edges to represent.*
3. *If $n \in \mathbb{N}$ and $n \geq 2$, then $\hat{D}_n$ needs at worst $7n + 8 + \frac{1-(-1)^n}{2}$ lattice edges to represent.*

4.4 Counting Lattice Edges in Lattice Graphs with Single D_n With C

Figure 4.4.1 is a lattice graph from a 4-regular path graph with single D_1 and C graphs. Let $k \in \mathbb{N}$. Suppose there are k number of C graphs. Noted that C graphs can be attached to both each of the 4-regular graphs, so Figure 4.4.1 is not the only way to attach C graphs. Since each C graph has 4 lattice edges, then the number of lattice edges needed can be calculated by just adding $4k$ to the number of lattice edges needed in $\hat{D}_1$.

Hence, a 4-regular path graph with single D_1 and k number of C graphs needs at worst $16+4k$ lattice edges to represent.

Figure 4.4.2 is a lattice graph from a 4-regular path graph with single D_2 and C graphs. Then by similar calculation, a 4-regular path graph with single D_2 and k number of C graphs needs at worst $20 + 4k$ lattice edges to represent.

The sequence starts at $n \geq 3$. For example, Figure 4.4.3 is a lattice graph from a 4-regular path graph with single D_7 and C graphs. Then by similar calculation, a 4-regular path graph with single D_1 and k number of C graphs needs at worst $58 + 4k$ lattice edges to represent.

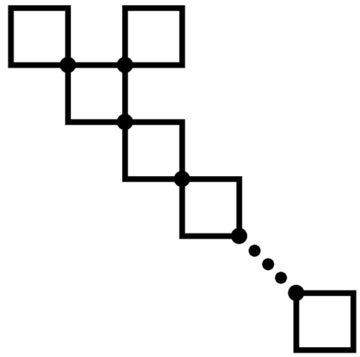

Figure 4.4.1.

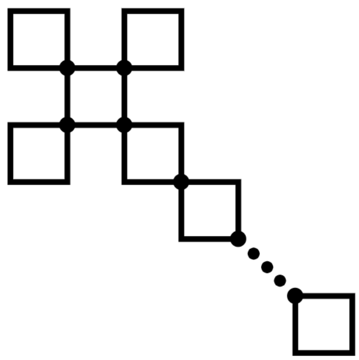

Figure 4.4.2.

Let $n \in \mathbb{N}$, and suppose $n \geq 3$. By similar calculation, a lattice graph from a 4-regular path graph with single D_n and C graphs needs at worst $7n + 8 + \frac{1-(-1)^n}{2} + 4k$ to represent.

Theorem 4.4.1. *Let $n, k \in \mathbb{N}$. Then,*

1. *A lattice graph from a 4-regular path graph with single D_1 and k number of C needs at worst $16 + 4k$ lattice edges to represent.*
2. *A lattice graph from a 4-regular path graph with single D_2 with k number of C needs at worst $20 + 4k$ lattice edges to represent.*
3. *If $n \in \mathbb{N}$ and $n \geq 3$, then a lattice graph from a 4-regular path graph with single D_n with k number of C needs at worst $7n + 8 + \frac{1-(-1)^n}{2} + 4k$ lattice edges to represent.*

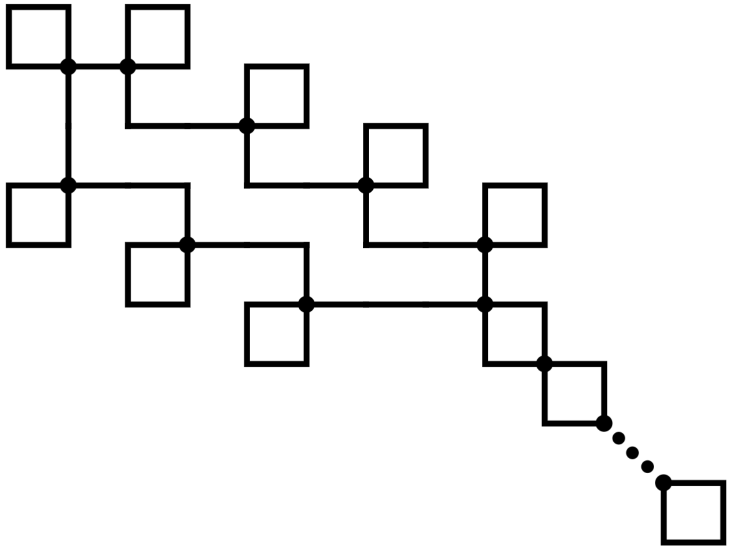

Figure 4.4.3.

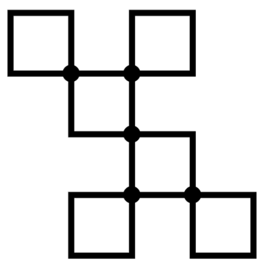

Figure 4.5.1.

4.5 Counting Lattice Edges in Lattice Graphs with D_i and D_j Without C

Figure 4.5.1 is a lattice graph from a 4-regular path graph with two D_1 and no C. Figure 4.5.1 has 24 lattice edges.

Figure 4.5.2 is a lattice graph from a 4-regular path graph with single D_1, D_2 and no C. Figure 4.5.2 has 34 lattice edges.

Figure 4.5.3 is a lattice graph from a 4-regular path graph with two D_2 and no C. Figure 4.5.3 has 40 lattice edges.

The sequence starts at $n \geq 3$. Figure 4.5.4 is a lattice graph from a 4-regular path graph with two D_3 and no C. Figure 4.5.4 has 52 lattice edges. The number of lattice edges is coming from

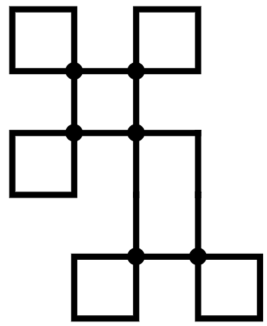

Figure 4.5.2.

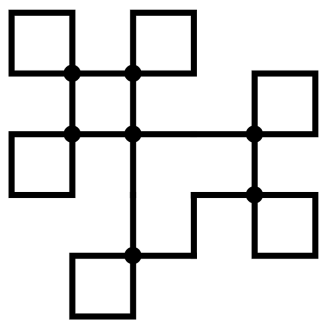

Figure 4.5.3.

adding the number of lattice edges of two $\hat{D}_3$ which has four ending loops in total, and subtract the number of lattice edges of two ending loops, since there can only be two ending loops.

Let $i, j \in \mathbb{N}$, suppose $n \geq 3$. Then a lattice graph from a 4-regular path graph with single D_i, D_j and no C needs at worst $7(i+j)+8+\frac{1-(-1)^i}{2}+\frac{1-(-1)^j}{2}$.

Theorem 4.5.1. *Let $i, j \in \mathbb{N}$. Then,*

1. *A lattice graph from a 4-regular path graph with two D_1 without C needs at worst 24 lattice edges to represent.*

2. *A lattice graph from a 4-regular path graph with single D_1 and D_2 without C needs at worst 36 lattice edges to represent.*

3. *A lattice graph from a 4-regular path graph with two D_2 without C needs at worst 40 lattice edges to represent.*

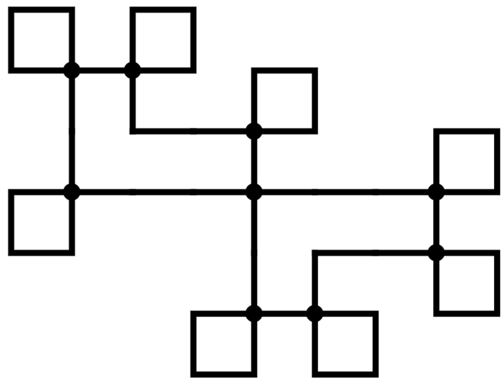

Figure 4.5.4.

4. *If $i \in \mathbb{N}$, and $i \geq 3$, then a lattice graph from a 4-regular path graph with single D_1, D_i without C needs at worst $7n + 20 + \frac{1-(-1)^n}{2}$ lattice edges to represent.*

5. *If $i \in \mathbb{N}$, and $i \geq 3$, then a lattice graph from a 4-regular path graph with single D_2, D_i without C needs at worst $7n + 24 + \frac{1-(-1)^n}{2}$ lattice edges to represent.*

6. *If $i, j \in \mathbb{N}$, and $i, j \geq 3$, then a lattice graph from a 4-regular path graph with single D_i, D_j without C needs at worst $7(i + j) + 8 + \frac{1-(-1)^i}{2} + \frac{1-(-1)^j}{2}$ lattice edges to represent.*

4.6 Counting Lattice Edges in Lattice Graphs with D_i and D_j With C

Let $k \in \mathbb{N}$. From previous section, we know that the addition of k number of C graphs can be calculated by just adding $4k$ to the number of lattice edges needed.

For example, Figure 4.6.1 is a lattice graph from a 4-regular path graph with two D_3 and three C. It has 64 lattice edges. Note that C graphs can be placed in between two D_n or at the end of each D_n graphs.

Let $k \in \mathbb{N}$. Figure 4.6.2 is a lattice graph from a 4-regular path graph with two D_1 and k number of C. The locations of C graphs are not going to change the lattice edges needed to represent. It needs at worst $24 + 4k$ lattice edges to represent.

Let $k \in \mathbb{N}$. Figure 4.6.3 is a lattice graph from a 4-regular path graph with single D_1, D_2 and k number of C. In this case, there are C graphs in between D_1 and D_2. It needs at worst $28 + 4k$ lattice edges to represent.

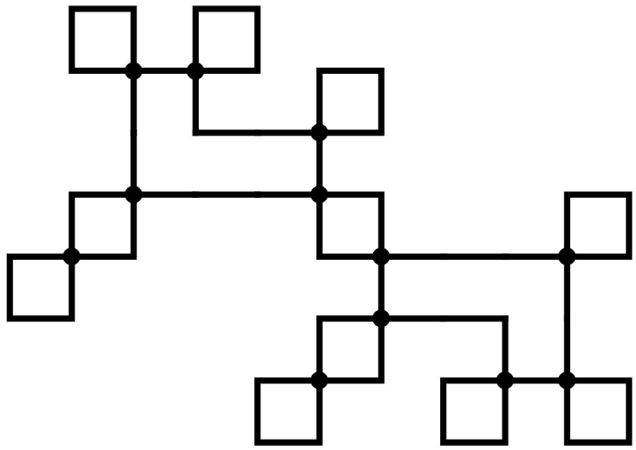

Figure 4.6.1.

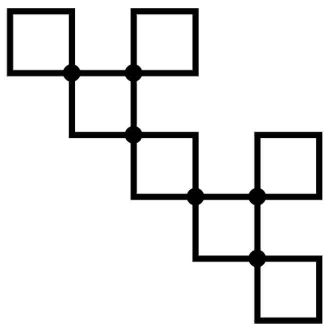

Figure 4.6.2.

If there is no C graphs in between D_1 and D_2, it need at worst $30 + 4k$ lattice edges to represent.

Let $k \in \mathbb{N}$. Figure 4.6.4 is a lattice graph from a 4-regular path graph with two D_2 and k number of C. In this case, there are C graphs in between two D_2. It needs at worst $32 + 4k$ lattice edges to represent.

If there is no C graphs in between two D_2, it need at worst $36 + 4k$ lattice edges to represent.

Theorem 4.6.1. *Let $i, j, k \in \mathbb{N}$. Then,*

1. *A lattice graph from a 4-regular path graph with two D_1 with k number of C needs at worst $24 + 4k$ lattice edges to represent.*
2. *A lattice graph from a 4-regular path graph with single D_1, D_2 with k number of C, and have C graphs in between D_1, D_2 needs at worst $28 + 4k$ lattice edges to represent.*

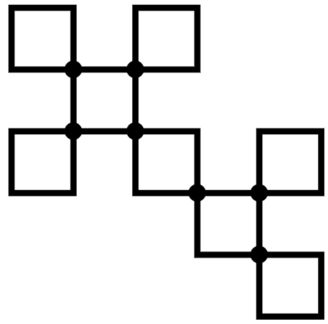

Figure 4.6.3.

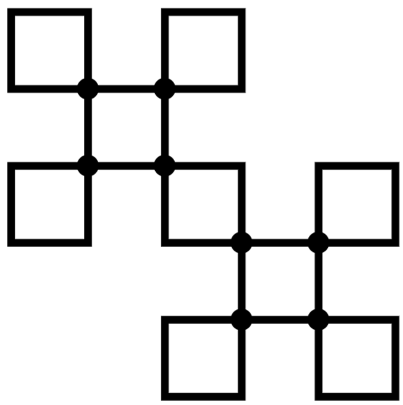

Figure 4.6.4.

3. *A lattice graph from a* 4*-regular path graph with single* D_1, D_2 *with* k *number of* C, *and no* C *graphs in between* D_1, D_2 *needs at worst* $30+4k$ *lattice edges to represent.*

4. *A lattice graph from a* 4*-regular path graph with two* D_2 *with* k *number of* C, *and have* C *graphs in between two* D_2 *needs at worst* $32+4k$ *lattice edges to represent.*

5. *A lattice graph from a* 4*-regular path graph with two* D_2 *with* k *number of* C, *and no* C *graphs in between two* D_2 *needs at worst* $36+4k$ *lattice edges to represent.*

6. *If* $i,j \in \mathbb{N}$, $i,j \geq 3$, *then a lattice graph from a* 4*-regular path graph with single* D_i, D_j *with* k *number of* C *needs at worst* $7(i+j)+8+\frac{1-(-1)^i}{2}+\frac{1-(-1)^j}{2}+4k$ *lattice edges to represent.*

5

Counting Lattice Edges in Lattice Graph from 4-Regular Cycle Graphs

5.1 Counting Lattice Edges in Lattice Graphs with C

To make a 4-regular cycle graph, there needs to be at least two C graphs. Figure 5.1.1 is a lattice graph that represents a 4-regular cycle graph with two C graphs. Figure 5.1.1 has 16 lattice edges.

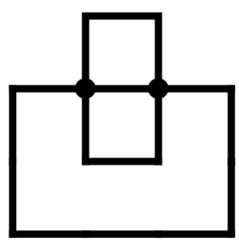

Figure 5.1.1.

Figure 5.1.2 is a lattice graph that represents a 4-regular cycle graph with three C graphs. Figure 5.1.2 has 16 lattice edges.

Figure 5.1.3 is a lattice graph that represents a 4-regular cycle graph with four C graphs. When the number of C graphs in the 4-regular graph is divisible by 4, the lattice graph can be represented by using only 4 lattice edges squares. Figure 5.1.3 has 16 lattice edges.

Figure 5.1.4 is a lattice graph that represents a 4-regular cycle graph with five C graphs. Figure 5.1.4 has 28 lattice edges.

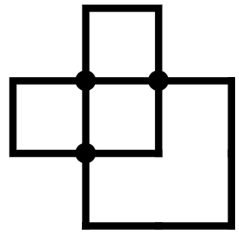

Figure 5.1.2.

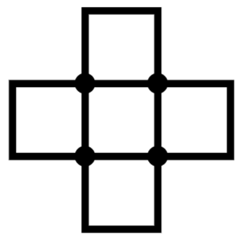

Figure 5.1.3.

Figure 5.1.5 is a lattice graph that represents a 4-regular cycle graph with six C graphs. Figure 5.1.5 has 28 lattice edges.

Figure 5.1.6 is a lattice graph that represents a 4-regular cycle graph with seven C graphs. Figure 5.1.6 has 32 lattice edges.

Figure 5.1.7 is a lattice graph that represents a 4-regular cycle graph with eight C graphs. This lattice graph can be represented by using only 4 lattice edges squares since it has eight C graphs. Figure 5.1.7 has 32 lattice edges.

Figure 5.1.8 is a lattice graph that represents a 4-regular cycle graph with nine C graphs. Figure 5.1.8 has 40 lattice edges.

Figure 5.1.9 is a lattice graph that represents a 4-regular cycle graph with ten C graphs. Figure 5.1.9 has 44 lattice edges.

Figure 5.1.10 is a lattice graph that represents a 4-regular cycle graph with eleven C graphs. Figure 5.1.10 has 48 lattice edges.

Figure 5.1.11 is a lattice graph that represents a 4-regular cycle graph with twelve C graphs. This lattice graph can be represented by using only 4 lattice edges squares since it has twelve C graphs. Figure 5.1.11 has 48 lattice edges.

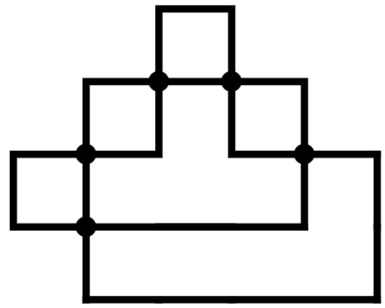

Figure 5.1.4.

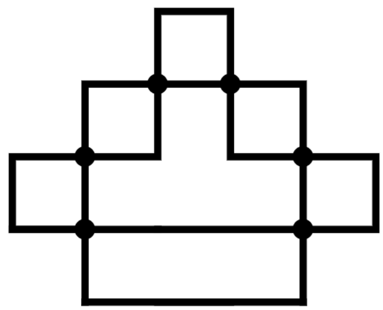

Figure 5.1.5.

Let $p \in \mathbb{N}$ be the number of C graphs in a lattice cycle graph with no D_k. Let C^p be the number of lattice edges in a lattice cycle graph with no D_k. By observation, the sequence is as following.

1. Let $n \in \mathbb{N}$ and suppose $p = 4n$. Then, $C^p = C^{4n} = 16n$
2. Let $n \in \mathbb{N}$ and suppose $p = 4n+1$. Then, $C^p = C^{4n+1} = 16n + 12$
3. Let $n \in \mathbb{N}$ and suppose $p = 4n+2$. Then, $C^p = C^{4n+2} = 4(4n+1) + 8 = 16n + 12$
4. Let $n \in \mathbb{N}$ and suppose $p = 4n+3$. Then, $C^p = C^{4n+3} = 4(4n+2) + 8 = 16n + 16$

Theorem 5.1.1. *Let $\bar{f} : \mathbb{Z} \to \mathbb{Z}$ be define by*

$$f(p) = \begin{cases} 0 & \textit{if } p = 4n \textit{ for some } n \in \mathbb{Z} \\ 12 & \textit{if } p = 4n+1 \textit{ for some } n \in \mathbb{Z} \\ 12 & \textit{if } p = 4n+2 \textit{ for some } n \in \mathbb{Z} \\ 16 & \textit{if } p = 4n+3 \textit{ for some } n \in \mathbb{Z} \end{cases}$$

Then $C^p = 16\lfloor \frac{p}{4} \rfloor + \bar{f}(p)$

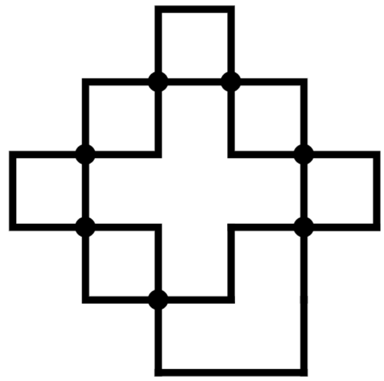

Figure 5.1.6.

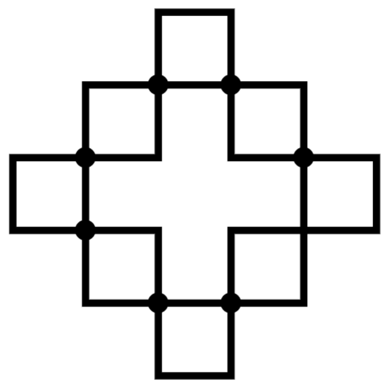

Figure 5.1.7.

5.2 Counting Lattice Edges in Lattice Graphs with Single D_n and C

Figure 5.2.1 is a lattice graph that represents a 4-regular cycle graph with single D_3 and eight C graphs. This lattice graph is created by attaching a D_3 to a 4-regular cycle graph with nine C graphs. Then one of the C graphs is removed and attach D_3. Figure 5.2.1 has 58 lattice edges.

Figure 5.2.2 is a lattice graph that represents a 4-regular cycle graph with single D_3 and eight C graphs. This lattice graph is created by attaching eight C graphs to a D_3 to from a cycle graph. Figure 5.2.2 has 58 lattice edges.

Figure 5.2.1 and Figure 5.2.2 have the same number of lattice edges, different way of attachment will not change the lattice edges needed to represent at worst. We know that $\hat{D}_n$ needs at worst $7n + 8 + \frac{1-(-1)^n}{2}$ lattice edges to represent. To attach the D_n graph, the end loops at each end are removed and one more C graph needs to be added to the 4-regular cycle graph

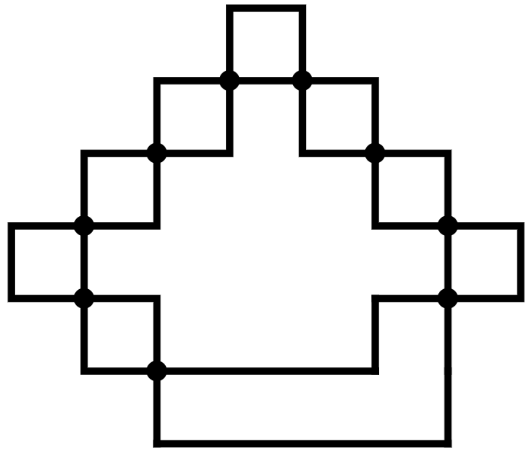

Figure 5.1.8.

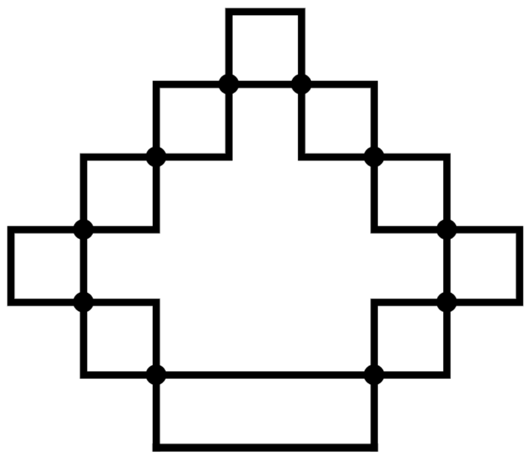

Figure 5.1.9.

represented by lattice graph. So a lattice graph from a 4-regular cycle graph with single D_n with k number of C will need $7n + 8 + \frac{1-(-1)^n}{2} - 8 + C^{p+1} = 7n + \frac{1-(-1)^n}{2} + C^{p+1}$ lattice edges to represent.

Theorem 5.2.1. *Let $n, k \in \mathbb{N}$. Then a lattice graph from a 4-regular cycle graph with single D_n with k number of C needs at worst $7n + \frac{1-(-1)^n}{2} + C^{p+1}$ lattice edges to represent.*

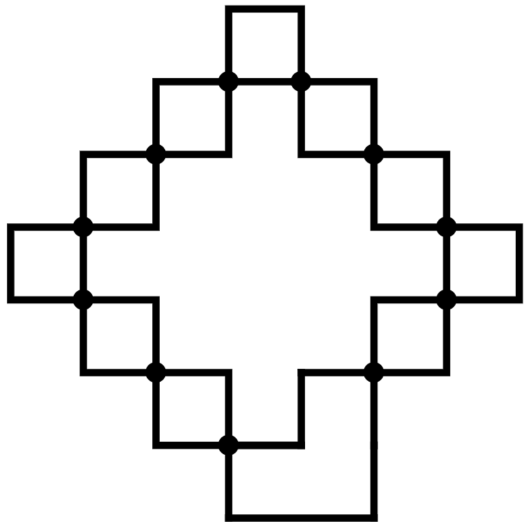

Figure 5.1.10.

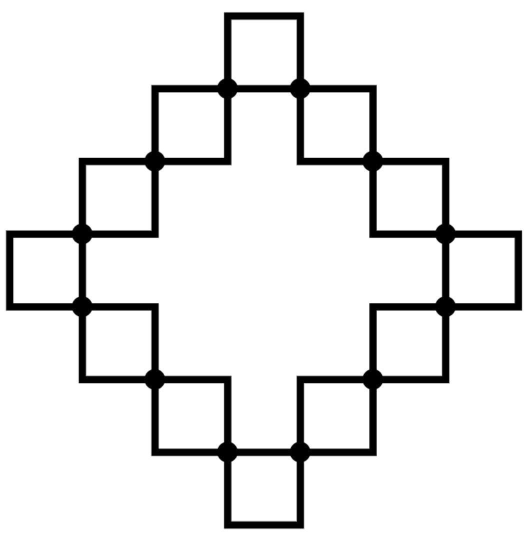

Figure 5.1.11.

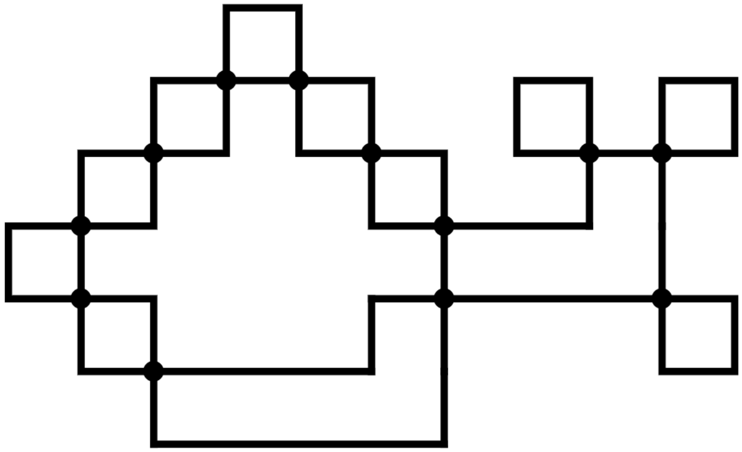

Figure 5.2.1.

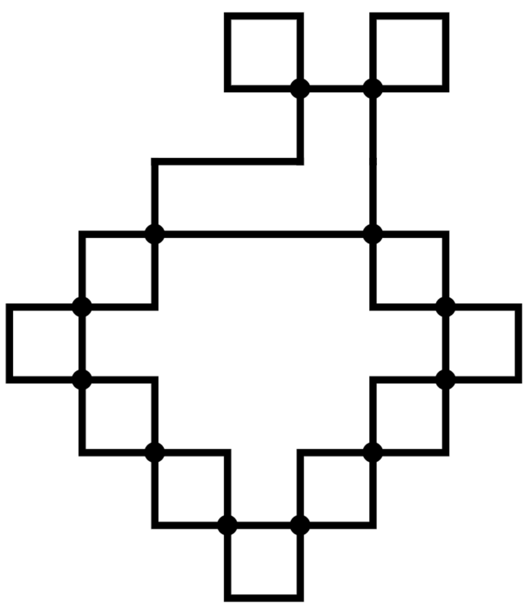

Figure 5.2.2.

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[1] Jonanthan L. Gross and Jay Yellen, *Graph Theory and its Applications*, 2nd ed., Chapman and Hall/CRC, 2005.