Rational Tilings of the Unit Square

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Abstract

A rational n-tiling of the unit square is a collection of n triangles with rational side length whose union is the unit square and whose intersections are at most their boundary edges. It is known that there are no rational 2-tilings or 3-tilings of the unit square, and that there are rational 4- and 5-tilings. The nature of those tilings is the subject of current research. In this project we give a combinatorial basis for rational n-tilings and explore rational 6-tilings of the unit square.

Contents

Acknowledgments

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Additional thanks to my board Ethan Bloch and James Belk – you each took the time to teach me something new and expand the scope of my project.

1

Introduction

We say that a tiling of the unit square is a collection of edges, vertices, and triangular tiles that fills up the unit square without gaps or overlaps. Our definition of a tiling is similar to a combinatorial definition of a tiling, but slightly more restrictive. In the combinatorial definition of a tiling, the tiles may be any shape and their union need not be a square. In our tilings, we specify the shape of our tiles and the fact that their union must be a square.

We say that a tiling is rational if each triangular tile has rational side lengths. We will explore the technical details in Chapter 2, where we give rigorous definitions of basic terms such as tilings and rational tilings as well as review the literature on questions of tiling the unit square with rational triangles.

In the literature, tiling the square with rational triangles is approached on a case by case basis based on the number of triangles. We follow that pattern, exploring the existence of rational tilings based on the number of triangles in a given tilings. We review previous results on 2-tilings, 3-tilings, 4-tilings, and 5-tilings prove several results that have been assumed in the literature but not been proven rigorously. In particular, we note that our question is a generalization of finding tilings of integer-sided squares with integer-sided triangles. We shall show that these investigations are equivalent and demonstrate how results from either can be applied to the other.

We then move away from the literature. In Chapter 3 we give a short mathematical background. We introduce the technique we will employ to study tiling the unit square with rational-sided triangles.

In Chapter 4 we discuss methods for finding rational tilings. We use projective geometry to create an algorithm for generating tilings with more than two triangles. Using this method, we will be able to provide several examples of 6-tilings.

In Chapter 5, we will explore the question of tiling the square with 6 rational triangles. We give several examples of 6-tilings and consider one class of tiling, which we call the Shark's Tooth Configuration. We consider several specializations of the Shark's Tooth Configuration.

2
Background

The intent of this chapter is to give the reader a short introduction to what is already known about rational tilings of the unit square. We define key terms and the review the literature of 2-tilings, 3-tilings, 4-tilings, and 5-tilings. We do not attempt to recreate the source materials but rather to demonstrate a unified summary of the current state of knowledge about rational tilings of the unit square.

2.1 Tilings and Configurations

The reader may be familiar with some notion of a tiling, either from a non-mathematical context or from a combinatorial background. There are many different definitions of a tiling and these definitions can vary greatly. Within the study of rational tilings of the unit square, however, there has not previously existed a rigorous definition of such a tiling. Previous literature has appealed to the reader's intuition to communicate what constitutes a tiling.

Below we give a rigorous definition of a tiling for our context, using the combinatorial definition of a plane tiling as a basis for our definition. We take a rational tiling of the square with side lengths one to be a specialization of a combinatorial plane tiling.

Definition 2.1.1. A **combinatorial plane tiling** $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, ...\}$ which cover the plane without gaps or overlaps. $\triangle$

The combinatorial definition of a tiling has three components: a countable collection of closed sets, a covering, and a packing (the no gaps or overlaps condition). Since we consider tilings of the square, the covering condition is trivially satisfied. Simply tile the plane with squares in the usual way.

The combinatorial plane tiling fails, however, to specify two important aspects of our notion of a tiling. These are: that all tiles must be triangular and that they somehow appear in a finite pattern whose union is a square. We must provide additional structure in order to obtain a precise definition that reflects what our tilings are. We give a different definition below.

Definition 2.1.2. Let S_t denote a square in $\mathbf{R}^2$ with side lengths t and corner vertices $(-\frac{t}{2}, -\frac{t}{2})$, $(\frac{t}{2}, -\frac{t}{2})$, $(-\frac{t}{2}, \frac{t}{2})$, and $(\frac{t}{2}, \frac{t}{2})$. A **tiling** of S_t is a collection of vertices $\mathcal{V} = \{\vec{v_1}, \vec{v_2}, ...\}$ such that $v_i \in \mathbf{R}^2$ and edges $\mathcal{E} = \{e_1, e_2, ...\}$ that form triangular tiles (closed sets) $\mathcal{T} = \{T_1, T_2, ...\}$ such that

1. The union of the triangular tiles is the square with side length t,

2. and the intersection of any pair $T_i, T_j \in \mathcal{T}$ where $i \neq j$ is at most their boundaries.

$\triangle$

So a tiling is a collection of vertices, edges, and tiles. The vertices are points in $\mathbf{R}^2$. The edges connect the vertices. The tiles are subsets of vertices and edges. To characterize a tiling, we give the tuple $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ where $\mathcal{V}$,$\mathcal{E}$, and $\mathcal{T}$ are the set of vertices, the set of edges, and the set of tiles, respectively, of a given tiling. Consider the following example.

Example 2.1.3. The sets $\mathcal{V}$,$\mathcal{E}$, and $\mathcal{T}$ for Figure 2.1.1 are

$$\mathcal{V} = \{A, B, C, D, E, F\}$$

$$\mathcal{E} = \{a, b, c, \overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}, \overline{AF}\}$$

$$\mathcal{T} = \{\triangle ABC, \triangle ACF, \triangle CEF, \triangle CDE\}$$

Then the $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ tuple for this tiling is

$$(\{A, B, C, D, E, F\}, \{a, b, c, \overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}, \overline{AF}\}, \{\triangle ABC, \triangle ACF, \triangle CEF, \triangle CDE\})$$

$\Diamond$

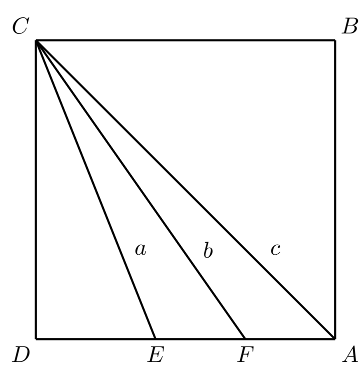

Figure 2.1.1. A 4-tiling.

This definition of this tiling more accurately depicts our situation since all tilings have triangular tiles and the union of their tiles is a square. To consider a tiling of the square with side lengths 1 (a translation of the unit square), simply let $t = 1$ and consider S_1. Although this definition does not describe the unit square, we will find it convenient to refer to a tiling of the unit square when we really mean a tiling of this form. The reader should observe that to make S_1 into the unit square, it suffices to translate each vertex of S_1 by $\left(\frac{1}{2}, \frac{1}{2}\right)$. We abuse language and say that a tiling of the unit square is a tiling of S_1.

When we begin to explore rational tilings, it will be useful to consider cases based on the number of triangles. In preparation for that, we define the following term.

Definition 2.1.4. An n-**tiling** is a tiling $T = (\mathcal{V}, \mathcal{E}, \mathcal{T})$ of S_1 in which $|\mathcal{T}| = n$. △

Intuitively, an n-tiling is a tiling of the unit square with n triangles. An example of a 4-tiling is given in 2.1.1. When we consider various tilings based on the number of tiles, we say that tilings with the same number of triangles are somehow comparable. We can say certain tilings are like other tilings. Likewise, we suggest that other tilings are unlike one another. We ask: in what other ways can we say that tilings are like or unlike one another?

First we have to determine what it means for two tilings to be "the same." Certainly two tilings are the same if they have the same (or at least isomorphic) vertex set, edge set, and tile set. So if $T_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$ and $T_1 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$ are tilings, then $T_1 = T_2$ if

$$\mathcal{V}_1 = \mathcal{V}_2,$$

$$\mathcal{E}_1 = \mathcal{E}_2,$$

$$\mathcal{T}_1 = \mathcal{T}_2.$$

But consider the two tilings given in Figure 2.1.2. These two tilings have the same tile set and the tiles are arranged with eachother in the same way. But it is not true that they have the same vertex, edge, and tile sets. It is obvious, however, that we could have re-labeled the vertices such that these tilings have the same vertex, edge, and triangle sets. The difference between these tilings is that one is rotated by $\frac{\pi}{2}$.

We want tilings to be the same under rotation and symmetry. The two tilings given in Figure 2.1.2 are examples of two tilings that can be rotated so that they have the same vertex, edge, and tile sets. Since we consider tilings to be the same under rotation and symmetry, we consider the group of rotations and symmetries of the square, D_4.

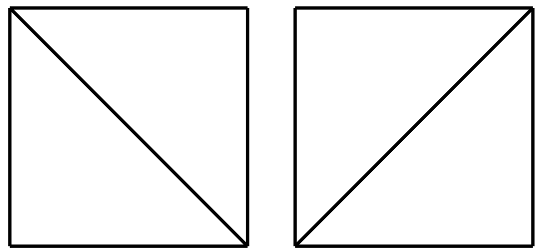

Figure 2.1.2. Two 2-tilings.

We must give an explicit definition of what it means for D_4 to act on a tiling. The reader may recall the standard rotation matrices on $\mathbf{R}^2$. These are matrices that rotate an arbitrary vector by some angle θ. The general form of these matrices

$$rot(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The rotation matrices for $\theta = \frac{\pi}{2}$ and $\theta = \pi$ are

$$rot\left(\frac{\pi}{2}\right) = R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$rot(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The reader should note that

$$R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = rot(\pi).$$

A similar matrix defines reflection across the line $y = x$ axis. It is

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can verify that this reflects across the line $y = x$ by seeing that this matrix fixes $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ but maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We find

$$S\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

Let $\vec{v}\in\mathbf{R}^2$. The standard way for D_4 to act on $\vec{v}$ is to equate $r\in D_4$ acting on $\vec{v}$ with multiplying $\vec{v}$ by R and we equate $s\in D_4$ acting on $\vec{v}$ with the product $S\vec{v}$. The rest of the matrices of the representation of D_4 on $\mathbf{R}^2$ can be obtained via matrix multiplication of the R and S matrices. That is applying $rs\in D_4$ is the same as multiplying by RS. We give examples of such an action below.

Example 2.1.5. Consider the vector $\begin{pmatrix}1\\1\end{pmatrix}\in\mathbf{R}^2$. Applying $r\in D_4$ is tantamount to multiplying

$$R\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$$

$\diamondsuit$

Example 2.1.6. Consider the vector $\begin{pmatrix}1\\0\end{pmatrix}\in\mathbf{R}^2$. Applying $rs\in D_4$ is tantamount to multiplying

$$RS\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$$

$\diamondsuit$

For readers who are familiar with representation theory, we note hat this is indeed a representation of D_4 on $\mathbf{R}^2$. Letting $\rho(r)=R$ and $\rho(s)=S$ defines a homomorphism $\rho: D_4\longrightarrow GL_2(\mathbf{R})$. So ρ is indeed a representation of D_4 on $\mathbf{R}^2$. We now define the action of D_4 on a tiling of the square with side lengths t.

Definition 2.1.7 (The Action of D_4 on a Tiling)**.** Let ρ be the standard representation of D_4 on $\mathbf{R}^2$ (given above). We say that **the action of D_4 on a tiling** T is given by applying D_4 to the vertex set of T. $\triangle$

Specifically, if $T = (\mathcal{V}, \mathcal{E}, \mathcal{T})$ is a tiling, then applying some $d \in D_4$ to T takes T to T' where $T' = (\mathcal{V}', \mathcal{E}, \mathcal{T})$ where $\mathcal{V}'$ is defined by

$$\mathcal{V}' = \{\rho(d)v \mid v \in \mathcal{V}\}.$$

Theorem 2.1.8. *The action of D_4 is an equivalence relation on the set of all tilings.*

Proof. Let $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$, $(\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$, and $(\mathcal{V}_3, \mathcal{E}_3, \mathcal{T}_3)$ be tilings of the square of size t. Consider the three properties of equivalence relation.

Symmetric Property. Assume that $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1) \sim (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$. Then there exists some $d \in D$ such that $d(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1) \cong (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$. Since $d \in D_4$, it follows that there exists $d^{-1} \in D_4$. Then $d^{-1}(\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2) = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$.

Reflexive Property. Apply the identity element to $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$.

Transitive Property. Assume $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1) \sim (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$ and $(\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2) \sim (\mathcal{V}_3, \mathcal{E}_3, \mathcal{T}_3)$. Then there exists $d \in D_4$ such that $d(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1) \cong (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$. There also exists $c \in D_4$ such that $c(\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2) \cong (\mathcal{V}_3, \mathcal{E}_3, \mathcal{T}_3)$. So apply cd to $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$. We have

$$cd(\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1) = c(\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2) = (\mathcal{V}_3, \mathcal{E}_3, \mathcal{T}_3).$$

Then D_4 is an equivalence relation on the set of all tilings. $\square$

Sometimes, however, we want a broader way to characterize similar tilings. For example, the tilings given in Figure 2.1.3 have different vertex sets, but their edges and tiles relate to each other in the same way. These tilings, however, are not the same by the definition we have already given. We must define a tiling configuration - an arrangement of triangles that is similar to another arrangement.

Definition 2.1.9. Two tilings $\sigma = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{T}_1)$ and $\tau = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{T}_2)$ are in the same **configuration** if

1. $|\mathcal{E}_1| = |\mathcal{E}_2|$,

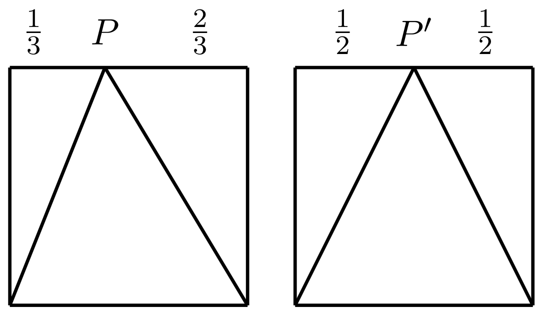

Figure 2.1.3. Two different 3-tilings.

2. $|\mathcal{T}_1| = |\mathcal{T}_2|$, and

3. there exists a function $f : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ that preserves the unit square and the collinearity of vertices.

$\triangle$

To show that the tilings given in Figure 2.1.3 are in the same configuration, we want to find some function f that maps P to P' while preserving the unit square and the number of triangles.

Example 2.1.10. Let

$$f(x,y) = \begin{cases} (x,y) & \text{if } y < \frac{1}{2} \\ (\frac{2}{3}x - \frac{1}{2}, y) & \text{if } y \geq \frac{1}{2} \text{ and } x < 0 \\ (\frac{4}{3}x - \frac{1}{6}, y) & \text{if } y \geq \frac{1}{2} \text{ and } x \geq 0. \end{cases}$$

Then $f(x,y)$ maps the right tiling in Figure 2.1.3 to the left tiling in Figure 2.1.3. $\lozenge$

Richard Guy gives an argument about the graph structure of such tilings to give necessary conditions for n-tilings to exist [5]. Although the argument becomes unwieldy for tilings with more than three triangles, we can use it to enumerate all possible 2-tiling configurations and 3-tiling configurations. We outline it below and will employ this argument in the proofs of Lemma 2.3.1 and Lemma 2.3.3.

Let V denote the number of vertices of the graph structure of an n-tiling. Let b denote the number of vertices on the boundary of the square (excluding the corners). Let i denote the number of vertices of the graph structure of an n-tiling in the interior of the square. Then for any n-tiling

$$V = 4 + b + i.$$

Let F denote the number of faces of the graph structure of an n-tiling. Then for any n-tiling $F = n + 1$.

Let E denote the number of edges of the graph structure of an n-tiling. Let a denote the number of internal angles that measure exactly π. Note that $a \leq i$ by definition. Combinatorially, any collection of n triangles must have $3n + a$ edges. So

$$2E = (3n + a) + (4 + b).$$

Since a tiling is a collection of vertices and edges (among other things), we can examine the graph structure of a tiling. In general, our tilings are far more restrictive than a simple graph structure can capture. In the case of simply counting tiles (three-cycles), using graph theory can be useful. Trivially, the graph structure of any tiling is connected. Since we specified that the overlap between any two triangles is at most their boundary, it follows that a tiling is planar. An example is given in Figure 2.1.4. Recall that the Euler Characteristic of a planar, connected graph is 2 [7]. For us, this means that $V + F = E + 2$.

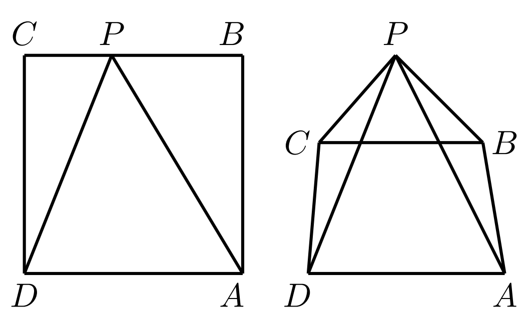

Figure 2.1.4. A 2-tiling and its graph.

Substitution yields

$$(4 + b + i) + (n + 1) = \frac{1}{2}[(3n + a) + (4 + b)] + 2$$

$$2[(4+b+i)+(n+1)] = [(3n+a)+(4+b)]+4$$

$$2(4+b+i)+2(n+1) = [(3n+a)+(4+b)]+4$$

$$8+2b+2i+2n+2 = 3n+a+4+b+4$$

$$2+b+2i = n+a.$$

Since $a \leq i$, it follows that

$$n+a \leq n+i \tag{2.1.1}$$

$$2+b+2i \leq n+i. \tag{2.1.2}$$

This argument is what will allow us to make claims about the number of n-tiling configurations for a given n. In Lemma 2.3.1 and Lemma 2.3.4 we will use this argument to enumerate all possible 2-tiling configurations and 3-tiling configurations respectively.

From a purely visual perspective, it seems clear that that not all tiling configurations can be generated in the same fashion. That is, some configurations may be created by adding edges to other configurations while others are somehow "minimal." That is: they are not subdivisions of some other tiling. This distinction is the key to understanding primitive and imprimitive tilings.

Definition 2.1.11. Let $k, n \in \mathbf{Z}$ such that $k > n$. A **subdivision** of an n-tiling is a k-tiling obtained by introducing new edges to the n-tiling such that the new object is still a tiling. $\triangle$

For example, the tiling given in Figure 2.1.6 is a subdivision of the tiling given in Figure 2.1.5. Another way to look at this, would be to note that removing edges A and C from the tiling in Figure 2.1.6 gives the tiling in Figure 2.1.5.

Definition 2.1.12. An n-tiling is **imprimitive** if it is a subdivision of an m-tiling where $m < n$. $\triangle$

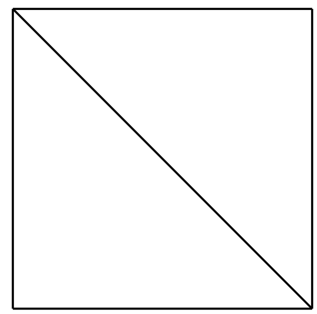

Figure 2.1.5. A 2-tiling.

Since the tiling in Figure 2.1.6 is a subdivision of the 2-tiling given in Figure 2.1.5, it follows that the tiling in Figure 2.1.6 is imprimitive.

Definition 2.1.13. An n-tiling is **primitive** if it is not imprimitive. $\triangle$

The tiling in Figure 2.1.5 is an example of a primitive tiling, since it is not a subdivision of another tiling. As we shall see later, all 2-tilings are primitive.

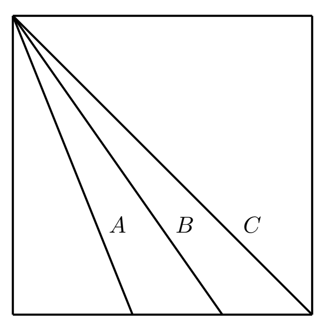

Figure 2.1.6. A subdivision of a 2-tiling.

2.2 Rational Triangles

A rational triangle is a triangle whose sides have rational length. Rational tilings are composed of sets of rational triangles. Understanding the conditions under which triangles are rational will help us to determine when tilings are rational. We give rigorous definitions below.

Definition 2.2.1. A **rational triangle** is a triangle whose side lengths are rational values. △

The notion of a rational triangle should not be surprising. It is valuable to pause here and observe one counter-intuitive property of a rational triangle: if a triangle T is rational, it does not follow that the area of T is rational.

Example 2.2.2. Consider the triangle given in Figure 2.2.1. Each side of this triangle has length 2. Then it has height $\sqrt{3}$. Then the area of the triangle is $A = \frac{1}{2}(2)\sqrt{3} = \sqrt{3}$. So the triangle given in Figure 2.2.1 is a rational triangle and does not have rational area. ♢

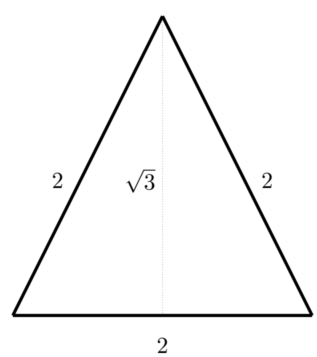

Figure 2.2.1. A rational triangle.

Definition 2.2.3. A **rational tiling** is an n-tiling $\mathcal{T}$ such that each triangle $T_1, T_2, ...T_n$ is a rational triangle. △

From our discussion so far, it is unclear which n-tilings are rational and which are not, but we can ask an equivalent question.

Question 1. Which n-tiling configurations yield rational n-tilings?

It is well-known that there are no rational tilings with 2 or 3 triangles. The question of 4 or 5 rational triangles is the subject of current research. We will explore rationality conditions for tilings with 6 triangles.

In the remainder of this chapter we review the literature on rational tilings of the unit square as well as a separate but related problem: tiling the integer square with integer-sided triangles. We will demonstrate that these problems are equivalent when we consider these tilings.

2.3 Tiling the Unit Square with Rational Triangles

It is obvious that there are no 1-tilings of the unit square. Thus we begin our discussion of rational n-tilings with 2-tilings and 3 tilings, We give well-known results about ratioanl 2-tilings and 3-tilings. We then consider current research into 4-tilings and 5-tilings.

2.3.1 Two-Tilings

We begin our study with 2-tilings, since 2 is the smallest value of n such that an n-tiling exists. Trivially, all 2-tilings must be primitive (since there are no 1-tilings to subdivide).

Whenever I introduce my problem to discrete mathematicians, the first question they ask is: “How many are there?” In general, it can be quite difficult to determine the total number of tiling configurations for a given n. For small values of n, however, we are able to prove very strong statements about the how many n-tiling configurations exist. For 2-tilings we can even say something stronger.

Lemma 2.3.1. *There exists only one 2-tiling.*

Proof. Consider Equation 2.1.1 and Equation 2.1.2 for $n = 2$. We have

$$2 + b + 2i = 2 + a \leq 2 + i.$$

$$b + 2i = a \leq i.$$

Since $n = 2$, it follows that $i = 0$. Then we have

$$b + 2i = a \leq 0.$$

Which means that $i = a = b = 0$. So the only possible 2-tiling configuration is given by drawing a line along the diagonal of the unit sqare. Up to rotation, there is only one way to do this. Then there is only one 2-tiling. □

An example is given in Figure 2.3.1. Recall that two n-tilings are the same if they are equivalent under the action of the dihedral group on four elements (D_4). Since there is only one 2-tiling, it is easy to check if that 2-tiling is a rational tiling.

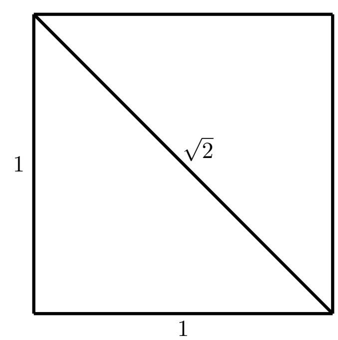

Figure 2.3.1. A 2-tiling has no rational solutions.

Claim 2.3.2. *There does not exist a rational* 2*-tiling.*

Proof. By Lemma 2.3.1 there is only one 2-configuration: the tiling given By the Pythagorean Theorem, the length of the diagonal of the unit square is $\sqrt{2}$. Then neither triangle of our 2-tiling is a rational triangle. □

For a visual version of this proof, consider Figure 2.3.1.

Although this result is trivial, it carries useful implications for n-tilings with larger values of n. It is easy to show that n-tilings such as the one given in Figure 2.3.2 are not rational tilings. Such is the power of understanding which tiling configurations are primitive.

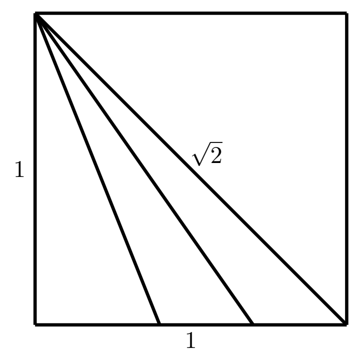

Figure 2.3.2. An imprimitive 4-tiling. Since the 2-tiling is not rational, it follows that this tiling is not rational either.

Each 2-tiling has many subdivisions and it is very easy to see how the rationality (or not) of imprimitive tilings depends on the rationality (or not) of the primitive tiling from which they are derived. Examples of a 2-tiling and a subdivision of that 2-tiling are given in Figure 2.3.1 and Figure 2.3.2, respectively.

2.3.2 Three-Tilings

Unlike 2-tilings, there are infinitely-many 3-tilings. As a result 3-tilings are the smallest n for which we can discuss tiling configurations which contain more that one tiling. As we shall see, there are three 3-tiling configurations: two imprimitive variations of the 2-tiling (see Figure 2.3.3 and Figure 2.3.4), and a primitive configuration that we will call the Mountain Configuration (see Figure 2.3.5). Since there are no rational 2-tilings, it follows that there are no rational 3-tilings that are derived from 2-tilings. Then the imprimitive 3-tiling configurations have no rational tilings. Then it is left to determine if the Mountain Configuration contains rational 3-tilings. Before we show that however, it is useful to prove that the configurations given above really were the only possible three-tilings.

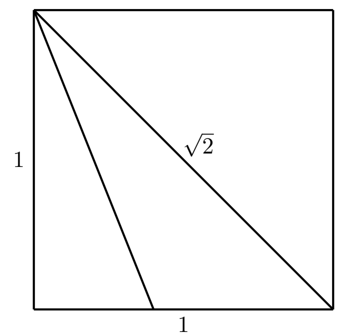

Figure 2.3.3. A 3-tiling in an imprimitive configuration.

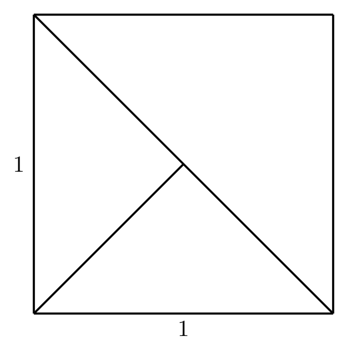

Figure 2.3.4. An imprimitive 3-tiling.

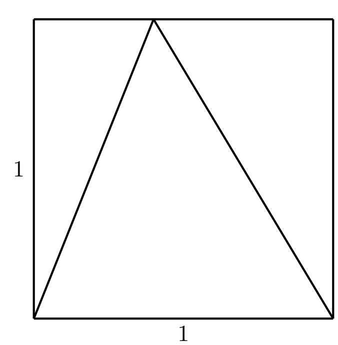

Figure 2.3.5. The Mountain Configuration.

Lemma 2.3.3. *The only primitive* 3*-tiling configuration is the Mountain Configuration. There are two other imprimitive* 3*-tilings.*

Proof. Consider Equation 2.1.1 and Equation 2.1.2 for $n = 2$. We have

$$2 + b + 2i = 3 + a \leq 3 + i.$$

$$b + 2i = a + 1 \leq i + 1.$$

Consider cases.

1. Let $i = 0$. Then $a = 0$ (since $a \leq i$). Then $b + 0 = 0 + 1 = 1$. Then there is one boundary vertex and no internal vertices. Since there are three triangles, it follows that the boundary vertex is the meeting place of two non-boundary edges or the endpoint of one non-boundary edge. The first type is the Mountain Configuration. The second type is an subdivision of the 2-tiling (an example of such a tiling is given in Figure 2.3.3).

2. Let $i = 1$. Then $b + 2 = a + 1 \leq 2$. So $b = a - 1 \leq 0$. Then $b = 0$ and $a = 1$. This configuration is imprimitve. An example of such a tiling is given by 2.3.4.

3. Let $i > 1$. Then

   $$b + 2i \leq i + 1.$$

   But there does not exist an $i \in \mathbb{N} - \{0, 1\}$ such that

   $$2i \leq i + 1.$$

   Since $b \in \mathbb{N}$, it follows that $i \not> 1$.

$\square$

This lemma shows us that the only primitive 3-tilings are those in the Mountain Configuration. Since there are no rational 2-tilings, it must be that if there are rational 2-tilings, they are tilings of the Mountain Configuration. The following theorem shows that there are no rational 3-tilings by examining a potential 3-tiling of the Mountain Configuration.

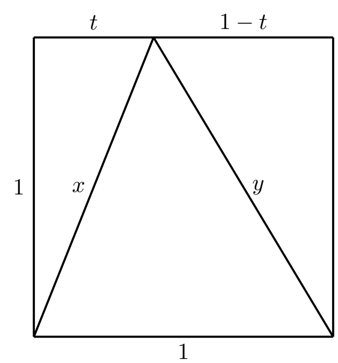

Figure 2.3.6. Labelled 3-tiling.

Theorem 2.3.4. *There does not exist a rational* 3*-tiling.*

Proof. By Lemma 2.3.3, it suffices to show that the Mountain Configuration does not yield any rational 3-tilings. This configuration may be labelled as in Figure 2.3.6. Then any rational tiling of this configuration must consist of $x, y, t \in \mathbf{Q}$ such that

$$1 + t^2 = x^2$$

$$1 + (1-t)^2 = y^2.$$

In particular $t \in \mathbf{Q} \cap (0, 1/2]$, since if $t > 1/2$, we can simply swap t and $1-t$ under the action of D_4. Consider two cases.

Case 1. Let $t = 1/2$. Then it must be that there exists $x \in \mathbf{Q}$ such that

$$x^2 = 1 + t^2 = 1 + \frac{1}{2}^2 = \frac{5}{4}.$$

Then $t = 1/2$ does not yield a rational tiling.

Case 2. Let $t \neq 1/2$. Suppose there exists a $t \in \mathbf{Q}$ that yields a rational tiling. Then there exists $(x_0, y_0) \in \mathbf{Q} \times \mathbf{Q}$ such that

$$0 = 1 + t^2 - x_0^2$$

$$0 = 1 + (1-t)^2 - y_0^2.$$

Taking the difference of these equations gives

$$0 = [1 + t^2 - x_0^2] - 1 + (1-t)^2 - y_0^2 = 2t^2 - 2t + 3 - x_0^2 - y_0^2.$$

The quadratic formula gives

$$t\frac{1}{2} \pm \frac{\sqrt{7 + 2x_0^2 + 2y_0^2}}{2}.$$

Recall that we specified $t \in (0, {}^1\!/\!{}_2) \cap \mathbf{Q}$. Then it suffices to consider

$$t = \frac{1}{2} - \frac{\sqrt{7 + 2x_0^2 + 2y_0^2}}{2}$$

where $0 < 7 + 2x_0^2 + 2y_0^2 < {}^1\!/\!{}_2$. Then

$$0 < 7 + 2x^2 + 2y^2 < 1$$

$$-7 < 2x_0^2 + 2y_0^2 < -6.$$

But it is also true that $1 < x_0 < 2$ and $1 < y_0 < 2$. By that observation

$$4 < 2x_0^2 + 2y_0^2 < 8.$$

It cannot be true that both $-7 < 2x^2 + 2y^2 < -6$ and $4 < 2x^2 + 2y^2 < 8$. Then there is no rational 3-tiling. □

So there are no rational 2-tilings or 3-tilings.

2.3.3 Four-Tilings

The study of 2-tilings and 3-tilings is in some sense trivial. There is only one 2-tiling and only one primitive 3-tiling configuration. Each of these tilings yields no rational tilings. The first cases of rational tilings appear among 4-tilings. Much of the study of 4-tilings comes from Guy, who pioneered the question of rational tilings of the unit square.

Guy asks a slightly more restrictive question than ours, however. Instead of asking how to tile the unit square with rational triangles, Guy tiles integer-sided squares with integer-sided triangles [3]. We call these integer tilings to distinguish them from the rational n-tilings described previously.

Definition 2.3.5. An **integer tiling** is a tiling of the integer-sided square with integer-sided triangles. △

The tiling given in Figure 2.3.7 is an integer tiling. To obtain an n-tiling, we must scale the integer tiling given in Figure 2.3.7 so that the side lengths of the square are equal to one. The resulting n-tiling is given in Figure 2.3.8.

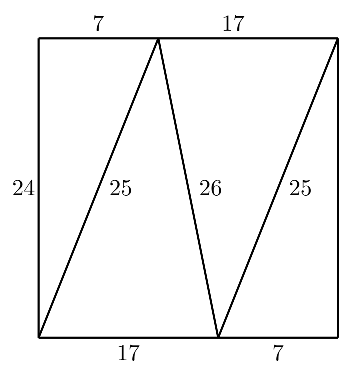

Figure 2.3.7. An example of the ν-configuration for an integer-sided square.

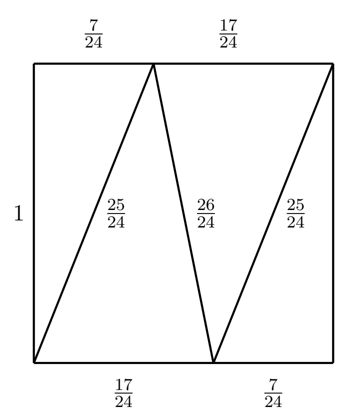

Figure 2.3.8. A rational tiling of the ν-configuration.

Theorem 2.3.6. *Let S be an n-tiling, then S is not an integer tiling.*

Proof. Assume S is an n-tiling. Assume S is an integer tiling. Then the triangles of T have integer side lengths. The longest possible edge length of a triangle in a square is the diagonal. The Pythagorean theorem tells us that the diagonal of the unit square has length $\sqrt{2}$. Then all the edges of triangles in S must have length less than or equal to $\sqrt{2}$. The only such positive integer is 1, but there is no way to tile the unit square with a collection of triangles all of whose edges have length 1. Thus S is not an integer tiling. □

Corollary 2.3.7. *Let T be an integer tiling, then T is not an n-tiling.*

Proof. Assume T is an integer tiling. Then T is a tiling of the square with side lengths t. By Theorem 2.3.6, it must be that $t \neq 1$. Then T is not a tiling of the unit square. Then t is not an n-tiling. □

Although rational n-tilings and integer tilings are distinct and contain no tilings in common, results for one can easily be modified to apply to the other. Recall that a tiling is a collection of vertices together with some other components (edges and triangular tiles). We want to know what happens when we apply a scalar matrix to the set of vertices of a tiling.

Definition 2.3.8. Let $T = (\mathcal{V}, \mathcal{E}, \mathcal{T})$ be a tiling of the square with side lengths t. We say that a tiling $T' = (\mathcal{V}', \mathcal{E}, \mathcal{T})$ is a **scaling** of T by a if $\mathcal{V}'$ can be obtained by applying the matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

to each of the vertices in $\mathcal{V}$. △

The reader should note that if we scale a tiling T of the square with side lengths t by a scalar matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

then the new square has side lengths ta. For example, scaling a tiling of the unit square by 2 will result in a tiling of the square with side lengths equal to 2.

Lemma 2.3.9. *Let (x_0, y_0) and (x_1, y_1) be points in $\mathbf{R}^2$. Let d be the distance between them. Let*

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Then the distance between $A(x_0, y_0)^T$ and $A(x_1, y_1)^T$ is ad.

Proof. Applying the matrix A to both points gives

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (ax_0, ay_0)$$

$$A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (ax_1, ay_1).$$

The distance between (ax_0, ay_0) and (ax_1, ay_1) is given by

$$\sqrt{(ax_0 - ax_1)^2 + (ay_0 - ay_1)^2} = \sqrt{[a(x_0 - x_1)]^2 + [a(y_0 - y_1)]^2}$$

$$= \sqrt{a^2(x_0 - x_1)^2 + a^2(y_0 - y_1)^2} = \sqrt{a^2[(x_0 - x_1)^2 + (y_0 - y_1)^2]}$$

$$= a\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} = ad.$$

□

Lemma 2.3.10. *Let $\vec{v}$ and $\vec{w}$ be vectors with angle θ between them. Then $a\vec{v}$ and $a\vec{w}$ have angle θ between them.*

Proof. We know that

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta.$$

By Lemma 2.3.9, scaling by a gives $a\vec{v}$ and $a\vec{w}$. Taking the dot product again gives

$$a\vec{v} \cdot a\vec{w} = |a\vec{v}||a\vec{w}| \cos\varphi.$$

Where φ is the angle between $a\vec{v}$ and $a\vec{w}$. Doing a little bit of algebra gives

$$a\vec{v} \cdot a\vec{w} = |a\vec{v}||a\vec{w}| \cos\varphi$$

$$= a^2\vec{v} \cdot \vec{w} = |a||\vec{v}||a||\vec{w}| \cos\varphi$$

$$= a^2\vec{v} \cdot \vec{w} = a^2|\vec{v}||\vec{w}| \cos\varphi$$

$$= \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\varphi.$$

But we already know that $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. Then $\varphi = \theta$. $\square$

From these two lemmas, it follows that scaling a tiling T by a matrix A preserves the ratios between edge lengths as well as the angles between edges. So scaling by an appropriate matrix allows us to go between rational tilings and integer tilings.

Specifically, if an n-tiling T is rational, then scaling by the least common multiple of the denominators of the edge lengths of T generates an integer tiling T by Lemma 2.3.9. Likewise, if S is an integer tiling of a square with side lengths s, then scaling by

$$\begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$

will scale an integer tiling (of an integer square) down to a rational tiling (of the unit square). We give an example of such a scaled integer tiling in Figure 2.3.8. We were able to produce the tiling in Figure 2.3.8 by scaling the tiling in Figure 2.3.7 by

$$\begin{pmatrix} \frac{1}{24} & 0 \\ 0 & \frac{1}{24} \end{pmatrix}.$$

Hereafter all tilings will be presented as n-tilings, although we can always scale between integer tilings and rational n-tilings.

According to Guy here are four primitive 4-tiling configurations. These are the ν-configuration (an example of such a tiling is given in Figure 2.3.9), the Δ-configuration (see Figure 2.3.12), the κ-configuration (see Figure 2.3.10), and the χ-configuration (see Figure 2.3.11). The χ-configuration is sometimes called the Four Distance Problem [3].

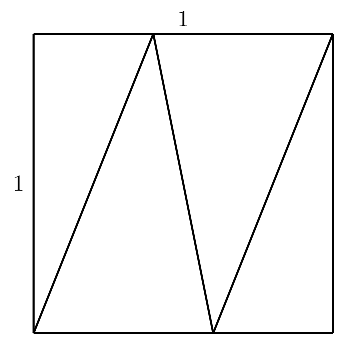

Figure 2.3.9. ν-configuration of a 4-tiling.

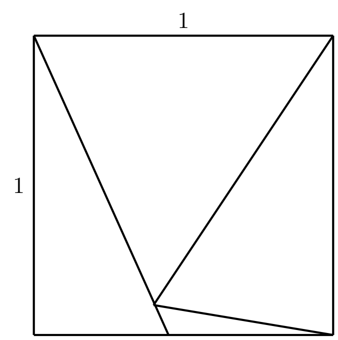

Figure 2.3.10. κ-configuration of a 4-tiling.

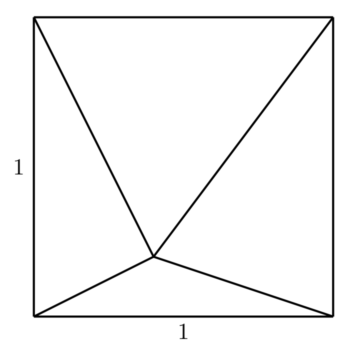

Figure 2.3.11. χ-configuration of a 4-tiling.

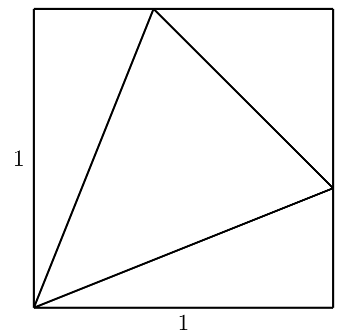

Figure 2.3.12. Δ-configuration of a 4-tiling.

Guy uses the theory of elliptic curves to show that the ν, Δ, and κ configurations yield infinitely-many rational tilings. He goes on to classify some of the solutions of these tilings. For example, Guy considers a specialization of the ν-configuration in which all the tilings have some form of rotational symmetry. That is, he considers a specialization of the ν-configuration whose tilings may be rotated by π and remain unchanged [5]. An example of such a tiling is given in Figure 2.3.8.

The tiling given in Figure 2.3.13 is an example of a rational 4-tiling in the κ configuration. Bremner and Guy prove that there are infinitely-many tilings of the κ configuration using the theory of elliptic curves [2]. Although it is interesting to note that they use the theory of elliptic curves, I will need a different approach when tackling the question of 6-tilings. As such, I will neglect to relay the details of how these results were obtained.

There is, however, a useful take-away: the Λ-configuration. This is not a true tiling configuration, as not all of its tiles are triangular. It is, however, a useful tool in understanding which tilings are rational. The Λ-configuration is obtained by considering the intersection of the κ- and χ- configurations. An example of the Λ-configuration is given in Figure 2.3.14.

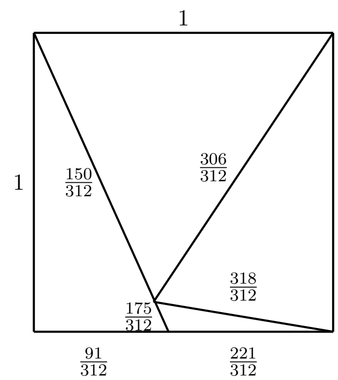

Figure 2.3.13. A rational tiling in the κ-configuration.

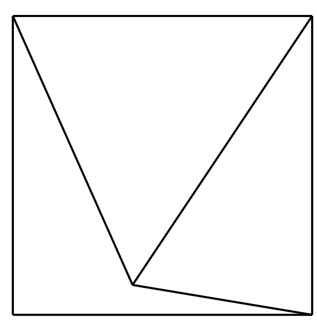

Figure 2.3.14. Λ-configuration of a 4-tiling.

Neither Bremner nor Guy is able to demonstrate the existence or nonexistence of rational tilings of the χ-configuration. This is left as an open problem.

2.3.4 Five-Tilings

Brady, Campbell and Nair indicate that there are 14 primitive 5-tiling configurations. They further classify these configurations into three collections: the simple Λ configurations, the ω-configuration (this collection contains only one configuration), and the sporadic configurations. [1].

Examples of simple Λ configurations are given in Figure 2.3.15 and Figure 2.3.16. Brady, Campbell and Nair show that all of the simple Λ configurations yield infinitely-many rational tilings at once. As with Guy's exploration of 4-tilings, they are able to use the theory of elliptic curves to demonstrate the existence of such tilings [1].

The ω-configuration is given its own classification. An example of an ω-configuration tiling is given in Figure 2.3.17. Like the simple Λ configurations, it has a similar structure to the Λ-configuration from Guy. Recall that the Λ-configuration is not a true configuration since it contains a tile that is not triangular. See Figure 2.3.14 for an example of the λ-configuration. Brady, Campbell, and Nair are able to show that this tiling yields rational infinitely-many rational solutions as well [1].

The 4 sporadic configurations are named the $\chi + \Lambda$-configuration (an example of such a tiling is given in Figure 2.3.18), the $Y + \Lambda$-configuration (see Figure 2.3.19), the Dragonfly configuration (see Figure 2.3.20), and the Super-X configuration (see Figure 2.3.21). Both the $\chi + \Lambda$-configuration and the $Y + \Lambda$-configuration are variations on the Λ-configuration introduced by Bremner and Guy [2]. Brady, Campbell, and Nair use arithmetic on elliptic curves in order to prove that each of these sporadic configurations yield infinitely-many rational tilings.

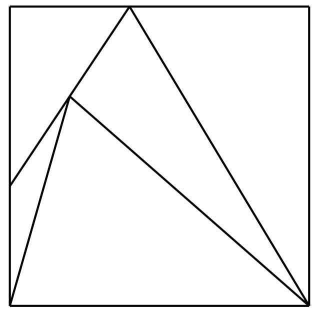

Figure 2.3.15. Simple Λ-configuration type (a).

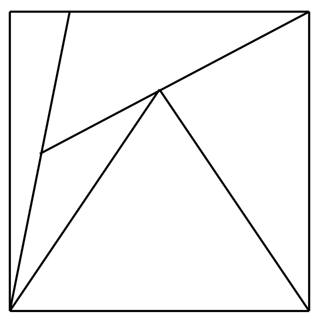

Figure 2.3.16. Simple Λ-configuration type (b)

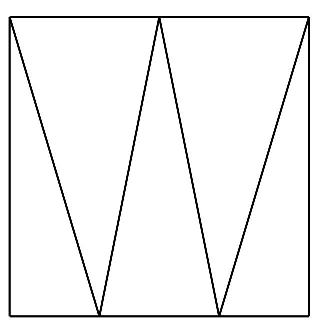

Figure 2.3.17. ω-configuration of a 5-tiling

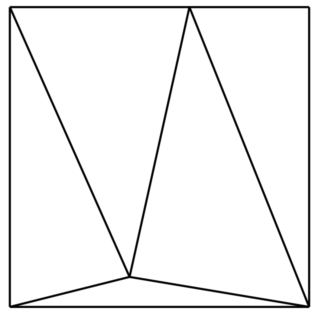

Figure 2.3.18. $\chi + \Lambda$-configuration of a 5-tiling.

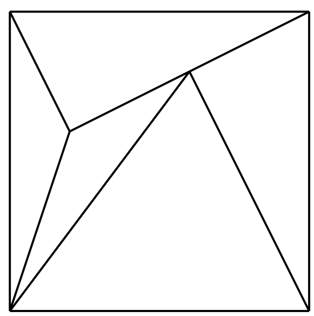

Figure 2.3.19. $Y + \Lambda$-configuration of a 5-tiling.

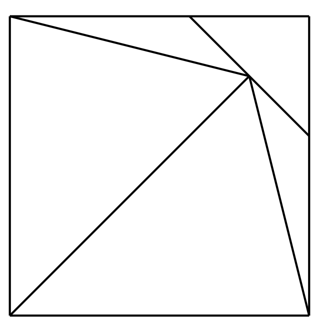

Figure 2.3.20. Dragonfly-configuration of a 5-tiling.

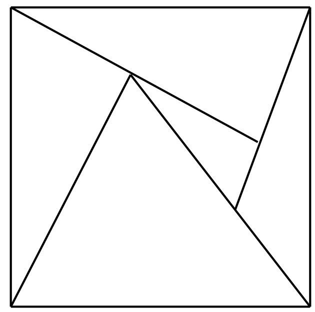

Figure 2.3.21. Super X-configuration of a 5-tiling.

At the end of their paper, Brady, Campbell, and Nair introduce 6-tilings. Although they do not provide a classification, they ask if rational 6-tilings exist and what form they might take.

3

Mathematical Background

3.1 Resultants

In the previous chapter, we saw that the question of finding rational tilings can sometimes be boiled down to finding rational points on certain curves. We will need to find common rational roots of two polynomials. So we ask, how can we find a common root between two polynomials?

The theory of resultants allows us to determine when two arbitrary polynomials have a common root. By cleverly utilizing resultants, we will be able to determine when certain tiling configurations have rational solutions or do not. Before examining resultants, however, we must introduce a little bit of notation.

Definition 3.1.1. Let k be a field. Let $f(x) \in k[x]$ be a polynomial. The **leading coefficient** of $f(x)$ is the coefficient of the highest degree term (leading term) of $f(x)$. It is denoted $l(f)$. $\triangle$

The leading coefficient of a polynomial is no doubt familiar to the reader and the definition we give here is the standard one. For example, the polynomial

$$f(x) = 5x^2 + 2x + 1$$

has leading coefficient 5. We say that $l(f) = 5$.

Definition 3.1.2. Let k be a field. Let $f(x), g(x) \in k[x]$ be polynomials. The **resultant** of f and g is

$$\mathrm{Res}(f, g) = [l(f)]^{\deg(g)} \prod_{f(x)=0} g(x).$$

△

Theorem 3.1.3. *Two polynomials have a common root if and only if their resultant is zero.*

Proof. Let $f(x)$ and $g(x)$ be polynomials. Assume $f(x)$ and $g(x)$ have a common root. Then there exists some x_0 such that $f(x_0) = g(x_0) = 0$. Then

$$[l(f)]^{\deg(g)} \prod_{f(x)=0} g(x) = 0.$$

Now assume the resultant of $f(x)$ and $g(x)$ is 0. Then

$$[l(f)]^{\deg(g)} \prod_{f(x)=0} g(x) = 0$$

Since $l(f) \neq 0$, it follows that there exists some x_0 such that $f(x_0) = g(x_0) = 0$. Then $f(x)$ and $g(x)$ share a root. □

Lemma 3.1.4. *Let $f(x)$ be a polynomial. Then* $\mathrm{Res}(f(x), f(x)) \neq 0$ *if and only if $f(x)$ is constant.*

Proof. Assume $\mathrm{Res}(f(x), f(x)) \neq 0$. Then f has no common roots with itself. Then f has no roots. So by the Fundamental Theorem of Algebra f must have degree 0. Then f is constant.

Now assume $f(x) = c$ where c is a constant. Then $\text{Res}(f(x), f(x)) = c^0 = 1$. Then $\text{Res}(f(x), f(x)) \neq 0$. $\square$

Example 3.1.5. Let $f(x) = x^2 + 1$. Clearly $f(x) \in \mathbf{Q}[x]$ but $f(x)$ is irreducible over $\mathbf{Q}$. So $f(x)$ has no roots in $\mathbf{Q}$. Taking the resultant of f with itself is 0, however, since f has roots over $\mathbf{Q}(i)$. $\Diamond$

Note that the resultant of f and g may be computed even if f and g have no roots in their base fields. For instance, a polynomial f may be irreducible over $\mathbf{Q}$ but since it has roots in some extension of $\mathbf{Q}$ (namely the splitting field of f), the resultant of f can be computed with any other polynomial.

Example 3.1.6. Let $f(x) = x^2 + 1$ and let $g(x) = x^3 + 2x^2 + x + 2$. Then

$$\text{Res}(f(x), g(x)) = 0.$$

$\Diamond$

Corollary 3.1.7. *Let $f(x), g(x) \in K[x]$. If $f(x)$ and $g(x)$ have a common root in K, their resultant is* 0.

Proof. Assume $f(x)$ and $g(x)$ have a common root in K. Then by Theorem 3.1.3, their resultant must be 0. $\square$

3.2 Projective Geometry and Facts about Rational Points on Curves

For the purposes of this project, we will only give background regarding $\mathbf{P}^2$.

We define the projective plane by the set of triples $[a : b : c]$ for all $a, b, c \in \mathbf{R}$ such that a, b, and c are not all 0. We define an equivalence relation E on this set by

$$\lambda[a : b : c] = [a : b : c]$$

for all $\lambda \in \mathbf{R}$. We call the equivalence classes of this equivalence relation "points." So

$$[1 : 0 : 1] = [2 : 0 : 2] = [3 : 0 : 3] = ...$$

is a point. A different point is

$$[1 : 1 : 1] = [\pi : \pi : \pi] = [150 : 150 : 150] = ...$$

We call this collection of all such equivalence classes (points) $\mathbf{P}^2$. Note that $\mathbf{R}^2 \subset \mathbf{P}^2$ where $\mathbf{R}^2$ is the set of points in $\mathbf{P}^2$ of the form $[a : b : 1]$. In other words $\mathbf{P}^2 = \mathbf{R}^3 / E$.

There are two ways in which we will use projective geometry in this project. First we will look at projective transformations on $\mathbf{P}^2$. Second we homogenize polynomials and put them into projective space. We will not give a complete overview, but rather remind the reader of some key facts that pertain to our problem. For more information, we direct the reader to Hindry and Silverman [4] and Silverman and Tate [6].

We now briefly consider projective transformations. As with linear transformations of $\mathbf{R}^3$, we can define projective transformations on $\mathbf{P}^2$ by 3×3 matrices with nonzero determinant. We define a similar equivalence relation F on the $GL_3(\mathbf{R})$ as the equivalence relation E that we defined on $\mathbf{R}^3$. We say that two matrices are in the same equivalence class of F if one is a scalar multiple of the other. So for any $\lambda \in \mathbf{R}$, we say

$$\begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \\ \lambda g & \lambda h & \lambda i \end{pmatrix} \quad \lambda \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Consider the following example.

Example 3.2.1. Consider$I_3 \in GL_3(\mathbf{R})$ where

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In projective space, this matrix has the property that for $a \in \mathbf{R}$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are in the same equivalence class under the equivalence relation F. $\Diamond$

Theorem 3.2.2. *Every projective quadrilateral has at least one vertex that is an affine point.*

Proof. Assume there exists a quadrilateral Q with no vertices that are affine points. Then all the vertices of the quadrilateral must have the form $[x : y : 0]$.

Recall that three points of $\mathbf{P}^1$ are collinear if their determinant is zero [6]. Observe that for any points of the form $[x : y : 0]$, we have

$$\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Then these three points are collinear. Since Q is a quadrilateral, it cannot be that three of the vertices are collinear. So Q must have a point of the form $[x, y, 1]$. That is Q must have an affine point. $\square$

Now we consider the homogenization of polynomials. To homogenize a polynomial $f(x, y)$ and put it into projective space, we must first determine the degree of $f(x, y)$. We denote the degree of $f(x, y)$ by $\deg(f)$. Then the homogenization $H_f(x, y, z)$ of $f(x, y)$ is given by

$$H_f(x, y, z) = z^{\deg(f)} f\left(\frac{x}{z}, \frac{y}{z}\right).$$

So to homogenize $f(x, y) = x^3 + y^2 + xy + 1$, we first note that $\deg(f) = 3$. Then

$$H_f(x, y, z) = z^3 f\left(\frac{x}{z}, \frac{y}{z}\right) = x^3 + y^2 z + xyz + z^3.$$

Once we put a curve into projective space, we are able to classify it by a term which we will call the genus of a curve. The genus of a curve relates the degree of a curve with the number of singularities. In algebraic geometry, it is standard to classify curves by their genus rather than by their degree. We compute the genus of a projective curve in Magma and so we will not include the formula. Instead, we give three theorems that characterize the number of rational solutions on a curve by the genus of the curve.

Theorem 3.2.3 ([4]). *Let C be a non-singular curve of genus 0 over $\mathbf{Q}$. Then C has either no rational points or infinitely many rational points.*

Theorem 3.2.4 (Mordell-Weil Theorem [4]). *Let C be a non-singular curve of genus 1 over $\mathbf{Q}$. Then C could have finitely many or infinitely-many rational points.*

Theorem 3.2.5 (Faltings [4]). *Let C be a non-singular curve of genus $g > 1$ over $\mathbf{Q}$. Then C has finitely many rational points.*

Since we hope to find rational n-tilings, understanding when curves have rational solutions can very useful. As we have already seen with our investigation into 3-tilings, our main technique in finding tilings will be to parameterize a given tiling by a curve. Once we have a parameterization of a tiling, we can classify it and use theorems from algebraic geometry to gain informational about the number of rational points on the curve.

3.3 Rationality Conditions for Lines, Points, and Slopes

When we defined n-tilings, we defined them by their vertices, edges, and tiles. In this section, we consider the restrictions that this definition puts on vertices and edges of a set of vertices and a related set of edges. We hope to make it clear that the rationality (or not) of the vetex set doe not always determine if a tiling is rational or not rational.

Question 2. Are two rational points on a line with rational slope always at rational distance from one another?

When $m = 0$, this question can be rephrased as: are two rational points on the real line at rational distance from one another. Since $\mathbf{Q}$ is closed under addition, it follows that rational points are always at rational distance from one another.

In general, however, this is not the case. There is an obvious counterexample. Figure 3.3.1 depicts a the function $f(x) = x$ on the interval $[0, 1]$. The distance between the two rational points $(0, 0)$ and $(1, 1)$ is $\sqrt{2}$. The reader may observe that this is the same as having a right triangle with two rational side lengths. It does not immediately follow that the third side is rational. Specifically, Figure 3.3.1 is equivalent to the the 45-45-90 right triangle given in Figure 3.3.2. If we place the triangle on the Cartesian plane in the usual way, the hypotenuse falls along the line $x = y$, extending between the origin and $(1, 1)$. Both are rational points, but their distance is irrational: $\sqrt{2}$. Furthermore, both tiles of the 2-tiling from the previous chapter are $45 - 45 - 90$ right triangles. It has already been shown that these are not rational triangles. See Figure 3.3.3.

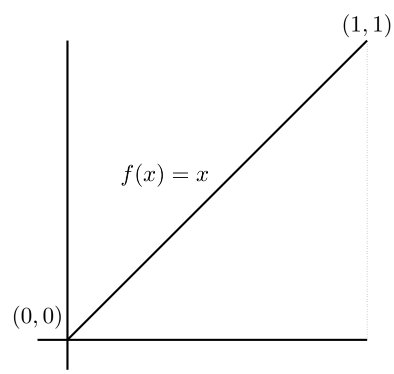

Figure 3.3.1. The line $f(x) = x$ on the interval $[0, 1]$.

So if we are given two rational points on a line with slope $m \neq 0$, it does not follow that the distance between them is rational. However, there are other conditions that we can demonstrate. For example, if we are given two rational points, then we can make a conclusion about the slope of the line between them.

Lemma 3.3.1. *Given two rational points, the line between them has rational slope.*

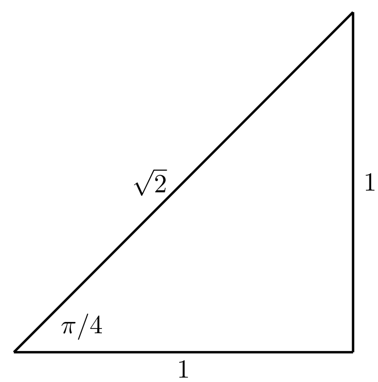

Figure 3.3.2. A $45-45-90$ right triangle.

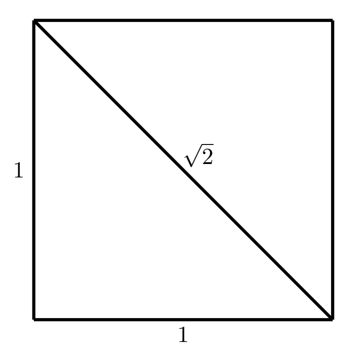

Figure 3.3.3. A 2-tiling has no rational triangular tiles.

Proof. Let (x, y) and (x_0, y_0) be rational points in $\mathbf{R}^2$. Then the slope of the line between them is

$$m = \frac{y - y_0}{x - x_0}.$$

Since every element of $\mathbf{Q}$ has a multiplicative and additive inverse in $\mathbf{Q}$, it follows that $m \in \mathbf{Q}$. □

Question 3. Assume that you are given a rational point x and a rational slope m. If a second point y is rational distance away from x on a line of slope m, does it follow that y is rational as well?

Consider the following counterexample. The distance between $(0, 0)$ and $(\sqrt{2}, \sqrt{2})$ on the line $y = x$ is

$$\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2 + 2} = \sqrt{4} = 2.$$

Then they have rational distance, which provides a counterexample to our hypothesis.

This result suggests that a triangle embedded in $\mathbf{R}^2$ may have rational side lengths even if the vertices of the triangle are not rational points. Thus it does not suffice to assume that the vertices of our tilings are rational points. In fact, there are only very limited conditions we can put on the coordinates of vertices of a tiling. Different strategies are needed to study such tilings. As we have seen, it can sometimes be more effective to study tilings from an algebraic perspective.

3.4 Rectangular Numbers

When constructing rational tilings of the unit square, we often find that tiling contain right triangles where one leg is an edge of the square (it has length 1) and the other leg is only part of an edge of the unit square. The triangle $\triangle ABA'$ in Figure 3.4.1 has this form. We want to create rational tilings, so it is vital that the hypotenuse of such a triangle have rational length. A rectangular number is a number that can serve as the non-1 leg of the right triangle.

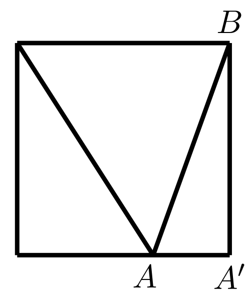

Figure 3.4.1. A 3-tiling.

Definition 3.4.1. Let $x, \alpha \in \mathbf{Q}$ such that $x^2+1 = \alpha^2$. Then x is a **rectangular number**. △

The fraction $\frac{3}{4}$, for example, is a rectangular number because

$$\left(\frac{3}{4}\right)^2 + 1 = \frac{9}{16} + \frac{16}{16} = \frac{25}{16} = \left(\frac{5}{4}\right)^2.$$

Lemma 3.4.2 (Rectangular Number Condition)**.** *Let $a, b, c \in \mathbf{Q}$. Then $a^2 + b^2 = c^2$ if and only if a/b is a rectangular number.*

Proof. Assume $a^2 + b^2 = c^2$. Then

$$\left(\frac{a}{b}\right)^2 + 1 = \frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{a^2+b^2}{b^2} = \frac{c^2}{b^2} = \left(\frac{c}{b}\right)^2.$$

So $\frac{a}{b}$ is a rectangular number. The reverse direction is obvious. □

This lemma draws out the connection between Pythagorean triples and rectangular numbers. A rephrasing of this lemma might replace the $a^2 + b^2 = c^2$ condition with the fact that a-b-c is a Pythagorean triple with c being the hypotenuse of the right triangle. By that phrasing, this lemma suggests that, for every rectangular number $\frac{a}{b}$, there exists a Pythagorean triple where $c^2 = a^2 + b^2$ and $\frac{a}{b} + 1 = \frac{c^2}{b^2}$.

4

Generating Primitive Tilings

The process of generating primitive tilings becomes more difficult as the number of tiles increases. For small values of n, such as 2 and 3, we are able to use a graph theoretic argument to enumerate all possible configurations. Since there are so few 2- and 3- tiling configurations, we can go through by hand and determine which tilings are primitive and which are not. For tilings with more tiles, it is impractical to apply this graph theoretic approach.

Instead, we can use projective geometry to generate new, primitive tilings.

Lemma 4.0.3. *Let $ABCD$ be a convex quadrilateral in projective space. Then one vertex can be mapped to* $[1:0:0]$.

Proof. Let $A = [a_1 : a_2 : a_3]$. Since $A \in \mathbf{P}^2$, it follows that $A \neq [0:0:0]$. Then there exists $a_i \in \{a_1, a_2, a_3\}$ such that $a_i \neq 0$. Consider cases.

1. Assume $a_1 \neq 0$. Then

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -\left(\dfrac{a_2 + a_3}{a_1}\right) & 1 & 1 \\ -\left(\dfrac{a_3}{a_1}\right) & 0 & 1 \end{pmatrix}.$$

2. Assume $a_2 \neq 0$. Then

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\left(\dfrac{a_1 + a_3}{a_2}\right) & 1 \\ 0 & -\left(\dfrac{a_3}{a_2}\right) & 1 \end{pmatrix}.$$

3. Assume $a_3 \neq 0$. Then

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -\left(\dfrac{a_1 + a_2}{a_3}\right) \\ 0 & 1 & -\left(\dfrac{a_2}{a_3}\right) \end{pmatrix}.$$

Note that for each case $\det(M) = 1$. □

Lemma 4.0.4. *Let $ABCD$ be a convex quadrilateral in projective space with one vertex $A = [1 : 0 : 0]$. Then one vertex can be mapped to $[0 : 1 : 0]$ while preserving the position of A.*

Proof. Let $B = [b_1 : b_2 : b_3]$. Since $B \in \mathbf{P}^2$, it follows that $B \neq [0 : 0 : 0]$. Then there exists $b_i \in \{b_1, b_2, b_3\}$ such that $b_i \neq 0$.

Furthermore if $b_1 \neq 0$, then either $B = A$ or there exists a $b_i \in \{b_2, b_3\}$ such that $b_i \neq 0$. If $B = A$, then $ABCD$ is not a quadrilateral. So it must be that $b_2 \neq 0$ or $b_3 \neq 0$. We consider cases.

1. Assume $b_2 \neq 0$. Consider cases.

   (a) Assume $b_2 + b_3 \neq 0$. Then

$$M = \begin{pmatrix} 1 & -\frac{b_1+b_3}{b_2} & 1 \\ 0 & 1 & 1 \\ 0 & -b_3 & b_2 \end{pmatrix}.$$

   Applying M to B gives

$$[b_1 - (b_1 + b_2) + b_2 : b_1 + b_2 : -b_3 b_2 + b_3 b_2] = [0 : 1 : 0].$$

(b) Assume $b_2 + b_3 = 0$. Then

$$M = \begin{pmatrix} 1 & -\frac{b_1+b_3}{b_2} & 1 \\ 0 & 1 & -1 \\ 0 & 1 & b_2 \end{pmatrix}.$$

Applying M to B gives

$$[b_1 - (b_1 + b_2) + b_2 : b_1 - b_2 : -b_3 b_2 + b_3 b_2] = [0 : 1 : 0].$$

2. Assume $b_3 \neq 0$. Then

$$B = [b_1 : b_2 : b_3] = \left[\frac{b_1}{b_3} : \frac{b_2}{b_3} : 1\right].$$

We abuse notation and say that $B = [b_1 : b_2 : 1]$. Consider two cases.

(a) Let $b_2 \neq -1$. Take

$$M = \begin{pmatrix} 1 & 1 & -(b_1 + b_2) \\ 0 & 1 & 1 \\ 0 & 1 & -b_2. \end{pmatrix}.$$

Applying M to B gives

$$[b_1 + b_2 - (b_1 + b_2) : b_1 + b_2 : b_2 - b_2] = [0 : 1 : 0].$$

(b) Let $b_2 = -1$. Take

$$M = \begin{pmatrix} 1 & 1 & -(b_1 + b_2) \\ 0 & 1 & -1 \\ 0 & 1 & 1. \end{pmatrix}.$$

Applying M to B gives

$$[b_1 + b_2 - (b_1 + b_2) : b_1 - 1 : -1 + 1] = [0 : 1 : 0].$$

□

Lemma 4.0.5. *Let $A = [1 : 0 : 0]$ and $B = [0 : 1 : 0]$. Let $ABCD$ be a convex quadrilateral in projective space. Then we can preserve the position of A and B while moving some other vertex to the point $[1 : 0 : 1]$.*

Proof. By Theorem 3.2.2, every quadrilateral in $\mathbf{P}^2$ must contain at least one affine vertex. Since we know that $ABCD$ has two vertices $A = [1 : 0 : 0]$ and $B = [0 : 1 : 0]$, it follows one of the remaining vertices must be an affine point. Let C be such a point. Then $c_3 \neq 0$. Consider cases.

1. Let $C = [0 : 0 : 1]$. Then
$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Let $C \neq [0 : 0 : 1]$. Since the corners of a quadrilateral must be distinct, it follows that $c_2 \neq 0$ or $c_3 \neq 0$. Consider sub-cases.

   (a) Let $c_2 = 0$. Then $C = [c_1 : 0 : 1]$ where $c_1 \neq 0$. Take
   $$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & c_1 \end{pmatrix}.$$

   (b) Let $c_1 = 0$. Then $C = [0 : c_2 : 1]$ where $c_2 = 0$. Take
   $$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (c) Assume $c_1 \neq 0$ and $c_2 \neq 0$. Then $C = [c_1 : c_2 : 1]$.
   $$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c_2 \\ 0 & 0 & c_1 \end{pmatrix}.$$

□

Lemma 4.0.6. *Let $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$ and $C = [1 : 0 : 1]$. Let $ABCD$ be a convex quadrilateral in projective space. Then we can preserve the position of A, B, and C with a linear transformation that maps D to $[0 : 1 : 1]$.*

Proof. Let $D = [d_1, d_2, d_3]$. Consider cases.

1. Let $d_1 = 0$. It follows that either $d_2 \neq 0$ or $d_3 \neq 0$. Without loss of generality, assume that $d_2 \neq 0$. Then since $ABCD$ is a quadrilateral, it must follow that $d_3 \neq 0$. must be distinct from the three other points of our quadrilateral, it follows that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{d_3}{d_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Let $d_1 \neq 0$. Since D is distinct from A, B, and C it follows that $d_2 \neq 0$ and $d_3 \neq 0$. Consider sub-cases.

   (a) Assume $d_1 = -d_3$. Then

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & \frac{d_3}{d_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (b) Asume $d_1 \neq -d_3$. Then

$$M = \begin{pmatrix} d_3 & 0 & -d_1 \\ 0 & \frac{d_3}{d_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Theorem 4.0.7. *All convex projective quadrilaterals can be mapped to the quadrilateral whose vertices are* $[0:1:0]$, $[1:0:0]$, $[1:0:1]$, *and* $[0:1:1]$.

Proof. In Lemma 4.0.3, Lemma 4.0.4, Lemma 4.0.5, and Lemma 4.0.6 we give the projective trasformations. Composing these projective transformations gives the appropriate map. □

Corollary 4.0.8. *Given an arbitrary convex quadrilateral in the projective plane, it can be mapped to the (affine) square whose vertices are* $[-\frac{1}{2} : -\frac{1}{2} : 1]$, $[\frac{1}{2} : -\frac{1}{2} : 1]$, $[-\frac{1}{2} : \frac{1}{2} : 1]$, *and* $[\frac{1}{2} : \frac{1}{2} : 1]$.

Proof. Let Q be the quadrilateral whose vertices are $[0 : 1 : 0]$, $[1 : 0 : 0]$, $[1 : 0 : 1]$, and $[0 : 1 : 1]$. By Theorem 4.0.7 any projective quadrilateral can be mapped Q. By the same process we can compose a reverse map that will take Q to any other quadrilateral. Since each matrix has nonzero determinant, it follows that each is invertible. Then there exists an inverse map from Q to the desired square. □

We can use this corollary to generate new primitive tilings by adjoining new triangles onto old tilings so that we have a new (convex) quadrilateral. Putting that quadrilateral into the projective plane means we can create a map it to the unit square. We call this process the projective transformation algorithm. Figure 4.0.1 and Figure 4.0.2 give two different ways to adjoin triangles to a two-tiling. Using our projective transformation, this can yield either a primitive or an imprimitive tiling. Likewise, Figure 4.0.3 and Figure 4.0.4 give examples of two alterations of the κ-configuration introduced by Guy. As before, one version is primitive and the other is imprimitive.

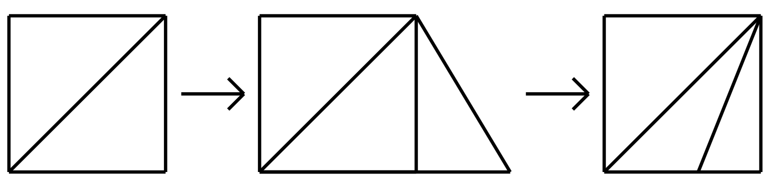

Figure 4.0.1. A projective transformation turns this 2-tiling into an imprimitive 3-tiling.

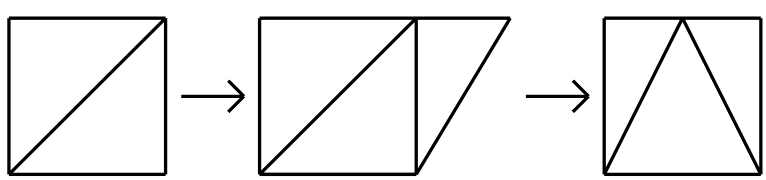

Figure 4.0.2. A projective transformation turns this 2-tiling into a primitive 3-tiling.

Brady, Campbell and Nair show that there are 14 primitive configurations of 5-tilings [1]. Ten of these configurations can be obtained from the Mountain Configuration, using the method we have described. Let us investigate one such configuration.

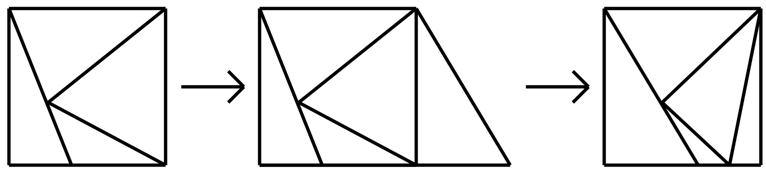

Figure 4.0.3. A projective transformation turns this primitive 4-tiling into a primitive 5-tiling.

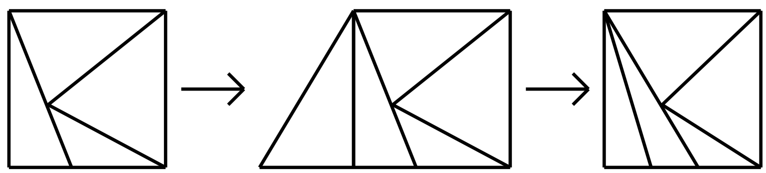

Figure 4.0.4. A projective transformation turns this primitive 4-tiling into an imprimitive 5-tiling.

Example 4.0.9. Projective Transformations and the ω-configuration. The ω-configuration is given by Brady et al[1]. In Figure 4.0.5, we show how the ω-configuration can be obtained from the ν-configuration introduced by Guy[5]. It is possible, however, to start with the 2-tiling and add triangles until we have the ω-configuration. We leave it to the reader to verify this should it be unclear. Furthermore, we can generate new primitive tilings by applying the projective transformation algorithm to the ω-configuration. An example is given in Figure 4.0.6. $\Diamond$

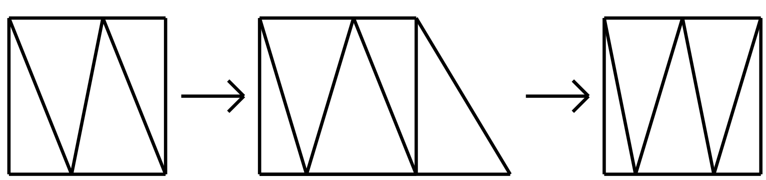

Figure 4.0.5. The ω-configuration can be obtained using projective linear transformations applied to the ν-configuration.

The projective transformation algorithm allows us to generate new tilings from old ones. As we have seen, it can be difficult to generate primitive tilings as the number of tiles increases. Although this algorithm does not guarantee new primitive tilings, it is

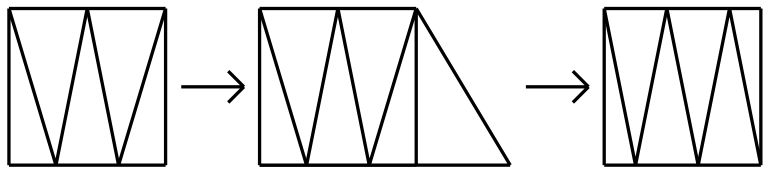

Figure 4.0.6. Applying the projective transformation algorithm to the ω-configuration can generate a primitive 6-tiling.

possible to determine when this process will yield primitive tilings and when it will not. This process is not obvious. We explore it below.

Question 4. When will the projective transformation algorithm yield primitive tilings?

Certainly we must begin with a primitive tiling. This is simply a combinatorial question. If we can remove an edge from a tiling and still have a tiling, then our projective transformation algorithm will not change that. So it is not possible that an imprimitive tiling will yield a primitive tiling.

In Figure 4.0.1 we give an example of a primitive tiling that yields an imprimitive tiling when we apply our algorithm. Likewise, in Figure 4.0.2 we demonstrate that the same primitive tiling can yield a primitive tiling. Careful application of our algorithm can determine which tiling outputs are primitive and which are not.

Consider the generic setup given in Figure 4.0.7. Lemma 4.0.7 tells us that we can map $A'BCD$ to $ABCD$ in the projective plane. Let's call the resulting quadrilateral Q. We want to know the conditions under which a primitive tiling of $ABCD$ will result in an imprimitive tiling of Q. Denote the primitive tiling of $ABCD$ by T_{ABCD} and denote the resulting tiling of Q by T_Q.

Observe that regardless of the tiling of $ABCD$, the tiling of Q must have the form given in Figure 4.0.8. If T_Q is primitive, then there exists an edge in the edge list of T_Q such that we can delete that edge and still have a tiling. But T_{ABCD} is primitive, so T_{ABCD} does not have such an edge. The only edge that T_Q and T_{ABCD} do not have in common

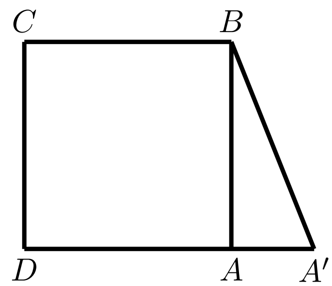

Figure 4.0.7. Two convex quadrilaterals: $ABCD$ and $A'BCD$.

is $\overline{A'B}$. It follows that if T_Q is imprimitive, then we must be able to delete $\overline{A'B}$ and still have a tiling.

Then there exists a vertex P and an edge $\overline{BP}$ such that $\triangle A'BP$ is a triangular tile of T_Q. See Figure 4.0.9. Then T_Q imprimitive if and only if T_{ABCD} has a triangular tile $\triangle ABP$. See Figure 4.0.10.

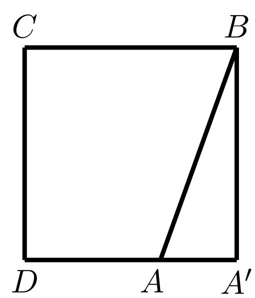

Figure 4.0.8. Quadrilateral Q.

The projective transformation algorithm is extremely useful to our study of rational tilings of the square. It allows us to simplify an otherwise long and laborious procedure. We should, however, give a small warning about the limitations of the algorithm. The algorithm tells us about the existence of tiling configurations. They give no information (as far as we have shown) about the rationality of n-tilings.

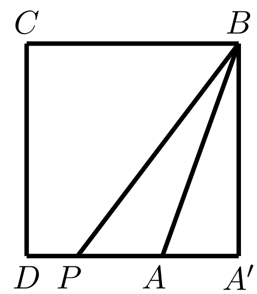

Figure 4.0.9. Generic form of an imprimitive T_Q.

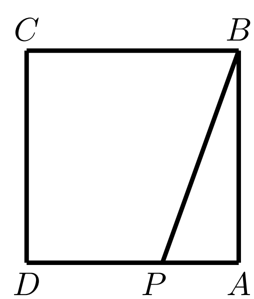

Figure 4.0.10. Generic form of a primitive T_{ABCD} that guarantees T_Q is imprimitive.

5
Six-Tilings

So far we have seen several examples of tilings. We have shown that there are no rational 2-tilings and no rational 3-tilings. We have seen that there are several primitive 4-tiling configurations and 5-tiling configurations that yield infinitely-many rational tilings.

We now turn to the open problem of 6-tilings. We will give several examples of primitive 6-tilings and then analyze one tiling in particular, which we will call the Shark's Tooth Configuration.

5.1 Six-Tiling Examples

The primitive 6-tilings given in Figure 5.1.1, Figure 5.1.2, Figure 5.1.3, and Figure 5.1.4 are generated from the $\chi + \Lambda$ configuration [1] using the projective transformation method from Chapter 4. Similarly, we can generate Figure 5.1.5 and Figure 5.1.6 from the $Y + \Lambda$-configuration. We can also generate Figure 5.1.7, Figure 5.1.8, and Figure 5.1.9 from the Dragonfly configuration. Each of the tilings given in these figures is representative of a primitive tiling configuration.

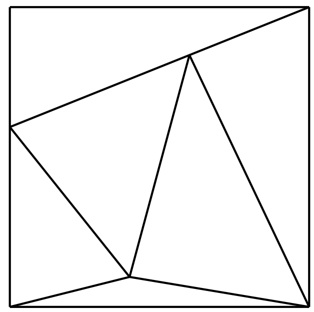

Figure 5.1.1. A primitive 6-tiling.

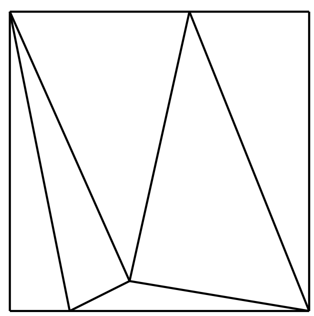

Figure 5.1.2. A primitive 6-tiling.

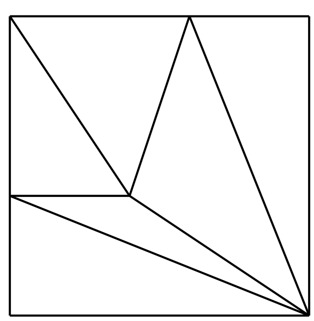

Figure 5.1.3. A primitive 6-tiling.

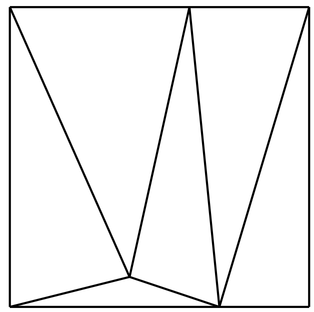

Figure 5.1.4. A primitive 6-tiling.

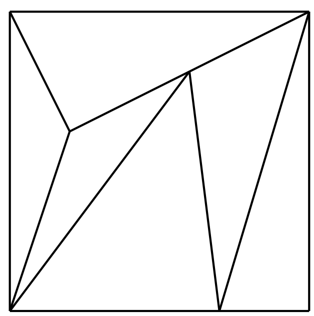

Figure 5.1.5. $Y + \Lambda$-configuration of a 5-tiling.

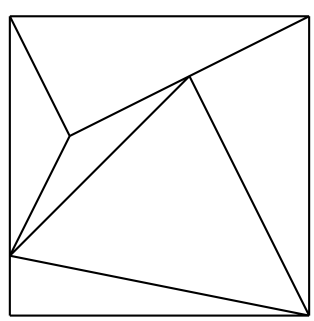

Figure 5.1.6. A primitive 6-tiling.

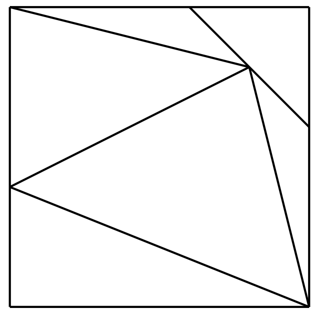

Figure 5.1.7. A primitive 6-tiling.

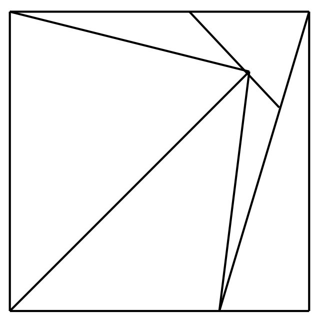

Figure 5.1.8. A primitive 6-tiling.

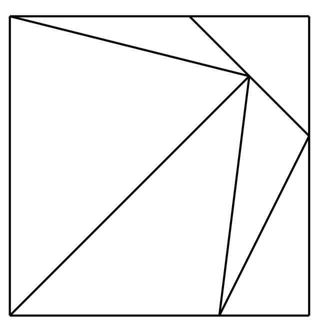

Figure 5.1.9. A primitive 6-tiling.

5.2 The Shark's Tooth Configuration

We now explore a particular primitive 6-tiling. We call this the Shark's Tooth Configuration. An example of a tiling of this configuration is given in Figure 5.2.1. Not only is this tiling primitive, but it has several nice specializations that makes the problem of finding rational solutions more approachable. We say that the Shark's Tooth Configuration has four points of variation. We call these variable s, x, d, and t. We denote tilings of this configuration by $ST(s, x, d, t)$. Before considering the general $ST(s, x, d, t)$, we explore two specializations $ST(s, 1/2, d, 1/2)$ and $ST(1/2, x, d, 1/2)$.

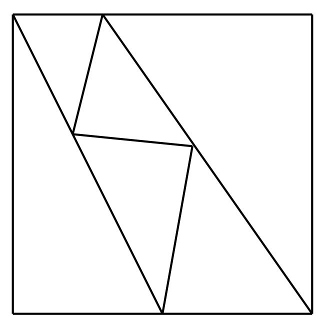

Figure 5.2.1. $ST(s, x, d, t)$

The most restrictive specialization of $ST(x, s, d, t)$ is given by $ST(1/2, x, d, 1/2)$. An example of such a tiling of this specialization is given in Figure 5.2.2. By the Pythagorean Theorem, it follows that for any tiling of this specialization $x = \sqrt{5}/2$. Then $ST(1/2, d, x, 1/2)$ has no rational tilings. See Figure 5.2.3.

Now consider $ST(s, 1/2, d, 1/2)$. An example of such a tiling of this specialization is given in Figure 5.2.4. As we shall see, this specialization has no rational tilings either. The conclusion, however, is not trivial and is proved below.

Theorem 5.2.1. *There are no rational tilings of the form $ST(s, 1/2, d, 1/2)$.*

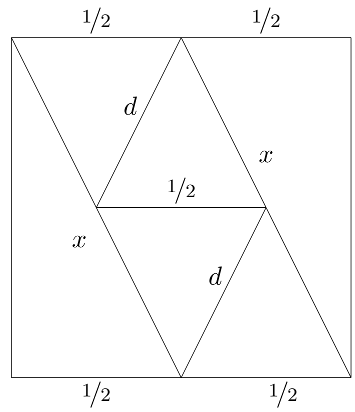

Figure 5.2.2. $ST(1/2, x, d1/2)$

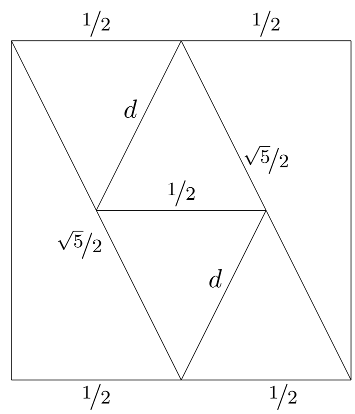

Figure 5.2.3. $ST(1/2, x, d, 1/2)$ with the x-edge labelled.

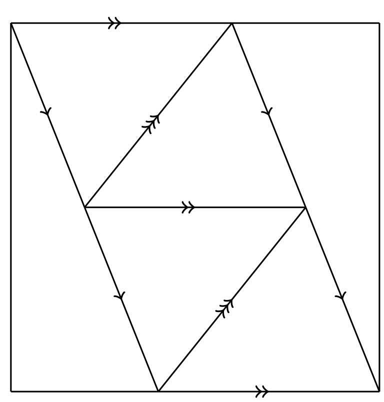

Figure 5.2.4. $ST(s, 1/2, d, 1/2)$ with parallel lines labelled.

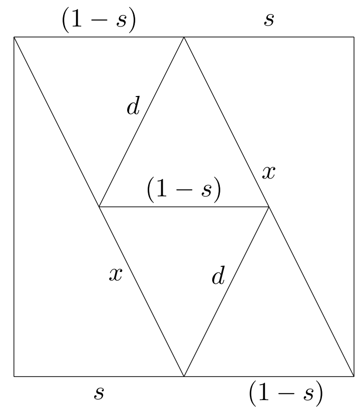

Figure 5.2.5. $ST(s, 1/2, d, 1/2)$ with edges labelled by their lengths.

Proof. Let

$$f(s, x, d, 1/2) = x^2 - (1 + s^2)$$

$$g(s, x, d, 1/2) = d^2 x - \frac{5}{4} s^2 x - \frac{3}{4} s x - s(1 - s).$$

Then a rational tiling of the unit square will be given by a common root (s_0, x_0, d_0) of f and g. By Theorem 3.1.7, the tuple (s_0, x_0, d_0) only gives a rational tiling if $\mathrm{Res}(f, g)(s_0, x_0, d_0) = 0$. Taking the resultant gives

$$\begin{aligned} \mathrm{Res}(f, g) &= \left(\tfrac{5}{4}^2\right) s^6 - \left(\tfrac{5}{2}\right) x^5 - \left[\left(\tfrac{5}{2}\right) d^2 - \tfrac{43}{16}\right] x^4 \\ &+ \left(2d^2 + \tfrac{7}{2}\right) x^3 + \left[d^4 + \left(\tfrac{5}{4}\right) d^2 - \tfrac{7}{8}\right] x^2 \\ &+ (-2d^2 - 1)x + 2 \end{aligned}$$

If we homogenize this polynomial we can interpret its zeros as defining an algebraic curve, the (affine) rational points of which give potential tilings. Homogenizing $\mathrm{Res}(f, g)$ yields

$$\begin{aligned} R[x : d : z] &= \tfrac{25}{16} x^6 - \tfrac{5}{2} z x^5 + \left(\tfrac{-5}{2} d^2 - \tfrac{43}{16} z^2\right) x^4 \\ &+ \left(2 z d^2 + \tfrac{7}{2} z^3\right) x^3 + \left(d^4 + \tfrac{5}{2} z^2 d^2 - \tfrac{7}{8} z^4\right) * x^2 \\ &+ \left(-2 z^3 d^2 - z^5\right) x + 2 z^6 \end{aligned}$$

defines an algebraic curve of genus 3. Computing the rational points using Magma shows that $[0 : 1 : 0]$ and $[-1 : 0 : 1]$ and $[1 : 0 : 1]$ are the only rational points on $R[x : d : z]$. Note that only two of these points are affine. The two affine points, however, require $x = \pm 1$. If $x = \pm 1$, then either $s = 0$ or $s = i\sqrt{2}$. Both values of s violate our conditions for rational tilings. Then there are no rational tilings of the $ST(s, 1/2, d, 1/2)$ specialization. $\square$

We now turn to the general case. Consider tilings of the form $ST(s, x, d, t)$. In Figure 5.2.6 we give an example of a tiling of the form $ST(s, x, d, t)$ and note which lines are parallel. These lines are parallel in all tilings of the form $ST(s, x, d, t)$.

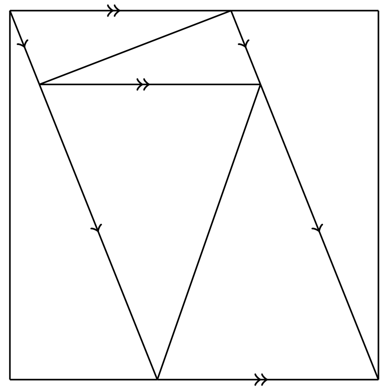

Figure 5.2.6. One tiling of the form $ST(s, x, d, t)$ with parallel lines labelled.

Theorem 5.2.2. *If there exists a rational tiling of $ST(x, s, d, t)$ given by x_0,s_0,d_0, and t_0, then there exists a rational point (s_0, d_0, t_0) on*

$$\begin{aligned} F(s,t,d) &= (-s^2-1)d^4+(2s^4-4s^3+(-2t^2+4)s^2-4s \\ &+ (-2t^2+2))d^2+(-s^6+4s^5+(3t^2-7)s^4+(-6t^2+8)s^3 \\ &+ (-t^4+5t^2-7)s^2+(-4t^2+4)s+(-t^4+2t^2-1)). \end{aligned}$$

Proof. Let

$$f(s, x, d, t) = x^2 - (1 + s^2)$$

$$g(s, x, d, t) = d^2x - (1 - s)^2x + t^2x - (1 - s)ts.$$

Setting $g(s, x, d, t) = 0$ and solving for x gives

$$x = \frac{(1-s)ts}{d^2 - (1-s)^2 + t^2}.$$

We substitute x into f and take the numerator. We have

$$\begin{aligned} F(s,t,d) &= (-s^2-1)d^4+(2s^4-4s^3+(-2t^2+4)s^2-4s \\ &+ (-2t^2+2))d^2+(-s^6+4s^5+(3t^2-7)s^4+(-6t^2+8)s^3. \\ &+ (-t^4+5t^2-7)s^2+(-4t^2+4)s+(-t^4+2t^2-1)) \end{aligned}$$

Since we want to find some s, t and d such that $f(x,s,t,d) = 0$ and $g(x,s,t,d)$ simultaneously, it follows that if there exists an $ST(s,x,d,t)$ tiling, then it must satisfy $F(s,t,d) = 0$. □

We would like to homogenize $F(s,t,d)$ and put it into projective space. Unfortunately to homogenize $F(s,t,d)$ and put it into $\mathbf{P}^2$, we need a polynomial in two variables and $F(s,t,d)$ has three variables. Define a function field $\mathbf{Q}(s)$. Then $F(s,t,d)$ (which we initially defined over $\mathbf{Q}$) can be re-defined as $F_s(t,d)$ over $\mathbf{Q}(s)$ which is

$$\begin{array}{rcc} F_s(t,d) & = & (-s^2-1)d^4 + (2s^4 - 4s^3 + (-2t^2+4)s^2 - 4s \\ & + & (-2t^2+2))d^2 + (-s^6 + 4s^5 + (3t^2-7)s^4 + (-6t^2+8)s^3. \\ & + & (-t^4+5t^2-7)s^2 + (-4t^2+4)s + (-t^4+2t^2-1)) \end{array}$$

Now we homogenize and put into projective space. We homogenize this polynomial and put it into projective space. We get

$$\begin{array}{rcc} H_f(t,d,z) & = & (-s^2-1)d^4 + (2z^2s^4 - 4z^2s^3 + (-2t^2+4z^2)s^2 - 4z^2s \\ & + & (-2t^2+2z^2))d^2 + (-z^4s^6 + 4z^4s^5 + (3z^2t^2 - 7z^4)s^4 \\ & + & (-6z^2t^2+8z^4)s^3 + (-t^4+5z^2t^2-7z^4) * s^2 \\ & + & (-4z^2t^2+4z^4)s + (-t^4+2z^2t^2-z^4)). \end{array}$$

We can now compute the genus of $H_f(t,d,z)$. The genus of $H_f(t,d,z)$ is 3. Magma is unable to produce a factorization of $H_f(t,d,z)$ over $\mathbf{Q}(s)$. Considering a specialization wher s is a rectangular number, however, produces a very different result.

Theorem 5.2.3. *The projective curve*

$$\begin{array}{rcc} H_f(t,d,z) & = & (-s^2-1)d^4 + (2z^2s^4 - 4z^2s^3 + (-2t^2+4z^2)s^2 - 4z^2s \\ & + & (-2t^2+2z^2))d^2 + (-z^4s^6 + 4z^4s^5 + (3z^2t^2 - 7z^4)s^4 \\ & + & (-6z^2t^2+8z^4)s^3 + (-t^4+5z^2t^2-7z^4) * s^2 \\ & + & (-4z^2t^2+4z^4)s + (-t^4+2z^2t^2-z^4)). \end{array}$$

defined over $\mathbf{Q}(s)$ *factors over* $\mathbf{Q}$ *when* s *is a rectangular number.*

Proof. Let s be a rectangular number. By Lemma 3.4.2, it follows that $s = {}^{a}\!/_{b}$ where $a^2+b^2=c^2$. Then $H_f(t,d,z)$ factors into

$$\left(t^2+d^2-\frac{a(b-a)}{bc}tz-\left(\frac{b-a}{b}\right)^2z^2\right)\left(t^2+d^2+\frac{a(b-a)}{bc}tz-\left(\frac{b-a}{b}\right)^2z^2\right).$$

Note that the coefficient of the tz term of the quadratics is $\pm\frac{a(b-a)}{bc}$. Since

$$c = \sqrt{a^2 + b^2},$$

it follows that if c is not a square, then the polynomial does not have a rational factorization. So our polynomial factors precisely if and only if s is a rectangular number. $\square$

In the previous proof, we found that $H_f(t, d, z)$ factors into

$$\left(t^2 + d^2 - \frac{a(b-a)}{bc}tz - \left(\frac{b-a}{b}\right)^2 z^2\right)\left(t^2 + d^2 + \frac{a(b-a)}{bc}tz - \left(\frac{b-a}{b}\right)^2 z^2\right).$$

precisely when s is a rectangular number of the form $s = a/b$ where $a^2 + b^2 = c^2$. Since $H(t, d, z)$ factors into a product of quadratic polynomials, we can take the discriminant of the affine portions. To obtain the affine quadratic, let $z = 1$. So we have

$$H_f(t, d, 1) = \left(t^2 + d^2 - \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right)\left(t^2 + d^2 + \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right).$$

We can compute now the discriminant of these polynomials in the usual way. We get

$$\operatorname{disc}\left(t^2 + d^2 - \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right) = 0^2 - 4(1)(1) = -4$$

$$\operatorname{disc}\left(t^2 + d^2 + \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right) = 0^2 - 4(1)(1) = -4.$$

By the characterization of quadratic polynomials, these are ellipses. Since these are ellipses, we can parameterize the rational points on these ellipses by finding one rational point. Observe that for both ellipses $(0, \frac{b-a}{b})$ is a rational point on the ellipse. We consider all lines of slope m that pass through the point $(0, \frac{b-a}{b})$ as well as a second point on our ellipse. By varying m, we obtain our parameterization of each of these ellipses. In this fashion, the ellipse

$$\left(t^2 + d^2 - \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right)$$

is parameterized by

$$d(m) = -\left(\frac{b-a}{b}\right)\frac{2m - \frac{a}{c}}{1 + m^2}$$

$$t(m) = \left(\frac{b-a}{b}\right)\frac{\frac{a}{c}m - m^2 + 1}{1+m^2}.$$

Likewise, the ellipse

$$\left(t^2 + d^2 + \frac{a(b-a)}{bc}t - \left(\frac{b-a}{b}\right)^2\right)$$

is parameterized by

$$d(m) = -\left(\frac{b-a}{b}\right)\frac{2m + \frac{a}{c}}{1+m^2}$$

$$t(m) = \left(\frac{b-a}{b}\right)\frac{-\frac{a}{c}m - m^2 + 1}{1+m^2}.$$

We leave it as an open conjecture whether or not there exist rational tilings of the general form $ST(s, x, d, t)$, but we observe that if such tilings existed, they would be rational points on these ellipses.

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