Factorization Lengths in Numerical Monoids

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Abstract

A numerical monoid M generated by $n_1, \ldots, n_k \in \mathbb{N}$ is a subset of $\mathbb{N}_0$ whose elements are non-negative linear combinations of the generators $n_1, \ldots, n_k$. The set of factorizations of an element $x \in M$ is the set of all the different ways to write that element as a linear combination of the generators. If $x = a_1 n_1 + a_2 n_2 + \cdots + a_k n_k$ for some $a_1, \ldots, a_k \in \mathbb{N}_0$, then the length of the factorization $a_1 n_1 + a_2 n_2 + \cdots + a_k n_k$ is given by $a_1 + a_2 + \cdots + a_k$. Since an element in a monoid can be written in different ways in terms of the generators, its set of factorization lengths may contain more than one element. In my project, I will focus on the maximum factorization length of an element x, denoted by $L(x)$, and the minimum factorization length of x, denoted by $l(x)$, and I will investigate for which numerical monoids M the conditions $L(x + n_1) = L(x) + 1$ and $l(x + n_k) = l(x) + 1$ hold for every $x \in M$.

Contents

Dedication

For my parents and grandparents.

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1
Introduction

The Fundamental Theorem of Arithmetic states that every natural number greater than one can be factored uniquely into a product of prime numbers up to ordering of the factors. The prime numbers are the constituents for the natural number system using multiplication. This project considers a different algebraic structure, namely, numerical monoids.

Instead of prime numbers, the constituents of a numerical monoid are called generators. "Factorization" in a numerical monoid means expressing an element as a sum of the generators. Unlike factorization in $\mathbb{N}$, factorization in numerical monoids need not be unique. A numerical monoid M generated by $n_1, \ldots, n_k$, denoted $M = \langle n_1, \cdots, n_k \rangle$, is a set of non-negative linear combinations $a_1 n_1 + \cdots + a_k n_k$ where $a_1, \ldots, a_k$ are non-negative integers. Since factorization is not unique, an element may have more than one factorization. The sum of the coefficients of a factorization $a_1 + \cdots + a_k$ is the length of that factorization. Because an element x may have more than one factorization, x may have more than one length. The maximum length of an element x is denoted $L(x)$ and the minimum length is denoted $l(x)$. In this project, I will be investigating the following research questions: for what numerical monoids $M = \langle n_1, \ldots, n_k \rangle$ do the two properties $L(x + n_1) = L(x) + 1$ and $l(x + n_k) = l(x) + 1$ hold for all $x \in M$?

It has previously been proven that given $M = \langle n_1, \ldots, n_k \rangle$, if $x > (n_1 - 1)n_k$, then $L(x+n_1) = L(x) + 1$, and if $x > n_{k-1}(n_k - 1)$, then $l(x + n_k) = l(x) + 1$.

However, when $x \leq (n_1 - 1)n_k$, $L(x+n_1)$ is not necessarily equal to $L(x)+1$. For example, let $M = \langle 9, 10, 23\rangle$. Note $41 \in M$ and $41 = 2 \cdot 9 + 0 \cdot 10 + 1 \cdot 23$ is the only factorization for 41, and hence the maximum length factorization of 41. Therefore $L(41) = 3$. Note that $41 + 9 = 50 \in M$ and the only two factorizations for 50 are $50 = 0 \cdot 9 + 5 \cdot 10 + 0 \cdot 23 = 3 \cdot 9 + 0 \cdot 10 + 1 \cdot 23$. Note $0 \cdot 9 + 5 \cdot 10 + 0 \cdot 23$ is the factorization with maximum length for 50. Then $L(41 + 9) = L(50) = 5 \neq 4 = L(41) + 1$. Therefore, there exists a numerical monoid M that contains an $x \in M$ such that $L(x + n_1) \neq L(x) + 1$. However, there do exist numerical monoids $\langle n_1, \ldots, n_k\rangle$ where $L(x + n_1) = L(x) + 1$ and $l(x + n_k) = l(x) + 1$ for every element $x \in \langle n_1, \ldots, n_k\rangle$. One example is the numerical monoid $\langle 6, 9, 20\rangle$, also known as the "Chicken McNugget" monoid.

In Chapter 2, we introduce the reader to basic concepts of number theory, monoids, numerical monoids, factorization sets, length sets, and elasticity. In Chapter 3, we cover previous results on maximum and minimum length in numerical monoids of k generators and arithmetical numerical monoids. In Chapter 4, we present our main results of this project. In Chapter 5, we present a summary of the results we have gotten as well as questions for the reader to consider. We conclude with the coding programs that were used to compute maximum and minimum length throughout this project.

2
Preliminaries

In this chapter, we cover the background information necessary for understanding the questions that this project investigates. In the first section of this chapter, we discuss basic results from number theory including the greatest common divisor of a set of integers as well as the Division Algorithm. In the second section, we cover the definitions of monoids and numerical monoids, and we provide some examples. In the third section, we explore the definition of the Frobenius number, the largest positive integer not in a numerical monoid. In the fourth section, we discuss factorization invariants, which are tools used to study non-unique factorizations in numerical monoids. Throughout this paper, $\mathbb{N}_0$ denotes $\{0, 1, 2, 3, \ldots\}$.

2.1 Basic Concepts in Number Theory

The following results from number theory are helpful for understanding the definitions in Section 2.2 of this project. First, a remark on notation.

Remark 2.1.1. Let a, $b \in \mathbb{Z}$. We say that $a \mid b$ if and only if $b = ax$ for some $x \in \mathbb{Z}$. $\Diamond$

The following definition is about the greatest common divisor of two non-zero integers.

Definition 2.1.2 ([3], Definition 2.2)**.** Let a and b be integers with at least one of them different from zero. The **greatest common divisor** of a and b, denoted $\gcd(a, b)$, is the positive integer d satisfying the following:

1. $d \mid a$ and $d \mid b$.

2. If $c \mid a$ and $c \mid b$, then $c \leq d$. △

We generalize this definition to a set of k integers.

Definition 2.1.3. Let $a_1, \ldots, a_k$ be integers, with at least one of them different from zero. The **greatest common divisor** of $a_1, \ldots, a_k$, denoted $\gcd(a_1, \ldots, a_k)$, is the positive integer d satisfying the following:

1. $d \mid a_1, \ldots,$ and $d \mid a_k$.

2. If $c \mid a_1, \ldots,$ and $c \mid a_k$, then $c \leq d$. △

We apply these definitions to the following lemma.

Lemma 2.1.4. *If $a, b, c \in \mathbb{Z}$ with at least one of them different from zero, then* $\gcd(\gcd(a, b), c) = \gcd(a, b, c)$.

Proof. Let $x = \gcd(a, b, c)$ and let $y = \gcd(\gcd(a, b), c)$. By Definition 2.1.3, we have $x \mid a$, $x \mid b$, $x \mid c$, $y \mid \gcd(a, b)$, and $y \mid c$. By the transitive property, since $y \mid \gcd(a, b)$, $\gcd(a, b) \mid a$, and $\gcd(a, b) \mid b$, we conclude that $y \mid a$ and $y \mid b$. Thus, $y \mid a$, $y \mid b$, and $y \mid c$, implies that $y \leq x$. Similarly, $x \mid a$, $x \mid b$, and $x \mid c$ implies that $x \leq y$. Since $x \leq y$ and $y \leq x$, $x = y$, that is, $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$. □

The following lemma shows that the greatest common divisor of two consecutive integers is 1.

Lemma 2.1.5. *If $a \in \mathbb{Z}$, then* $\gcd(a, a + 1) = 1$.

Proof. Let $d = \gcd(a, a + 1)$. Thus $d \mid a$ and $d \mid a + 1$. This implies $a = dm$ and $a + 1 = dn$ for some $m, n \in \mathbb{Z}$. By substitution, $dn = dm + 1$, and so $dn - dm = 1$. By factoring, we have

$d(m-n) = 1$. We know that $d, m, n \in \mathbb{Z}$, and the only factors of 1 are 1 and -1, so either $d = 1$ and $m - n = 1$, or $d = -1$ and $m - n = -1$. Since $d > 0$, we conclude that $d = 1$. Hence, $\gcd(a, a+1) = 1$. □

We introduce the following definition about integers a and b whose greatest common divisor is 1.

Definition 2.1.6. Two integers a and b are **relatively prime** if $\gcd(a, b) = 1$. △

We generalize this definition to a set of k integers.

Definition 2.1.7. A set of integers $\{n_1, \ldots, n_k\}$ is **relatively prime** if $\gcd(n_1, \ldots, n_k) = 1$. △

The last two results we present in this section are a special case of Bezout's Lemma and the Division Algorithm.

Theorem 2.1.8 ([3], Theorem 2.4)**.** *Let n_1, n_2 be integers, not both zero. Then n_1 and n_2 are relatively prime if and only if there exist a, $b \in \mathbb{Z}$ such that $1 = an_1 + bn_2$.*

We will generalize this theorem to a set of k integers.

Theorem 2.1.9. *Let n_1, $\ldots$, n_k be integers, not all zero. Then n_1, $\ldots$, n_k are relatively prime if and only if there exist a_1, $\ldots$, $a_k \in \mathbb{Z}$ such that $1 = a_1n_1 + \cdots + a_kn_k$.*

Theorem 2.1.10 ([3], Theorem 2.1)**.** ***Division Algorithm****. Given integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = qb + r$ with $0 \leq r < b$.*

2.2 Monoids and Numerical Monoids

In this section, we define a monoid and numerical monoid.

Definition 2.2.1. A **monoid** $(M, \star)$ is a set M, closed under the binary operation $\star$, such that the following axioms are satisfied:

1. For all a, b, $c \in M$, $(a \star b) \star c = a \star (b \star c)$ (i.e., $\star$ is associative).

2. There is an element $i \in M$ such that for all $x \in M$, $i \star x = x = x \star i$ (i is called the identity under $\star$). $\triangle$

One example of a monoid is a group. We present the definition of a group, and then a remark.

Definition 2.2.2 ([7], Definition 4.1)**.** A **group** $(G, \star)$ is a set G, closed under the binary operation $\star$, such that the following axioms are satisfied:

1. For all a, b, $c \in G$, $(a \star b) \star c = a \star (b \star c)$ (i.e., $\star$ is associative).

2. There is an element $i \in G$ such that for all $x \in G$, $i \star x = x = x \star i$ (i is called the identity under $\star$).

3. For each $x \in G$, there is an element $x^{-1} \in G$ such that $x \star x^{-1} = i = x^{-1} \star x$ (x^{-1} is called the inverse of x).

Remark 2.2.3. Every group is a monoid; not every monoid is a group. $\Diamond$

Another example of a monoid is the set of matrices under addition.

Example 2.2.4. Fix $k \in \mathbb{N}$. Let

$$\mathcal{M} = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} : a_{ij} \in \mathbb{Z}; 1 \leq i, j \leq k \right\}.$$

Let A, $B \in \mathcal{M}$. Note that

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \\ b_{k1} & \cdots & b_{kk} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1k} + b_{1k} \\ \vdots & & \\ a_{k1} + b_{k1} & \cdots & a_{kk} + b_{kk} \end{bmatrix} \in \mathcal{M}.$$

Hence, M is closed under addition. Note that $0 \in \mathbb{Z}$, so

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix} \in \mathcal{M}.$$

Therefore, $\mathcal{M}$ has an additive identity.

Note that since addition is associative in $\mathbb{Z}$, we conclude that addition in $\mathcal{M}$ is associative.

Hence, $(\mathcal{M}, +)$ is a monoid. $\Diamond$

Example 2.2.5. Some examples of monoids are $(\mathbb{N}_0, +)$, $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, +)$, and $(\mathbb{Z}, \cdot)$. Note that $\mathbb{N}_0$ and $\mathbb{Z}$ are closed under addition, and $\mathbb{Z}$ and $\mathbb{N}$ are closed under multiplication. Therefore each of these sets is closed under its binary operation. Note that addition and multiplication are both associative. For the additive monoids $(\mathbb{N}_0, +)$ and $(\mathbb{Z}, +)$, the identity element is 0, and for the multiplicative monoids $(\mathbb{N}, \cdot)$ and $(\mathbb{Z}, \cdot)$, the identity element is 1. $\Diamond$

In this project, we study a particular type of monoid, known as a numerical monoid, which is a subset of $\mathbb{N}_0$.

Definition 2.2.6 ([8], Definition 1.1)**.** Let $n_1, \ldots, n_k$ be a collection of relatively prime positive integers. The **numerical monoid** M **generated by** $n_1, \ldots, n_k$ is the set of all non-negative linear combinations of $n_1, \ldots, n_k$:

$$M = \langle n_1, \ldots, n_k \rangle = \{a_1 n_1 + \cdots + a_k n_k \mid a_1, \ldots, a_k \in \mathbb{N}_0\}. \qquad \triangle$$

Example 2.2.7. Let $M = \langle 3, 5 \rangle$. Each element in M is a non-negative linear combination of the generators 3 and 5, as follows:

$$0 = 0 \cdot 3 + 0 \cdot 5,$$

$$3 = 1 \cdot 3 + 0 \cdot 5,$$

$$5 = 0 \cdot 3 + 1 \cdot 5,$$

$$6 = 2 \cdot 3 + 0 \cdot 5,$$

$$8 = 1 \cdot 3 + 1 \cdot 5,$$

$$9 = 3 \cdot 3 + 0 \cdot 5,$$

and so on. Thus $M = \{0, 3, 5, 6, 8, 9, 10, 11, 12, \ldots\}$. $\Diamond$

Remark 2.2.8. Note that $2 \notin \langle 3, 5 \rangle$ because 2 cannot be written as a non-negative linear combination of 3 and 5. $\Diamond$

Example 2.2.9. The numerical monoid generated by 1 is $\langle 1 \rangle = (\mathbb{N}_0, +)$. $\Diamond$

Example 2.2.10. A famous numerical monoid is the Chicken McNugget monoid. The Chicken McNugget monoid is $\langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, \ldots\}$. $\Diamond$

If a numerical monoid appears to have k generators, but one of the generators can be written as a non-negative linear combination of the other generators, then the numerical monoid is generated by $k-1$ elements. A numerical monoid M is minimally generated if none of its generators can be written as a non-negative linear combination of any of the other generators.

Definition 2.2.11 ([9], Corollary 2.9)**.** Let M be a numerical monoid generated by $n_1, \ldots, n_k$. Then $\{n_1, \ldots, n_k\}$ is a **minimal system of generators** of M if and only if $n_{i+1} \notin \langle n_1, \ldots, n_i \rangle$ for all $1 \leq i \leq k-1$. $\triangle$

Remark 2.2.12. The numerical monoid $\langle 3, 4, 7 \rangle$ appears to have three generators, but since $7 = 1 \cdot 3 + 1 \cdot 4$ and $1 \in \mathbb{N}_0$, this numerical monoid has two generators, i.e. $\langle 3, 4, 7 \rangle = \langle 3, 4 \rangle$. $\diamondsuit$

The following lemma and proof show that every numerical monoid is a monoid. We will show that a numerical monoid M is closed under addition, the elements in M are associative under addition, and there is an additive identity element in M.

Lemma 2.2.13. *Let $M = \langle n_1, \ldots, n_k \rangle$. Then $(M, +)$ is a monoid.*

Proof. Let x, $y \in M$. By definition, $x = a_1 n_1 + \cdots + a_k n_k$ for some $a_1, \ldots, a_k \in \mathbb{N}_0$ and $y = b_1 n_1 + \cdots + b_k n_k$ for some $b_1, \ldots, b_k \in \mathbb{N}_0$. Observe:

$$\begin{aligned} x + y &= (a_1 n_1 + \cdots + a_k n_k) + (b_1 n_1 + \cdots + b_k n_k) \\ &= a_1 n_1 + b_1 n_1 + \cdots + a_k n_k + b_k n_k \\ &= (a_1 + b_1) n_1 + \cdots + (a_k + b_k) n_k. \end{aligned}$$

Note that $a_1 + b_1, \ldots, a_k + b_k \in \mathbb{N}_0$ because $\mathbb{N}_0$ is closed under addition. Therefore, M is closed under addition.

Note that addition in $\mathbb{N}_0$ is associative.

The identity element with respect to addition is 0 and $0 \in \mathbb{N}_0$. Note that $0 = 0 \cdot n_1 + 0 \cdot n_2 + \cdots + 0 \cdot n_k$, and hence $0 \in M$. Thus, we conclude that 0 is the identity of $(M, +)$.

Therefore, every numerical monoid is a monoid. $\square$

One specific type of numerical monoid is the arithmetical numerical monoid. This numerical monoid is generated by a finite subset of an arithmetic sequence.

Definition 2.2.14. ([8], Remark 3.6) A numerical monoid is an **arithmetical numerical monoid** if it is generated by an arithmetic sequence $n, n+k, n+2k, \ldots, n+dk$ with $\gcd(n, k) = 1$ and $1 \leq d < n$. △

Remark 2.2.15. If $d \geq n$, then the generators n, $n + k$, …, $n + dk$ do not form a minimal system of generators. ♢

Example 2.2.16. An example of an arithmetical numerical monoid is $\langle 3, 5, 7\rangle$. In this example, $n = 3$, $k = 2$, and $d = 2$. ♢

The following lemma is another description of a numerical monoid M. First, we will show that M is an additive subset of $\mathbb{N}_0$. Then, we will show that M is cofinite in $\mathbb{N}_0$.

Lemma 2.2.17 ([8], Remark 1.2)**.** *A numerical monoid is a cofinite additive subset of* $\mathbb{N}_0$*.*

Proof. Let M be a numerical monoid generated by n_1, …, $n_k \in \mathbb{N}$. By definition, $M = \langle n_1, \ldots, n_k\rangle = \{a_1n_1 + \cdots + a_kn_k \mid a_1, \ldots, a_k \in \mathbb{N}_0\}$. Note that $n_1, \ldots, n_k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$, so every element in M is a non-negative integer. Therefore, $M \subseteq \mathbb{N}_0$.

Note that the binary operation on M is addition.

By definition of numerical monoid, $\gcd(n_1, \ldots, n_k) = 1$. By Theorem 2.1.9, $1 = b_1n_1+\cdots+b_kn_k$ for some b_1, …, $b_k \in \mathbb{Z}$. Let $s = \max(|b_1|, \ldots, |b_k|) \in \mathbb{N}_0$, and let $z = sn_1 + \cdots + sn_k$. Note that $z \in M$. Let $t \in \mathbb{N}_0$ such that $0 \leq t < n_1$. Then $z + t = sn_1 + \cdots + sn_k + t$. Note $t = t \cdot 1 = t(b_1n_1 + \cdots + b_kn_k)$, so we compute

$$\begin{aligned} z + t &= s(n_1 + \cdots + n_k) + t(b_1n_1 + \cdots + b_kn_k) \\ &= (s + b_1t)n_1 + \cdots + (s + b_kt)n_k. \end{aligned}$$

Since $s = \max(|b_1|, \ldots, |b_k|)$, we know $s + b_1t$, …,$s + b_kt \in \mathbb{N}_0$. Thus $z + t \in M$ for every $0 \leq t < n_1$.

Let $x \in \mathbb{N}$ and assume that $x \geq z$. By the Division Algorithm, there exist q, $r \in \mathbb{Z}$ such that $x - z = qn_1 + r$, and $0 \leq r < n_1$. So $x = qn_1 + r + z$. Note $qn_1 \in M$ and $r + z \in M$ by the previous argument. So by Lemma 2.2.13, $x \in M$. Therefore, every integer $x \geq z$ is in M. Hence, M has a finite complement in $\mathbb{N}_0$, i.e., M is cofinite in $\mathbb{N}_0$.

Therefore, every numerical monoid M is a cofinite additive subset of $\mathbb{N}_0$. □

The previous lemma allows us to give an equivalent definition.

Definition 2.2.18 ([4], page 2)**.** A **numerical monoid** (also called a numerical semigroup) is a subset $M \subset \mathbb{Z}_{\geq 0}$ of the non-negative integers that

1. is closed under addition, i.e., whenever a, $b \in M$, we also have $a + b \in M$, and
2. has finite complement in $\mathbb{Z}_{\geq 0}$. △

2.3 The Frobenius Number of a Numerical Monoid

Since M is a cofinite additive subset of $\mathbb{N}_0$, there is a finite number of elements in $\mathbb{N}_0$ that are not in M. Therefore, there must be a maximum element not in M. This element is known as the Frobenius number of M.

Definition 2.3.1 ([5], Definition 2.2)**.** Let $n_1, \ldots, n_k$ be relatively prime positive integers. The **Frobenius number** of $\langle n_1, \ldots, n_k \rangle$ is the largest positive integer F such that $F \notin \langle n_1, \ldots, n_k \rangle$.

△

Example 2.3.2. Let $M = \langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \ldots\}$. Observe that for the first few elements, the difference between consecutive elements is greater that one. However, every natural number greater than or equal to 8 is in M. Thus, 7 is the Frobenius number of M. ♢

The following theorem states a formula for determining the Frobenius number F of a numerical monoid of two generators. We will show in the proof that every natural number greater than F is in M and that $F \notin M$ to conclude that F is the Frobenius number of M.

Theorem 2.3.3. [10] *Let $M = \langle n_1, n_2 \rangle$. The Frobenius number F of M is $n_1 n_2 - n_1 - n_2$.*

Proof. Let $M = \langle n_1, n_2 \rangle$, and let $F = n_1 n_2 - n_1 - n_2$.

Let $x \in \mathbb{N}$, and suppose $x > F$. By Theorem 2.1.8, $1 = an_1 + bn_2$ for some a, $b \in \mathbb{Z}$. Thus

$$x \cdot 1 = x(an_1 + bn_2) = (ax)n_1 + (bx)n_2. \tag{2.3.1}$$

Let $a_1 = ax$ and $b_1 = bx$, and note that a_1, $b_1 \in \mathbb{Z}$.

Let $t \in \mathbb{Z}$ be the smallest integer such that $b_1 + tn_1 \geq 0$, i.e. $t = \lceil \frac{-b_1}{n_1} \rceil$. We compute

$$\begin{aligned} x &= a_1 n_1 + b_1 n_2 \\ &= a_1 n_1 - n_1 t n_2 + b_1 n_2 + n_1 t n_2 \\ &= (a_1 - tn_2)n_1 + (b_1 + tn_1)n_2. \end{aligned}$$

We will now prove that $a_1 - tn_2 \geq 0$ in order to prove that $(a_1 - tn_2)n_1 + (b_1 + tn_1)n_2$ is a factorization for $x \in M$. We know that $x \geq F + 1 = n_1 n_2 - n_1 - n_2 + 1$. So

$$x = (a_1 - tn_2)n_1 + (b_1 + tn_1)n_2 \geq n_1 n_2 - n_1 - n_2 + 1.$$

This implies

$$(a_1 - tn_2)n_1 \geq n_1 n_2 - n_1 - n_2 + 1 - (b_1 + tn_1)n_2. \tag{2.3.2}$$

Since t is the smallest integer such that $b_1 + tn_1 \geq 0$, we know that $b_1 + (t-1)n_1 < 0$. This implies that $n_1 > b_1 + tn_1$, or equivalently, $n_1 - 1 \geq b_1 + tn_1$. Therefore, $-(n_1 - 1)n_2 \leq -(b_1 + tn_1)n_2$. It follows from equation (2.3.2) that

$$\begin{aligned} (a_1 - tn_2)n_1 &\geq n_1 n_2 - n_1 - n_2 + 1 - n_2(n_1 - 1) \\ &= n_1 n_2 - n_1 - n_2 + 1 - n_1 n_2 + n_2 \\ &= 1 - n_1 > -n_1. \end{aligned}$$

From $(a_1 - tn_2)n_1 > -n_1$, it follows that $(a_1 - tn_2) > -1$ and thus $(a_1 - tn_2) \geq 0$. Since x is an arbitrary element greater than F, $x \in M$ for every $x > F$.

We will now show by contradiction that $F \notin M$. Suppose $F = n_1 n_2 - n_1 - n_2 \in M$. By definition, $n_1 n_2 - n_1 - n_2 = cn_1 + dn_2$ for some c, $d \in \mathbb{N}_0$. Using modulo n_1, we obtain that

$-n_2 \equiv dn_2 \pmod{n_1}$, which means that $dn_2 + n_2 = qn_1$ for some $q \in \mathbb{Z}$. Since $\gcd(n_1, n_2) = 1$, $(d+1)n_2 = qn_1$ implies that n_1 divides $d+1$. Thus $d+1 = sn_1$ for some $s \in \mathbb{N}$ because $d+1$ is strictly positive. Note that $sn_1 \geq n_1$, and so $d = sn_1 - 1 \geq n_1 - 1$.

Similarly, using modulo n_2, we compute $-cn_1 \equiv n_1 \pmod{n_2}$, which means that $cn_1 + n_1 = rn_2$ for some $r \in \mathbb{Z}$. Since $\gcd(n_1, n_2) = 1$, $(c+1)n_1 = rn_2$ implies that n_2 divides $c+1$, and so $c + 1 = yn_2$ for some $y \in \mathbb{N}$. Note that $yn_2 \geq n_2$, and so $c = yn_2 - 1 \geq n_2 - 1$. Thus

$$\begin{aligned} n_1n_2 - n_1 - n_2 &= cn_1 + dn_2 \\ &\geq (n_2 - 1)n_1 + (n_1 - 1)n_2 \\ &= n_1n_2 - n_1 + n_1n_2 - n_2. \end{aligned}$$

From $n_1n_2 - n_1 - n_2 \geq n_1n_2 - n_1 + n_1n_2 - n_2$, it follows that $0 \geq n_1n_2$, which is a contradiction because $n_1, n_2 \in \mathbb{N}$. Hence $F = n_1n_2 - n_1 - n_2 \notin M$.

Thus, every natural number greater than F is in M and $F \notin M$. Therefore, $F = n_1n_2 - n_1 - n_2$ is the Frobenius number of $\langle n_1, n_2 \rangle$. □

2.4 Factorization Invariants

Factorization invariants are an essential tool in studying factorizations in numerical monoids. In this section, we focus on factorizations, factorization sets, length, length sets, and elasticity. Recall that "factorizations" in numerical monoids mean an expression of an element as a sum of the generators. We will see that we can write an element in different ways as a sum of the generators. Note that factorizations in numerical monoids are not unique. The reader can find the following definitions in [5].

Definition 2.4.1. Let $M = \langle n_1, \ldots, n_k \rangle$, and let $x \in M$. The **factorization set** of x is

$$Z(x) = \{(a_1, \ldots, a_k) \in \mathbb{N}_0^k \mid x = a_1n_1 + \cdots + a_kn_k\}. \qquad \triangle$$

Since factorizations in numerical monoids are not unique, factorization sets may have more than one element.

Example 2.4.2. Let $M = \langle 3, 5\rangle$. Note that

$$\begin{aligned} 30 &= 10 \cdot 3 + 0 \cdot 5 \\ &= 5 \cdot 3 + 3 \cdot 5 \\ &= 0 \cdot 3 + 6 \cdot 5. \end{aligned}$$

Hence $30 \in M$ can be written as a linear combination of the generators in three different ways and so 30 admits three different factorizations. The factorization set for the element 30 is $Z(30) = \{(10, 0), (5, 3), (0, 6)\}$. ♢

Taking the sum of the coefficients of a factorization of an element gives us the length of that factorization. The set of all the lengths of an element is known as the length set of that element.

Definition 2.4.3. Let $M = \langle n_1, \dots, n_k\rangle$, and let $x \in M$. The **length of a factorization** $(a_1, \dots, a_k) \in Z(x)$ is $a_1 + \cdots + a_k$. △

Definition 2.4.4. Let $M = \langle n_1, \dots, n_k\rangle$, and let $x \in M$. The **length set** of x is

$$\mathcal{L}(x) = \{|(a_1, \dots, a_k)| \in \mathbb{N}_0 : (a_1, \dots, a_k) \in Z(x)\} . \qquad \triangle$$

We denote the maximum length factorization by $L(x)$ and the minimum length factoriztion by $l(x)$.

Example 2.4.5. Let $M = \langle 3, 5\rangle$. Since $Z(30) = \{(10, 0), (5, 3), (0, 6)\}$, $\mathcal{L}(30) = \{6, 8, 10\}$. Note that $L(30) = 10$ and $l(30) = 6$. ♢

We will now show that the factorization set and length set of an element are finite.

Lemma 2.4.6. *Let $M = \langle n_1, \dots, n_k\rangle$, and let $x \in M$. The factorization set $Z(x)$ and length set $\mathcal{L}(x)$ are finite.*

Proof. Let $M = \langle n_1, \dots, n_k\rangle$ and let $x \in M$. By definition, $x = a_1n_1 + \cdots + a_kn_k$ for some $a_1, \dots, a_k \in \mathbb{N}_0$. So $(a_1, \dots, a_k) \in Z(x)$ and $a_1 + \cdots + a_k \in \mathcal{L}(x)$. By definition, $a_1n_1 \leq x$, $a_2n_2 \leq x$, $\dots$, and $a_kn_k \leq x$. This implies $0 \leq a_1 \leq \left\lfloor \frac{x}{n_1} \right\rfloor$, $0 \leq a_2 \leq \left\lfloor \frac{x}{n_2} \right\rfloor$, $\dots$, $0 \leq a_k \leq \left\lfloor \frac{x}{n_k} \right\rfloor$. Let $m_1 = \left\lfloor \frac{x}{n_1} \right\rfloor$, $m_2 = \left\lfloor \frac{x}{n_2} \right\rfloor$, $\dots$, $m_k = \left\lfloor \frac{x}{n_k} \right\rfloor$. Hence, $|Z(x)| \leq (m_1 + 1)(m_2 + 1)\cdots(m_k + 1)$.

Since $(m_1+1)(m_2+1)\cdots(m_k+1)$ is finite, so is $Z(x)$. Note that $|Z(x)| \geq |L(x)|$, so $L(x)$ is also finite. Hence, the factorization set and length set of an element $x \in M$ are finite. $\square$

We conclude this section with one last factorization invariant. In addition to factorization sets and length sets, another factorization invariant is the elasticity of an element and the elasticity of a numerical monoid. The elasticity of an element measures how "spread out" the factorizations of that element are, and the elasticity of a numerical monoid measures how "spread out" the factorizations of all the elements of that numerical monoid can get.

Definition 2.4.7. Let $M = \langle n_1, \ldots, n_k \rangle$, and let $x \in M$. The **elasticity** of x is

$$\rho(x) = \frac{L(x)}{l(x)}. \qquad \triangle$$

Example 2.4.8. Let $M = \langle 3, 5 \rangle$. Since $\mathcal{L}(30) = \{6, 8, 10\}$, we see that $L(30) = 10$, and $l(30) = 6$, and so $\rho(30) = \frac{10}{6} = \frac{5}{3}$. $\Diamond$

Definition 2.4.9. Let $M = \langle n_1, \ldots, n_k \rangle$. The **elasticity of M** is

$$\rho(M) = \sup\{\rho(x) \mid x \in M\}. \qquad \triangle$$

Theorem 2.4.10 ([5], Theorem 2)**.** *Let $M = \langle n_1, \ldots, n_k \rangle$. The elasticity of M is*

$$\rho(M) = \frac{n_k}{n_1}.$$

Proof. Let $x \in M$, and let $a_1 n_1 + \cdots + a_k n_k$ be an arbitrary factorization for x. Recall that $n_1 < \cdots < n_k$. We compute

$$\begin{aligned}
\frac{x}{n_k} &= \frac{a_1 n_1 + \cdots + a_k n_k}{n_k} \\
&= \frac{n_1}{n_k} a_1 + \cdots + \frac{n_k}{n_k} a_k \\
&\leq a_1 + \cdots + a_k \\
&\leq \frac{n_1}{n_1} a_1 + \cdots + \frac{n_k}{n_1} a_k \\
&= \frac{a_1 n_1 + \cdots + a_k n_k}{n_1} \\
&= \frac{x}{n_1}.
\end{aligned}$$

This implies $L(x) \leq \frac{x}{n_1}$ and $l(x) \geq \frac{x}{n_k}$. Also, $\rho(M) \geq \rho(n_1 n_k)$ because $\min \rho(n_1 n_k) = n_1$ and $\max \rho(n_1 n_k) = n_k$. Therefore, $\rho(x) = \frac{n_k}{n_1}$. $\square$

Example 2.4.11. Let $M = \langle 3, 5 \rangle$. So $\rho(M) = \frac{5}{3}$. $\diamondsuit$

3

Past Results on Maximum and Minimum Factorization Lengths

The following two chapters concentrate on two very important properties that are the central focus of this project. We introduce Property 1: $L(x+n_1) = L(x)+1$ and Property 2: $l(x+n_k) = l(x) + 1$ for an arbitrary $\langle n_1, \ldots, n_k \rangle$. In this chapter, we cover previous results on maximum and minimum factorization lengths for numerical monoids generated by k elements as well as factorization lengths and length sets of arithmetical numerical monoids generated by $d + 1$ elements. In the first section, we cover previous results that state that if x is "large" enough, then both properties hold. For the purposes of the proofs, we will be assuming that for generators $n_1, \ldots, n_k$, that $n_1 < \ldots < n_k$.

3.1 Numerical Monoids Generated by k Elements

The following results by Christopher O'Neill, Thomas Barron, and Roberto Pelayo state that if $M = \langle n_1, \ldots, n_k \rangle$, $x \in M$, and $x > (n_1 - 1)n_k$, then Property 1 is fulfilled. If $x > n_{k-1}(n_k - 1)$, then Property 2 is fulfilled. The following lemmas guide the reader through the proof of the first theorem that discusses Property 1.

Lemma 3.1.1 ([2], Lemma 4.1)**.** *Let $k \geq 0$, and fix $c_1, \ldots, c_r \in \mathbb{Z}$ with $r \geq k$. Then there exists $T \subsetneq \{1, \ldots, r\}$ satisfying $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \pmod{k}$.*

Proof. For each $j \in \{0, \ldots, r\}$, define $s_j = \sum_{n=1}^{j} c_n$. The sequence s_0, s_1, …, s_r has length $r+1 > k$. Since there are $r+1 > k$ integers, and there are k equivalence classes modulo k, by the pigeonhole principle, there exist i and j with $0 \leq i < j \leq r$ such that $s_i \equiv s_j \pmod{k}$. Let $T = \{1, \ldots, i, j+1, \ldots, r\} \subsetneq \{1, \ldots, r\}$. Since $s_i \equiv s_j \pmod{k}$, we see that $c_0 + \cdots + c_i \equiv c_0 + \cdots + c_j \pmod{k}$ and so $c_0 + \cdots + c_i - c_0 - \cdots - c_j = kt$ for some $t \in \mathbb{Z}$. This implies $-(c_{i+1} + \cdots + c_j) = kt$, which is equivalent to $c_1 + \cdots + c_i + c_{j+1} + \cdots + c_r \equiv c_1 + \cdots + c_r \pmod{k}$. This allows us to conclude that $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \pmod{k}$.

Therefore, there exists $T \subsetneq \{1, \ldots, r\}$ satisfying $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \pmod{k}$. □

Lemma 3.1.2 ([2], Theorem 4.3)**.** *Let $M = \langle n_1, \ldots, n_k \rangle$, let $x \in M$, and write $x = a_1 n_1 + \cdots + a_k n_k$ for some a_1, …, $a_k \in \mathbb{N}_0$. If $a_2 + \cdots + a_k \geq n_1$, then $a_1 + \cdots + a_k < L(x)$.*

Proof. We have $x = a_1 n_1 + \cdots + a_k n_k$ for some $a_1, \ldots, a_k \in \mathbb{N}_0$. This implies $a_2 n_2 + \cdots + a_k n_k \equiv x \pmod{n_1}$. Suppose that $a_2 + \cdots + a_k \geq n_1$. Note that

$$a_1 n_1 + \cdots + a_k n_k = n_1 + \cdots + n_1 + \cdots + n_k + \cdots + n_k.$$

Consider the set with $a_2 + \cdots + a_k$ integers taken from $\{n_1, \ldots, n_k\}$. Note that $a_2 + \cdots + a_k \geq n_1$, and so by Lemma 3.1.1, we know there exist $b_2, \ldots, b_k$ such that

1. $0 \leq b_i \leq a_i$ for all $i > 1$,
2. $\sum_{i=2}^{k} b_i < \sum_{i=2}^{k} a_i$, and
3. $\sum_{i=2}^{k} b_i \equiv \sum_{i=2}^{k} a_i \equiv x \pmod{n_1}$.

By the conditions above, we see that there exists $b_1 \geq 0$ such that $b_1 n_1 + \cdots + b_k n_k$ is a factorization for x. Since $b_1 n_1 + \cdots + b_k n_k = a_1 n_1 + \cdots + a_k n_k$, we see that $b_1 n_1 - a_1 n_1 = a_2 n_2 - b_2 n_2 + \cdots + a_k n_k - b_k n_k$, which allows us to compute $(b_1 - a_1) n_1 = (a_2 - b_2) n_2 + \cdots + (a_k - b_k) n_k > (a_2 - b_2) n_1 + \cdots + (a_k - b_k) n_1$. This implies that $(b_1 - a_1) n_1 > (a_2 - b_2) n_1 + \cdots + (a_k - b_k) n_1$, and dividing both sides by n_1 we conclude that $b_1 - a_1 > a_2 - b_2 + \cdots + a_k - b_k$. Hence, $b_1 + \cdots + b_k > a_1 + \cdots + a_k$. This shows that $(b_1, \ldots, b_k) \in Z(x)$ is a factorization with length greater than the length of $(a_1, \ldots, a_k) \in Z(x)$. Thus, if $a_2 + \cdots + a_k \geq n_1$, then $a_1 + \cdots + a_k < L(x)$. □

The following lemma states that if $x > (n_1 - 1)n_k$, then the coefficient of the first generator of the maximum factorization length must be strictly positive.

Lemma 3.1.3 ([2], Theorem 4.3)**.** *Let $M = \langle n_1, \ldots, n_k \rangle$, and let $x = a_1n_1 + \cdots + a_kn_k \in M$ be the maximum factorization length of x. If $x > (n_1 - 1)n_k$, then $a_1 > 0$.*

Proof. Suppose $(a_1, \ldots, a_k) \in Z(x)$ and that $x > (n_1 - 1)n_k$. By definition, $x = a_1n_1 + \cdots + a_kn_k$ and note from Lemma 3.1.2, since $L(x) = a_1 + \cdots + a_k$, then $a_2 + \cdots + a_k < n_1$. If $a_1 = 0$, we compute $x = a_2n_2 + \cdots + a_kn_k < a_2n_k + \cdots + a_kn_k = (a_2 + \cdots + a_k)n_k \leq (n_1 - 1)n_k$. This implies $(n_1 - 1)n_k < (n_1 - 1)n_k$, which is a contradiction, because $0 \not< 0$. Therefore, a_1 must be strictly positive. □

The following theorem and proof discuss Property 1 being fulfilled for all $x > (n_1 - 1)n_k$ in a numerical monoid generated by k elements. Utilizing the lemmas so far in this section, the first half of the proof shows that $L(x - n_1) \geq L(x) - 1$, and is done directly. The second half of the proof, showing that $L(x - n_1) \not> L(x) - 1$, is done by contradiction.

Theorem 3.1.4 ([2], Theorem 4.2)**.** *Let $M = \langle n_1, \ldots, n_k \rangle$, and let $x \in M$. If $x > (n_1 - 1)n_k$, then $L(x - n_1) = L(x) - 1$.*

Proof. Suppose that $x = a_1n_1 + \cdots + a_kn_k$ is the maximum length factorization for $x \in M$. By Lemma 3.1.1, $a_2 + \cdots + a_k < n_1$. By Lemma 3.1.3, since $x > (n_1 - 1)n_k$, $a_1 > 0$. Since a_1 is strictly positive, we know that $(a_1 - 1, \ldots, a_k)$ is a factorization for $x - n_1$. Thus, $L(x - n_1) \geq a_1 - 1 + a_2 + \cdots + a_k = a_1 + a_2 + \cdots + a_k - 1 = L(x) - 1$. Hence, $L(x - n_1) \geq L(x) - 1$.

Suppose that $L(x - n_1) > L(x) - 1$. Since $a_1n_1 + a_2n_2 + \cdots + a_kn_k$ is the maximum length factorization for $x \in M$, this implies $a_1 - 1 + a_2 + \cdots + a_k > a_1 + \cdots + a_k - 1$, which implies $0 > 0$. Therefore, $L(x - n_1) \geq L(x) - 1$, but $L(x - n_1) \not> L(x) - 1$. Hence, $L(x) - 1 = L(x - n_1)$ for all $x > (n_1 - 1)n_k$. □

Remark 3.1.5. Let $M = \langle n_1, \ldots, n_k \rangle$, and let $x \in M$. The statement $L(x + n_1) = L(x) + 1$ is equivalent to the statement $L(x) - 1 = L(x - n_1)$. ◊

Proof. Let $z = x + n_1$. Then $L(x + n_1) = L(z)$ and $L(x) = L(z - n_1)$. $\square$

The following theorem is about a certain condition on x, and Property 2 being fulfilled.

Theorem 3.1.6 ([2], Theorem 4.3). *Let $M = \langle n_1, \ldots, n_k \rangle$ and let $x \in M$. If $x > n_{k-1}(n_k - 1)$, then $l(x + n_k) = l(x) + 1$.*

The proof for this theorem is similar to the proof for Theorem 3.1.4 and hence omitted.

3.2 Factorization Lengths in Arithmetical Numerical Monoids

The following results by Jeff Amos, Scott T. Chapman, Natalie Hine, and João Paixão discuss factorization lengths and length sets of arithmetical numerical monoids generated by $d + 1$ elements. We present a lemma on an element x in an arithmetical numerical monoid utilizing the Division Algorithm.

Lemma 3.2.1 ([1], Lemma 2.1). *Let $M = \langle n, n + k, \ldots, n + dk \rangle$. If $x \in M$, then $x = c_1 n + c_2 k$ for some c_1, $c_2 \in \mathbb{N}_0$ and $0 \le c_2 < n$.*

Proof. Let $x \in M$. By definition, $x = a_1 n + a_2(n + k) + \cdots + a_{d+1}(n + dk)$ for some a_1, $\ldots$, $a_{d+1} \in \mathbb{N}_0$. It follows that

$$\begin{aligned} x &= a_1 n + a_2(n + k) + a_3(n + 2k) + \cdots + a_{d+1}(n + dk) \\ &= a_1 n + \cdots + a_{d+1} n + a_2 k + 2a_3 k + \cdots + d a_{d+1} k \\ &= (a_1 + \cdots + a_{d+1})n + (a_2 + 2a_3 + \cdots + d a_{d+1})k. \end{aligned}$$

Let $d_1 = a_1 + \cdots + a_{d+1}$, and let $d_2 = a_2 + 2a_3 + \cdots + d a_{d+1}$. By the Division Algorithm, there are unique q, $r \in \mathbb{N}_0$ such that $d_2 = qn + r$ with $0 \le r < n$. So $x = d_1 n + (qn + r)k = (d_1 + qk)n + rk$. Let $c_1 = d_1 + qk$ and $c_2 = r$ to get the desired result. $\square$

In using the fact that an element x in an arithmetical numerical monoid can be rearranged so that it is in the form $x = c_1 n + c_2 k$ such that $0 \le c_2 < n$, we can determine its length set, which then helps us determine its maximum and minimum lengths. The following theorem is about the length set of an element x in an arithmetical numerical monoid generated by $d+1$ elements.

Theorem 3.2.2 ([1], Theorem 2.2)**.** *Let $M = \langle n, n+k, \ldots, n+dk \rangle$. Suppose $x = c_1 n + c_2 k \in M$ for some $0 \le c_2 < n$. Then*

$$\mathcal{L}(x) = \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}.$$

The following proof shows that $\mathcal{L}(x) = \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}$ by letting an element $y \in \mathcal{L}(x)$ and proving that $y \in \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}$ in order to show $\mathcal{L}(x) \subseteq \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}$. Then we let $y \in \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}$ and prove that $y \in \mathcal{L}(x)$ in order to show that $\left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\} \subseteq \mathcal{L}(x)$. This ultimately proves that $\mathcal{L}(x) = \left\{ c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \le m \le 0 \right\}$.

Proof. Let $y \in \mathcal{L}(x)$. This means that $y = a_1 + \cdots + a_{d+1}$ for some $(a_1, \ldots, a_{d+1}) \in Z(x)$. By definition,

$$\begin{aligned} x &= a_1 n + a_2(n+k) + \cdots + a_{d+1}(n+dk) \\ &= (a_1 + \cdots + a_{d+1})n + (a_2 + 2a_3 + \cdots + d a_{d+1})k \\ &= yn + (a_2 + 2a_3 + \cdots + d a_{d+1})k. \end{aligned}$$

Hence, $x \equiv yn \pmod{k}$.

We also know that $x \equiv c_1 n \pmod{k}$ since $x = c_1 n + c_2 k$. By symmetry transitivity of congruence modulo k, $yn \equiv c_1 n \pmod{k}$.

Since $\gcd(n,k) = 1$, $y \equiv c_1 \pmod{k}$. So $y - c_1 = kp$ for some $p \in \mathbb{Z}$, or equivalently, $y = c_1 + kp \in c_1 + k\mathbb{Z}$. Therefore, $\mathcal{L}(x) \subseteq c_1 + k\mathbb{Z}$.

Let $y = c_1 + kp$. Note that

$$\begin{aligned} yn &= (a_1 + \cdots + a_{d+1})n \\ &\le x = a_1 n + a_2(n+k) + \cdots + a_{d+1}(n+dk) \\ &\le a_1(n+dk) + \cdots + a_{d+1}(n+dk) \\ &= y(n+dk). \end{aligned}$$

This shows that $yn \leq x \leq y(n+dk)$, or $(c_1+kp)n \leq x \leq (c_1+kp)(n+dk)$. By algebraic computations,

$$\left\lceil \frac{\frac{x}{n+dk} - c_1}{k} \right\rceil \leq p \leq \left\lfloor \frac{\frac{x}{n} - c_1}{k} \right\rfloor .$$

Observe that

$$\begin{aligned}
\frac{\frac{x}{n+dk} - c_1}{k} &= \frac{\frac{c_1 n + c_2 k}{n+dk} - \frac{c_1(n+dk)}{n+dk}}{k} \\
&= \frac{c_1 n + c_2 k - c_1 n - c_1 dk}{(n+dk)k} \\
&= \frac{(c_2 - c_1 d)k}{(n+dk)k} \\
&= \frac{c_2 - c_1 d}{n+dk}.
\end{aligned}$$

So $c_1 + k\left\lceil \frac{\frac{x}{n+dk} - c_1}{k} \right\rceil = c_1 + k\left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil$. Observe that

$$\frac{\frac{x}{n} - c_1}{k} = \frac{\frac{c_1 n + c_2 k}{n} - c_1}{k} = \frac{c_1 n + c_2 k - c_1 n}{nk} = \frac{c_2 k}{nk} = \frac{c_2}{n}.$$

Since $c_2 < n$, $\left\lfloor \frac{c_2}{n} \right\rfloor = 0$. So $c_1 + k\left\lfloor \frac{\frac{x}{n} - c_1}{k} \right\rfloor = c_1$. So $\left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \leq p \leq 0$. Therefore,

$$\mathcal{L}(x) \subseteq \left\{ c_1 + kp \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \leq p \leq 0 \right\}.$$

Let $z = c_1 + kp$ for some $p \in \mathbb{Z}$, and such that $\left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \leq p \leq 0$. Define $m = \frac{x - nz}{k}$ and note $m = c_2 - pn \in \mathbb{Z}$. By the Division Algorithm, there exists q, $r \in \mathbb{N}$ such that $m = qd + r$ and $0 \leq r < d$. We compute

$$\begin{aligned}
x &= nz + k\left(\frac{x - nz}{k}\right) \\
&= nz + km \\
&= n(c_1 + kp) + km \\
&= n(q + 1 + c_1 + kp - 1 - q) + k(qd + r) \\
&= qn + n + c_1 n + kpn - n - qn + qdk + kr \\
&= (c_1 + kp - 1 - q)n + (n + rk) + q(n + dk)
\end{aligned}$$

where $0 \leq r < d$. This is a factorization for x of length $(c_1 + kp - 1 - q + 1 + q = z = c_1 + kp$. Therefore, $z = c_1 + kp \in \mathcal{L}(x)$. Hence,

$$\left\{c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n + dk} \right\rceil \leq m \leq 0\right\} \subseteq \mathcal{L}(x).$$

Therefore, $\mathcal{L}(x) = \left\{c_1 + km \mid \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil \leq m \leq 0\right\}$. $\square$

The following corollaries are a result of the previous theorem.

Corollary 3.2.3. *Let $M = \langle n, n+k, \ldots, n+dk\rangle$. Suppose that $x = c_1 n + c_2 k \in M$ for some c_1, $c_2 \in \mathbb{N}_0$ and $0 \leq c_2 < n$. Then $L(x) = c_1$.*

Corollary 3.2.4. *Let $M = \langle n, n+k, \ldots, n+dk\rangle$. Suppose that $x = c_1 n + c_2 k \in M$ for some c_1, $c_2 \in \mathbb{N}_0$ and $0 \leq c_2 < n$. Then $l(x) = c_1 + k \left\lceil \frac{c_2 - c_1 d}{n+dk} \right\rceil$.*

4

Main Results on Maximum and Minimum Factorization Lengths

In the previous chapter, we saw that both properties hold for numerical monoids generated by k elements as long as x is "large" enough. We also saw that if M is an arithmetical numerical monoid, we can determine the length set, maximum length, and minimum length of an element in M. The research question that this paper investigates is for what numerical monoids $\langle n_1, \ldots, n_k \rangle$ do the properties hold for all x?

In this chapter, we study numerical monoids that satisfy both properties for all x as well as numerical monoids that contain an element in which at least one property is not satisfied. In the first, second, and third sections, we discuss numerical monoids generated by one, two, and three elements respectively that satisfy the properties. In the fourth section, we discuss a certain class of numerical monoid where each numerical monoid in the class contains at least one element that does not satisfy Property 1. In addition, the section also covers which elements do not satisfy Property 1. The fifth section discusses the "smallest" numerical monoid with an element that does not satisfy Property 1.

4.1 The Numerical Monoid Generated by One Element

The numerical monoid generated by one element is $(\mathbb{N}_0, +) = \langle 1 \rangle$. This numerical monoid is the only numerical monoid generated by one element; any other set consisting of non-negative linear combinations of any integer greater than one has an infinite complement in $\mathbb{N}_0$. Since $\langle 1 \rangle$ only has one generator, there is only one factorization for each element of $\langle 1 \rangle$. Therefore, both properties hold for all $x \in \langle 1 \rangle$. We present these results in the form of remarks.

Remark 4.1.1. Let $M = \langle 1 \rangle$. Then $L(x+1) = L(x) + 1$ for all $x \in M$. $\diamondsuit$

Remark 4.1.2. Let $M = \langle 1 \rangle$. Then $l(x+1) = l(x) + 1$ for all $x \in M$. $\diamondsuit$

4.2 Numerical Monoids Generated by Two Elements

In the following section, we prove that every numerical monoid of two generators satisfies both properties for all elements.

Theorem 4.2.1. *Let $M = \langle n_1, n_2 \rangle$. Then $L(z + n_1) = L(z) + 1$ for all $z \in M$.*

In the following proof, we suppose that $z \in M$ and $L(z) = a_1 + b_1$. In order to show that $a_1 + 1 + b_1$ is the maximum length of $z + n_1$, we will prove that there is a factorization length $a_1 + 1 + b_1 \in \mathcal{L}(x)$ and that $a_1 + 1 + b_1 \geq x + y$ for an arbitrary factorization $xn_1 + yn_2 = z + n_1$.

Proof. Let $z \in M$, and suppose that $L(z) = a_1 + b_1$. Note that since $z = a_1 n_1 + b_1 n_2$, it follows that $z + n_1 = (a_1 + 1)n_1 + b_1 n_2$, and hence, $a_1 + 1 + b_1 \in \mathcal{L}(z + n_1)$.

Consider an arbitrary factorization $xn_1 + yn_2$ of $z + n_1$ for some x, $y \in \mathbb{N}_0$. It follows that $z = (x-1)n_1 + yn_2$. Since $x \in \mathbb{N}_0$, $x - 1 \geq -1$. We will now consider two cases: $x - 1 = -1$ and $x - 1 \geq 0$.

Case 1: Suppose that $x - 1 = -1$, which implies that $x = 0$. It follows that

$$yn_2 = xn_1 + yn_2 = z + n_1 = (a_1 + 1)n_1 + b_1 n_2.$$

Hence,

$$(y - b_1)n_2 = (a_1 + 1)n_1.$$

Note that $a_1 \geq 0$, so $a_1 + 1$ is strictly positive. We then have

$$\frac{(y - b_1)n_2}{(a_1 + 1)n_2} = \frac{(a_1 + 1)n_1}{(a_1 + 1)n_2},$$

and so

$$\frac{y - b_1}{a_1 + 1} = \frac{n_1}{n_2} < 1,$$

where the inequality holds because $n_1 < n_2$. Thus $a_1 + 1 > y - b_1$, or equivalently, $a_1 + 1 + b_1 > y = x + y$.

Case 2: Suppose that $x - 1 \geq 0$. It follows that $(x-1)n_1 + yn_2$ is a factorization of z, and since we already know that $a_1n_1 + b_1n_2$ is the factorization of maximum length, we conclude that $x - 1 + y \leq a_1 + b_1$, which means that $a_1 + b_1 + 1 \geq x + y$.

In both cases, $(a_1 + 1)n_1 + b_1n_2$ is the factorization of maximum length for $z + n_1$, and thus $L(z + n_1) = a_1 + 1 + b_1 = L(z) + 1$.

Hence, $L(z + n_1) = L(z) + 1$ for all $z \in \langle n_1, n_2 \rangle$. □

A similar theorem holds for the minimum length of elements in a numerical monoid generated by two elements.

Theorem 4.2.2. *Let $M = \langle n_1, n_2 \rangle$. Then $l(z + n_2) = l(z) + 1$ for all $z \in M$.*

The proof of this theorem is similar to the proof above and hence omitted.

4.3 Numerical Monoids Generated by Three Elements

In the following section, we discuss arithmetical numerical monoids generated by three elements satisfying both properties. The following theorem utilizes Corollary 3.2.3 as well as the Division Algorithm in order to prove Property 1.

Theorem 4.3.1. *Let $M = \langle n, n+k, n+2k \rangle$ be an arithmetical numerical monoid. Then $L(x + n) = L(x) + 1$ for all $x \in M$.*

Proof. Let $M = \langle n, n+k, n+2k \rangle$, and let $x \in M$. By definition, $x = an + b(n+k) + c(n+2k)$ for some a, b, $c \in \mathbb{N}_0$. Note $x = (a+b+c)n + (b+2c)k$ and $x + n = (a+1+b+c)n + (b+2c)k$. We now consider three cases:

Case 1: Suppose that $0 \leq b + 2c < n$. By Corollary 3.2.3, $L(x) = a + b + c$ and $L(x + n) = a + b + c + 1$.

Case 2: Suppose that $b+2c=n$. Then $x=(a+b+c)n+(b+2c)k=(a+b+c+k)n+0\cdot k$ and $x+n=(a+b+c+1+k)n+0\cdot k$. Note $0<n$, so by Corollary 3.2.3, $L(x)=a+b+c+k$ and $L(x+n)=a+b+c+k+1$.

Case 3: Suppose that $b+2c>n$. By the Division Algorithm, there exist unique q, $r\in\mathbb{N}$ such that $b+2c=qn+r$ and $0\le r<n$. So

$$\begin{aligned} x &= (a+b+c)n+(b+2c)k \\ &= (a+b+c)n+(qn+r)k \\ &= (a+b+c+qk)n+rk \end{aligned}$$

and $x+n=(a+b+c+1+qk)n+rk$. So by Corollary 3.2.3, $L(x)=a+b+c+qk$ and $L(x+n)=a+b+c+1+qk$.

In each case, we see that $L(x+n)=L(x)+1$. Therefore $L(x+n)=L(x)+1$ for all $x\in\langle n,n+k,n+2k\rangle$. □

The following theorem is about the minimum length of elements in an arithmetical numerical monoid generated by three elements.

Theorem 4.3.2. *Let $M=\langle n,\ n+k,\ n+2k\rangle$. Then $l(x+(n+2k))=l(x)+1$ for all $x\in M$.*

The proof for this theorem is similar to the one above and hence omitted.

4.4 A Class of Counterexamples

Certain classes of numerical monoids contain at least one element for which the properties do not hold. If M is of the form $\langle n,n+1,2(n+1)+1\rangle$ and $n>3$, then there is at least one element that does not satisfy Property 1.

Remark 4.4.1. The numerical monoid $\langle 3,4,9\rangle$ is of the form $\langle n,n+1,2(n+1)+1\rangle$ with $n=3$. Note that $9=3\cdot 3+0\cdot 4$ and $0,3\in\mathbb{N}_0$, so $9\in\langle 3,4\rangle$. Hence, $\langle 3,4,9\rangle=\langle 3,4\rangle$ is a numerical monoid generated by two elements. Since $\langle 3,4,9\rangle=\langle 3,4\rangle$ is a numerical monoid with two generators, by Theorem 4.2.1, $L(x+3)=L(x)+1$ for all $x\in\langle 3,4,9\rangle$. ◊

If $n \leq 3$, the numerical monoid only has one or two generators depending on whether $n = 1$ or $1 < n \leq 3$. We start with a lemma that states that if $n > 3$, then the generators n, $n+1$, and $2n+3$ form a minimal system of generators.

Lemma 4.4.2. *If $n > 3$, then the integers n, $n+1$, $2(n+1)+1$ form a minimal system of generators.*

Proof. Suppose that $n > 3$, and suppose that n, $n+1$, $2(n+1)+1$ do not form a minimal system of generators. This means that $2(n+1)+1 = an + b(n+1)$ for some a, $b \in \mathbb{N}_0$. Thus

$$2n + 3 = (a+b)n + b. \tag{4.4.1}$$

It follows that $3 - b = (a+b-2)n$. If $0 \leq b \leq 3$, then $0 \leq 3 - b \leq 3$, which implies that $0 \leq (a+b-2)n \leq 3$. If $a+b-2 = 0$, then from equation (4.4.1), it follows that $b = 3$ and so $a = -1$, which is a contradiction. If $a+b-2 > 0$, then $(a+b-2)n > 3$, and since $n > 3$, we still reach a contradiction. If $b \geq 4$, then $a+b-2 \geq a+4-2 = a+2 \geq 2 > 0$. So $a+b-2$ is strictly positive, n is strictly positive, and $3-b < 0$ because $b \geq 4$. Since $3-b = (a+b-2)n$, we reach a contradiction because the product of two positive integers cannot be a negative integer. Therefore, if $n > 3$, then n, $n+1$, $2(n+1)+1$ form a minimal system of generators. $\square$

Now that we have shown that n, $n+1$, and $2n+3$ form a minimal system of generators for all $n > 3$, we know that $\langle n, n+1, 2n+3 \rangle$ is a numerical monoid generated by three elements (and not two).

The following lemma states that if M is of this form $\langle n, n+1, 2n+3 \rangle$ and $n > 3$, then M contains at least one element that does not satisfy Property 1.

Lemma 4.4.3. *Let $n > 3$, and let $M = \langle n, n+1, 2n+3 \rangle$. Then there exists an element $x \in M$ such that $L(x+n) \neq L(x) + 1$.*

Proof. Let $x = 2n+3 \in M$. Since M is minimally generated by n, $n+1$, and $2n+3$, $L(x) = 1$. Note that $x + n = 3(n+1)$, and since $n+1$ is a generator, $L(x+n) \geq 3 > 2 = L(x) + 1$. Therefore, there exists an element $x \in M$ such that $L(x+n) \neq L(x) + 1$. $\square$

The following two theorems are about the smallest and largest $x \in M$ such that Property 1 is not satisfied. We first introduce some lemmas regarding maximum length of the first few elements of M that only have one factorization. In each of these lemmas, we assume that $n > 3$.

Lemma 4.4.4. *Let $M = \langle n, n+1, 2n+3\rangle$. Then $L(n+n) = 2 = L(n) + 1$.*

Proof. We evaluate $L(2n)$ as follows. Suppose that $2n = an + b(n+1) + c(2n+3)$ for some a, b, $c \in \mathbb{N}_0$. This implies that

$$(2 - a - b - 2c)n = b + 3c. \tag{4.4.2}$$

Since a, b, $c \in \mathbb{N}_0$, $0 \leq 2 - a - b - 2c \leq 2$. We now consider the following three cases:

Case 1: Suppose that $2 - a - b - 2c = 2$. Then $a = b = c = 0$, which implies that $2n = 0$, but since $n \neq 0$, this is not possible.

Case 2: Suppose that $2 - a - b - 2c = 1$. Then $a + b + 2c = 1$. It follows that $c = 0$. If $a = 1$ and $b = 0$, then $n = 0$, which is not possible since $n > 3$. If $a = 0$ and $b = 1$, then $n = 1$, which is also not possible since $n > 3$.

Case 3: Suppose that $2 - a - b - 2c = 0$. This means that $a + b + 2c = 2$. If $c = 1$, then $a = b = 0$; from equation (4.4.2), we see that $0 = 2$, which is an obvious contradiction. If $c = 0$, then $a + b = 2$, and so $a = 1$ and $b = 1$, or $b = 2$ and $a = 0$, or $a = 2$ and $b = 0$. If $a = 1$ and $b = 1$, then $0 = 1$. If $a = 0$ and $b = 2$, then $0 = 2$. Both are contradictions, so neither can be true. If $a = 2$ and $b = 0$, then $0 = 0$, which confirms that $a = 2$, $b = 0$, and $c = 0$ is the only factorization for $2n \in \langle n, n+1, 2n+3\rangle$. Therefore, $L(n+n) = 2 = L(n) + 1$. $\square$

Lemma 4.4.5. *Let $M = \langle n, n+1, 2n+3\rangle$. Then $L(n+1+n) = 2 = L(n+1)+1$, $L(2n+n) = 3 = L(2n)+1$, $L(2n+1+n) = 3 = L(2n+1)+1$, and $L(2n+2+n) = 3 = L(2n+2)+1$.*

The proof is similar to the proof of the previous lemma and hence omitted.

Theorem 4.4.6. *Let $M = \langle n, n+1, 2n+3\rangle$. The smallest element $x \in M$ such that $L(x+n) \neq L(x) + 1$ is the third generator.*

Proof. We saw in the proof of Lemma 4.4.3 that when $x = 2n+3$, $L(x+n) \neq L(x) + 1$.

The only elements in M smaller than $2n+3$ are 0, n, $n+1$, $2n$, $2n+1$, and $2n+2$. Note $L(0+n) = 1 = L(0)+1$. By Lemma 4.4.4 and Lemma 4.4.5, we know that $L(n+n) = 2 = L(n)+1$, $L(n+1+n) = 2 = L(n+1)+1$, $L(2n+n) = 3 = L(2n)+1$, $L(2n+1+n) = 3 = L(2n+1)+1$, and $L(2n+2+n) = 3 = L(2n+2)+1$.

Therefore, $x = 2(n+1)+1$ is the smallest element $x \in M$ for which $L(x+n) \neq L(x)+1$. $\square$

The following proposition prepares the reader for the next theorem, which states that the maximum element $x \in M$ that does not satisfy Property 1 is the Frobenius number of $\langle n, n+1 \rangle$.

Proposition 4.4.7. *Let $M = \langle n, n+1, 2n+3 \rangle$. Then $L(n^2-n-1) = n-3$.*

Proof. Note that $n^2-n-1 = (n-4)(n+1)+2n+3$. Hence, $n-4+1 = n-3 \in \mathcal{L}(n^2-n-1)$. We consider an arbitrary factorization $(a,b,c) \in Z(n^2-n-1)$. Then $n^2-n-1 = an+b(n+1)+c(2n+3)$. Factoring n, we have

$$(a+b+2c-n+1)n+b+3c+1 = 0. \tag{4.4.3}$$

Since $c \in \mathbb{N}_0$, we know $c-1 \geq -1$. We will now consider two different cases:

Case 1: Suppose that $c-1 = -1$, and so $c = 0$. From equation (4.4.3), we have

$$(a+b-n+1)n = -1-b. \tag{4.4.4}$$

Note that $-1-b < 0$. This implies $a+b-n+1 < 0$, or equivalently, $a+b < n-1$, thus $a+b \leq n-2$.

Suppose $a+b = n-2$. From equation (4.4.4), $(n-2-n+1)n = -1-b$, and so

$$\frac{-1}{-1-b} = \frac{1}{n}.$$

This implies $n = 1+b$. Since $0 > a+b-n+1$, we see that $0 > a+b-n+1 = a+b-1-b+1 = a$, which implies $0 > a$. This is a contradiction since $a \in \mathbb{N}_0$. Therefore, since $a+b \leq n-2$ and $a+b \neq n-2$, we conclude that $a+b+c = a+b \leq n-3$.

Case 2: Suppose that $c-1 \geq 0$. This implies that $c \geq 1$. From equation (4.4.3), we compute $(a+b+2c-n+1)n = -1-b-3c$.

Note $-3c-b-1<0$. This implies that $a+b-n+1+2c<0$, thus $a+b+2c<n-1$, which allows us to conclude $a+b+2c \leq n-2$.

Since $c \geq 1$, $a+b+c<a+b+2c \leq n-2$. This implies $a+b+c<n-2$, which allows us to conclude $a+b+c \leq n-3$.

In either case, $a+b+c \leq n-3$. Since $n-3 \in \mathcal{L}(n^2-n-1)$ and $n-3 \geq a+b+c$ for all $(a,b,c) \in Z(n^2-n-1)$, we conclude that $L(n^2-n-1)=n-3$. □

In the following theorem, we prove that the largest element $x \in M$ such that Property 1 does not hold is the Frobenius number F of $\langle n, n+1\rangle$. We first compute the Frobenius number F of $\langle n, n+1\rangle$, next we show that $F \in \langle n, n+1, 2n+3\rangle$, then we use the previous proposition to conclude that $L(n^2-n-1)=n-3$, and finally we show that for all $x>F$, Property 1 is satisfied.

Theorem 4.4.8. *Let $M=\langle n,\ n+1,\ 2(n+1)+1\rangle$. The largest element $x \in M$ such that $L(x+n) \neq L(x)+1$ is the Frobenius number F of $\langle n, n+1\rangle$.*

Proof. The Frobenius number F of $\langle n, n+1\rangle$ is $F=n(n+1)-(n+1)-n=n^2-n-1$.

Note that $(n-4)(n+1)+2(n+1)+1=n^2-4n+n-4+2n+3=n^2-n-1$ and that $n-4 \in \mathbb{N}_0$ because $n>3$. Hence $n^2-n-1 \in M$.

By Proposition 4.4.7, $L(n^2-n-1)=n-3$.

Note that $n^2-n-1+n=n^2-1$, and $L(n^2-1)=L((n-1)(n+1)) \geq n-1>n-2= L(n^2-n-1)+1$. Hence, $L(n^2-n-1+n) \neq L(n^2-n-1)+1$. Let $x \in \langle n, n+1, 2(n+1)+1\rangle$ and suppose $x>F$. This means that $x \in \langle n, n+1\rangle$ and by Theorem 4.2.1, $L(x+n)=L(x)+1$ for all $x \in \langle n, n+1\rangle$. Therefore, the largest element $x \in M$ such that $L(x+n) \neq L(x)+1$ is the Frobenius number F of $\langle n, n+1\rangle$. □

4.5 The Smallest Counterexample

In this section, we will focus our attention on the smallest numerical monoid that has an element that does not satisfy Property 1. Recall that elements n_1, n_2, n_3 must form a minimal system of generators, and $\gcd(n_1, n_2, n_3) = 1$.

Note that since there only exists one numerical monoid of one generator, and we know that both properties hold for that numerical monoid, we know that the smallest numerical monoid must have more than one generator. Moreover, by Theorems 4.2.1 and 4.2.2, we know that the properties hold for all numerical monoids of two generators. Thus, the smallest numerical monoid must have more than two generators.

Recall from the introduction that the properties do not hold for all $x \in \langle 9, 10, 23 \rangle$, which is a numerical monoid of three generators. This implies that the smallest numerical monoid that does not satisfy Property 1 has three generators.

In order to define what we mean by "smallest" numerical monoid, we introduce the following notation. Let $\mathcal{C}_3$ denote the set of numerical monoids generated by 3 elements for which there exists at least one element that does not satisfy Property 1:

$$\mathcal{C}_3 = \left\{ \langle n_1, n_2, n_3 \rangle \mid \text{there exists } x \in \langle n_1, n_2, n_3 \rangle \text{ such that } L(x + n_1) \neq L(x) + 1 \right\}.$$

Define function $\mathcal{S}_3 : \mathcal{C}_3 \to \mathbb{N}$ by

$$\mathcal{S}_3 \left(\langle n_1, n_2, n_3 \rangle \right) = n_1 + n_2 + n_3 \quad \text{for all } \langle n_1, n_2, n_3 \rangle \in \mathcal{C}_3.$$

We introduce the reader to the following definition of what we mean by "smallest" numerical monoid in $\mathcal{C}_3$.

Definition 4.5.1. We say that M is the **smallest numerical monoid in** $\mathcal{C}_3$ if $\mathcal{S}_3(M) \leq \mathcal{S}_3(N)$ for all $N \in \mathcal{C}_3$. $\triangle$

Theorem 4.5.2. *The smallest numerical monoid in* $\mathcal{C}_3$ *is* $\langle 4, 5, 11 \rangle$.

Proof. Consider the numerical monoid $M = \langle n, m, k\rangle$. Since M must be the smallest numerical monoid in $\mathcal{C}_3$, the second generator must be $m = n+1$. If $n = 1$, then $\langle 1, 2, k\rangle = \langle 1\rangle \notin \mathcal{C}_3$. If $n = 2$, then $\langle 2, 3, k\rangle = \langle 2, 3\rangle \notin \mathcal{C}_3$.

If $n = 3$, then the third generator k for $\langle 3, 4, k\rangle$ must be 5 because the Frobenius number of $\langle 3, 4\rangle$ is 5 and if $k > 5$, then $\langle 3, 4, k\rangle = \langle 3, 4\rangle$. However, by Theorem 4.3.1, Property 1 holds for all elements in the arithmetical numerical monoid $\langle 3, 4, 5\rangle$.

If $n = 4$, then the third generator k for $\langle 4, 5, k\rangle$ must satisfy the condition $6 \leq k \leq 11$ because the Frobenius number of $\langle 4, 5\rangle$ is 11. Note that if $k = 6$, then Property 1 holds by Theorem 4.3.1, and if $k = 8, 9$, or 10, then $\langle 4, 5, 8\rangle = \langle 4, 5\rangle$, $\langle 4, 5, 9\rangle = \langle 4, 5\rangle$, and $\langle 4, 5, 10\rangle = \langle 4, 5\rangle$ so $\langle 4, 5, k\rangle = \langle 4, 5\rangle$ in these cases. If $k = 7$, then all elements in the numerical monoid $\langle 4, 5, 7\rangle$ satisfy Property 1 by direct computation (see Appendix A). If $k = 11$, then the numerical monoid $\langle 4, 5, 11\rangle$ is of the form $\langle n, n+1, 2n+3\rangle$ with $n > 3$, so by Theorem 4.4.3, not all elements in $\langle 4, 5, 11\rangle$ satisfy Property 1. Note that $\mathcal{S}_3(\langle 4, 5, 11\rangle) = 20$.

If $n = 5$, then the third generator k for $\langle 5, 6, k\rangle$ must satisfy $7 \leq k \leq 19$ because 19 is the Frobenius number of $\langle 5, 6\rangle$. However, if $k \geq 10$, then $\mathcal{S}_3(\langle 5, 6, k\rangle) = 5 + 6 + k > 20$. The arithmetical numerical monoid $\langle 5, 6, 7\rangle$ satisfies both properties by Theorem 4.3.1, and the numerical monoids $\langle 5, 6, 8\rangle$ and $\langle 5, 6, 9\rangle$ satisfy Property 1 by direct computation (see Appendix A).

If $n \geq 6$, then we must have $k \geq 8$, and so $\mathcal{S}_3(\langle n, n+1, k\rangle) = n + n + 1 + k \geq 21 > 20$.

Hence, the smallest numerical monoid $M \in \mathcal{C}_3$ is $\langle 4, 5, 11\rangle$. □

In the previous section, we saw that the smallest element in a numerical monoid that does not satisfy Property 1 is the third generator, and the largest element in a numerical monoid that does not satisfy Property 1 is the Frobenius number of the first two generators. In this section, we showed that $\langle 4, 5, 11\rangle$ is the smallest counterexample. Here, we present one last remark on the only element in $\langle 4, 5, 11\rangle$ that does not satisfy Property 1.

Remark 4.5.3. The Frobenius number of $\langle 4, 5\rangle$ is 11, and 11 is the third generator of $\langle 4, 5, 11\rangle$. We see that 11 is the only element $x \in \langle 4, 5, 11\rangle$ such that $L(x+4) \neq L(x) + 1$. ◊

5

Future Work

In the previous chapter, we proved results that describe the maximum and minimum factorization lengths in numerical monoids generated by up to three elements. We explored numerical monoids generated by one, two, and three elements in which both properties hold as well as a class of numerical monoids generated by three elements in which Property 1 does not hold. In addition, we explored which elements of this class of counterexamples do not satisfy Property 1. We also concluded that the smallest element that does not satisfy Property 1 is the third generator and the largest element that does not fulfill Property 1 is the Frobenius number of the numerical monoid generated by the first two generators.

We observed that, given a numerical monoid generated by three elements, there are elements in the numerical monoid for which Property 1 does not hold. Aside from the smallest numerical monoid in $\mathcal{C}_3$, we observe that the second smallest element of a numerical monoid for which Property 1 does not hold is the sum of the second and third generators (see Appendix A). Aside from the smallest and second smallest numerical monoids in $\mathcal{C}_3$, we also observed that the third smallest element is the sum of twice the second generator and the third generator (see Appendix A). This pattern continues until we reach the Frobenius number of the numerical monoid generated by the first two elements. Note that the larger the numerical monoid, the more counterexamples the numerical monoid has.

The following are some interesting questions to investigate in the future.

Question 1. Fix $n > 3$, and let $M = \langle n, n+1, 2n+3 \rangle$. In addition to $2n+3$ and the Frobenius number of $\langle n, n+1 \rangle$, which other elements in M do not satisfy Property 1? How many elements do not satisfy Property 1? Which elements do not satisfy Property 2? How many elements do not satisfy Property 2?

Conjecture 5.0.1. *Let $M = \langle n, n+1, 2(n+1)+1 \rangle$. Then there exist $n-3$ elements $x \in M$ of the form $x = k(n+1) + 2n + 3$ with $0 \leq k < n-4$ such that $L(x+n) \neq L(x)+1$.*

Question 2. In addition to $\langle n, n+1, 2n+3 \rangle$, what other classes of numerical monoids M are in $\mathcal{C}_3$? Which elements in M do not satisfy Property 1 and/or Property 2? How many elements in M do not satisfy these properties?

Question 3. What classes of numerical monoids generated by four or more elements satisfy Property 1 and Property 2 for all elements in the monoid? What classes do not satisfy the two properties? Which elements do not satisfy the properties? How many elements do not satisfy the properties?

Appendix A

Computer Software and Programs for Numerical Monoids

In this project we wrote python programs to compute maximum and minimum lengths in a numerical monoid. Recall that a numerical monoid is also known as a numerical semigroup. We used `numericalsgps` package [6] using the following software:

```
gap.eval('SetPackagePath("numericalsgps", "/home/user/numericalsgps");')
load("/home/user/NumericalSemigroup.sage")
gap('LoadPackage("numericalsgps");')
```

When we run these programs, the following is returned:

```
''
true
true
```

Recall that a numerical monoid is also known as a "numerical semigroup." Once the package has been installed, we can start using the Numerical Semigroup feature. The following example shows how to create the numerical semigroup $\langle 3, 5\rangle$ and how to print the factorization set for the element 30.

```
M=NumericalSemigroup([3,5])

print M.Factorizations(30)
```

When run, the output is the following:

```
[[10, 0], [5, 3], [0, 6]]
```

In order to get the length set, we run the following:

```
M=NumericalSemigroup([3,5])

print M.LengthSet(30)
```

This returns:

```
[6, 8, 10]
```

In order for the program to print the maximum length of element 30, we input the following:

```
M=NumericalSemigroup([3,5])

print max(M.LengthSet(30))
```

which returns

```
10
```

and in order for the program to print the minimum length of element 30, we input:

```
M=NumericalSemigroup([3,5])

print min(M.LengthSet(30))
```

which returns

```
6.
```

Utilizing these tools, we present the python code that we wrote in order to perform calculations of maximum and minimum lengths of elements of numerical semigroups of two and three generators.

The following program evaluates the maximum lengths of numerical monoids of two generators a and b. It first evaluates the greatest common divisor to confirm it is 1, and then it turns a and b into generators of a numerical semigroup. Each element i in the numerical semigroup that is less than or equal to the product of a and b is evaluated to see if Property 1 holds. If there is an element i in a numerical semigroup such that Property 1 does not hold, then the program returns 'False,' i, a, b. Note the first generator is always less than the second generator.

```
a=2
while a<20:
    b=a+1
    while b<20:
        if gcd(a,b)==1:
            M = NumericalSemigroup([a,b])
            i=0
            while i<a*b+1:
                if M.Contains(i):
                    if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
                        print "False", i,a,b
                i=i+1
        b=b+1
    a=a+1
```

In order to evaluate minimum length and determine whether Property 2 holds, the same program is written, aside from line 10, which commands the program to evaluate the minimum length of i and $b+i$:

```
    if 1+min(M.LengthSet(i))!=min(M.LengthSet(b+i)):
```

The following program evaluates the maximum lengths of numerical semigroups generated by three elements a, b, and c. This class of numerical semigroup such that the difference between the generators is the same, is an arithmetical numerical monoid generated by three elements. This program evaluates the greatest common divisor of a, b, and c to confirm it is one, and then proceeds to turn them into generators of a numerical semigroup. It evaluates each element i from 0 up to the product of the first and last generators. If there is an element i such that Property 1 does not hold, the program returns 'False,' i, a, b, c.

```
j=1
while j<10:
    a=3
    while a<10:
        b=a+j
        c=b+j
        if gcd(gcd(a,b),c)==1:
            i=1
            M = NumericalSemigroup([a,b,c])
            while i<a*c+1:
                if M.Contains(i):
                    if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
                        print "False", i,a,b,c
                i=i+1
        a=a+1
    j=j+1
```

In order to evaluate minimum length and whether Property 2 holds, the same program is written, except for line 10, which commands the program to evaluate elements i up to the product of the last two generators, and line 12, which commands it to evaluate minimum length of i and $c+i$:

```
while i<b*c+1:

if 1+min(M.LengthSet(i))!=min(M.LengthSet(c+i)):
```

In addition to writing programs that did not return any counterexamples, we also present some programs that were written that did return some counterexamples. This inspired the sections on a class of counterexamples, as well as showing what the smallest counterexample is. The following program evaluates numerical semigroups of three generators where the difference between the first two generators is one and the difference between the second two generators increases with every loop:

```
a=1
while a<10:
    j=1
    while j<10:
        b=a+1
        c=b+j
        if gcd(gcd(a,b),c)==1:
            i=0
            M = NumericalSemigroup([a,b,c])
            while i<a*c+1:
                if M.Contains(i):
                    if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
                        print "False", i,a,b,c
                i=i+1
        j=j+1
    a=a+1
```

When the above program runs, the following is the output:

```
False 11 4 5 11
False 13 5 6 13
False 19 5 6 13
False 19 5 6 14
False 22 6 7 11
False 15 6 7 15
False 22 6 7 15
False 29 6 7 15
False 22 6 7 16
False 29 6 7 16
False 33 7 8 13
```

```
False 17 7 8 17
False 25 7 8 17
False 33 7 8 17
False 34 7 8 17
False 41 7 8 17
False 28 8 9 14
False 37 8 9 14
False 46 8 9 15
False 41 9 10 16
False 51 9 10 16
False 51 9 10 17
False 61 9 10 17
```

Observe how many of the numerical monoids with counterexamples are generated by elements of the form n, $n+1$, $2(n+1)+1$. This inspired the next program to be written. The following program evaluates numerical monoids of three generators where the difference between the first two generators is one and the third generator is twice the second generator summed with 1.

```
a=2
while a<11:
    b=a+1
    c=2*b+1
    if gcd(gcd(a,b),c)==1:
        i=0
        M = NumericalSemigroup([a,b,c])
        while i<a*c+1:
            if M.Contains(i):
                if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
                    print "False", i,a,b,c
            i=i+1
        c=c+1
    a=a+1
```

When this program is run, the following is the output:

```
False 11 4 5 11
False 13 5 6 13
False 19 5 6 13
False 15 6 7 15
False 22 6 7 15
False 29 6 7 15
False 17 7 8 17
False 25 7 8 17
False 33 7 8 17
False 34 7 8 17
False 41 7 8 17
False 19 8 9 19
False 28 8 9 19
False 37 8 9 19
False 38 8 9 19
False 46 8 9 19
```

```
False 47 8 9 19
False 55 8 9 19
False 21 9 10 21
False 31 9 10 21
False 41 9 10 21
False 42 9 10 21
False 51 9 10 21
False 52 9 10 21
False 61 9 10 21
False 62 9 10 21
False 71 9 10 21
False 23 10 11 23
False 34 10 11 23
False 45 10 11 23
False 46 10 11 23
False 56 10 11 23
False 57 10 11 23
False 67 10 11 23
False 68 10 11 23
False 69 10 11 23
False 78 10 11 23
False 79 10 11 23
False 89 10 11 23
```

This output confirmed my observation. In addition to figuring out a class of counterexamples, we observe that the first element listed for each numerical monoid is the third generator. We also observe that the largest element with a counterexample is the Frobenius number of the numerical monoid generated by the first two generators. Lastly, we observe that the first counterexample from both programs is $\langle 4, 5, 11\rangle$. This inspired the section on the smallest counterexample.

One of the key aspects of proving that the largest element in a numerical monoid $\langle n, n+1, 2n+3\rangle$ such that Property 1 does not hold is showing that $L((n-4)(n+1)+2n+3) = L(n^2-n-1) = n-3$. The following program was written to confirm that this was the case.

```
a=4
while a<51:
    b=a+1
    c=2*b+1
    M = NumericalSemigroup([a,b,c])
    if max(M.LengthSet(a*b-a-b))!=a-3:
        print 'False'
    a=a+1
```

The following programs were written to aid in proof by computation, when proving that $\langle 4, 5, 11\rangle$ is the smallest numerical monoid with a counterexample.

```
a=4
b=5
c=7
i=0
M = NumericalSemigroup([a,b,c])
while i<a*c+1:
    if M.Contains(i):
        if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
            print "False", i,a,b,c
    i=i+1

a=5
b=6
c=8
i=0
M = NumericalSemigroup([a,b,c])
while i<a*c+1:
    if M.Contains(i):
        if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
            print "False", i,a,b,c
    i=i+1

a=5
b=6
c=9
i=0
M = NumericalSemigroup([a,b,c])
while i<a*c+1:
    if M.Contains(i):
        if 1+max(M.LengthSet(i))!=max(M.LengthSet(a+i)):
            print "False", i,a,b,c
    i=i+1
```

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