Spring 2019

What Causes Black Holes to Spin?

Mac B. Selesnick
Bard College

What Causes Black Holes to Spin?

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Mac Selesnick

Annandale-on-Hudson, New York
May, 2019

Abstract

Black holes are recently at the cutting edge of cosmological and astrophysical research. Both experiment and theory are leading to surprising conclusions on the physical properties of black holes and their affects on space and time. In this project, I set out to explore the origin and mechanics of a black hole's spin, that is, its internal angular momentum. What causes a black hole to spin in the first place is rich and nuanced. In order to make this project accessible and focused I explore the process of a minor merger, a collision between two black holes, one large and one comparatively small. Working from the ground up, I provide the necessary background information on General Relativity, and guide the reader to explore the subject from scratch. I determine how the spin is altered by a minor merger, making a few necessary approximations along the way. This project provides a foundation for further exploration into the subject of black hole spin. There are many other causes and ways in which spin can be altered that are not touched upon here. Future work could examine larger black hole mergers, accretion disk accumulation, and initial conditions of a collapsing star. This project gives a theoretical model of black hole motion. As black holes are now being observed and pictured, the calculations I do and others can be tested, validated, and observed. Future predictions and experiments on black hole spin is an exciting prospect for the future.

Contents

Dedication

For Popop, I know he'd be proud to see me here today.

Acknowledgments

There are many people I would like to thank for making this project and my education possible. I am grateful to Bard College, and the science division for a fantastic education, and for making this project a possibility. I would like to thank Janet Stetson for welcoming me into the community before I was even a student here, and for helping to make the transition into college manageable. Thank you to James Cox Chambers for the support you have always shown to me and my family. Thank you to all the physics faculty, Matt, Antonios, Paul, and Hal for helping to guide me through such a rigorous and challenging curriculum. Without all of your constant support, I would have never been able to take on the subject. I would especially like to thank Paul, for convincing me to become a physics major in the first place and nurturing my passion and helping to develop my ability for it, and Hal for being an extremely engaging, helpful, open, approachable, and knowledgeable advisor and professor. I couldn't have asked for better advising and guidance during the rigor of senior year. I need to say a special thank you to Christopher Lafratta. While I did not ultimately end up becoming a chemistry major, he alone inspired me to become passionate about the field of STEM. He was always beyond kind and approachable, and was the ideal professor to have my first semester of college. Without him, I would not be in the place I was today. I would like to thank my amazing friends who made these last four years some of the best of my life. Anna Sofia, Cooper, Suzanne, Ellie, Fyodor, Lucy, Katlin, Chase, Tong, Hannah, and Eric. Thank you all for the wonderful memories and consistent support. I'd like to thank Sarah for all the support and love over this last year. Finally, I have endless gratitude towards everyone in my family and all of their unconditional love, Bof, Ivan, Haifa, Connie, Anya, James, Anne Margaret, Nadia, Chapin, Danielle, Ayla, Joslin, Marina, and last but not least my fantastic parents.

1

Special Relativity and Spacetime

Along with Quantum Mechanics, General Relativity completely restructured our view of the physical world, exposing us to elements that are not seen in every day experience. A famous implication of General Relativity's was the existence of black holes. Black holes were determined to be a theoretical possibility before they were ever actually observed. What makes their existence possible, and perhaps Einstein's most crucial formulation, was the notion of spacetime. Until the twentieth century, time was thought to be an independent quantity, existing entirely separately from the three spatial dimensions. Before General Relativity was even developed, it was discovered that space and time are linked, and that the geometry of the universe is defined by four dimensions, one of them being time. This generalizes Euclidean geometry, which is by nature, three dimensional. Euclidean geometry does not account for time as another dimension and coordinate. Spacetime is what allows the possibility of black holes, which form due to the effect gravity has upon spacetime. In this chapter, spacetime is examined in detail within the context of Special Relativity. The form spacetime takes independent of a gravitational field is referred to as flat spacetime. Having a grasp of what exactly spacetime is, and what its properties are is an important step towards understanding fundamental properties of black holes.

1.1 Reference Frames and The Invariant Quantity

1.1.1 Reference Frames

The idea of reference frames are very common to all who work in physics. Relativity was a known concept before the time of Einstein. Galileo emphasized that the laws of physics would be universally upheld in any moving frame of constant velocity. But, the simple conclusion of light's universal speed, no matter what reference frame it is observed in, led Einstein to realize that fundamental properties of objects and events could be changed based upon their observations in frames of reference with varying velocities. The most surprising and important consequence of this is that time is not invariant or universal. Time varies between inertial reference frames, meaning there is no universal clock that ticks independently of of all motion, events, and occurrences. The inconsistency between durations of time in different frames is most easily noticed at speeds near the speed of light, explaining why such effects went unnoticed for so long.

Given that time was known not to tick independently of casual motion, a need arose to be able to plot it graphically, so it could be compared throughout different reference frames. Finding the location of one particle with respect to another means that both of the times in their own separate frames of reference would need to be calculated in order to make accurate predictions of their motion with respect to both frames. Because of this sort of issue, Herman Minkowski began to plot events on diagrams, defining an event as a moment in space and time. Of course, to plot this event you would need coordinates of both time and space. Since there are three spatial dimensions, x, y, and z, a four-dimensional geometry is needed which would have coordinates of space and time, t, x, y, and z. This line of thought led to the inception of spacetime, which plots points as events that occur in both space and time, and allows for an analytical geometry of spacetime. This geometry treats time as it would any other coordinate, that is it is variable and dependent on the motion of the frame being studied. For years, the spatial coordinates x, y, and z had special mathematical transformations to translate them from one reference frame to another. Now new transformations exist in order to account for time differences between different reference frames. To get sense of how these frames a pictured, observe Figure 1.1.

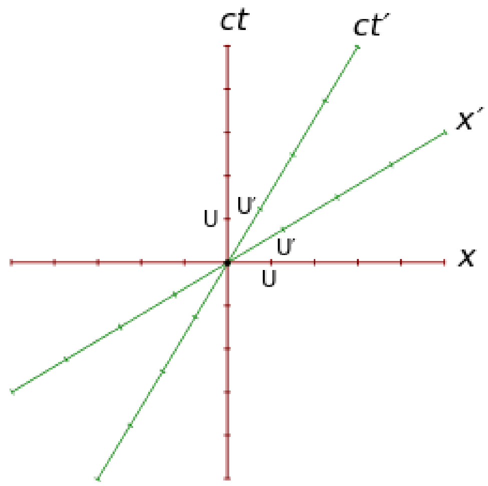

Figure 1.1.1. Plot of time axis vs position (x). Two reference frames are plotted independently, a unprimed frame and a primed frame. Events are plotted on graphs such as these.

1.1.2 *The Spacetime Invariant*

While much changes in between reference frames, events still do occur in every one of them. Most importantly, one mathematical quantity does not change no matter which reference frame it is measured in. A universal quantity in all reference frames is referred to as an invariant. This particular invariant quantity determines the very nature of spacetime. As we will soon see, it is what differentiates flat spacetime from other forms of spacetime that we will encounter in future chapters. The spacetime invariant is defined by the following

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1.1.1}$$

This equation resembles the infinitesimal line element of Euclidean geometry, $ds^2 = dx^2 + dy^2 + dz^2$, otherwise known as the equation for an infinitesimally short straight line, or the shortest distance between two points. Just as this axiom defines many of the properties of Euclidean geometry, the spacetime invariant defines the properties of flat spacetime. The ds^2 above defines the shortest displacement between two points in space and time. The major difference between Euclidean geometry and flat spacetime is the negative time component, which ensures the geometry is uniquely determined, and plots coordinates with time as well as space. The negative sign gives time its unique distinction, separating it as a different quantity then the three spatial dimensions. The interpretation of the invariant is that it represents an infinitesimal displacement in space and time.

1.2 Four Vectors and The Metric of Flat Spacetime

1.2.1 Four Vectors

Since geometry is redefined under Special Relativity as four-dimensional, four-dimensional vectors are needed, which include both space and time. Four-vectors carry this property, and any relation or algebraic manipulation rules for them are invariant, similar to the general laws of physics. The scalar product of two four-vectors, analogous to the dot product is

$$\boldsymbol{a} \cdot \boldsymbol{b} = -a_t b_t + a_x b_x + a_y b_y + a_z b_z \tag{1.2.1}$$

Here the subscripts t, x, y, and z specify the vector's components for time and the three spatial directions. As with three-vectors, all components of special relativistic four vectors are mutually orthogonal, as is evident in Eq. (2). Separating a four-vector into its components, one can see that direction is represented with $\boldsymbol{e}_\alpha$ where the α subscript can take any of the index values t, x, y, or z. The magnitude of each component is denoted by a^α, with alpha being again an index. Indices such as α, are either free and can be chosen to take each value freely, or, when repeated once up and once down, represent a summation over all values. With this in mind, the scalar product of two vectors can be re-expressed as the following

$$\boldsymbol{a} \cdot \boldsymbol{b} = (\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta) a^\alpha b^\beta \tag{1.2.2}$$

This equation is a double sum over α and β. The scaler $(\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta)$ represents the scalar product of the basis vectors. In flat spacetime, the basis vectors are simply (-1,0,0,0), (0,1,0,0), (0,0,1,0) and (1,0,0,0) attached to t, x, y and z respectively. These define the directions of each component, and point along the orthogonal coordinate axis t, x, y and z. This scalar product is defined as $\eta_{\alpha\beta}$. This quantity is a matrix and its usage in this situation will become clear shortly.

1.2.2 The Metric

Using $\eta_{\alpha\beta}$ to re-express the scalar product of two four vectors in flat spacetime, we get

$$\boldsymbol{a} \cdot \boldsymbol{b} = \eta_{\alpha\beta} a^{\alpha} b^{\beta} \tag{1.2.3}$$

The indexes again refer to a summation over each of the four components of space and time. This linear combination of orthogonal components specifies an invariant quantity in four-dimensional spacetime. Armed with four-vectors, scalar products, and Einstein's spacetime invariant, a "metric" of flat spacetime can be defined. The metric is, in fact, the previously defined quantity η. From the indices η can be expressed as a matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2.4}$$

The matrix is diagonal and symmetric, where the four components (e^t, e^x, e^y, e^z) represent the spaces in order from top down and left to right. This shows that multiplying one matrix element by another different element (eg e^t and e^x) would yield 0. This displays the orthogonal characteristics of scalar products in flat spacetime. This metric represents the crucial property of flat spacetime, the invariant displacement as predicted by Special Relativity. Knowing the definition of ds^2, it can be written as

$$ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{1.2.5}$$

Which, if expanded using the definition of η, would yield $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. As illustrated, the metric of flat spacetime also ensures the correct minus sign for the time component of a four vector scalar product.

1.3 Moving towards General Relativity

A known fact often quoted by the layman is that black holes severely warp the fabric of spacetime. This chapter has provided the means for such a statement to make sense at all. Because of Einstein, it was known that space and time were connected. This connection is defined by the metric, η, and the invariant spacetime from equation (6). Knowing that there exists a four-dimensional geometry that links both space and time, serves to partially decode the above statement about black holes. The other part of that sentence, which must be illuminated is how spacetime can be severely warped. Special Relativity incorrectly assumed spacetime always took the form specified in this chapter, that is the invariant displacement would always be determined by the same metric η. As will be seen in the next chapter, spacetime can be changed, most specifically it can have different metrics. What these metrics represent is curvature, the straight line in spacetime specified by ds^2 will not always be straight. The next chapter will examine geodesics, as the shortest paths particles will take in spacetime, and often these paths will not be straight lines, instead they will be curved. This forces new metrics to be defined, which drastically alter the properties of spacetime that were just discussed in this chapter. What causes these alterations of curvature, as Einstein discovered, is gravity. Specifically, sources of mass cause spacetime to curve and alter the metric and invariant line element of flat spacetime. Since black holes are such massive objects, the curvature produced by them is going to be all the more dramatic. As will also be seen, the metric can contain other properties of spacetime itself, such as energy or angular momentum. Deducing the angular momentum of a rotating black hole and comparing it to the angular momentum of particles in the affected spacetime will greatly help answer questions behind the mechanics of black hole spin.

2

General Relativity and Curved Spacetime

2.1 How Gravity Curves Spacetime

2.1.1 The Equivalence Principle

There were two primary observations that led to the discovery of curved spacetime. The first is a known property of photons, the fact that they are massless. If this was true, then, of course, photons would have to remain unaffected by the force of gravity as understood from Newton's Law, due to the fact that mass is the quantity which determines the magnitude of the Newtonian gravitational force. The second was a novel observation by Einstein, often referred to as his "happiest thought." Using similar logic to the principles that define inertial frames, he deduced that there were no experiments one could do to distinguish between a frame affected by a constant gravitational force, versus a frame undergoing constant acceleration. These observations remain the same for any object, since the rate of an object's acceleration due to gravity its not dependent on its intrinsic mass. If these two states of motion really are the same, the following thought experiment explains a behavior of light that shouldn't be physically possible under Newton's theory of gravity. The following set up was considered; a ray of light passing horizontally through the interior of a two-dimensional frame accelerating upwards at a constant rate. The light travels exactly in a straight line in the reference frame inside the ship. However, if it continues in a straight line the frame will have traveled upwards, meaning the beginning of the lights path will be lower then the end of its path. From the point of view of an outside frame, it will appear that the light has travelled downwards in the acceleration frame. According to Einstein, this exact same phenomena would be observed in a frame subject to a uniform downward pointing gravitational field. Therefore, light must somehow be affected by the gravitational force. This

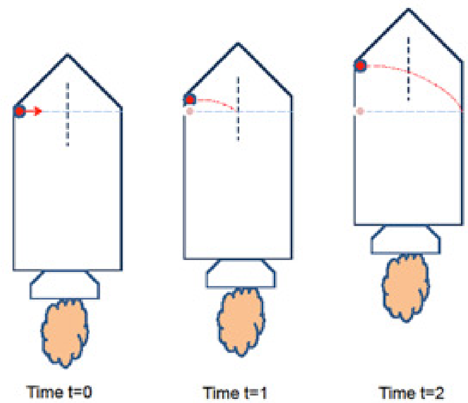

Figure 2.1.1. This figure illustrates the idea of light bending in a gravitational field. In the situation with constant acceleration, the lights path is shown to be curved in the frame of the ship. If this situation really is equivalent exactly to a constant gravitation field, the same effect would have to be observed even if the ship was not acceleration upwards.

simple thought experiment has profound consequences. It implies that there is a fundamental flaw in the Newtonian theory of gravity, which was thought to be universally accurate and verifiable before Einstein put together the theory of General Relativity.

2.1.2 Geodesics

The motion of particles in spacetime must be thought of entirely differently if the force of gravity is to be reformulated. Gravity, in fact, is not a basic field as once thought, it can no longer share all the analogous properties with Electricity and Magnetism. The solution to this issue is complex, and its full derivation is beyond the scope of this project. However, broadly speaking, spacetime is shown to be curved by sources of mass, meaning it must be represented by a different metric then the flat spacetime one that was shown in Chapter 1. Particles move throughout all of spacetime on geodesics, basically the shortest distances in spacetime between its points. This is commonly known as a straight line in one of the crucial axioms of flat Euclidean geometry. In Classical Mechanics, there exists a form of a geodesic principle that arrises through the Calculus of Variations. Hamilton's Principle states that the motion of the system in a given time interval is such as to maximize or minimize the time integral of the Lagrangian, which is known as the action integral. In other words, particles follow the path of the extrema of the time integral between between two points in space in Lagrangian Mechanics. A similar principle exists in General Relativity where the principle determining a particles motion through spacetime is as follows: "The worldline of a free test particle between two timelike separated points extremizes the proper time between them." Note that this differs from Lagrangian Mechanics, as now the action integral refers to the particle's proper time. The particle's proper time, as you will recall from Chapter 1, is the time as measured within its own frame of reference. The idea of using a test particle is a mixed blessing. Test particles in General Relativity are dramatically approximated as massless, so as to neglect the effects they may have gravitationally on spacetime. A planet orbiting a sun for instance may even be treated as a test particle. While this paves the way for more simple elegant calculations and conceptual understanding, the effect the particle itself has on spacetime is certainly a point of interest. A notable example of the shortcomings of such approximations is the famous two body problem, often studied in Newtonian Mechanics. In General Relativity, both bodies in orbit around each other would need attention as proper particles of mass significant enough to warp spacetime. Proceding for now with the idea of test particles, the variational principle can be utilized. Starting with the proper time path length

$$\tau_{AB} = \int_A^B d\tau \tag{2.1.1}$$

The relation from Chapter 1 between proper time and path length ds can be utilized to replace the proper time $d\tau$. Additionally, in general any path length in any coordinate basis is defined by

$$d\tau^2 = -ds^2 = -g(x)_{\alpha\beta}dx^\alpha dx^\beta \tag{2.1.2}$$

These relations give

$$\tau_{AB} = \int_A^B (-g(x)_{\alpha\beta}dx^\alpha dx^\beta)^{1/2} \tag{2.1.3}$$

If this function is parameterized arbitrarily by the variable σ such that σ=0 at A and σ=1 at B the proper time is then

$$\tau_{AB} = \int_A^B (-g(x)_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma})^{1/2} \tag{2.1.4}$$

Finding the Legrange Equations for this integral can be utilized to derive the motion of certain particles. Specific examples will follow later on. This is really all that is needed in order to be able to start solving for equations of motion for a test particle, as long as integrals for the proper time can be determined.

2.1.3 Local Spacetime

Locally speaking, that is infinitesimally, there exists a coordinate transformation that transforms any general metric of spacetime to that of flat spacetime. This is an idea that a lot of people may have good intuition for in an analogous situation. If you have space that is defined by a spherical shape, an infinitesimal area on that sphere will be flat locally, a reason why the earth would appear flat from our perspective. The local properties of curved spacetime should appear indistinguishable from those in flat spacetime. A curved metric is defined in general by the symbol $g_{\alpha\beta}(x)$. In the flat spacetime considered in Chapter 1, that of Special Relativity, basis vectors that specified the directions of each quantity t,x,y,z were all orthogonal to each other, giving the relation in cartesian coordinates

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1.5}$$

However, a coordinates basis of a general metric with basis vector components e_α and e_β are not necessarily orthogonal to each other. The scaler product of any two vectors is still defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = g_{\alpha\beta}a^\alpha b^\beta \tag{2.1.6}$$

And the relation

$$\boldsymbol{e}_\alpha(x) \cdot \boldsymbol{e}_\beta(x) = g_{\alpha\beta}(x) \tag{2.1.7}$$

still holds and defines the metric. The differences from Minkowski's flat spacetime are contained within the metric, $g_{\alpha\beta}(x)$, as opposed to $\eta_{\alpha\beta}$. The scalar product still gives you the infinitesimal distance between two points in spacetime, and the scalar product of coordinate basis vectors still give the definition of the metric. As stated before, mass warps the fabric of spacetime, and therefore changes the flat spacetime metric to a general one, causing a difference in the motion of objects that manifests as the force of gravity. The path length of the object in spacetime follows the variational principle discussed in section 2.1.2. But, infinitesimally speaking, in any localized point, there exists a coordinate transformation that will yield the metric of flat spacetime. Stated mathematically,

$$g'_{\alpha\beta}(x'_p) = \eta_{\alpha\beta} \tag{2.1.8}$$

Here x'_p represents the localized point. A nice analogy can be made by taking a infinitesimally localized point on a sphere, which can be treated as flat despite being on a curved surface.

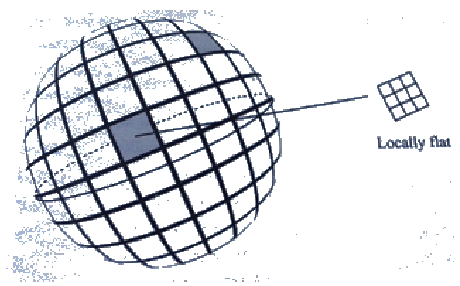

Figure 2.1.2. If we take the analogy of the curved surface of the earth, we have to isolate lots of little square patches, each patch being pretty near flat. The small patch pictured here can be pictured as such. This would very much explain why the earth appears as flat from our perspective!

This fact also has the implication that at every point there are three space dimensions and one time dimension. The neighborhood of this point must be small enough so that the derivatives of the metric vanish at it, and, of course, this form of the metric cannot hold for all of spacetime, or else spacetime would be flat. But this transformation has a powerful ability to allow us to work locally in flat coordinates with the simple metric of flat spacetime.

Finally, an important relation that will be heavily called up is the dot product of the four velocity vector with itself. The four velocity vector accounts for all four dimensions of space and time, and is defined in general as the derivative of the position vector with respect to the proper time of a moving frame. In any case

$$\boldsymbol{u} \cdot \boldsymbol{u} = g_{\alpha\beta} u^\alpha u^\beta = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -1 \tag{2.1.9}$$

Since $d\tau^2 = -ds^2$. This relation is referred to as the normalization of four velocity and often comes in handy in simplifying calculations.

2.2 Spherically Symmetric Star Geometry

As heavily implied by the preceding sections, black holes will curve the fabric of spacetime dramatically. Black holes are made up of a large amount of mass, which curves spacetime such that a new metric is needed to describe regions around them. For regions far enough away from the blackhole, beyond a radius called the Schwarzschild radius, this spacetime can be represented by the Schwarzschild geometry. Understanding the Schwarzschild geometry is the first step towards understanding the fundamentals of black holes, and how they affect spacetime.

2.2.1 The Schwarzschild Metric

The metric of this curved spacetime, defined exclusively for regions outside of spherically symmetric stars is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{2.2.1}$$

The quantities involved will be explored shortly. This metric arrises out of the Einstein equations. Theoretically and experimentally the Schwarzschild metric is tested and observed through the orbits of test particles. Some effects that will be discussed can be measured to small degrees in the solar system. The representation of the metric above is in MLT units, where the speed of light $c = 1$, and the gravitational constant, $G = 1$.

2.2.2 Properties of Schwarzschild Geometry

Being that the line element takes the form of Eq. (3) the metric, $g_{\alpha\beta}(x)$, is

$$g_{\alpha\beta}(x) = \begin{pmatrix} -(1-\frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{2.2.2}$$

The first thing to notice about this metric is that it is independent of time. This means the metric is symmetric under displacement in the coordinate time t. The second is that the metric is spherically symmetric, which follows from the fact that the angular metric is $r^2 d\theta^2 + r^2 \sin^2 d\phi^2$, the same as that of a sphere. Third, the large quantity M is simply the total mass of the source of curvature, which does not depend on how the mass is radially distributed on the star. Finally it is worth now defining the Schwarzschild radius. Here we briefly restore the units of G and c in order to get a physical picture of this radius. At $r = \frac{2GM}{c^2}$, or $r = 2M$ under MLT units, it can be seen that the quantity in front of dr^2 blows up. This indicates a coordinate singularity suggesting that the Schwarzschild coordinates do not accurately represent that particular radius. Therefore, Schwarzschild coordinates are only used for regions outside or inside of that radius, and other ones are needed to make meaningful statements about it. Often the Schwarzschild radius can be smaller then that of the actual star in question, so the Schwarzschild metric adequately describes the spacetime outside of stars. Being that a black hole collapses all of its mass to effectively zero radius, different kinds of coordinates will be needed to represent there properties accurately. Another singularity exists at $r = 0$, as one can see by looking at the metric. Unlike the Schwarzschild radius, this singularity is believed to be a physically accurate description of the region, rather then a fault in the coordinates. However, like with stars, any region outside of the Schwarzschild radius, or inside of it before r=0, a blackhole can be accurately represented by Schwarzschild geometry.

2.3 Particle Motion Around Spherically Symmetric Bodies

Particle motion is best studied by examining the orbits of test particles following their geodesic paths. Even particles as large as a planet can be effectively treated as a test particle, as long as they are orbiting large enough bodies.

2.3.1 Quantities conserved in Schwarzschild Geometry

As mentioned before, the metric is independent of both t and ϕ. According to Noether's Theorem, [2] these symmetries have conserved quantities associated to them. The symmetries can be described more precisely by the vectors, $\boldsymbol{\zeta}$ (1,0,0,0) and $\boldsymbol{\kappa}$ (0,0,1,0). Taking the scaler product of these with the four velocity $\boldsymbol{u}$, we get two new quantities which are called e and l

$$-\boldsymbol{\zeta}\cdot\boldsymbol{u} = (1-\frac{2M}{r})\frac{dt}{d\tau} = e \tag{2.3.1}$$

$$\boldsymbol{\kappa}\cdot\boldsymbol{u} = r^2\sin^2(\theta)\frac{d\phi}{d\tau} = l \tag{2.3.2}$$

The quantity e represents the energy per unit mass, and gives conservation of energy for a free particle in General Relativity. In the formulation of four-vector mechanics, the energy of a particle is represented by a four-vector. Is is redefined as the time component of a particle four-vector momentum. As the four momentum is $p = Mu^{\alpha}$, the time component is simply $-Mu^t$. This also implies that $E = -\boldsymbol{\zeta}\cdot\boldsymbol{p}$. Dividing by mass on both sides gives $-\boldsymbol{\zeta}\cdot\boldsymbol{u} = e$, meaning $e = E/M$, a relation that holds true in all spacetime. Similarly, l represents the angular momentum per unit rest mass, and the relation is similarly analogous to the conservation of angular momentum. These quantities make calculations relating to particle orbits far more tractable. To check the validity of representing them as energy and angular momentum, one can check to see that in flat space, E=mu^t, the known special relativistic total energy per unit rest mass. At low velocities, l would be equivalent to angular momentum per unit rest mass.

2.3.2 Effective Potential

It is useful to examine the motion of a particle's orbit to see the significant effect of curved spacetime. Crucially, the conservation of angular momentum implies that the orbits lie in a plane, which is analogous to classic Newtonian Theory. The particle remains in a plane such that ϕ=0. Then one can reorient the coordinates such that θ=$pi/2$. It is helpful to consider the universal normalization of four velocity to supply us another integral for the geodesic equation, that being

$$\boldsymbol{u}\cdot\boldsymbol{u} = g_{\alpha\beta}u^{\alpha}u^{\beta} = -1 \tag{2.3.3}$$

This can be used to express the three nonzero components of the four velocity in terms of the conserved quantities l and e. Expanding the above Eqn,

$$-(1-\frac{2M}{r})(u^t)^2 + (1-\frac{2M}{r})^{-1}(u^r)^2 + r^2(u^{\theta})^2 = -1 \tag{2.3.4}$$

By expanding out using the definitions of four-velocity from earlier in the chapter, we get

$$-\frac{1}{(1-\frac{2M}{r})}e^2+\frac{1}{(1-\frac{2M}{r})}(\frac{dr}{d\tau})^2+\frac{l^2}{r^2} \tag{2.3.5}$$

Rearranging algebraically we end up with expression for $\frac{(e^2-1)}{2}$.

$$\frac{(e^2-1)}{2}=\frac{1}{2}(\frac{dr}{d\tau})^2+V(r) \tag{2.3.6}$$

where importantly, $V(r)$ is the radial effective potential for an orbiting particle. It is referred to as radial since the only variable it depends upon is r all the rest are the constant conserved quantities e and l and then the mass. Writing out $V(r)$ entirely,

$$V(r)=-\frac{M}{r}+\frac{l^2}{2r^2}-\frac{Ml^2}{r^3} \tag{2.3.7}$$

This quantity is very similar to the Newtonian effective potential, differing only by an additional factor of $\frac{Ml^2}{r^3}$. In fact, with limits of large enough r, the final term disappears, reducing the entire expression to its classical counterpart. The differences between Newtonian and Schwarzschild potentials is pictured below

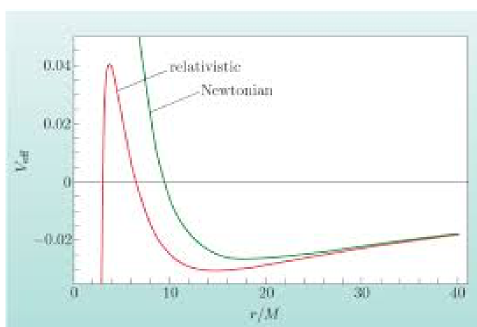

Figure 2.3.1. The vertical axis represents the potential at certain radii on the horizontal axis. The classical vs Schwarzschild general relativistic potentials are labeled. Likely the newtonian curve is familiar to students of classical mechanics. The potential in GR goes to zero at small radii, exactly the opposite of the Newtonian!

Since we now have this potential to work with, techniques for treating orbits by effective potentials in the Newtonian mechanics can be applied to the orbits in the Schwarzschild geometry.

2.3.3 Stable Circular Orbits

A concept that is crucial for orbiting bodies around a black hole is the innermost stable circular orbit, or “ISCO.” These occur at the minima of the effective radial potential, which would be found by taking the derivative of the potential with respect to r. This gives us

$$r_{\text{min}} = \frac{l^2}{2M}\left[1 + \sqrt{1 - 12(\frac{M}{l})^2}\right] \tag{2.3.8}$$

As can be seen above, when $\frac{l}{M}$ is equal to $\sqrt{12}$, $r = 6M$, the smallest value that r_{min} can take. This is the ISCO. This orbit will become very important for exploring the properties of black hole spin later. We often approximate the last point a particle exists with a certain angular momentum before collapsing into the black hole at the ISCO. This will give us crucial information on the resulting spin afterwards.

2.3.4 The Relevance of This Chapter

While this chapter goes pretty deep information of spacetime that will be true without the presence of a black hole, it still holds significant relevance in the chapters to come. Conserved quantities in spacetime will be of primary importance, particularly the angular momentum and energy per unit rest masses, l and e. These quantities will end up functioning as the backbone of exploring the mechanics of a black hole's spin. Since we will look into mergers, the collision between two black holes, the inner-most stable circular orbit, R_{ISCO} will be important. The ISCO radius plays a huge part in determining initial conditions of a black hole orbiting another, which helps determine the final spin of the resulting combined black hole. Finally, an introduction to the way in which calculations are made and motion is studied in General Relativity is enormously helpful investigating the deeper physics behind black hole spin. Black holes mold spacetime into that of the Schwarzschild metric, so using it as our example was not arbitrary. With a stronger grasp of General Relativity, and how spacetime is affected by mass, we can move to explore qualitative and quantitative properties of black holes and their effects on spacetime.

3
How Black Holes Affect Spacetime

In the previous two chapters, it has been made clear why discussion of spacetime and General Relativity are so important. Spacetime is curved by mass, and there is no source of gravitation more dramatic then that of a black hole. The properties of a black hole will alter the metric of spacetime, for select few points in space. However, chapter two holds extra relevance, as Schwarzschild geometry correctly represents the region outside a black hole's event horizon. After this, the topic will move to how the angular momentum of a rotating black hole affects spacetime, providing more clues on ways in which a black hole can gain or loose its spin.

3.1 A Conceptual Introduction

While what lies at the center of a black hole remains a mystery, it is often comparatively more straightforward to make empirical calculations on black hole properties, and their affect on spacetime. Any star reacts to the effects of its own intrinsic gravity, created just by the fact that they exist as masses in what would otherwise be flat spacetime. When a star is compressed due to gravity, the fusion of the inner core gasses push back. The fusion inside the core depletes whichever elements constitute it, leading to the creation of new elements. In stars with mass lower then 10 solar masses, the core eventually overwhelms the force of gravity, and the star's ultimate fate is an explosion, creating a supernova. However, if the mass 10 solar masses or higher, the force of gravity will overwhelm that of the core, exhausting the core's supply of gas entirely. With nothing to push back against gravity, the star continuously collapses with no end. Consequently, the interior of a black hole contains a region of spacetime where a large compact mass is compressed into an infinitesimally small volume. This region, with essentially zero volume and huge quantities of mass is called a "singularity." The singularity is encoded within Schwarzschild coordinates, existing at the point $r = 0$, as seen in Chapter 2. How could it be physically possible for so much mass to be contained within such a small region? In fact, General Relativity predicts that the density approaches infinity, which appears physically impossible. Unfortunately, constructing a theory that addresses such questions has remained out of reach. Black holes contain regions called event horizons, the last possible surface in spacetime where light can escape the black hole's gravitational pull. The radius of this surface is also encoded as a singularity in the Schwarzschild coordinates, but the singularity can be removed by switching coordinates. For radii less then that of the event horizon, no form of matter or electromagnetic radiation can escape, the gravitational field, the curvature of spacetime, is too strong. Many measurable properties of black holes are available to us for study, for example the mass-energy and angular momentum. These quantities are also easily discernible with mathematics when examining test particles placed in the spacetime around the black hole. The metric created by

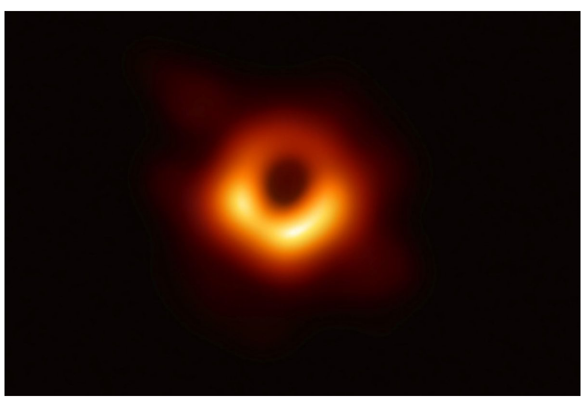

Figure 3.1.1. This is the first real image of a black hole, which was published a couple weeks ago. The event horizon is scene as the ring of light around the darkness of the interior, from which no light can escape.

a black hole can be analyzed perfectly, as can the infinitesimal line element, ds^2, allowing us to

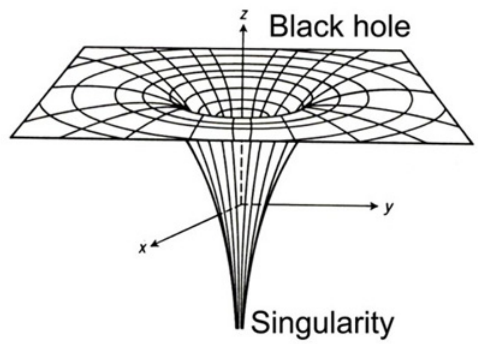

Figure 3.1.2. As can be seen, spacetime is curved more and more as the singularity is approached. Eventually the origin implies a source of infinite curvature.

study geodesics. This project will focus on said properties, in particular the angular momentum, and this chapter will explain the General Relativistic effects black holes have on spacetime.

3.2 Schwarzschild Blackholes

3.2.1 The Metric of Spherically Symmetric Black Holes

A Schwarzschild black hole gives a spherically symmetric metric, and are applicable for regions outside the event horizon of a black hole, and inside of it. However, two coordinate singularities arise at the regions previously discussed, at radii $r = 2M$ and $r = 0$. The singularity $r = 2m$ is not a singularity in the geometry of spacetime, but just in Schwarzschild coordinates. Therefore, new coordinates will suffice to correct this, one being Eddington-Finkelstine coordinates. To make the transformation, the time coordinate of the Schwarzschild metric is redefined for a new one v.

$$t = v - r - 2Mlog(\frac{r}{2m} - 1) \tag{3.2.1}$$

Then the Schwarzschild line element becomes

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{3.2.2}$$

This is still the same geometry represented by the Schwarzschild line element, but with a different set of coordinates used for labeling points. Since $r = 2M$ no longer signifies an infinite spacetime element, the two regions, $r > 2M$ and $r < 2M$ are now smoothly connected, with a regular continuos set of values across that boundary. Without the term $(1 - \frac{2M}{r})^{-1}dr^2$ the singularity disappears, so Eddington-Finkelstine coordinates excellently describe regions in and outside of the event horizon, at least before $r = 0$. At large enough radius r, the geometry becomes flat, which is what one would naturally assume. The point $r = 0$, as discussed, is a real singularity of infinite spacetime curvature and gravitational force.

3.2.2 The Behavior of Light

Light can be used to exemplify a crucial property of blackholes, that nothing can escape from the event horizon. Examining a radial light ray, with a world line such that, ds^2=0, we see

$$-(1-\frac{2M}{r})dv^2 + 2dvdr = 0 \tag{3.2.3}$$

From this it's easy to see that v=const. is one solution to this equation, since in that case dv=0. Eq. (1) also shows that as time increases, r must decrease, or else v would not be a constant. Therefore, the light rays are ingoing, moving towards $r = 0$, the center of the blackhole. However, Eq. (3) has another solution

$$-(1-\frac{2M}{r})dv + 2dr = 0 \tag{3.2.4}$$

This solution is useful as it allows the dv/dr to be solved and then integrated

$$v - 2(r + 2Mlog(\frac{r}{2M} - 1) = constant \tag{3.2.5}$$

This equation reveals that if $r > 2M$, r must increase. By similar logic, r decreases if $r < 2M$. Therefore, $r > 2M$ light rays are outgoing, and $r < 2M$ light rays are ingoing, since $t = r + constant$. This shows crucially that just via the spacetime geometry, light rays inside the event horizon must always be ingoing, and therefore never escape from the region! With every point $r > 2M$ one radial light ray, v=const. which was one solution to Eq. (3) moves inward to smaller values of r. The other moves outward to larger and large values of r as shown by eq(5). Both solutions for a light $r < 2M$ light ray, are ingoing light rays with decreasing r. The surface $r = 2M$, is the event horizon, and it has just been proved that light rays inside that region cannot ever be outgoing. The event horizon $r = 2m$ also allows for a solution where light rays remain stationary, effectively hovering at the event horizon. These are all defining features of a black hole's geometry.

3.3 Geometry of Event Horizon and the Singularity

As has been shown, once something has crossed the event horizon it cannot cross back. To get a sense of the area of the event horizon, one can examine a slice where the coordinate v is constant. Simply putting in $r = 2M$ and v=const. into Eq. (2) yields a two dimensional metric

$$d\Sigma^2 = (2M)^2 d\theta^2 + \sin^2(\theta)d\phi^2) \tag{3.3.1}$$

It might be hard to see, but this defines the familiar geometry of a sphere with constant area

$$A = 16\pi M^2. \tag{3.3.2}$$

The area of a sphere from basic geometry is equal to πr^2. From the definition of the constant 2-D metric above, $2M$ is the constant r coordinate. Therefore plugging in $2M$ for the radius r yields Eq. (7).This area corresponds to the area of the event horizon. As long as the entire geometry is time independent, then A does not change with the coordinate v. However, A clearly has dependance on M, as the mass of a spherically symmetric black hole increases, so does the area of it's event horizon.

3.4 A Brief Summary

The qualitative and quantitative pictures of black holes have now been reconciled. Now that we can study black hole geometry with the aid of the spherically symmetric metric, we can make calculations on its properties. The coordinate singularity $r = 2M$ has been defined as the event horizon, and spacetime geometry tells us the relationship between it's area and the black holes mass. It is also apparent now why light cannot escape the event horizon. Consequently, all matter coming into the black hole is trapped within the surface indefinitely which will indeed have consequences on its mass-energy and if it is rotating, angular momentum. Rotating black holes are the next important area of study, and understanding the effects rotating Black Holes have on spacetime will be the last bit of information needed mathematically and theoretically to discern the origins of a black holes spin.

4
Rotating Black Holes

Now that there is a clear quantitative understanding of black hole, the core of this project can be explored. The question posited was what makes Black Holes spin? The final step towards addressing this is understanding the mechanics of a black hole with spin. Rotating Black Holes are remarkably similar to stationary Schwarzschild Black Holes, depending simply on just the total mass and total angular momentum. However, the fact that the Black Hole contains the extra property of angular momentum means that spacetime is affected in a different way. This chapter will explore the new metric that arrises as a result, as well as the motion of test particles in regions of spacetime affected.

4.1 Kerr Spacetime Geometry

The new line element is specified with Kerr geometries, discovered in 1963

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Marsin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2 + \frac{2Ma^2rsin^2\theta}{\rho^2})sin^2\theta d\phi^2 \tag{4.1.1}$$

where the following quantities are defined below

$$a = \frac{J}{M}, \rho^2 = r^2 + a^2cos^2\theta, \Delta = r^2 - 2Mr + a^2 \tag{4.1.2}$$

This leads to some interesting conclusions. First of all, it's easy to see that the metric is independent of t and ϕ. As with Schwarzschild geometry, both of these symmetries correspond to conserved quantities. The metric is therefore stationary and axisymmetric, which as you will recall from Chapter 2, has two associated vectors of symmetry

$$\eta^\alpha = (0, 0, 0, 1) \tag{4.1.3}$$

$$\zeta^\alpha = (1, 0, 0, 0) \tag{4.1.4}$$

However, since there is an explicit dependance on θ by g_{tt} and g_{rr}, the metric is not spherically symmetric. Secondly, if $a = 0$ then the metric reduces to the Schwarzschild metric, which is to be expected for a non rotating version of this.

4.2 Singularities in the Kerr Metric

Singularities can be found which correspond to the origin and the event horizon. If $\rho = 0$ which would occur when $r = 0$ and $\theta = \pi/2$ a singularity occurs, corresponding to what is thought to be the real place of infinite spacetime curvature. Otherwise, the quantity $\Delta = 0$ would clearly cause a singularity, but this is a coordinate singularity. This would happen at

$$r = M + \sqrt{M^2 - a^2} \tag{4.2.1}$$

and

$$r = M - \sqrt{M^2 - a^2} \tag{4.2.2}$$

Eq.(4) corresponds to the event horizon, and becomes the familiar Schwarzschild radius if $a = 0$. The total angular momentum of a black hole is limited by the square of its mass. This can be seen in Eq. (4) which exists only for $a < M$. If $a = M$ then $J = M^2$. Matter that falls into a Black Hole carries it's own angular momentum, and due to the conservation of angular momentum, the black hole J will increase, until it is limited by $J = M^2$. Black Holes often get very close to this extreme limit, and the parameter a will be very relevant in tackling any question relating to black hole spin.

4.3 Rotating Black Hole Orbits

4.3.1 Conserved Quantities and the Effective Potential

Unlike in the Schwarzschild metric, the total angular momentum is not conserved, only the component of angular momentum along the symmetry axis. Some orbits however are confined to the $\theta = \pi/2$ plane. In this specific plane, the Kerr metric becomes

$$ds^2 = -(1 - \frac{2M}{r})dt^2 - \frac{4aM}{r}d\phi dt + \frac{r^2}{\Delta}dr^2 + (r^2 + a^2 + \frac{2Ma^2}{r})d\phi^2 \tag{4.3.1}$$

Again similar to the procedure in Chapter 2, there are quantities that correspond to conserved energy per unit rest mass and angular momentum per unit rest mass, arriving from the $\theta = \pi/2$ metrics independence of t and ϕ. These have the same symmetry vectors e and l. Evaluating these quantities at infinity, away from sources of curvature show there identities as such. Given that $e = \boldsymbol{\zeta} \cdot \boldsymbol{u}$ and $l = \boldsymbol{\eta} \cdot \boldsymbol{u}$, just like chapter 2, e and l can be expressed in the following way

$$-e = g_{tt}u^t + g_{t\phi}u^\phi \tag{4.3.2}$$

and

$$l = g_{t\phi}u^t + g_{\phi\phi}u^\phi \tag{4.3.3}$$

Which if solved for u^t and u^ϕ yield

$$\frac{dt}{d\tau} = \frac{1}{\Delta}((r^2 + a^2 + \frac{2Ma^2}{r})e - \frac{2Ma}{r}l) \tag{4.3.4}$$

and

$$\frac{d\phi}{d\tau} = \frac{1}{\Delta}((1 - \frac{2M}{r}) + \frac{2Ma}{r}e) \tag{4.3.5}$$

These relations give us the tools to sole for an effective potential, which governs the radial motion of the test particles. Like the Schwarzschild metric, this can be found with the fact that $u \cdot u = 1$ and that u^θ=0. Solving $u \cdot u = 1$ with all these relations in mind means that $dr/d\tau$ can be found. Explicitly defining a radial potential

$$\frac{e^2 - 1}{2} = (\frac{dr}{d\tau})^2 + V(r, e, l) \tag{4.3.6}$$

where

$$V(r,e,l) = -\frac{M}{r} + \frac{l^2 - a^2(e^2-1)}{2r^2} - \frac{M(l-ae)^2}{r^3} \tag{4.3.7}$$

This effective potential is particularly useful for examining orbits of constant radius, in which case it will only depend on such radii R, and the conserved quantities of energy and angular momentum, e and l. This makes the process of orbital calculations tractable around rotating black holes. As will become clear in Chapter 5, studying such orbits will be a crucial piece of determining how black holes spin.

4.3.2 Radial Plunge Orbits

The conserved quantities e and l are again crucial to relate to four velocity vectors, such as $d\phi/dr$. Similar to the radial plunge calculation done in Chapter 2, another can be done with the Kerr Metric.

$$\frac{d\phi}{dr} = \frac{(d\phi/d\tau)}{(dr/d\tau)} = -\frac{2Mar}{\Delta}[\frac{2M}{r}(1-\frac{a^2}{r^2})]^{1/2} \tag{4.3.8}$$

Integrating this expression would show a relation between r and ϕ, specifically the angle $\Delta\phi$ swept out in falling to radius r. What turns out to be interesting about particle orbits is the fact that the radii of photons and particles are different depending on whether or not they are rotating with the black hole or in the opposite direction. Positive values of l correspond to rotation with the blackhole, and the opposite applies for particles counterrotating with respect to the black holes. Particles due indeed become dragged around by Black Hole rotation, indicating that rotating black holes cause rotational frames which particles are susceptible to in their motion. As will be elucidated in the next chapter, any generalized equation dealing with particle orbits around black holes, should take into account all possible motion with respect to the black holes spin.

4.4 Putting The Pieces Together

Up until now, this project has reviewed the basics of General Relativity, and black hole physics. The idea is that an undergraduate student studying physics, with a solid background in Classical Mechanics and math would be able to follow along the whole time. The next chapter finally moves into more uncharted territories, and partially gets to the very question that this project set out to answer. Chapter 1 showed how Einsteins theory Special Relativity reformulated the scientific communities idea of the nature of time. Time could not be treated as a quantity independent of spacial dimensions, so the conceptual idea of spacetime was accepted geometrically and algebraically in dealing with any event. Chapter 2 showed how spacetime was not always in the form as it took in Chapter 1, that it could be affected and changed by sources of mass. This was taken by Einstein as the explanation for gravity, that the "force" of gravity was nothing more then the motion of objects in spacetime, and that when spacetime was curved by mass, the geodesics of any object would be altered. Not lost however, as the ability to treat spacetime as flat locally. Also examined in that chapter was a direct case study which made an example of how large sources of mass curve spacetime, namely large spherically symmetrical Schwarzschild stars. In Chapter 3, black holes were explained with the new tools developed from General Relativity, and many properties turned out to be very similar to those of the Schwarzschild metric. The orbits of particles in Chapters 2 and 3 were studied, as well as conserved quantities familiar to us as energy and angular momentum. Finally, this chapter extended the knowledge built up towards rotating black holes, and the orbits of particles as well as conserved quantities were examined again. It turns out that some specific orbits around Kerr black holes will contribute to it's spin, even if the black hole is rotating extremely slowly. Chapter 5 will reveal how the result of two black holes merging into one will create negative spin, i.e. spin down. There is even a specific calculation that can be done that relates how much the final black hole will have spun down to the accumulated mass. Now that we know all we do about rotating black holes, what properties can be examined on events that affect its spin?

5
A Source of Spin

As a vast array of current research will indicate, there are many factors that can lead to a black hole's spin. According to Gammie et al, spin can arise due to initial conditions of a collapsing star, accretion along the Event Horizon, and mergers.[1]This project will focus on the case of minor mergers, a small black hole colliding with a larger one that is slowly rotating. The collision of two black holes, an event that was experimentally observed rather recently using gravitational waves, leads to the formation of a single more massive black hole, which has a new angular momentum J. Remarkably, everything studied up until this chapter, is all that will be needed to make a fairly good model of how a black hole's spin changes due to the accretion of a smaller source of mass. Crucial to this calculation will be the conserved quantities, l and e, as well as the Schwarzschild radius R. You will recall that l and e are the angular momentum and energy of a test particle and are per unit rest mass of this particle. Allowing the merger itself to contain all the angular momentum of the system is an approximation that will be made here. This unfortunately ignores more recent discoveries, and treats the gravitation radiation emission during the process as negligible. Thankfully this factor emitted is very small, so ignoring it leads to a reasonable approximation.[1]

5.1 The Set Up

5.1.1 Initial Conditions

To set up this problem we take a relatively small Kerr black hole with mass m and imagine it orbitally plunging into a larger Kerr black hole with mass M. The ratio of the two masses is then of course $q = \frac{m}{M}$, which we will assume is a number far smaller then 1. The larger black hole also spins slowly, with a value of $j << 1$. The lower case j is known as the reduced spin and is equal to $\frac{J}{M^2}$. This is because j works ideally as a dimensionless quantity, and its use will become very apparent as the calculation proceeds.Before attacking the calculation, it is useful to reexamine the relevant quantities, these should now be familiar from previous chapters. This entire situation, in fact, is very similar to radial plunge orbits of test particles, the difference being of course that the smaller black hole cannot be treated as a massless. It will, however, still serve to treat it as a test particle, meaning its affect on spacetime will be approximated as negligible. Like the test particles examined before, its orbit will end up being the ISCO, the innermost stable circular orbit. It makes sense to find the ISCO of the smaller orbiting black hole, as this will be the last moment that the two black holes coexist independently before the plunge and combination. In Chapter 4, we found an effective radial potential $V(r, e, l)$ from conserved quantities, for the orbit of any particle subjected to a Kerr metric. For an orbit to be circular, it will need to follow a path of constant radius r. Therefore, the radial acceleration should vanish. If examining the ISCO, r must be a constant radius R. Knowing the angular momentum and energy per rest mass comes exactly from the calculations done in Chapter 4. Also, from Chapter 4, we recall the formula $\frac{e^2-1}{2} = \frac{dr}{d\tau} + V(R, e, l)$ where V is the effective radial potential function.

5.1.2 Inclination Angles

The last bit of setup that must be acknowledged is the inclination angle i. The inclination angle is the angle between the direction of a large masses spin and the direction of the orbital angular momentum of the smaller orbiting body. The quantity $\mu = \cos(i)$ is defined which elucidates the behavior's of masses around Kerr black holes. If $\mu = 1$, the angle would be 0, meaning the orbiting body is rotating exactly in the same direction as the large mass's spin. In the opposite case, $\mu = -1$, the smaller mass would be rotating exactly in the opposite direction as the large mass's spin. Bardeen was able to find expansions for R_{ISCO}, e and l for the cases where $\mu = \pm 1$.[4] For the case of two black holes, the total energy per mass is defined as $\frac{E}{mM}$, the total energy of the system divided by the product of the mass of both black holes. This quantity is simply the total energy of the system divided by the product of small black hole mass times the larger mass, very similar to e which accounted for only one source of mass. Similar quantities are defined for $\frac{L}{mM}$, which relates to l in the same way, and $\frac{R_{\text{ISCO}}}{M}$ which is the ISCO radius divided by the mass of the large black hole.

5.1.3 Re-expressing Quantities

For the radial acceleration to be constant then,

$$\frac{\partial V(r,e,l)}{\partial r} = 0 \tag{5.1.1}$$

Taking the derivative of $V(r,e,l)$ with respect to r we have

$$\frac{\partial V(r,e,l)}{\partial r} = \frac{\partial}{\partial r}(-M/r + l^2 - a^2(e^2-1)/2r^2 - M(l-ae)^2/r^3) \tag{5.1.2}$$

and setting the result equal to zero

$$-\frac{M}{r^2} - \frac{l^2 - a^2(e^2-1)}{r^3} + \frac{3M(l-ae)^2}{r^4} = 0 \tag{5.1.3}$$

Note that all r terms here refer to R_{ISCO} since we are defining a minimum of the effective potential. Since $(dr/d\tau)^2 = V(R,e,l) - e^2 - 1/2$, the entire equation must therefore be equal to zero when r is constant. Therefore

$$r(-\frac{M}{r} + \frac{l^2 - a^2(e^2-1)}{2r^2} - \frac{M(l-ae)^2}{r^3}) - \frac{e^2-1}{2} = 0 \tag{5.1.4}$$

These two relations are clearly quadratic for the variables e and l, meaning they can be solved with familiar algebraic techniques.Using the quadratic equations and substitutions we can solve for e and l separately in terms of r only, where $r = R_{\text{ISCO}}$. Finally, $r = R_{\text{ISCO}}$ is found by plugging in the results for e and l into Eq. (3) and solving for r. Bardeen was indeed able to to express the stable circular radius in terms of the spin parameter a, and l and e in terms of a the mass of the large black hole M and R_{ISCO}. [4] The following relations, were found by solving E and L independently. Also, from here onwards, we shall define new constant quantities $e* = \frac{e}{M} = \frac{E}{mM}$ and $l* = \frac{l}{M} = \frac{L}{mM}$, where m is the mass of the small black hole and M is the mass of the large black hole. The larger mass was encoded in the equations and was largely divided out on the other side of the equation, with the exception being the $\frac{a}{M}$ terms. The smaller was already encoded in the constants e and l from their already established definitions.

$$\frac{r}{M} = 3 + Z_2 \pm [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \tag{5.1.5}$$

,

$$e* = \frac{E}{mM} = \frac{1 - 2v^2 \pm (\frac{a}{M})v^3}{\sqrt{1 - 3v^2 \pm 2(\frac{a}{M})v^3}} \tag{5.1.6}$$

and

$$l* = \frac{L}{mM} = \pm rv \frac{1 \pm 2(\frac{a}{M})v^3(\frac{a}{M})^2 v^3}{\sqrt{1 - 3v^2 \pm 2(\frac{a}{M})v^3}} \tag{5.1.7}$$

where $v = \sqrt{\frac{M}{r}}$,

$$Z_1 = 1 + (1 - (\frac{a}{M})^2)^{1/3}[(1 + (\frac{a}{M}))^{1/3} + (1 - (\frac{a}{M}))^{1/3}] \tag{5.1.8}$$

and,

$$Z_2 = (3(\frac{a}{M})^2 + Z_1^2)^{1/2} \tag{5.1.9}$$

5.2 The Minor Merger Calculation

With all of this set up, the calculation can begin. Taking the change in total angular momentum before and after the collision, assuming the momentum transfer is thought to be entirely closed within the merger system we find

$$\Delta(\boldsymbol{J}^2) = \boldsymbol{J_f}^2 - \boldsymbol{J_i}^2 = (\boldsymbol{J} + \Delta\boldsymbol{J})^2 - \boldsymbol{J}^2 = 2\boldsymbol{J} \cdot \Delta\boldsymbol{J} + (\Delta\boldsymbol{J})^2 \tag{5.2.1}$$

The above equation just refers to the familiar subtraction of initial angular momentum from the final, after the black holes merge.[1] The event of the merger is driven by gravitational waves, a topic which is out of the scope of this paper.[1] However, the inspiral can be approximated as a progression of circular orbits with fixed inclination angles. Therefore, the final moment before the two holes merge occurs when the smaller one is at the ISCO. Then , the total initial orbital angular momentum of the smaller black hole can be evaluated at the ISCO. With this in mind, $(\Delta \boldsymbol{J})^2$ can be set approximately equal to $(ml * M)^2$, where $l*$ is the total angular momentum divided by the product of the masses of both black holes. Recall that earlier in this chapter, this variable $l*$ was specified as $l* = L/mM$, and similarly $e*$ is referring to E/mM. This takes care of one term in Eq. (10), recasting it in terms of the familiar conserved quantity l, and the masses of both black holes.

The requirement that j is extremely small means that high order terms in j can be neglected. To deal with the other term in Eq. (10), recall Eq. (5). We replace $(\frac{a}{M})$ with j since $j = \frac{J}{M^2}$ and $J = \frac{a}{M}$. This means that j is a dimensionless representation of the angular momentum of the orbiting body. Recasting the Z terms with j we get

$$Z_1 = 1 + (1 - j^2)^{1/3}[(1 + j)^{1/3} + (1 - j)^{1/3}] \tag{5.2.2}$$

and

$$Z_2 = (3j^2 + Z_1^2)^{1/2} \tag{5.2.3}$$

Taylor expanding Z_1 and keeping terms only up to j^2 gives

$$Z_1 = 3 - (8/9)j^2 \tag{5.2.4}$$

and then plugging this in for Z_2

$$Z_2 = 3 - (7/6)j^2 \tag{5.2.5}$$

A reminder is relevant here that that $j = \frac{J}{/}M^2$. From these expansions we see that,

$$\frac{r}{M} = 6 - \frac{4\sqrt{2}}{\sqrt{3}} j\mu + \mathcal{O}(j^2) \tag{5.2.6}$$

Doing similar expansions for Eq. (6) and (7), it can be seen that

$$\frac{L}{mM} = 2\sqrt{3}\mu - \frac{2\sqrt{2}}{3} j\mu^2 + \mathcal{O}(j^2) \tag{5.2.7}$$

and,

$$\frac{E}{mM} = \frac{2\sqrt{2}}{3} - \frac{1}{18\sqrt{3}} j\mu + \mathcal{O}(j^2) \tag{5.2.8}$$

As will soon be shown, these equations allow for a new convenient way to express the change in angular momentum ΔJ.

One very noticeable caveat in the derivations of the Eqs. (15), (16) and (17), is that they contain the variable μ which comes out of nowhere. Recall that this relates to the inclination angle i. The original Bardeen equations for l, e and R_{ISCO} were solved only for $\mu = \pm 1$,[5] which is why they were not present before. The factor of μ accounts for all possible inclination angles taken on by the smaller black hole with respect to the spin of the larger one. Elucidating their appearance in Eq. (9), (10), and (11) would be a useful point of further study, but unfortunately I ave not had the chance to do it.

Using Eq. (16), we can now solve for the other part of Eq. (10), the overall change in angular momentum

$$2\boldsymbol{J} \cdot \Delta \boldsymbol{J} = 2jM^2 \Delta \boldsymbol{L} = 2jM^3 m 2\sqrt{3}\mu - \frac{2\sqrt{2}}{3} j\mu^2 \tag{5.2.9}$$

To account for the possibility of any inclination angle, we take an average from 1 to -1

$$\int_{-1}^{1} (2jM^3 m 2\sqrt{3}\mu - \frac{2\sqrt{2}}{3} j\mu^2) d\mu = \frac{-4\sqrt{2}M^3 m j^2}{9} \tag{5.2.10}$$

Next up, we define $e * m$ as the change in mass of the large black hole. This change is meant to model the accretion of mass from the smaller black hole. For convience, which will become apparent later we also multiply $\boldsymbol{J}$ by M, giving

$$\frac{M\Delta J^2}{e * m} = \frac{M d(j^2 M^4)}{dM} = \frac{d(j^2 M^4)}{d\ln M} \tag{5.2.11}$$

noting that $j = J/M^2$. Now putting in both pieces for Eq. (10)

$$\frac{d(j^2 M^4)}{d\ln M} = \frac{M}{e{*}m} \left[\frac{4\sqrt{2}M^3 m j^2}{9} + l *^2 m^2 M^2 \right] \tag{5.2.12}$$

By solving for $\frac{d\ln j}{d\ln M}$ utilizing the product and chain rule, the above becomes

$$\frac{d\ln j}{d\ln M} = -2 - \frac{2\sqrt{2}}{9e*} + \frac{l *^2 m}{2e * Mj^2} \tag{5.2.13}$$

Next one only needs to substitute for $e*$ and $l*$ to find the remarkably simple result

$$\frac{d\ln j}{d\ln M} = -\frac{7}{3} + \frac{9m}{\sqrt{2}Mj^2} \tag{5.2.14}$$

Remarkably, the second term can be ignored under limits where $j^2/q >> 27$. Separating variables and integrating both sides then yields

$$\ln j = -\frac{7}{3}\ln M + A \tag{5.2.15}$$

Here A is an arbitrary constant of integration. Since a constant can be cast as any value, it is useful to specify as well under a logarithmic function. After this one can simplify the expression using familiar logarithmic algebra.

$$\ln j = -\frac{7}{3}\ln M + \ln A = \ln M^{-7/3} + \ln A = \ln AM^{-7/3} \tag{5.2.16}$$

Rearranging a bit more algebraically we arrive at the relationship between the black hole spin j and the accumulation of mass M

$$j = AM^{-7/3} \tag{5.2.17}$$

But what is the integration constant and what does it represent? As usual, this sort of problem is solved with boundary conditions. Since $A = \frac{j}{M^{-7/3}}$, and we know initially $j = jo$ the initial angular momentum, and $M = Mo$, the initial mass, Eq. (26) takes the form

$$j = \frac{jo}{Mo^{-7/3}}M^{-7/3} \tag{5.2.18}$$

This equation offers us a powerful tool to insert a value of mass accumulated from a merging black hole and yield a value of the angular momentum. It also proves that the process of a minor merger spins down a black hole. As mass is accumulated, the spin decays with a negative root power. The more minor mergers occur, the less effect they have on the large black hole's spin. This decay in spin yields the curve pictured below, where the initial value jo starts out at a number much less then 1, as specified in the original set up. See figure 5.2.1 As can be seen, the allowed range of j from 0-1 decays swiftly as mass accumulates. Intuitively, this result seems to

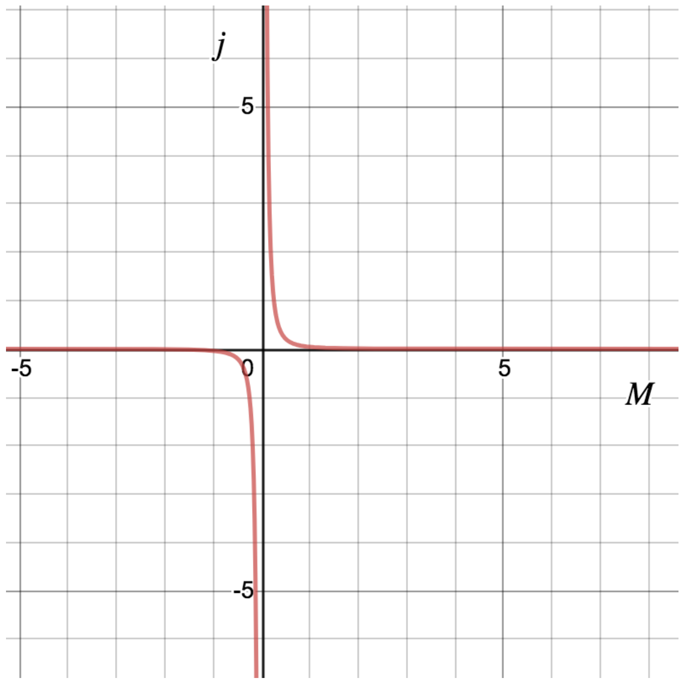

Figure 5.2.1. Plot of j vs M. The region that should be concentrated on is from 0-1. The initial spin jo begins at a very small value where jo is much less then 1.

make sense. As mass is accreted on to the black hole, it spins less rapidly. This shows that the larger mass's spin slows down.

Major mergers, a topic not touched on in this project, have been shown to spin black holes up, suggesting that the previous intuition may have limited validity. Also, the second term in Eq. (22) cannot be ignored if the mass ratio q is a significantly large number. Being that we assumed the mass ratio was small, that is, the smaller black hole was of much less mass comparatively, it was ok to ignore it. However, a more generalized look would also aim to solve this system for larger q values, meaning Eq. (22) would have to be solved with the second term as included.

During this project we narrowed focus to examine the mechanics of spin due to a minor merger. If the small black hole is small enough, then I have shown that a minor merger spins the black hole down. This successfully addresses the minor merger process in many valid cases, giving an accurate if at times approximated result.

6
Conclusion

From all of this, it can be seen that these mergers will spin down a black hole if one of them is significantly smaller then the other. This has led this project to a partial success, with some approximations and reliance on previous research, one valid answer to the question of the black hole spin has been answered. However, this work still merely scratches the surface of a bigger picture. As this paper has illustrated, many familiar laws of physics change within the context of General Relativity. Parameters that represent energy and angular momentum come from symmetry vectors of their respective metrics, and despite being a close analogy to classical energy and angular momentum, do have different origins, being that they are divided by the mass of an object. Noether's Theorem classically relates conservation of energy and angular momentum to rotational coordinate displacement symmetries, and coordinate independence of time. These symmetries in General Relativity appear within the metric, but do they behave the same way when spacetime is curved rather then flat? From Chapter 4, we do know that $l = \frac{L}{M}$ is conserved, and can indeed be related to the total angular momentum $J = \frac{a}{M}$. Another crucial approximation was neglecting the loss of energy and angular momentum that occur through the emission of gravitational waves. Accounting for this would greatly enrich the merger model, and being that gravitational radiation was picked up from a real life merger already, it should be a manageable task for future work. Also, being that the rotation of the large black hole was so small, $j << 1$, this allowed for the possibility of expanding any function involving j to first order only. If the large black hole started out rotating significantly, it would dramatically change the end result of the calculation. The inclination angle μ noticeably was not a part of the Bardeen equations, which only expanded the parameters e, l, and R_{ISCO} for $\mu = \pm 1$. A detailed calculation on how the μ terms make it into the final equations for l, e and R_{ISCO} would be useful for the sake of thoroughness and validity. Finally, there are many other sources of black hole spin. One can examine the accumulation of accretion disks, mergers of black holes of more or less equal size, and initial conditions of a rotating Schwarzschild star before it's collapse and transformation into a black hole. As this paper functions partially as a tool for grasping General Relativity from the ground up, there was not enough much time to examine all of these cases throughly. Future work in this topic should aim to neglect or further justify the approximations made, and explore other causes of black hole spin.

Bibliography

[1] Charles F. Gammie, Stuart L. Shapiro, and Jonathan C. McKinney, *Black Hole Spin Evolution*, Astrophys.J. **602** (2004), 312–319.

[2] Noether Emmy, *Invariant Variation Problems*, Transport Theory and Statistical Physicsl **405** (1918)), 235-257.

[3] James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Addison Wesley, Santa Barbara, 2003.

[4] James M. Bardeen, William H. Press, and Saul A. Teukolsky, *Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation*, Astrophysical Journal **178** (1972), 347–370.

[5] Scott A. Hughes and Roger D. Blandford, *Black Hole Mass and Spin Coevolution By Mergers*, The Astrophysical Journal **585** (2003), 101–104.