Lagrangian Cobordisms of Legendrian Pretzel Knots with Maximal Thurston-Bennequin Number

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Abstract

In the study of Legendrian knots, which are smoothly embedded circles constrained by a differential geometric condition, an actively-studied problem is to find conditions for the existence of Lagrangian cobordisms, which are Lagrangian surfaces whose slices resemble specific Legendrian knots at each end. Any topological knot has infinitely many distinct Legendrian representatives, which are partially distinguished by the Thurston-Bennequin number tb, an integer invariant of Legendrian isotopy which is bounded above. We demonstrate a family of knots where each has a maximal-tb representative K admitting a Lagrangian cobordism from a stabilized Legendrian unknot, a property which guarantees the existence of a similar cobordism from stabilized unknots to any representatives resulting from stabilization of K.

Dedication

Dedicated to my 10^{th} and 11^{th} grade math teachers, George Palen and Masha Albrecht, respectively. Mr. Palen gave me challenging open-ended problems to work on when I was bored with the curriculum, and Ms. Albrecht introduced me to calculus — how one can learn calculus without immediately falling in love with mathematics is beyond me.

Acknowledgments

Most of all, I am grateful to my wonderful advisor Caitlin Leverson for introducing me to this beautiful topic, and for her continued support, guidance, and patience during these strange months.
I have also benefited much from the many forgiving friends to whom I have tried, and failed, to explain what a Legendrian knot is.

Contents

1 Introduction

Mathematical knots are conceptually similar to real twists of thread. But physical knots are of interest because of the friction between strands, while mathematical knots have no such interaction. Thus the primary question of interest in mathematical knot theory is the identification, classification, and equivalence of knots. By studying the "universe" of knots, we study the topology of a 3-dimensional manifold, most commonly Euclidian 3-space $\mathbb{R}^3$. Knot theory also has a number of real-world applications in biology and physics, such as in the knotting of DNA, in the folding of proteins, and in fluid dynamics.

We normally define two knots as equivalent if they are smoothly isotopic. But this is not the only possible equivalence relation, or even the only interesting one. Another that we will make use of in this paper is the existence of a cobordism — a smooth 2-manifold having two knots as its boundary. The existence of a cobordism is a much weaker condition than knot equivalence, so it divides the universe of knots into classes [FM66].

We can also use knot theory to study spaces with more structure than the normal $\mathbb{R}^3$. In particular, by the association of a certain plane field (see Definition 2.2.1) with $\mathbb{R}^3$, we are able to study *contact manifolds*. Contact geometry is a rich and actively-studied field, with broad applications to physics, including geometric optics and classical mechanics. The knots of interest living in contact manifolds are *Legendrian knots*, whose tangent vectors lie on the plane field. This gives rise to the equivalence relation of Legendrian equivalence, which is *strictly finer* than that of smooth equivalence. The structure of these equivalence classes is nontrivial but somewhat understood: given a Legendrian representative K, additional representatives with lower Thurston-Bennequin number (defined in Subsection 2.2.3) may be obtained by adding additional local twists.

Moreover, each contact manifold has a canonically associated 4-manifold, equipped with a similar differential condition. Analogous to Legendrian curves in contact 3-manifolds are Lagrangian surfaces in symplectic 4-manifolds, which allow us to define a similar notion of cobordism for Legendrian knots. This relation is in a sense finer than that of smooth cobordism, as the existence of a Lagrangian cobordism implies the existence of a smooth one, but it is not an equivalence relation on the set of Legendrian knots as it is not symmetric [Cha15].

Smooth cobordisms have been extensively studied, but much less is known about their Lagrangian counterparts. There are several known necessary conditions for the existence of cobordisms, some of which we will briefly mention here. First, the existence of a Lagrangian cobordism from K to $\emptyset$ is mutually exclusive with the existence of a Lagrangian cobordism from $\emptyset$ to K, a result of Gromov [Gro85]. The existence of a Lagrangian cobordism from K_- to K_+ also gives information about many invariants of K_- and K_+ (see [Pan17], [Cha+20], [BLW19]). A particularly useful result of Chantraine gives a simple condition on the values of the classical invariants (Subsection 2.2.3) of K_- and K_+ [Cha10]. In addition to these obstructions, there are diagrammatically-defined sufficient conditions (see [BST15], [Lin16], [GSY21]), though it is nontrivial to use these conditions to make general positive statements about the existence of cobordisms.

The main result of this thesis is an infinite family of knots, $\{P_n\}$, each of which has a maximal-tb Legendrian representative K_n admitting a Lagrangian cobordism from a suitably stabilized Legendrian unknot U_n. In fact, this allows us to construct cobordisms from stabilizations of U_n to any stabilization of K_n. In some cases, such as P_1, all Legendrian representatives are stabilizations of the maximal representative. We state the theorem here; the proof of this is the focus of Chapter 3.

Theorem A. *Let* $P_n = P(3, -3, n)$*, for* n *an integer, and define* $\overline{\text{tb}}\, P_n$ *to be the maximal value of the Thurston-Bennequin number over all Legendrian representatives of* P_n*. Then there exists a Legendrian representative* K *of* P_n*, and a Legendrian unknot* U *with* $\text{tb}\, K = \text{tb}\, U = \overline{\text{tb}}\, P_n$*, such that there is a decomposable Lagrangian concordance from* U *to* K*.*

This thesis is organized as follows. In Chapter 2, we provide background material on the theory of Legendrian knots: in Section 2.1, introducing knots in the smooth context; in Section 2.2, laying out the basics of Legendrian knots, their representations, and useful invariants; and in Section 2.3, defining smooth and Lagrangian cobordisms and briefly summarizing known results about Lagrangian cobordisms. In Chapter 3, we motivate and prove Theorem A. In Chapter 4, we discuss other classes of knots which are promising candidates for results similar to Theorem A (i.e., whose maximal-tb representatives might admit Lagrangian cobordisms from the unknot). Finally, in Appendix A, we provide and explain the Mathematica code that we used to determine $\overline{\text{tb}}$ for the $P(3, -3, n)$ family.

2
Background

2.1 Smooth Knots

We are interested in defining a certain class of knots called Legendrian knots. Before we proceed let us give a definition for knots in the smooth setting and for some of their properties, as well as touch on how we represent knots.

Definition 2.1.1. A **knot** is a (smoothly) embedded S^1 in $\mathbb{R}^3$. Two knots are said to be **equivalent** if there exists a (smooth) isotopy of $\mathbb{R}^3$ taking one knot to the other.

We require that knots be smooth for two reasons. The first is because non-smooth knots can be wild (pathological). But one can also exclude the possibility of pathological behavior by defining knots to be finite polygonal chains, so the second reason for knots to be smooth is so that we may make use of their derivatives. In particular, in Section 2.2 we will define Legendrian knots with a condition on their tangent vectors — thus our knots must be differentiable.

An additional piece of information that we sometimes include with a knot is **orientation**: the "direction" that the strand runs. Knots with orientation are called **oriented**. An oriented knot has two possible orientations.

Further, define a **link** to be the union of finitely many disjoint knots. Links are a natural generalization of knots, and knots are exactly the links with one component. Links behave for the most part the same as knots, though this is not always the case: for example, an oriented link with c components has 2^c possible orientations.

We generally represent knots by *diagrams*, which are projections of the knot onto a plane, marked at each double point to indicate which strand passes over the other. Furthermore, diagrams which have no points of intersection of three or more strands, have only a finite number of double points, and in which the strands at a double point are not locally parallel, are called *regular diagrams*. Some regular diagrams can be seen in Figure 2.1.1.

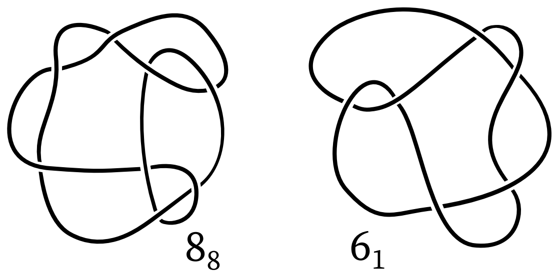


Figure 2.1.1. Knot diagrams. See the end of Subsection 2.1.1 to understand what these numbers mean.

Any diagram of a tame (non-wild) knot can be approximated by a regular diagram [MK96]. Moreover, regular diagrams contain enough information to reconstruct the original knot (up to isotopy). For these reasons it is very convenient to represent knots by regular diagrams, and we will make use of them here frequently. Any reference to knot diagrams should be assumed to refer to regular diagrams.

2.1.1 Classification of Knots

Given that diagrams record the entire topology of a knot, it is intuitively reasonable that we should be able to determine knot equivalence just by looking at diagrams. We are able to do so by classifying the ways that a diagram of a knot can change when the knot undergoes smooth isotopy. In particular, there are three types of diagrammatic "moves" which correspond to smooth isotopy. They are called the **Reidemeister moves**, pictured in Figure 2.1.2.

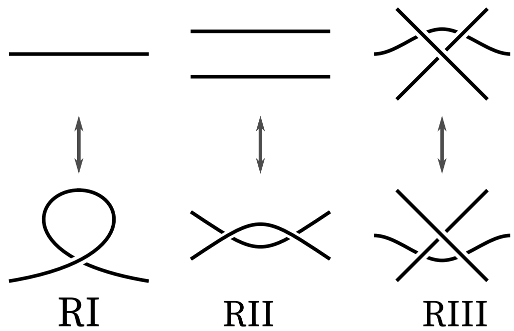


Figure 2.1.2. The smooth Reidemeister moves. Reflections and rotations of these three relations are also included.

Theorem 2.1.2 ([Rei27], [AB26]). *Let K and K' be knots, with D a diagram for K and D' a diagram for K'. Then K and K' are smoothly isotopic if and only if D and D' are related by a finite sequence of Reidemeister moves (Figure 2.1.2) and planar isotopy.*

Though Reidemeister moves completely determine smooth isotopy, it remains difficult in practice to determine when two knots are equivalent, which makes it a useful project to look at other ways of classifying knots. Since the origins of interest in knot theory, an ongoing project has been to create a complete table of distinct (small) knots. There are two important simplifications we can make to reduce the number of distinct knots required for such an atlas.

First, given two knots, we can construct a new knot by splicing them together. This construction is called a **connected sum**, and the connected sum of K and L is written $K\#L$. The connected sum is well-defined (note that the same does not hold for multi-component links), as it is topologically invariant with respect to how K and L are spliced. The properties of a connected sum such as $K\#L$ can usually be determined based on knowledge about K and L. Thus it suffices to catalogue the knots which are not connected sums; these are called **prime** knots.

Second, we can construct a new knot by changing the crossings of a diagram. In particular, given a diagram D for a knot K, we can construct a new diagram $m(D)$ by switching the overstrand and the understrand at every crossing. Then there exists some possibly new knot K' for which $m(D)$ is a regular diagram. The resulting knot K' is independent of the choice of D, and it is referred to as the **mirror** of K, or $m(K)$. Note that $m(m(K)) = K$. We say that K is **amphichiral** if

$m(K) = K$, but this is not the case in general. For example, the trefoil and its mirror, shown in Figure 2.1.3, are not equivalent.

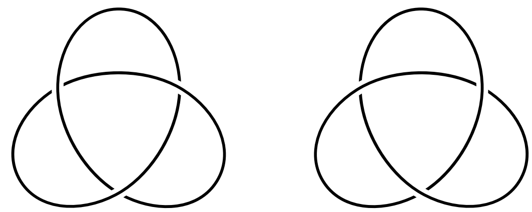


Figure 2.1.3. The left-handed trefoil 3_1 and the right-handed trefoil $m(3_1)$.

Such tables are organized according to the **crossing number**, which is defined for a knot K as the minimum number of crossings over all regular diagrams of K. There are finitely many prime knots of a given crossing number, and within tables they are ordered arbitrarily (although there is an agreed-upon numbering of prime knots up to 10 crossings) and numbered. For more information on knot tables, see [HTW98].

For example, a reference to the knot 6_1 as in Figure 2.1.1 should be read as the *1st* prime knot with a crossing number of *6*. This is called **Alexander-Briggs notation**, named for and following the convention of an important early knot table [AB27]. Be careful when comparing Alexander-Briggs notation between sources: there is little consistency with regards to which knot is the mirror, so a reference to 6_1 must be read as a reference to the *two* knots 6_1 and $m(6_1)$.

2.2 Legendrian Knots

2.2.1 Contact Geometry

In order to define Legendrian knots we begin by defining a certain plane field on $\mathbb{R}^3$.

Definition 2.2.1. At each $(x, y, z) \in \mathbb{R}^3$ we define the **standard contact structure** ξ, seen in Figure 2.2.1, by

$$\xi(x, y, z) = \text{span}\{\partial_y, \partial_x + y\partial_z\}.$$

Note that ξ can be thought of as a linear combination of the partial derivatives of the coordinate functions, as these are the basis vectors of the tangent space $\mathrm{T}_{(x,y,z)}\,\mathbb{R}^3$, of which ξ is a 2-dimensional subspace.

Every plane field is the kernel of the 1-form given by its normal. In the case of the standard contact structure ξ, this one form is referred to as α, and it is given by $\alpha = dz - y\,dx$.

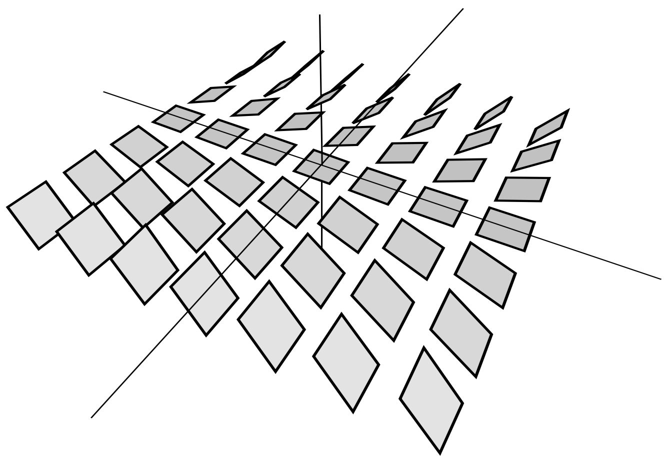


Figure 2.2.1. The standard contact planes in $\mathbb{R}^3$. Diagram from S. Schonenberger.

As we travel in the positive y-direction, the plane $\xi(p)$ gets steeper and steeper in the ∂x direction. In fact, the planes twist so much that there is no 2-dimensional surface everywhere tangent to ξ, or even tangent to ξ in any nonempty open set (see [Boo03]). Such a plane field is called **completely non-integrable**. In general, a 3-manifold equipped with a completely non-integrable plane field is called a **contact 3-manifold**.

To see why ξ is completely non-integrable, note that by the Frobenius Theorem (see [Boo03]), non-integrability is equivalent to the condition that $[X,Y](p) \notin \xi(p)$, where $[X,Y]$ is the Lie bracket of the vector fields $X = \partial_y$ and $Y = \partial_x + y\partial_z$ which together span ξ (in this case, $[X,Y] = \partial_z$). Intuitively, the Lie bracket measures the instantaneous direction of travel along an infinitesimal loop that heads first in the X direction, then the Y direction, then the $-X$ direction, and then the $-Y$ direction. If there were a surface tangent to ξ, then such a walk would remain in the surface: that is, the Lie bracket would be contained in the tangent plane.

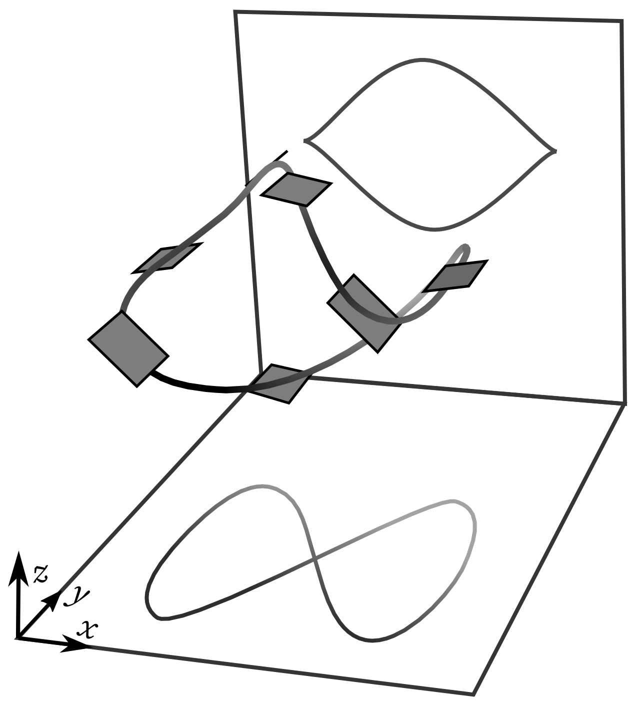


Figure 2.2.2. A 3D rendering of a Legendrian unknot, showing the contact planes and the shadow of the knot in the xz- and xy-planes. Diagram used with permission from Joshua Sabloff, "What is a Legendrian Knot" [Sab09].

Although no surface can be everywhere tangent to ξ, there are many *curves* which run tangent to ξ, such as in Figure 2.2.2. Such a curve is called Legendrian, leading to the following definition.

Definition 2.2.2. Let $K : (0,1) \to \mathbb{R}^3$ be a smooth curve. We say K is **Legendrian** if K is everywhere tangent to ξ. That is, at all $t \in (0,1)$,

$$K'(t) \in \xi(K(t)).$$

As a knot is a smooth embedding of the circle in $\mathbb{R}^3$, so a Legendrian knot is a Legendrian embedding of the circle in $(\mathbb{R}^3, \xi)$. But the equivalence relation under which one defines a knot is as important as the curve itself, and so we will define an analogous relation for Legendrian knots.

Definition 2.2.3. Let K and K' be Legendrian knots. We say K and K' are **Legendrian equivalent** if there exists a smooth function $\phi : [0,1] \to \mathbb{R}^3$ such that $\phi(0) = K$, $\phi(1) = K'$, and for every $t \in [0,1]$, $\phi(t)$ is a Legendrian knot.

The only difference between this definition and Definition 2.1.1 is that the intermediate knots must also be Legendrian. We frequently use the terms **equivalent** and **isotopic** to refer to knots that are either Legendrian equivalent or smoothly equivalent. Generally, when referring to Legendrian knots, we mean Legendrian equivalence, but when confusion may arise we will explicitly use the term **smoothly equivalent/isotopic**.

By definition, all Legendrian knots are smooth knots, and it is clear that there are representatives of smooth knots which are not Legendrian. Nonetheless, Legendrian knots are plentiful. In fact, any smooth knot can be continuously approximated by a Legendrian knot (a visual demonstration of this fact will be given in Section 2.2.3, once we have defined stabilization; for a rigorous proof see [Gei06]).

Such an approximation is smoothly isotopic to the target curve, and thus there exist Legendrian representatives of any smooth knot type.

2.2.2 Front Diagrams

How do we represent and record Legendrian knots? Legendrian knots are no different from smooth knots in that they are embeddings of S^1 in $\mathbb{R}^3$, and so it is convenient to represent them by diagrams.

Unlike smooth knots, Legendrian knots contain geometric as well as topological information, and so we want our diagrams to record that information too. But because the geometric condition that Legendrian knots satisfy is not invariant under rotation, we must be careful to distinguish which plane we are projecting onto to create a diagram.

There are two projections which are used to represent Legendrian knots, seen in Figure 2.2.2. The first is the **Lagrangian projection**, which is projection onto the xy-plane. This projection is useful in defining certain algebraic invariants, but we will not need it here.

Instead, we restrict our examination to the **front projection**, which is projection onto the xz-plane such that the positive y direction points *into* the page. This orientation agrees with the right-hand rule, where $z = x \times y$: the positive x direction points to the east, the positive z

direction points north, and then the right-hand rule tells us that the positive y direction points into the page.

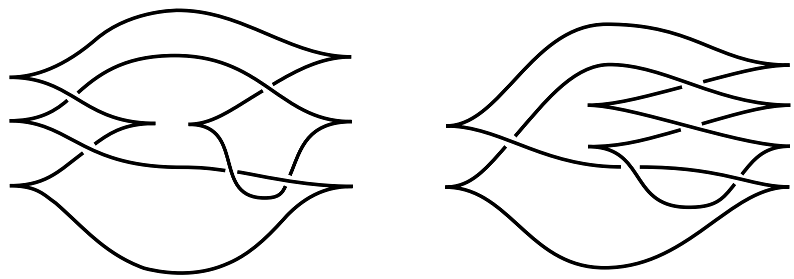


Figure 2.2.3. Front projections for some Legendrian representatives of $m(5_2)$. This specific pair is known as the Chekanov Examples.

Front projections, Figure 2.2.3, contain enough information to recover the exact geometry of the original knot. This is because the Legendrian condition is a requirement on the tangent vector based on the y-coordinate. Recall that a curve K is Legendrian if $\alpha = dz - y\,dx$ vanishes on $\mathrm{T}_p\,K$ for all $p \in K$. Thus we have $dz - y\,dx = 0$ and so

$$y = \frac{dz}{dx}.$$

This explains the cusps we see on the left and right local maxima of front diagrams: Since the part above the cusp (which has a negative slope, in the case of a right cusp) and the part below the cusp (which has a positive slope at a right cusp) have to meet, the slopes of each must be equal at the cusp. Note that while these points are cusps in the projection, they are smooth in the 3-dimensional knot.

Moreover, in a front diagram there is no need to mark the overstrand at a crossing: unambiguously, the strand with a more negative slope goes over the other. In this thesis, we have chosen to mark the overstrands in front diagrams for ease of viewing.

2.2.3 Classical Invariants

Any knots which are Legendrian equivalent are also smoothly equivalent by definition, so let us ask whether the converse is true. As intuition (and Figure 2.2.3) may suggest, the answer is no. Moreover, since there exist Legendrian representatives of any smooth knot, the relation of Legendrian equivalence “refines” the equivalence classes of smooth knots into a larger number of Legendrian equivalence classes. The structure of this refinement is nontrivial, but we will begin to answer it by looking at Legendrian equivalence of front diagrams.

We saw earlier that the equivalence of smooth knots corresponds to equivalence of diagrams under the Reidemeister moves. The smooth Reidemeister moves do not preserve Legendrian equivalence, but there exists a similar set of three diagrammatic moves, Figure 2.2.4, which determine Legendrian equivalence of front diagrams.

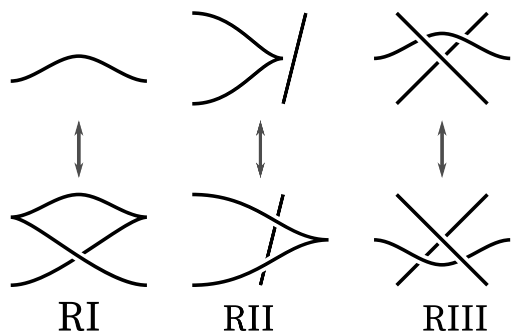


Figure 2.2.4. The **Legendrian Reidemeister moves**. Vertical and horizontal reflections of these moves are also allowed, as long as the crossings are corrected so that the overstrand has the more negative slope.

Theorem 2.2.4 ([Świ92]). *Let K and K' be Legendrian knots, with D a front diagram for K and D' a front diagram for K'. Then K and K' are Legendrian isotopic if and only if D and D' are related by a finite sequence of Legendrian Reidemeister moves (Figure 2.2.4) and planar isotopy through Legendrian knots.*

These moves correspond to restricted versions of the smooth Reidemeister moves. Unfortunately, as with the smooth Reidemeister moves, it is difficult in practice to determine equivalence of knots using these rules. Nonetheless, they are useful in the construction of more practical invariants, as invariance under all three Reidemeister moves is equivalent to invariance under Legendrian isotopy.

There are two numerical invariants which can be easily defined in and computed from the front projection, and they are called the **classical invariants**. Before we define them, we need to define the writhe of a diagram.

To define the **writhe** $w(D)$ of a diagram D, we assign a sign to each crossing in D, using the right-hand rule as seen in Figure 2.2.5. If as you travel along the overstrand, the understrand goes from right to left, then the crossing is positive; and if the understrand runs from left to right, then the crossing is negative. The writhe is the sum of the signs of the crossings.

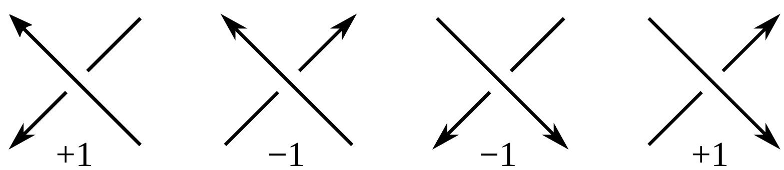


Figure 2.2.5. The sign at a crossing.

Writhe is *not* an invariant of knots, Legendrian or smooth. Because the RI move adds a crossing without changing the orientation of other crossings, it changes the writhe. Thus $w(D)$ is invariant under RII and RIII moves only. We also note that while writhe is well-defined for unoriented *knots*, it is orientation-dependent for links with more than 1 component. Moreover, it allows us to define the following invariant.

Definition 2.2.5. Let K be a Legendrian knot, and D a front diagram for K. Let $c_r(D)$ be the number of right cusps in D, and $w(D)$ the writhe of D. Define the **Thurston-Bennequin number**, or tb, as

$$\mathrm{tb}(K) = w(D) - c_r(D).$$

It is a simple matter to show that the Thurston-Bennequin number is an invariant of Legendrian knots.

Proposition 2.2.6. *The Thurston-Bennequin number is an invariant of Legendrian knots.*

Proof. Let D be a front diagram for a Legendrian knot K. It suffices to show that $\mathrm{tb}(D)$ is unchanged under the Legendrian Reidemeister moves.

1. RI adds one right cusp and one crossing. Regardless of the orientation of the segment before the move, the new crossing is easily seen to have sign +1, and therefore

$$\mathrm{tb}(D') = (w(D) + 1) - (\mathrm{c}_r(D) + 1) = \mathrm{tb}(D).$$

2. RII adds two crossings, but they have opposite signs, so the writhe remains the same.

3. RIII moves two crossings, but their signs remain unchanged.

□

Definition 2.2.7. Let K be a Legendrian knot and D a front diagram for K. Given an orientation (t increasing), we say a cusp is **upward-pointing** (resp. **downward**) if $\frac{dz}{dt} > 0$ near the cusp (resp. < 0). Let $\mathrm{c}_u(D)$ be the number of upward-pointing cusps and $\mathrm{c}_d(D)$ be the number of downward-pointing cusps. Define the **rotation number** of K as

$$\mathrm{r}(K) = \frac{1}{2}(\mathrm{c}_d(D) - \mathrm{c}_u(D)).$$

Although $\mathrm{r}(K)$ is only well-defined for *oriented* K, it is defined up to multiplication by ± 1 for unoriented knots.

Proposition 2.2.8. *The rotation number is an invariant of Legendrian knots.*

Proof. As before, let D be a front diagram for K. Neither RII nor RIII change the orientation or the number of cusps. On the other hand, RI creates a pair of cusps, one pointing upward and one pointing downward. □

The classical invariants, tb and r, do a good job of distinguishing certain types of Legendrian knots (see [EF08], [EH01]), though there are known to be pairs of smoothly-isotopic Legendrian knots which have the same tb and rotation number but which are not Legendrian equivalent, such as Chekanov's examples [Che02] in Figure 2.2.3. Nonetheless, they reveal a great deal about how the Legendrian equivalence classes of a smooth knot are structured.

Figure 2.2.6 shows four Legendrian unknots. The top left has $\mathrm{tb} = -1$ and the others have $\mathrm{tb} = -2$, so the first is not isotopic to the rest. But all of the unknots with $\mathrm{tb} = -2$ are mutually

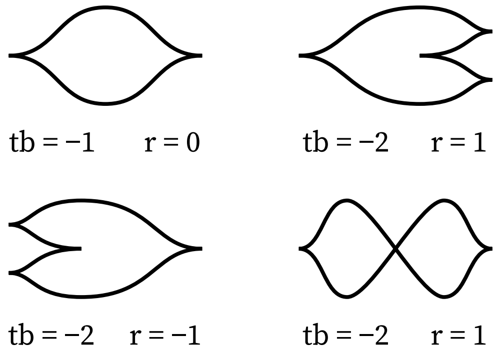


Figure 2.2.6. A selection of Legendrian unknots.

isotopic, and they can be obtained from the first by means of a so-called **stabilization**, shown in Figure 2.2.7.

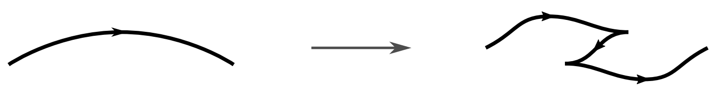


Figure 2.2.7. A "positive" stabilization, increasing the rotation number.

As mentioned in Subsection 2.2.1, the stabilization can be used to construct continuous Legendrian approximations of smooth knots. We show two examples of how this may be done in Figure 2.2.8. Note that the first example demonstrates how to create a Legendrian approximation of a crossing in which the overstrand has the more *positive* slope. With Legendrian approximation and the stabilization move in hand, it is clear that there are infinitely many distinct Legendrian representatives of each smooth knot.

A stabilization decreases the tb by 1, and changes the rotation number by ± 1 depending on the orientation. Moreover, the stabilization is itself a smooth isotopy, so it preserves topological knot type. Thus for any knot, the tbs of its Legendrian representatives are unbounded below, and the rotation numbers are unbounded both above and below.

A classical result of Bennequin is that for any topological knot, the tb is bounded above [Ben83], and therefore the **maximal Thurston-Bennequin number** is a smooth knot invariant, which we denote $\overline{\text{tb}}(K)$. For example, the maximal tb for the unknot is -1, as seen in Figure 2.2.6, and all other representatives are (Legendrian isotopic to) stabilizations of the maximal-tb unknot

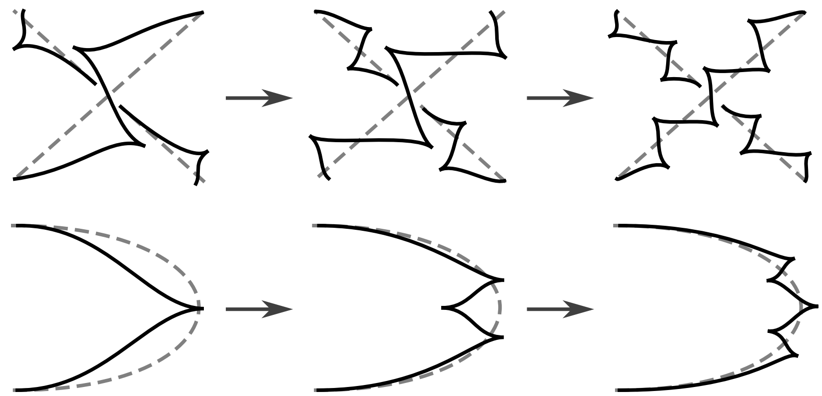


Figure 2.2.8. Approximating smooth knots with stabilized Legendrian knots.

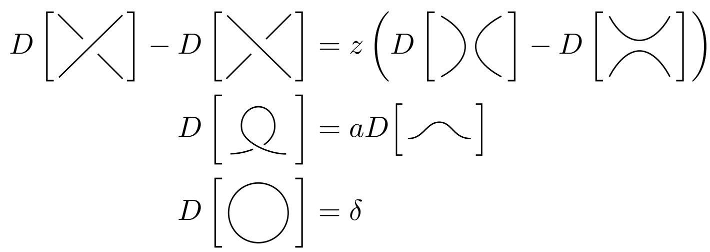
[CN15]. It is possible to determine whether a Legendrian knot is a stabilization using algebraic invariants such as the Chekanov-Eliashberg DGA [Che02], but this does not in general determine $\overline{\mathrm{tb}}$: for example, there exists a Legendrian representative of $m(10_{139})$ with $\mathrm{tb} = -17$ and $\mathrm{r} = 4$ which is not a stabilization of the maximal-tb representative, which has $\mathrm{tb} = -16$ and $\mathrm{r} = 1$ [CN15]. Thus we have to look elsewhere to determine $\overline{\mathrm{tb}}$.

2.2.4 Polynomial Invariants and Skein Relations

Many useful knot invariants take the form of Laurent polynomials (i.e., polynomials having both positive and negative exponents). Typically, these are defined recursively using skein relations, which give an algebraic relationship between the polynomials of knots differing only at a crossing. For certain such relations the resulting polynomial can be shown to be not only unique, but invariant under smooth isotopy.

In particular, we are interested in the Kauffman polynomial, as it gives an upper bound on the maximal tb of a smooth knot type. We define the Kauffman polynomial in terms of the L-polynomial, which is an invariant only under regular isotopy (RII and RIII moves only). There are many varying definitions of the Kauffman polynomial in the literature. We use here a variant known as the Dubrovnik polynomial (it was discovered in the city of Dubrovnik in then-Yugoslavia [Kau90]), and we refer to its normalized version as the Kauffman polynomial. It is known that the Dubrovnik polynomial and other formulations of the Kauffman polynomial are interconvertible, a result which was supposedly discovered by W.B.R Lickorish and communicated via postcard [Kau90].

Let T be a knot diagram or an oriented link diagram, and define the **Dubrovnik polynomial** D recursively via the following skein relations, where $\delta = \frac{a-a^{-1}}{z} + 1$.



(2.2.1)

(2.2.2)

(2.2.3)

The first two equations are local, indicating a relation between the Dubrovnik polynomials of diagrams which are identical except for the substitution of the indicated figures. The third equation normalizes the recurrence relation by determining the Dubrovnik polynomial of an unknot diagram without crossings. The choice of δ for the polynomial of the unknot diagram with no crossings is natural; it arises from setting $D(\emptyset) = 1$ for the empty link.

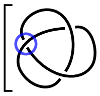
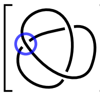
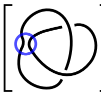
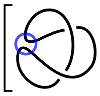
Yet such a polynomial is certainly not a topological invariant: the RI move corresponds to multiplication by $a^{\pm 1}$, seen in (2.2.2). Thus we normalize the D-polynomial by the writhe to get the Kauffman polynomial $Y(K)$, since the same RI move which corresponds to a multiplication by a in the Dubrovnik polynomial increases the writhe by 1. This is why we require that K be oriented if it is a link and not a knot: recall that writhe is well-defined for unoriented knot diagrams but not for unoriented *link* diagrams in general.

Definition 2.2.9. Let K be a knot or an oriented link and T a diagram for K. Define the **Kauffman polynomial** of K by

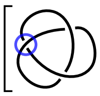
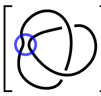
$$Y(K) = a^{-w(T)} D(T).$$

The polynomial, which is a Laurent polynomial in a, x, is an invariant under smooth isotopy [Kau90].

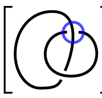
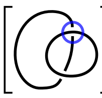
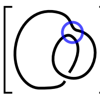
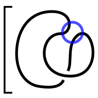
As an example, we show here the computation of the Kauffman polynomial of the trefoil. Using (2.2.1), we have

$D\Big[$  $\Big] - D\Big[$  $\Big] = z\Bigg(D\Big[$  $\Big] - D\Big[$  $\Big]\Bigg).$

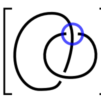
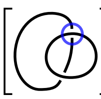
Two of these diagrams are simply twisted unknots, so we have

$D\Big[$  $\Big] = a\delta + z\Bigg(D\Big[$  $\Big] - a^{-2}\delta\Bigg).$ (2.2.4)

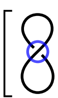
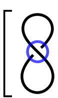
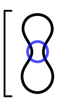
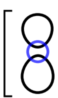
We have expressed the D-polynomial of this diagram for the trefoil in terms of the D-polynomial of a simpler diagram, and so we now compute this simpler polynomial. We once again use the first relation.

$D\Big[$  $\Big] - D\Big[$  $\Big] = z\Bigg(D\Big[$  $\Big] - D\Big[$  $\Big]\Bigg),$

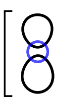
and thus

$D\Big[$  $\Big] = D\Big[$  $\Big] + z\left(a\delta - a^{-1}\delta\right).$ (2.2.5)

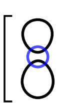
Finally we compute the D-polynomial of this 2-component unlink. Recall that although the D-polynomial is not a knot invariant, it is invariant under RII and $RIII$ moves — that is, we may safely move the top loop away from the bottom.

$D\Big[$  $\Big] - D\Big[$  $\Big] = z\Bigg(D\Big[$  $\Big] - D\Big[$  $\Big]\Bigg),$

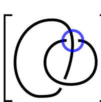
and thus

$a^{-1}\delta - a\delta = z\Bigg(\delta - D\Big[$  $\Big]\Bigg).$

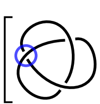
Rearranging, we have

$D\Big[$  $\Big] = \dfrac{a^{-1}\delta - a\delta}{z} + \delta = \delta^2.$

Returning to (2.2.5), we have

$D\Big[$  $\Big] = \delta^2 + z(a\delta - a^{-1}\delta) = \delta^2 + z\delta(a - a^{-1}).$

Finally, we return to (2.2.4), where we have

$D\Big[$  $\Big] = a\delta + z\delta^2 + z^2\delta(a - a^{-1}) - za^{-2}\delta.$ (2.2.6)

Normalizing by the writhe, which is readily seen to be 3, we have at last the Kauffman polynomial of the trefoil knot (for simplicity, we have left δ as a symbol rather than expanding it):

$$Y(3_1) = a^{-2}\delta(z^2+1) + a^{-3}\delta^2 z - a^{-4}\delta z^2 - a^{-5}\delta z.$$

The degree of the framing variable a in the Kauffman polynomial gives rise to an upper bound on the tb, which was first proved by Rudolph [Rud90]. The version we use here is due to Tabachnikov [Tab97], though we use the notation of [Rut06]. More information on this bound and its history can be found in [Fer02].

Theorem 2.2.10 (Kauffman Bound [Tab97])**.** *If K is a Legendrian knot, then*

$$\overline{\text{tb}}\, K \leq -\deg_a Y(K),$$

where $deg_a P$ denotes the maximum exponent of a in the polynomial $P(a,z)$.

This bound is very useful: there is no method in general for computing the maximal tb of a knot, but it is a fairly straightforward matter to compute its Kauffman polynomial. The bound fails to be sharp in some known cases (e.g., [Fer02]) but there are also classes of knots for which it is known to be sharp. These include positive knots, most torus knots, 2-bridge links, and most 3-twist pretzel links. That the Kauffman bound is sharp for many 3-twist pretzel links is a theorem of Ng:

Theorem 2.2.11 ([Ng01])**.** *Suppose $p_1, p_2, p_3 > 0$. Then the Kauffman bound is sharp for the pretzel links $P(p_1,p_2,p_3)$, $P(-p_1,p_2,p_3)$, and $P(-p_1,-p_2,-p_3)$, and for $P(-p_1,-p_2,p_3)$ when $p_1 \geq p_2 \neq p_3+1$.*

We will make use of this theorem in our main result.

2.3 Lagrangian Cobordisms

2.3.1 Symplectic Geometry

Cobordisms are objects originating in smooth knot theory — in essence, surfaces having specific knots as boundary — which define a fundamental relation between types of knots. We define them here in the context of smooth knots, and then we will be able to define an analogous type of cobordism for Legendrian knots by requiring that it be everywhere tangent to a certain differential form.

Definition 2.3.1. Let K and K' be smooth knots. Define a **cobordism** as a smoothly embedded 2-manifold with boundary A in $\mathbb{R}^3 \times I$ such that the "bottom" ($t = 0$, where t is the coordinate in I) edge of A is K and the "top" ($t = 1$) edge of A is K'. That is,

$$A \cap \left(\mathbb{R}^3 \times \{0\}\right) = K \times \{0\}$$
$$A \cap \left(\mathbb{R}^3 \times \{1\}\right) = K' \times \{1\}$$

We extend this definition by defining a **symplectic manifold** in much the same way we defined a contact manifold: as a manifold equipped with an appropriate differential form. Recall that a 2-form ω on a manifold X is **closed** if $d\omega = 0$ (i.e., its exterior derivative vanishes) and **nondegenerate** if for any $\vec{v} \neq 0$, $\omega(\vec{v}, \vec{w}) \neq 0$ for all $\vec{w} \in \mathrm{T}_p X$.

Definition 2.3.2. Let X be a 4-dimensional smooth manifold and ω a 2-form on X that is closed and nondegenerate. Then the pair (X, ω) is said to be a **symplectic 4-manifold**.

Analogous to Legendrian curves in contact manifolds are Lagrangian surfaces in symplectic manifolds.

Definition 2.3.3. Let (X, ω) be a symplectic 4-manifold and L a smoothly embedded 2-manifold in X. We say L is **Lagrangian** if for all $p \in L$, ω vanishes on $\mathrm{T}_p L$.

Given a contact manifold $(Y, \ker \alpha)$ there is a canonically associated symplectic manifold $(Y \times \mathbb{R}, d(e^t\alpha))$, where t is the coordinate on the attached copy of $\mathbb{R}$. This allows us to put Legendrian knots into symplectic manifolds as slices of Lagrangian submanifolds. Thus a Lagrangian cobordism is a Lagrangian surface that is also a cobordism, with a few extra conditions required.

Definition 2.3.4. Let K_- and K_+ be Legendrian links, and L a (orientable, exact) Lagrangian manifold in $(\mathbb{R}^3 \times \mathbb{R}, d(e^t\alpha))$. We say L is a **Lagrangian cobordism** from K_- to K_+ if for some

$T > 0$,

$$L \cap \left(\mathbb{R}^3 \times (-\infty, -T)\right) = K_- \times (-\infty, -T)$$
$$L \cap \left(\mathbb{R}^3 \times (T, \infty)\right) = K_+ \times (T, \infty)$$
$$L \cap \left(\mathbb{R}^3 \times [-T, T]\right) \text{ is compact}$$

and there exists a function $f : L \to \mathbb{R}$ such that $df = e^t \alpha|_{\mathrm{T}\,L}$ and f is constant for $t \geq T$ and $t \leq -T$.

We call a L a **concordance** if K_- and K_+ are knots and L has genus zero.

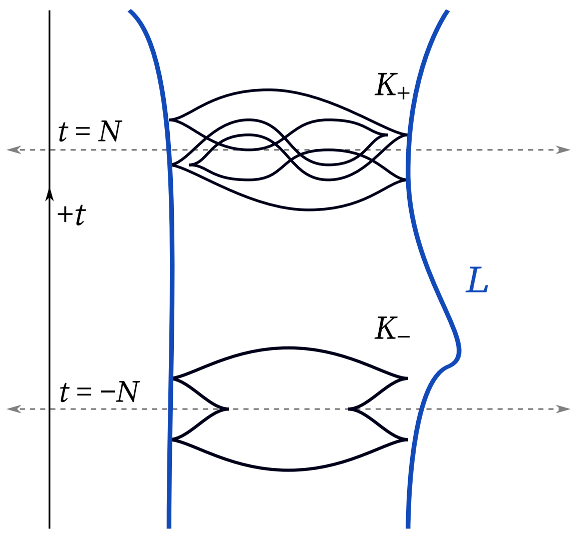


Figure 2.3.1. A visualization of a Lagrangian cobordism L as an infinite cylinder which looks like K_- for $t \leq -N$ and K_+ for $t \geq N$. Each slice along the t axis is an entire $\mathbb{R}^3$. In this diagram K_- is the unknot and K_+ is $m(6_1)$.

Smooth cobordisms have boundary, and the knots K_- and K_+ together form that boundary. Lagrangian cobordisms do not have boundary: rather, we think of the cobordism as “looking like” the knots below and above some t-value, respectively. But this difference is largely inconsequential. More importantly, we have added an asymmetric geometric condition on the cobordism, and the result is that the relation "there exists a Lagrangian cobordism from K_- to K_+" is not symmetric [Cha15].

We make the distinction between cobordisms with genus and concordances because the genus of a cobordism gives a lot of information about the two knots at its ends. In particular, if L is a cobordism from K_- to K_+,

$$\mathrm{r}(K_-) = \mathrm{r}(K_+) \text{ and } \mathrm{tb}(K_+) - \mathrm{tb}(K_-) = 2g(L).$$

This is a result of Chantraine [Cha10]. Thus a Lagrangian concordance can only exist between two knots with equal rotation number and tb.

2.3.2 Decomposable Cobordisms

In general, Lagrangian cobordisms are difficult to find. However, there are several conditions in which they are known to exist. Of interest is a certain set of moves which may be easily defined on front diagrams, the Reidemeister moves among them, such that a Langrangian cobordism exists between knots related by them.

Theorem 2.3.5 ([BST15]). *Suppose K_- and K_+ are Legendrian knots. If the front diagram of K_+ can be obtained from the front diagram of K_- by a finite sequence of handle moves (Figure 2.3.2) and Legendrian Reidemeister moves (Figure 2.2.4), then there exists a Lagrangian cobordism from K_- to K_+.*

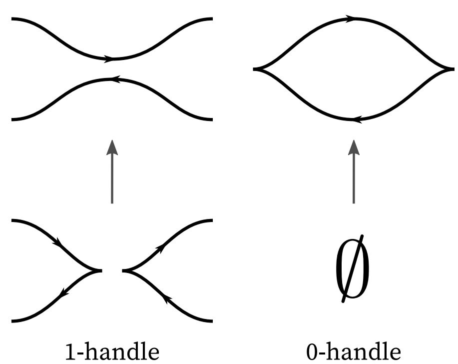


Figure 2.3.2. The handle moves. Note that these moves are one-directional, unlike the Reidemeister moves. The addition of a one-handle is sometimes referred to as a *pinch move*. The addition of a zero-handle corresponds to adding an unlinked unknot with maximal tb to the diagram.

We refer to a cobordism that is a result of a sequence of these moves as **decomposable**. We can represent these cobordisms visually via a sequence along increasing t of front diagrams of Legendrian links, each of which is obtained from the previous by a short (that is, easy to see) sequence of the decomposable moves. For example, Figure 2.3.3 shows a "movie" for a decomposable cobordism from a double-stabilized unknot to a Legendrian representative of 6_1.

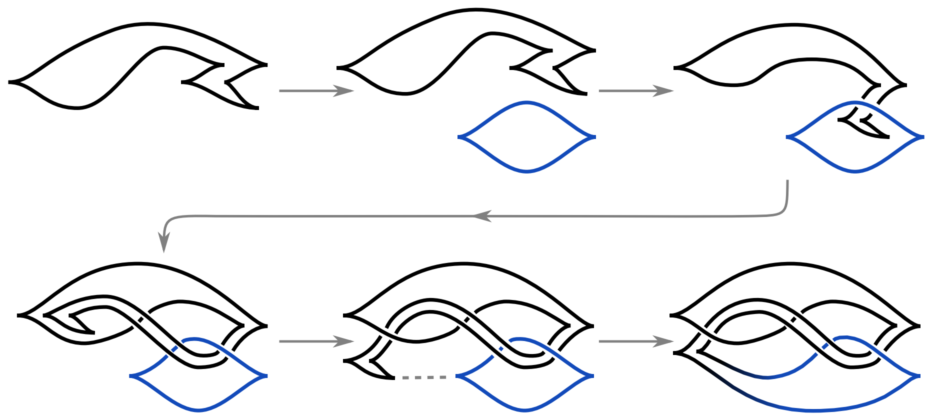


Figure 2.3.3. Constructing a Lagrangian cobordism from the unknot to a Legendrian 6_1.

It is known that not all Lagrangian cobordisms are decomposable. Moreover, there exist pairs of links K_-, K_+ such that a Lagrangian cobordism exists from K_- to K_+ but no such decomposable cobordism exists. Specifically, Lin showed [Lin16] that there exists a Lagrangian cobordism from the unknot to the empty set, despite the fact that no decomposable move can give $\emptyset$ from a nonempty Legendrian. It is not known whether this is the case when $K_+ \neq \emptyset$.

A number of other obstructions are known to exist. First, any Lagrangian cobordism yields a topological cobordism if the cylindrical ends are truncated, so any obstructions to the existence of topological cobordisms also obstructs the existence of Lagrangian cobordisms.

We also note the existence of obstructions to and from specific links. M. Gromov showed [Gro85] that if there exists a Lagrangian cobordism from $\emptyset$ to K, there does not exist a Lagrangian cobordism from K to $\emptyset$. Further, if there exists a Lagrangian concordance from K to the unknot U, then K is itself an unknot, a result of Cornwell, Ng, and Sivek [CNS16]. Other obstructions are derived from knot Floer homology [BLW19], normal rulings [CNS16], and the Chekanov-Eliashberg DGA [Pan17]. Though these obstructions provide useful information, the search for new obstructions is ongoing.

3

The $P(3, -3, n)$ family

3.1 Motivation

There is no general method of finding Lagrangian cobordisms, and there are very few sufficient conditions for their existence. A natural way of refining this question is to restrict either K_- or K_+. In particular, in this thesis we examine under what conditions there exists a decomposable cobordism from the unknot U to some Legendrian knot K. This choice is not arbitrary: it is unclear whether any of the known obstructions give information about the existence of such cobordisms when $\overline{\text{tb}}\, K \leq -1$.

A good candidate for this search is the class of knots called ribbon knots, which we define here.

Definition 3.1.1. A knot K is said to be **ribbon** if K bounds a smoothly embedded disk $d : D \to \mathbb{R}^3$ with only ribbon singularities. That is, every region of self-intersection of d is an arc $A \in \mathbb{R}^3$ such that the preimage of A, $d^{-1}(A)$, consists of two arcs in D of which one is within the interior of D and the other has its endpoints on the boundary of D.

This definition is more clear alongside a ribbon diagram, Figure 3.1.1.

Ribbon knots are a natural class to try to find Lagrangian cobordisms to, as topologically there always exist smooth cobordisms between the unknot and any ribbon knot (recall that the existence of a Lagrangian cobordism from K_- to K_+ implies the existence of a smooth cobordism

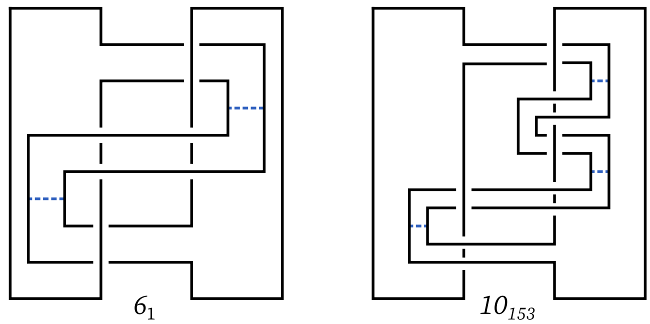


Figure 3.1.1. Diagrams for some ribbon knots, with the arcs of self-intersection marked with dashed lines. The forms of these ribbons come from [Kaw96].

between the two). In the case of Legendrian knots, Leverson and Etnyre have shown [EL] that that ribbon knots admit decomposable Lagrangian cobordisms from sufficiently stabilized unknots. For a ribbon knot with one band, we can start from a twisted Legendrian unknot, add a second unknot with a 0-handle, and then use Legendrian isotopy to "pass" the tip of the first unknot through the two loops however desired, before finally using a 1-handle to join the ribbon tip to the second unknot, thus closing the knot.

The cobordism created by these moves is necessarily a concordance: Each 0-handle adds a separate component to the link, and the 1-handle only adds genus if between two points on the same component. Thus to construct such a cobordism to a Legendrian ribbon knot K, we have to start with a Legendrian unknot U with $\operatorname{tb} U = \operatorname{tb} K$. Given the topological knot type of K, both U and K must certainly have $\operatorname{tb} \leq \overline{\operatorname{tb}}\, K$. It is an open question when this can be achieved with equality: that is, when a cobordism can be constructed from a stabilized unknot to a maximal-tb Legendrian ribbon knot.

3.2 Constructions of Cobordisms

The main result of this project, Theorem A, is the demonstration of an infinite family of knots, each of which has a maximal-tb Legendrian representative admitting such a Lagrangian concordance from a Legendrian unknot. We restate the theorem here.

Theorem A. *Let $P_n = P(3,-3,n)$ (the knot shown in Figure 3.2.1), for n an integer. Then there exists a Legendrian representative K_n of P_n, and a Legendrian unknot U_n with $\operatorname{tb} K_n = \operatorname{tb} U_n = \overline{\operatorname{tb}}\, P_n$, such that there is a decomposable Lagrangian concordance from U_n to K_n.*

Figure 3.2.1 shows a diagram of a knot in this family. For example, $P(3,-3,0) = 3_1 \# m(3_1)$. Further known examples are $P_1 = 6_1$; $P_2 = 8_{20}$; $P_3 = 9_{46}$; $P_4 = 10_{140}$, and $P_5 = 11n_{139}$. We also note that $P_{-n} = m(P_n)$ [Kaw96].

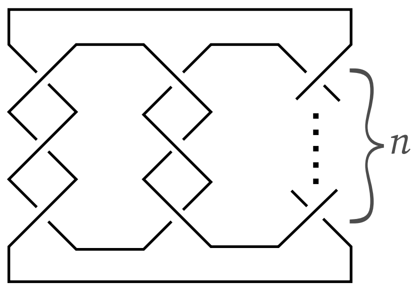


Figure 3.2.1. The pretzel knot $P(3,-3,n)$. On the right, there are n left half-twists if n is positive, and $|n|$ *right* half-twists if n is negative.

We break the proof of this theorem into the following two lemmas.

Lemma 3.2.1.

$$\overline{\operatorname{tb}}\, P_n = \min\{-n-4, -1\}.$$

Lemma 3.2.2. *For each P_n, there exists a Legendrian representative K_n of P_n and a stabilized Legendrian unknot U_n such that*

$$\operatorname{tb} K_n = \operatorname{tb} U_n = \min\{-n-4, -1\},$$

and there exists a decomposable Lagrangian concordance from U_n to K_n.

The proof of Theorem A follows trivially from Lemmas 3.2.1 and 3.2.2, which we prove here.

Proof of Lemma 3.2.1. We first compute the degree of a in the Kauffman polynomial of P_n; as this allows us to obtain a bound on $\overline{\operatorname{tb}}\, P_n$ by Theorem 2.2.10. We use Lu and Zhong's method [LZ08] for computing the Kauffman polynomial of a pretzel knot, and find that $-\deg_a Y(P_n)$ is given by $\min\{-n-4,-1\}$. We implemented this computation in Mathematica; for details see Appendix A.

By Theorem 2.2.11 (Ng), the Kauffman bound is sharp for the pretzel link $P(3,-3,n)$ except when $n=0$ or $n=\pm 2$. That is, $\overline{\text{tb}}\,P_n = \min\{-n-4,-1\}$ for $n \neq 0$ and $n \neq \pm 2$. We will deal with these cases presently.

In the case where $n=0$, $P_0 = 3_1 \# m(3_1)$. The Thurston-Bennequin number of a connected sum is well known [Tor03], [EH03]. In particular,

$$\overline{\text{tb}}\,K_1 \# K_2 = \overline{\text{tb}}\,K_1 + \overline{\text{tb}}\,K_2 + 1.$$

Thus $\overline{\text{tb}}\,P_0 = -6+1+1 = -4$ as desired [CN15].

In the case where $n = \pm 2$, we have $P_2 = 8_{20}$ and $P_{-2} = m(8_{20})$, and therefore $\overline{\text{tb}}\,P_2 = -6 = -2-4$ and $\overline{\text{tb}}\,P_{-2} = -2 = 2-4$ as desired [CN15].

$\square$

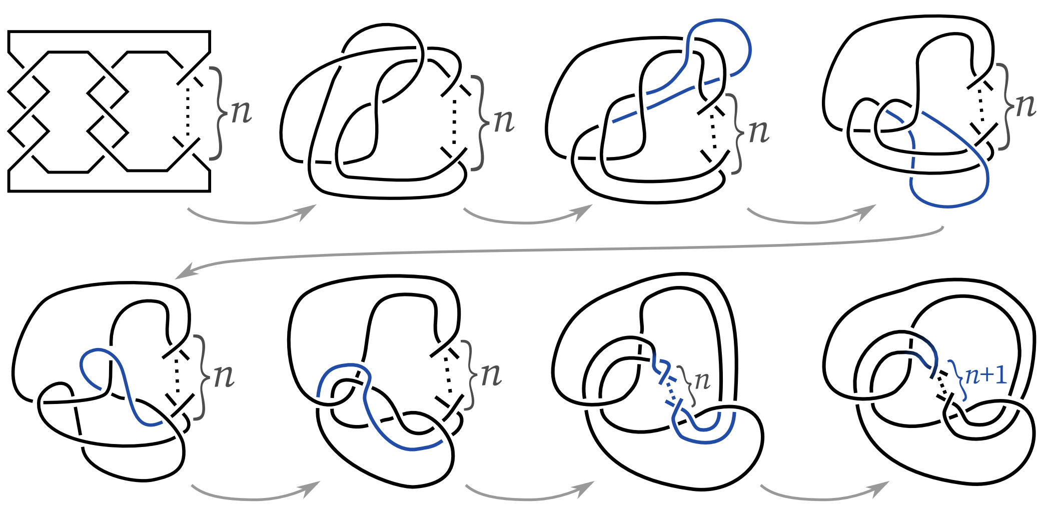


Figure 3.2.2. Smooth isotopy from the pretzel diagram for $P(3,-3,n)$ to a twisted ribbon.

Proof of Lemma 3.2.2. Depending on the value of n, there are three cases.

In each, we first construct a suitable unknot U_n with the desired tb, using stabilizations or RI moves to add a total of $n-1$ half-twists, and we show the form of the desired Legendrian representative K_n. Figure 3.2.2 verifies that K_n is in fact smoothly isotopic to $P(3,-3,n)$.

We then use the decomposable moves to describe a decomposable Lagrangian cobordism from U_n to K_n. A more detailed example of this construction is seen in Figure 2.3.3.

$n \geq -1$: The desired tb is $-n-4$. We start with an unknot stabilized $n+1$ times in order to add $n+1$ left half-twists, as seen on the left in Figure 3.2.3. We add a 0-handle, and then use the Legendrian RII and RIII moves to thread the twisted band through both loops, before connecting them with a 1-handle.

As this cobordism is in fact a concordance, we have $\operatorname{tb} U_n = \operatorname{tb} K_n$, so it suffices to check that $\operatorname{tb} U_n = -n-4$. The diagram for U_n has three right cusps and $n+1$ crossings, each of which is negative. Thus $\operatorname{tb} U_n = -(n+1) - 3 = -n-4$ as desired.

Note that in the case $n = -1$, the band is untwisted, as in Figure 2.3.3.

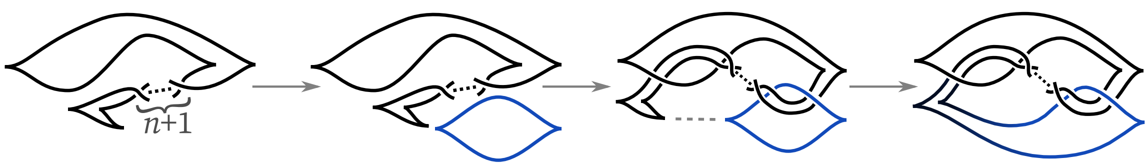


Figure 3.2.3. Cobordism movie for constructing K_n, where $n \geq 0$.

$n = -2$: Recall that $P_{-2} = m(8_{20})$, and its maximal tb is -2. In the twisted band we will have 1 *right* twist.

The diagram for the unknot U_{-2}, on the left, has 3 right cusps and a single crossing with positive sign. Thus $\operatorname{tb} U_{-2} = 1 - 3 = -2$.

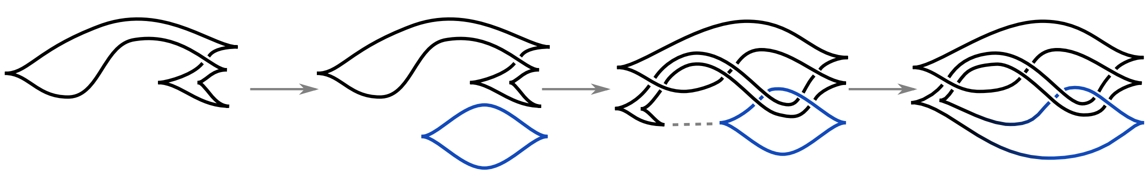


Figure 3.2.4. Construction of K_{-2}.

$n \leq -3$: The desired tb is -1. Using the RI move we can add as many right half-twists as we like (i.e., $|n| - 1$) to our U_n before we make the pinch move.

Note that in Figure 3.2.5 below, there are a total of $|n|-1$ right half-twists, but one of them "stays behind" when we pass the ribbon tip through the loops.

The diagram for U_n has $|n|$ right cusps and $|n|-1$ positive crossings, so $\operatorname{tb} U_n = (|n|-1) - |n| = -1$ as desired.

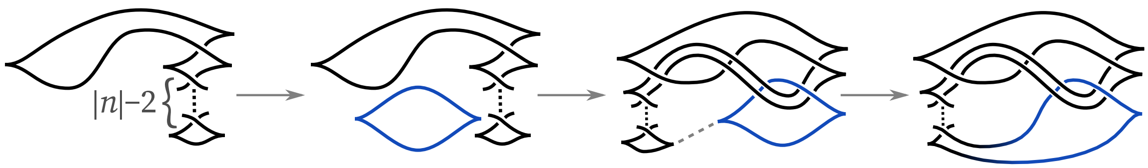


Figure 3.2.5. Construction of K_n, where $n \leq -3$.

□

For a few specific n, Theorem A suffices to show that all Legendrian representatives of P_n admit Lagrangian cobordisms from the unknot.

Corollary 3.2.3. *All Legendrian representatives of P_1, P_3, P_{-3}, and all but (possibly) one of P_{-1} admit decomposable Lagrangian concordances from stabilized Legendrian unknots.*

Proof. Recall that if there exists a Lagrangian cobordism from K_- to K_+, then there exists a Lagrangian cobordism from $S_-(K_-)$ to $S_-(K_+)$ and from $S_+(K_-)$ to $S_+(K_+)$, where S_- and S_+ denote positive and negative stabilization respectively. Thus it suffices to check that all Legendrian representatives of P_n are either K_n or obtained from K_n by repeated stabilization.

For P_1 and $P_{\pm 3}$, all stabilized Legendrian representatives are known to be stabilizations of a single representative with maximal tb [CN15]. This representative is necessarily K_n.

In the case of P_{-1}, all stabilized representatives are stabilizations of *either of two* maximal-tb representatives. That is, from *either* maximal representative, all non-maximal representatives may be obtained by stabilization. Thus one of these representatives is K_{-1}, but it is unclear whether the other admits a Lagrangian concordance from the unknot.

□

4
Future Work

The question of interest is whether there exist Lagrangian cobordisms from the unknot to all Legendrian ribbon knots. It suffices in this case to show that this is true for all nonstabilized representatives. The representatives with maximal tb are a subset of the nonstabilized representatives. In this paper we showed that there exist Lagrangian cobordisms from the unknot to *at least one* representative with maximal tb for a large family of ribbon pretzel knots, but our solution does not give information about the answer to this question in general.

Our solution made use of a parameterization of ribbon pretzel knots, and we computed the maximal tb using an explicit formula for the Kauffman bound. There is another class of ribbon knots for which this approach might work — 2-bridge knots. The Kauffman bound is known to be sharp for these knots [Ng01], and an algorithm exists for computing it [LZ06], much like the algorithm we used for pretzel knots. Moreover, many of the prime knots for which we have failed to explicitly find maximal-tb cobordisms from the unknot are 2-bridge: for example, 8_8 and 8_9. In fact, there are several known families of 2-bridge ribbon knots, and it is conjectured that these families include all 2-bridge ribbon knots [Lam06].

Yet in the case of the pretzel knots $P(3, -3, n)$, our proof relied on an explicit stabilized unknot for each n such that each cobordism could be constructed using the exact same sequence of moves.

If a similar method could be used to prove the existence of such cobordisms for 2-bridge knots, the first step would be to find a single explicit example — which so far we have failed to do.

However, finding a positive answer in general is much more difficult due to the wide variety of ways that the equivalence classes of Legendrian knots can be organized. For example, Menasco and LaFountain give an example of a knot having *nonstabilized* representatives with the same tb and rotation number as stabilized representatives [LM08]. There exist knots with infinitely many pairs of distinct Legendrian representatives (K, K') such that $\operatorname{tb} K = \operatorname{tb} K'$ and $\operatorname{r} K = \operatorname{r} K'$ (in particular, this is true of P_{-4}). More examples of the strangeness of Legendrian equivalence classes may be readily found in [CN15]. Yet the takeaway is that the nonstabilized Legendrian representatives of a topological knot type can have little in common, making it difficult to to prove anything about them.

Appendix A

Mathematica Code for Kauffman Bound Computation

This code is also available as a Mathematica notebook at [Wal21]. Throughout, the pretzel knot $P(a, b, c)$ is encoded by the list `{a, b, c}`.

We obtain Lu and Zhong's version of the Dubrovnik Polynomial using the algorithm from [LZ08]. After marking some shorthand and writing out the base change matrix M, we directly implement Lu and Zhong's formula for the Dubrovnik polynomial in the function `LuZhong[q]`.

```
ai  := 1/a;
si  := 1/s;
d   := (a - ai)/(s - si) + 1;
di  := 1/d;
M    = {
     {   (si - di*si - di*ai) / (s + si),
         (-si - di*s + di*ai) / (s + si),
         di},
     {   (-s - di*si - di*ai) / (s + si),
         (s - di*s + di*ai)   / (s + si),
         di},
     {   (si*d + a - di*si - di*ai) / (s + si),
         (s*d - a - di*s + di*ai)   / (s + si),
         di}
     };

LuZhong[q_] :=
     d * M[[3]] . Table[
         Times @@ (M[[j]] . {s, -si, ai}^#1 &) /@ q, {j, 3}
     ]
```

Now we compute the "standard" Dubrovnik polynomial, as Lu and Zhong's version has $s - s^{-1}$ instead of z throughout. To do this we need to rewrite the equation in the variables a, z where

$z = s - s^{-1}$. This doesn't affect the degree of a, but it makes it easier to check that the coefficients of the relevant powers of a are not identically zero.

```
Dubrovnik[q_] := LuZhong[q] /. Solve[z == s - si, s][[1]] // Simplify
```

Now we will normalize the Dubrovnik polynomial to get the Kauffman Y polynomial. But first, we need the writhe. For the family we are interested in, the writhe is easy to compute.

```
Writhe[{3, -3, n_}] := -n
Kauffman[q_] := Simplify[Dubrovnik[q] * ai^Writhe[q]]
```

As we know, we can use the Kauffman polynomial to get an upper bound on the maximal Thurston-Bennequin number. Using Rutherford's version [Rut06] of Tabachnikov's bound [Tab97], we have

```
TBBound[q_] := (-Exponent[Kauffman[q], a, Max]) // Simplify
```

Finally, we can allow Mathematica to crunch the terms:

```
$Assumptions = {n ∈ Integers};
TBBound[{3, -3, n}]
```

and the result is `-Max[1, 4 + n]` which is of course $\min\{-1, -4-n\}$ as desired.

Is there any n such that coefficient of a^{n+4} or of a vanishes in the Kauffman polynomial of $P(3,-3,n)$? It is simple to check that this is not the case, verifying the above expression for $\deg_a$.

```
P = Kauffman[{3, -3, n}] // Expand // PowerExpand // Apart // Expand;
coeff1 = Coefficient[P, a, 1] // FullSimplify
coeffn4 = Coefficient[P, a, n+4] // FullSimplify;
Solve[coeff4n == 0, n]
```

On the one hand, `coeff1 == 1/z`, which is certainly nonzero. Moreover, `coeff4n == 0` only in the following condition, which is not satisfied for any integer constant n:

$$n = \frac{\ln\left((2+16z^2+20z^4+8z^6+z^8+4z\sqrt{4+z^2}+10z^3\sqrt{4+z^2}+6z^5\sqrt{4+z^2}+z^7\sqrt{4+z^2})/2\right)}{\ln 2 - \ln\left(z(-z+\sqrt{4+z^2})-2\right)}$$

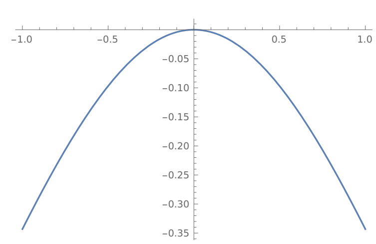


Figure A.0.1. A plot of the real part of the above expression, showing that it is not satisfied by any integral constant n.

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