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Chase-Escape on Sparse Networks

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Abstract

Chase-escape is a competitive growth process in which prey spread through an environment while being chased and consumed by predators. The environment is typically modeled by a graph—such as a lattice, tree, or clique—and the species by particles competing to occupy sites. It is arguably more natural to study these dynamics in heterogeneous environments. To this end, we consider chase-escape on a canonical sparse random graph called the Erdős-Rényi graph. We show that if prey spreads too slowly then both species quickly die out. On the other hand, if prey spreads fast enough, then coexistence occurs. Concrete bounds are given for the location of the threshold. Simulation evidence is provided.

Contents

Dedication

For Kiwi, my greatest confidant.

Acknowledgments

Only possible because of:

Matthew Junge for the guidance and patience it took to realize this project.

Stan for his unwavering love and support.

My wonderful and caring friends, those at Bard, in my hometown, and from my childhood.

My parents for their faith in me, especially when I lacked faith in myself. Mom and dad, thank you. You have shaped the person I am today, and I am so proud. Words aren't enough.

1
Introduction

The study of interacting particle systems is relatively young with work first being published in 1970. Five years later, four classic models and general axioms of existence and uniqueness had been established. Liggett's book, *Interacting Particle Systems*, published in 1985 served to ground the subject as a well-developed academic discipline [16]. Since then, research in the field has substantiated a considerable amount of literature in the study of applied probability.

As one of the most broadly-ranging fields of probability, the interacting particle system appears as a model in a number of disciplines including physics, biology, computer science, sociology, economics, and ecology. The model is structured with particles interacting in an environment in which the nature of particle motion or behavior defines the system. The model lends itself to different disciplines as the particles and environment come to represent real world populations and spaces. Although this definition is simple, it is surprisingly challenging to make nontrivial, rigorous claims about these models. This leaves some basic questions unanswered while others have been solved utilizing complicated techniques.

Despite these difficulties, the simplicity of the interacting particle system is what allows for so many fields to effectively use it in understanding natural phenomena. In fact, it is generally understood that complicating models in order to make situations more realistic does not change much in the overall behavior of the model. It follows that simple models are able to encapsulate all the necessary information about a system despite nuances that may seem necessary to include. In this way, interacting particle systems are often used as 'toy models,' a deliberately simplistic model where details are removed to concisely explain a mechanism or structure.

Where specifics of real world details differentiate systems into separate academic fields, a single toy model can be used to understand and study general behavior and long run events. Researchers sort interacting particle systems into 'universality classes' where systems with roughly the same behavior constitute classes. Often classes are made up of structures taken from multiple disciplines. Once a claim has been proven about a class, that claim holds for all systems in the class and can be used as fact in exploring specific structures in the context of their own academic field.

Interacting particle systems often model stochastic phenomena. Stochastic refers to random processes, so in the context of interacting particles, stochastic details the nature of particle behavior. When particles interact randomly, exogenous questions come to light like the role of the environment in evaluating particle fitness. Inherently these systems describe outcomes that are nondeterministic, which is in line with many observable natural phenomena we aim to study. Nondeterministic systems highlight the process rather than the outcome and emphasize the relationship between particles and their environment with respect to time.

Stochastic growth models are interpreted as systems where particles strive to occupy space. Growth models are a specific type of interacting particle system where a critical aspect of particle behavior is the creation of new particles. One such system is known as the **contact process** [16], which behaves such that if a site on the graph is occupied, meaning the site hosts a particle rather than being empty, then adjacent unoccupied sites probabilistically become populated with new particles. This concept is used in a biological setting modeling the spread of an infection, where the occupation of a site represents an infected party. With or without the notion of particle death, the contact process dictates that the number of particles occupying sites grows with time. Attention is then focused on the specific behavior that determines *how* the number of particles grows.

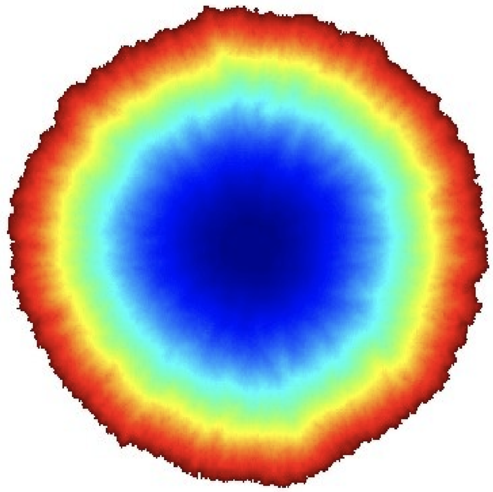


Figure 1.0.1. The Richardson Growth Model on $\mathbb{Z}^2$ [20]

A classical one-type stochastic growth model is known as **the Richardson growth model** introduced in 1971. Informally, the Richardson growth model begins with an occupied site, and adjacent sites become occupied at a rate proportional to the number of nearest neighbors that are occupied. A simulation of this growth on the square integer lattice is shown in Figure 1.0.1. In this image color represents the time at which a site became occupied, blue being the initial particles and red being the most recent ones. We see in this image some of the most interesting elements of Richardson growth like the rich randomness resulting in a coarse boundary and the growth tendency towards a limiting shape. Richardson proved with the introduction of the model that particles converge to a deterministic region. In other words, after conducting one-type Richardson growth, the occupied sites on the lattice look like a shape. He did not show what exactly that shape is, though he conjectured intuitively that it might be a circle or disk. Modern computing power allows for more extensive simulation studies that suggest the process grows faster along the axes, countering this conjecture and suggesting a slightly elliptical limiting shape. Surprisingly, though this growth process is about as straightforward as can be, not much beyond simple existence theorems have been rigorously proven, and our understanding of this model has not improved much since the mid 1980s.

A natural extension of one-type Richardson growth is the two-type model. The two species are represented by red and blue particles where each type aims to occupy space in the environment. This constitutes a **competitive** system. Both populations spread independently, and territory is occupied on a first-come, first-served basis; once a site on the graph is colored, it remains that color forever. With this, a new parameter is brought into consideration: speed. Since the rate of red and blue speeds are relative, we fix blue particles spread and control red particle spread with a parameter. We see a large scale image of competitive growth in Figure 1.0.2, which preserves the questions of a limiting shape and coarse boundary, as well as introduces new questions of species coexistence and fitness. In this model, simple existence theorems address these new questions, but, again, no rigorous claims are made about the competitive growth model [11].

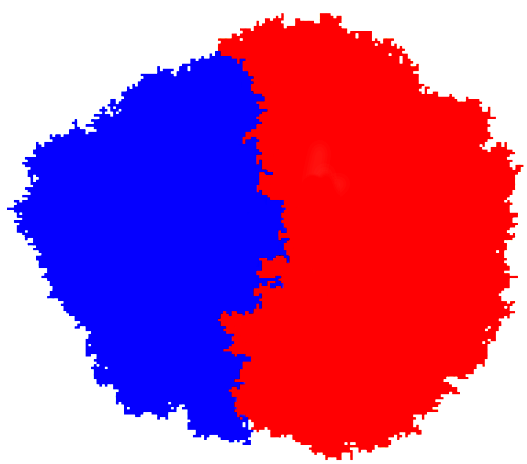


Figure 1.0.2. Competitive Richardson Growth on $\mathbb{Z}^2$

Figure 1.0.3 shows other types of competitive growth models where populations do not follow Richardson growth; the images correspond to the rules of particle growth given by their respective systems.

Chase-escape is a type of competitive growth where the expansion of one species both depends on and hinders the expansion of the other. In chase-escape, red particles behave in line with classic Richardson growth while blue particles ‘chase’ red, only spreading to red occupied adjacent sites. Detailed in the following chapter, chase-escape is a model for several natural

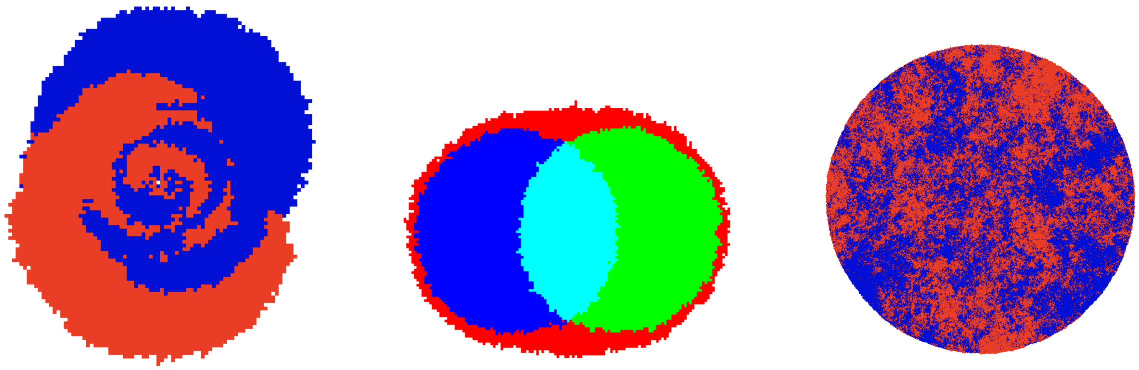


Figure 1.0.3. Competitive Growth Models

and social phenomena. Predator-prey and host-parasite dynamics are easy to visualize: take the graph to be a forest where chase-escape describes a parasite spreading through the roots of the trees. An application that is useful in contextualizing the results of this paper is the rumor scotching model, where red particles represent a rumor spreading through a social network and blue particles represent the suppression of that rumor.

Figure 1.0.4 shows chase-escape on the square lattice. The randomness of particle motion maintains the coarse boundary present in the one-type system and preserves questions of a limiting shape. Take note of the small uncolored patches that lie within largely colored sections inherent because of the asymmetric nature of red and blue growth. This asymmetry, that red

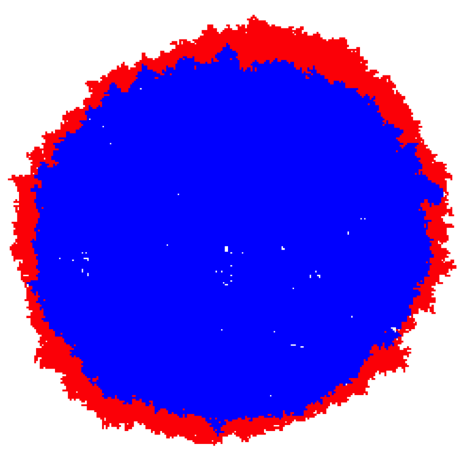


Figure 1.0.4. chase-escape on $\mathbb{Z}^2$

occupies uncolored sites while blue occupies red sites, describes the relationship between the two populations and sets the stage for research in the subject. With this relationship in mind, we look to see how varying relative growth speed and environment representation affects overall population dynamics.

2
Chase-escape

2.1 Definiton

Chase-escape is a stochastic growth process in which competing species spread through an environment. The environment is typically modeled by a graph, a collection of points with edges connecting some subset of them. The species are typically red and blue colored particles occupying graph vertices. Sites can be in one of three states $\{r, b, w\}$, where r indicates red occupation, b indicates blue occupation, and w indicates no occupation. Chase-escape's initial configuration typically has a single r site with an adjacent b site and all other sites w. Red particles spread to uncolored adjacent sites following a Poisson process with rate-λ exponential distribution; blue particles spread to red occupied adjacent sites at rate-1. The rate is *not* a time duration, it is a measure of how often color transitions occur. For more details see Section 2.2.

Transitions of blue particles to red occupied sites substantiate a predatory dynamic where red particles' transition to unoccupied sites can be seen as prey escape. We denote adjacent vertices in ordered pairs such that (r, w) transition to (r, r) and (b, r) transition to (b, b). Note (b, w) and (w, w) have no transitionary effect because of the predefined growth behavior.

The behavior of red and blue growth have offsetting factors: blue's survival depends on but hinders red's survival. In this way chase-escape dynamics give insight into how the environment influences the spread and survival of these species. The graph selected to model an environment plays a significant role in our understanding of the relationship between populations, especially in using chase-escape as a toy model. Mathematicians question how results vary across graph types and how to justify or account for this variation.

Chase-escape dynamics naturally lend to questions about the limiting behavior. Because blue can not expand without the presence of red particles, two natural states occur at infinity: blue completely consumes red and exists as a finite shape on a fraction of the graph, or red and blue continue growing with both populations present. The former is known as extinction (of red particle types) and the latter coexistence. The conditions that allow for either of these outcomes are interesting on any graph when imposing chase-escape dynamics.

2.2 Exponential Distribution

Chase-escape relies on the **exponential distribution**, denoted $\mathrm{Exp}(\lambda)$, which is a probability distribution that is used to describe time between events in a Poisson process. The **Poisson point process** is a counting method that is used to track events that occur at random. Counting processes are often used to model events known as arrivals. At the top of Figure 2.2.1 we see events or arrivals occurring in time with $N(t)$ being the number of events that occur from time $[0, t]$ beneath. The Poisson process highlights that the arrivals, $T_1, T_2, \ldots$ occur randomly rather than with regular, even spacing. To rigorously define the Poisson process, let $\lambda > 0$ be fixed. Then the counting process is a Poisson point process with rate or intensity λ if the distribution of the number of arrivals in an arbitrary interval depends only on the the length of the interval rather than the location of the interval (on the real line).

The probability density function for the exponential distribution is

$$f(x) = \frac{1}{\beta} e^{-(x-\mu)/\beta} \qquad x \geq \mu \ ; \ \beta > 0$$

where μ is the location parameter (where the distribution is centered on the x-axis) and β is the scale parameter (the spread or stretch of the distribution), see Figure 2.2.2. The scale parameter is often referred to as λ where $\lambda = \frac{1}{\beta}$. The case where $\mu = 0$ and $\beta = 1$ is known as the **standard**

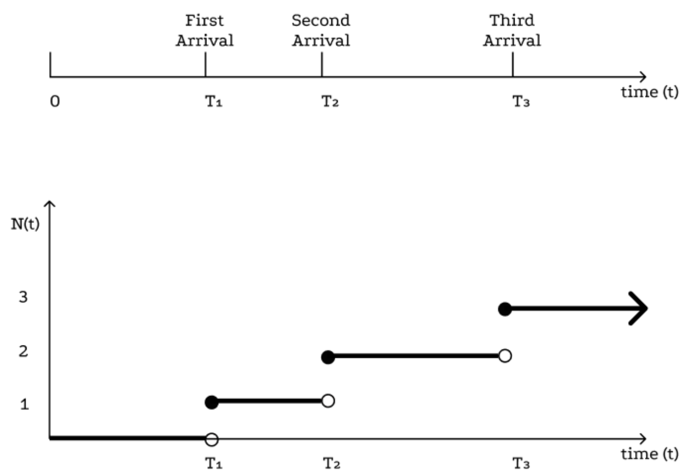


Figure 2.2.1. A possible realization and the corresponding sample path of a counting process

exponential distribution whose density function is

$$f(x) = e^{-x} \qquad x \geq 0.$$

Exponential distributions are important in building continuous time **Markov chains**, a model where the information needed to predict future states is encoded in the model's present state. In this way, exponentials are often used in answering questions regarding passage times like "how long do I have before my car's transmission dies?" or "how much time will it take for the next earthquake to hit California?". If you consider the answers to these questions to be unknown, then the time elapsed between states is a **random variable**, say X, such that if $X = \text{Exp}(\lambda)$, then the probability that the elapsed time is less than some real value x is given by $P(X < x) = e^{-x}$ according to the density function of the standard exponential distribution. Following our example, if we want to know the probability that our car's transmission dies in the next 30 days, then $P(X < 30) = e^{-30}$ in the standard case.

The exponential distribution has a **memoryless property**. This property means the distribution "forgets" what comes before it, such that knowing the time observed since the last event neither increases nor decreases the probability of the next event happening. For example if we observe hurricane in the Atlantic, the probability of the next hurricane occurring in a week, a

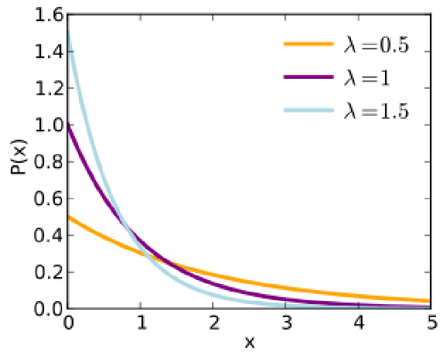


Figure 2.2.2. Exponential distribution density function

month, a year, or ten years are all equal. The memoryless property is unique to the exponential distribution and is critical in considering how long it takes sites to become colored in the context of chase-escape dynamics.

2.3 Phase Transition: Infinite Space

Let $\mathcal{R} = \{\text{all sites that have been red occupied}\}$ and $\mathcal{B} = \{\text{all sites that have been blue occupied}\}$. Let the events that blue and red occupy sites on the graph at infinity be:

$$A = \{|\mathcal{R}| = \infty\} \quad \text{and} \quad B = \{|\mathcal{B}| = \infty\}. \tag{2.3.1}$$

We define the following **phases** for chase-escape:

$$\begin{gathered} \text{Coexistence: } P_\lambda(B) > 0 \\ \text{Extinction: } P_\lambda(A) = 0. \end{gathered} \tag{2.3.2}$$

The subscript λ refers to the rate-λ spread of red particles relative to the rate-1 spread of blue particles. It follows from the definition of growth in chase-escape that $B \subseteq A$, which allows for concise definitions of coexistence and extinction.

A natural question to ask on a graph is how the red speed λ affects resulting phases. We define

$$\lambda_c^-(G) = \inf\{\lambda \colon P(A) > 0\} \qquad \text{and} \qquad \lambda_c^+(G) = \sup\{\lambda \colon P(A) = 0\} \tag{2.3.3}$$

to be critical speeds of λ where phase transitions occur on a graph. In words, λ_c^- is the slowest red expansion rate for which coexistence occurs, and λ_c^+ is the fastest red expansion rate for which extinction occurs. It is important to note that the question of whether $\lambda_c^- = \lambda_c^+$ remains open, although intuitively we expect the equality to hold. It is also noteworthy that the critical speed depends on the chosen graph, emphasizing how varying the graph impacts the particle dynamics. For any graph, we are interested in whether or not phases occur as well as how the process of growth transitions between them. In this paper, take

$$\lambda_c(G) \colon = \inf\{\lambda \colon P(A) > 0\} \tag{2.3.4}$$

to be the slowest red expansion rate for which coexistence occurs.

When initially approaching chase-escape on a graph, the question of criticality is an obvious one. Exploring criticality gives insight on phase transitions and accounts for much of the existing research. To look closely at phase transition, consider the path (the integer number line), a one dimensional, infinite, homogenous space.

Lemma 2.3.1 (**[DJT 2018]** [9])**.** $\lambda_c^-(\mathbb{Z}) = 1$ *and* $P_1(A) = 0$.

Proof. Since the process evolves independently in the positive and negative directions, it suffices to prove that red survives on $\{-1, 0, 1, 2, \ldots\}$ with a blue particle at 1 and a red particle at 0 initially. Let R_t be the number of sites ever occupied by the red particles up to time t and let B_t be the corresponding quantity for blue particles. Define $\tau_n = \inf\{t \colon R_t + B_t = n + 2\}$, and $\tau_n = \infty$ if there is no such t. Let $D - t$ be the distance between the rightmost red and blue sites at time t and $S_n = D_{\tau_n}$. Notice that (S_n) is a nearest-neighbor random walk, starting at $S_0 = 1$, where 0 is an absorbing state. Due to the independence of red and blue passage times, we have

$$p := P_\lambda(S_{n+1} = S_n + 1) = P(\mathrm{Exp}(\lambda) \leq \mathrm{Exp}(1)) = \frac{\lambda}{\lambda + 1}. \tag{2.3.5}$$

When $\lambda \leq 1$, the extinction of red is equivalent to the above p-biased random walk visiting zero. This is well known to be a.s. finite. □

This lemma shows that the critical speed λ_c that allows for coexistence is 1 on the path, when red and blue move with equal passage rates. This is done by showing that when red moves slower than 1, it is completely consumed, resulting in a finite colored component. Because it is unknown if $\lambda_c^- = \lambda_c^+$, this does *not* say that red will survive at rates faster than one. In other words, this proof shows that extinction occurs for $\lambda \leq 1$, but not that coexistence occurs for $\lambda > 1$, this requires additional proof.

2.4 Phase Transitions: Finite Space

In looking to apply chase-escape dynamics on finite space, we must reimagine our definitions of phase transitions. We maintain that $\mathcal{R} = \{\text{all sites that have been red occupied}\}$ and $\mathcal{B} = \{\text{all sites that have been blue occupied}\}$. Given a sequence of graphs $\mathcal{G} = (G_n)_{n=1}^{\infty}$ with $|G_n| = n$, we say that **coexistence occurs** if there exists some $\delta > 0$ such that

$$\liminf_{n\to\infty} P(|\mathcal{R}(G_n)| > \delta_n) > 0;$$

otherwise, **extinction** occurs. That is to say coexistence occurs when there is a non-zero probability that the number of red sites on a given graph occupies a positive fraction of the graph's total sites. We say that **strong extinction** occurs if

$$\limsup_{n\to\infty} \mathrm{E}\,|\mathcal{R}(G_n)| < \infty.$$

This means that as the graph gets larger the number of colored sites remains finite. Note strong extinction implies extinction.

3
Erdős-Rényi Graphs

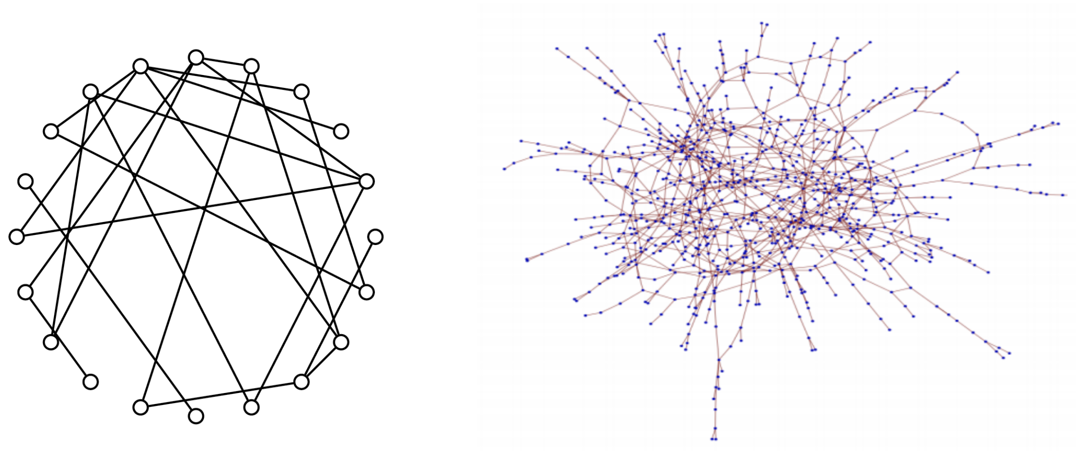
The Erdős-Rényi graph, denoted $G(n, p)$, is a random graph ideal for mathematically modeling social networks. Social networks are defined as a set of people with a pattern of ties between them. Families, friendships, and business relationships all constitute social networks that have been studied sociologically. The social science perspective concerns itself with networks through data-driven investigations of living or historical communal structures, where theoretical graph analysis is used to target centrality or the influence of specific events or actors. Since the dot-com boom with a rise of academic research about the internet, mathematicians have become interested in the features of networks, especially probabilistic and statistical network properties such as graph density and degree. The concept of the random network introduced by Erdős and Rényi in 1959 has become a cornerstone in discrete mathematics, especially in building models of society [18].

As a facet of social networks, rumor spreading is omnipresent. The way in which a rumor is spread follows the well defined behavior of chase-escape. Rumors are propagated by a "spreader" population (red particles) to "ignorants" (uncolored sites). Another population of "stiflers" (blue particles) are only active once they've been contacted by a spreader. This forms a competitive model where spreaders seek to increase the number of people in the network who know the rumor while stiflers seek to scotch it [6]. It is clear that rumor scotching falls into the category of chase-escape applications, though results from infinite or homogenous graphs are unreliable in considering society as the space or setting of this behavior. Moving to study chase-escape on the Erdős-Rényi graph is a natural transition in understanding rumor scotching as a spreading phenomena.

3.1 Construction and Definition

The Erdős-Rényi graph is simple to define. It is constructed by probabilistically connecting n vertices. That is to say for any pair of nodes, there is an edge connecting them with probability p. Both the local and the global picture are shown in Figure 3.1.1; for the local picture (3.1.1(a)), we have $n = 20$.

It follows from this definition of constructing the Erdős-Rényi network that the connectedness of the graph varies with the edge inclusion probability. If it is very unlikely of connecting nodes with edges, what results is many scattered nodes with some small clusters. On the other hand, if the probability of connecting nodes is high what results is something very dense, one large cluster with some smaller clusters and scattered nodes surrounding it. We take the edge inclusion probability $p = \mu/n$ where the value μ determines the connectedness or density of the network. Normalizing by the size of the graph n insures sparsity.



(a) $G(n, p)$ for small n (b) $G(n, p)$ for large n

Figure 3.1.1. Erdős-Rényi Random Graph

In addition to being the natural move in considering the rumor scotching application, the Erdős-Rényi graph is an intuitive transition from previous work studying chase-escape on the tree and the complete graph. This is largely because of the similarities between these graphs and Erdős-Rényi's general structure. The tree is very similar to the Erdős-Rényi's local picture, differing in finiteness and cyclicality. The complete graph is related being the result of an Erdős-Rényi network with edge inclusion certainty, where all n nodes are connected, each having degree n, detailed further in the following chapter.

Despite being the natural shift in the context of current literature and the social network application, the Erdős-Rényi graph is a complicated space. This complication is primarily due to the formation of cycles, detailed in Section 3.4. Though the similarities between the tree and Erdős-Rényi's local picture drive our research, the difficulties brought about by the presence of cycles causes a tree approximation of $G(n, p)$ to fail.

3.2 Branching Process

Branching processes became popular in probability literature in considering reproduction and population growth. The question was posed, "how many male children [on average] must each generation of a family have in order for the family name to continue in perpetuity?". This was answered by H. W. Watson and Francis Galton in a joint paper titled *On the Probability of the Extinction of Families* in 1875 [22]. The **Galton-Watson** branching process begins with a single person at time 0. After one unit of time, that person produces some number of children. If there are no children, the population is dead; if there are one or more children, each child produces some number of children in the next time step. We assume that the distribution of the number of children is the same in every generation and that the production of children is mutually independent. That is to say that the distribution is **independent and identically distributed (i.i.d)**. This distribution is called the **offspring distribution**.

Formally, take two discrete time parameters $t \geq 0$ and s. Let $\zeta_i^t \geq 0$, and t be i.i.d. integer-valued random variables. Let the sequence Z_t for $t \geq 0$ be defined by $Z_0 = 1$ (the initial person) and

$$Z_{t+1} = \begin{cases} \zeta_1^{t+1} + \cdots + \zeta_{Z_t}^{t+1} & \text{if } Z_t > 0 \\ 0 & \text{if } Z_t = 0. \end{cases} \tag{3.2.1}$$

Z_t is a Galton-Watson process where Z_t is the number of people in the t^{th} generation. Let μ be the average number of children each individual gives birth to.

Theorem 3.2.1 (**[DUR 2007]** [7]). *If $\mu < 1$ then $Z_t = 0$ for sufficiently large t.*

Proof. $E(Z_t/\mu^t) = E(Z_0) = 1$, so $E(Z_t) = \mu^t$. Now $Z_t \geq 1$ on $\{Z_t > 0\}$ so
$P(Z_t > 0) \leq E(Z_t; Z_t > 0) = E(Z_t) = \mu^t \to 0$ exponentially fast if $\mu < 1$. □

This result follows intuitively, that the species will die out if each individual gives birth on average to less than on child. This result holds for $\mu = 1$.

Theorem 3.2.2 (**[DUR 2007]** [7]). *If $\mu > 1$ then $P(Z_t > 0$ for all $t) > 0$.*

This shows that when $\mu > 1$, the limit of Z_t/μ^t is nonzero, and that the population continues to grow for all t. This result is critical in our exploration of a giant component in the next section.

3.3 Giant Component

The **giant component** is a prominent feature of the Erdős-Rényi graph. In the study of networks, a giant component, also referred to as the connected component containing 1, is a connected cluster that contains a sizable fraction of the entire graph's vertices. Erdős and Rényi discovered that there is a threshold for the appearance of the giant component.

To detail the model, let $V = \{1, 2, \ldots, n\}$ be the set of all vertices. For $1 \leq x < y \leq n$, let $\eta_{x,y} = 1$ with independent probability p and 0 otherwise such that $\eta_{x,y} = 1$ indicates an edge between x and y and $\eta_{x,y} = 0$ indicates that x and y are not connected by an edge. Note $\eta_{x,y} = \eta_{y,x}$. Take $p = \mu/n$ to be the probability of connecting two vertices with an edge. When $\mu < 1$ all components are small, the largest having only $O(\log n)$ vertices. In other words, there exists some finite constant C such that the largest component in the network is less than or equal to $C \log n$ as $n \to \infty$. Moreover, the expected number of children in generation k is μ^k, which converges to 0 exponentially fast. A similar argument is made for $\mu = 1$, showing that there is no giant component for $\mu \leq 1$ following [DUR 2007] (3.2.1).

The giant component emerges for all larger values. For $\mu > 1$, there is a positive probability that the branching process does not die out. When several sites have surviving branching processes, they combine to form the giant component. This occurs when two clusters that have grown to size $n^{1/2+\varepsilon}$ intersect, which happens with probability $1 - o(n^{-1})$. The error term, $o(n^{-1})$ gives high probability that all clusters will behave as expected. For $\mu > 1$ clusters that don't die out grow like $\mu^k = n$. Thus $k = \frac{\log n}{\log c}$. In physics literature, k is referred to as the "diameter" of the cluster, or the distance between two randomly chosen points on the giant component. Thus the giant component emerges for $\mu > 1$ following [DUR 2007] (3.2.2).

3.4 Cluster Growth

Using results from the branching process, we discuss the growth of the giant component. Following from the SIR epidemic interpretation of a growing cluster, let $S_0 = \{2, 3, \ldots, n\}, I_0 = \{1\}$, and $R_0 = \emptyset$ where S_t are susceptible, I_t are infected, and R_t are removed sites. These sets grow as follows:

$$\begin{aligned} R_{t+1} &= R_t \cup I_t, \\ I_{t+1} &= \{y \in S : \eta_{x,y} = 1 \text{ for some } x \in I_t\}, \\ S_{t+1} &= S_t - I_{t+1}. \end{aligned} \tag{3.4.1}$$

Redefine $\zeta^t_{x,y}, t \geq 1, 1 \leq x, y \leq n$ to be independent, and =1 with probability μ/n, and 0 otherwise. Let $Z_0 = 1, S^\mu_t = \{1, 2, \ldots, n\} - S_t$ and

$$Z_{t+1} = \sum_{z \in I_t, y \in S_t} \eta_{x,y} + \sum_{x \in I_t} \sum_{y \in S^c_t} \zeta^t_{x,y} + \sum_{x=n+1}^{n+Z_t-|I_t|} \sum_{y=1}^{n} \zeta^t_{x,y}. \tag{3.4.2}$$

The third term here represents children of individuals that are not in I_t. The second term, denoted B_t, represents extra births that occur as a result of $|S_t| < n$. The first term represents the number of births that occur that are not matched by an increase in cluster size. These births are called **collisions**, where

$$C_{t+1} = \sum_{x \in I_t, y \in S_t} \eta_{x,y} - |I_{t+1}| \geq 0.$$

Note $Z_t \geq |I_t|$. By definition, Z_t is a branching process with offspring distribution Binomial$(n, \mu/n)$ [7]. A binomial distribution is used to model events with two possible outcomes, in this case to have children or not. The parameter n indicates the number of independent people, and $p = \mu/n$ is the probability of each person having children. Our case, where the binomial distribution has $n \to \infty$ and $p = \mu/n \to 0$, we have a Poisson(μ) distribution. This is critical in applying chase-escape dynamics to $G(n, \mu/n)$.

3.5 Connection to Chase-Escape

The size of $G(n, \mu/n)$ is dominated by a Galton-Watson tree with Poi(μ) offspring distribution denoted $\mathbb{T}_\mu$. Because of results known on the tree, this indicates that $\lambda_c(G(n, \mu/n))$ follows $\lambda_c(\mathbb{T}_\mu)$, motivating our research. However, collisions cause cycles and leads to the failure of a tree approximation of $G(n, \mu/n)$, complicating the relationship between the two critical speeds.

4
Literature Review

The study of chase-escape in mathematical research typically considers the process on homogenous graphs such as the lattice, tree or clique. It is important to remember that chase-escape serves as a model to be used in applications where competitive growth systems appear as natural or social phenomena. Of course homogenous space is quite dissimilar from many environments where these phenomena occur. For some applications, like the growth of bacteria on a rock or the spread of disease through a forest, the complications of boundary constraints and sparsity have little bearing on the relevance of the model's results; however other applications, like the rumor scotching process, can not overlook the incompatibilities between infinite, homogenous space and the environment true to circumstance without consequence. As previously mentioned, the selection of the graph as a representation of real-life space significantly influences our understanding of population dynamics, for chase-escape behavior remains well defined throughout graph types.

Though chase-escape on infinite, homogenous space substantiates most of the literature, there are few rigorous results beyond statements of existence in these settings.

4.1 Lattice and Oriented Lattice

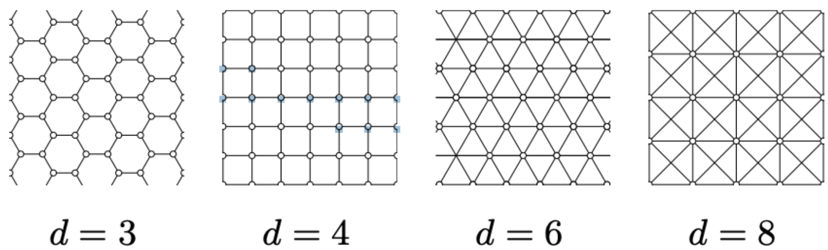
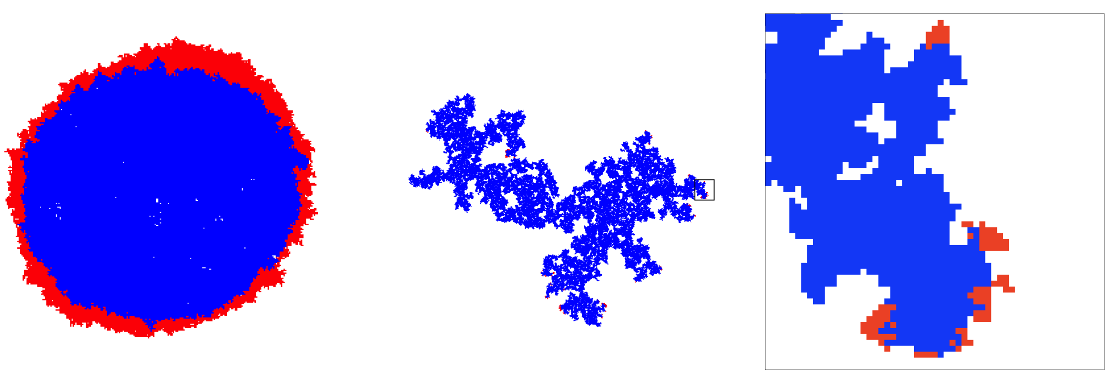


Figure 4.1.1. Types of two dimensional lattices

Consider the two dimensional lattice where the degree of the lattice, d, is the number of nearest neighbors for each vertex (4.1.1). It is natural to begin the study of chase-escape on a new environment by considering the critical speed where phase transition occurs. Simulations show that λ_c exists on the four degrees of lattices studied and that $\lambda_c \approx 1/2$ on $\mathbb{Z}^2$ (shown in Figure 4.1.2). The critical value λ_c decreases with the degree of the lattice. Authors note that this result is expected because red particles have more directions to escape from blue particles as the degree of the lattice increases [21]. Thus, with more directions to escape blue, red is able to move slower without being completely consumed. We see this relationship between degree of the graph and critical speed hold in research conducted on the d-ary tree.



(a) $\lambda_r = 1$ (b) $\lambda_r = \frac{1}{2}$ (c) Boundary of $\lambda_r = \frac{1}{2}$

Figure 4.1.2. Coexistence on $\mathbb{Z}^2$

A $d-$dimensional lattice is said to be oriented if a rule determines directionality to edges connecting arbitrary vertices. The $d-$dimensional oriented lattice is denoted $\overrightarrow{\mathbb{Z}^d}$. In the context of chase-escape, particles on the oriented lattice can only occupy neighboring sites along oriented edges. Authors prove on this graph the relationship we noted earlier, that red can move stochastically slower than blue and escape with positive probability given that d is large enough [9]. This result contributes to our understanding of λ_c, the critical speed for red's escape, highlighting that λ_c is a function of the graph and its dimension. Simulation and rigorous findings that red can move slower that blue on the lattice and oriented lattice indicate that these environments are conductive to red's survival. It is important to note the graph's influence on population survival dynamics, especially in considering the assignment of environment in applications of the chase-escape model.

4.2 Tree

A d-ary tree of degree $d \geq 2$, denoted $\mathbb{T}_d$, is a rooted, infinite, acyclic, connected graph such that every non-root node has exactly d children. In considering chase-escape on $\mathbb{T}_d$, we include the additional configuration that has the root red with one extra blue vertex attached to it (see Figure 4.2.1). At time $t = 0$, these are the only colored sites leaving the rest of the tree uncolored. Exploring phase transitions on this space, Kordzakhia provides the first favorable rigorous results for chase-escape. Kordzakhia shows $\lambda_c = (2d-1) + \sqrt{(2d-1)^2 - 1}$, a concise form for the pivotal speed of red that gives the value λ_c for all d-ary trees [14]. This allows with the property proven on the oriented lattice, that red can coexist with blue moving at significantly slower passage rates.

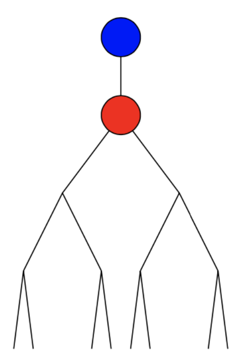


Figure 4.2.1. Initial configuration of chase-escape on a binary tree

Though Kordzakhia explicitly stated $\lambda_c(\mathbb{T}_d)$, he did not explain what happens when $\lambda = \lambda_c(\mathbb{T}_d)$. This was answered later in the more general setting of Galton-Watson trees with mean degree d. The Galton-Watson tree is not a homogenous space, though it is rooted, infinite, acyclic and connected. Bordenave shows that red does not survive on any Galton-Watson tree at criticality [3], meaning coexistence occurs only when $\lambda < \lambda_c$. With this, Bordenave shows that phase transition only depends on growth rate of the tree, generalizing Kordzakhia's result on homogenous trees to infinite space.

4.3 Complete Graph

The complete graph is a finite, homogenous space holding the property that every pair of vertices is connected by an edge. The complete graph with n vertices is denoted K_n (4.3.1). Inherently, all vertices on the complete graph have the same degree n, and this is an incredibly dense space. This graph is the first we see used to model a social network in the rumor scotching application

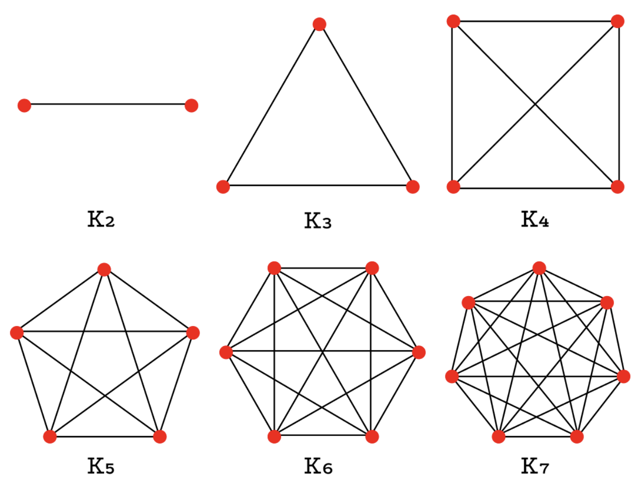


Figure 4.3.1. Complete Graphs

of chase-escape dynamics. This assumes that, in the process of spreading and scotching a rumor, every person in the network knows every other person. Though this is not entirely true of how social networks appear in the real world or in much of the literature, it is certainly more accurate than an infinite space.

On the complete graph we see that the rumor scotching process converges to the birth-and-assassination process and is related to predator-prey dynamics. The birth-and-assassination process was developed in 1989 by Aldous and Krebs [1] as a well-defined branching procedure in which particle death clocks do not start ticking until their parent dies. More precisely, Bordenave shows that the birth-and-assassination process is the scaling limit of the rumor scotching process in this space [2]. Informally, this says that rumor scotching behaves like birth-and-assassination when you zoom out on the complete graph.

The SIR (susceptible, infected, removed) model is an epidemiological model that explores how an infectious disease spreads using chase-escape dynamics. A variant of this model studied on the complete graph reveals three phase transitions: coexistence, extinction, and escape. Because of the removal or death feature of this variant, a colored site does not necessarily remain colored forever; instead, a site can 'die', becoming uninhabitable by either population. This introduces an escape phase, where blue particles can be completely closed in a set of these dead sites, allowing red to expand to uncolored sites without being chased. Kortchemski shows that phase transitions occur for $\lambda \in (0,1), \lambda = 1$, and $\lambda > 1$ and discusses the expected size of populations at criticality [15].

Results of the complete graph have been driven by explorations of chase-escape applications across disciplines. These findings emphasize how chase-escape as an interacting particle system is a toy model. These contexts (birth-and-assassination, rumor scotching, predator-prey, SIR) have little in common when considering the specifics of their attributes; however, the results found on the complete graph are similar across applications.

4.4 Gilbert Graph

The Gilbert graph, denoted $G(\eta_t, \delta_t)$, is also known as a random geometric graph or distance graph, introduced in 1961 [10]. It is constructed by placing points randomly throughout $\mathbb{R}^2$

and connecting nodes with an edge if their pairwise distance is within a predefined radius (see Figure 4.4.1). It differs from the Erdős-Rényi network in that $G(n, p)$ is purely combinatorial where the construction of the Gilbert graph depends on nodes' relative position. The Gilbert graph is effective in modeling many social networks, especially those demonstrating community structure.

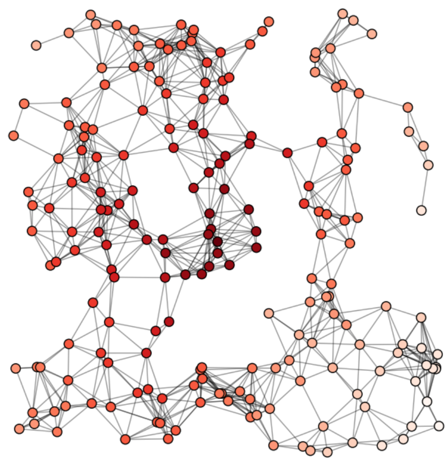


Figure 4.4.1. Portion of a Gilbert Graph

Research about chase-escape on the Gilbert graph is driven by a computer science application. Authors consider the Gilbert graph to model device-to-device networks where chase-escape defines a process of malware infection. In order to remove the malware, a population of white knights is deployed. Once a white knight treats a device, it too becomes a white knight. The units infected with malware are modeled by red particles and the white knights by blue particles, clearly falling in line with chase-escape dynamics.

On the Gilbert graph we see the favorable rigorous results about infinite phase transitions [12]. Authors define three phases: extinction, local survival, and global survival. These phases are presented with corresponding passage rates. The authors show that global survival occurs when there is positive probability of an infinite component with large enough infection rate, and that it can not occur if the graph is too sparse, the infection spreads too slowly, or if there are too many white knights present. The use of λ^+, the upper bound on the infection's critical speed, in forming a component with large infection rate informs the upper bound of the threshold presented in our results.

4.5 Dense Erdős-Rényi

Research done on the dense Erdős-Rényi network extends from one-type rumor spreading models. That is the pairwise process of rumor spreading between ignorants and spreaders. Authors modify the one-type system to classical chase-escape dynamics with the introduction of stifler population. Authors consider the proportion of uncolored sites on $G(n,1)$, the complete graph, giving closed forms for the asymptotic variance and other variance properties. These results can be approximated for $G(n,p)$ with p fixed, verified through Monte Carlo simulations, though they are not suitable for scale-free networks. In particular, the distribution of the final fraction of uncolored sites converges to the theoretical result for homogenous mixtures as the network becomes denser. Simulations reveal similar behavior for $G(n,p)$, showing that coexistence always occurs because the graph is so dense, though these results are non-rigorous [5].

5

Preliminaries

The previous chapter follows the prominent feats of work to date on chase-escape. This chapter serves to contextualize our results through specific points in the existing literature. This starts with Kordzakhia's rigorous study of chase-escape on the d-ary tree. As mentioned in Section 3.4, the growth of Erdős-Rényi's giant component follows a Poisson(μ) Galton-Watson tree. This informs our initial conjecture that $\lambda_c(G(n, \mu/n)) = \lambda_c(\mathbb{T}_\mu)$. However, the failure of the tree approximation of $G(n, \mu/n)$ causes a split into a critical threshold with lower and upper bounds proven.

5.1 Lower Bound

First we consider the method in developing the closed form of $\lambda_c(\mathbb{T})$ originally found in [KOR 2005] [14] as it is shown in a shorter form in [DJT 2018]:

Theorem 5.1.1 (Reproven [DJT 2018] [9]**).** *On the d-ary tree with* $d \geq 2$*,*

$$\lambda_c(\mathbb{T}) = 2d - 1 - 2\sqrt{d^2 - d} \sim \frac{1}{4d},$$

and $P_{\lambda_c}(A) = 0$.

Proof. We consider the initial configuration in which the root is red and a special vertex $\mathfrak{b}$ attached to the root is blue. First note that if $\lambda > 1$, then the distance between red and blue along an arbitrary path to ∞ is equivalent to chase escape on $\mathbb{Z}$. By Lemma 2.3.1 we have $P_\lambda(A) > 0$ in this case.

Now, suppose that $\lambda \leq 1$. Let R_n be the number of sites at distance n that are ever colored red and $R = \sum_{n=1}^{\infty} R_n$ be the total number of sites of colored red. Notice that red survives a.s. if and only if it occupies infinitely many sites. Thus, $P_\lambda(A) = P_\lambda(R = \infty)$. We show that $P_\lambda(R = \infty) = 0$ for λ small enough by proving $ER < \infty$.

For any vertex $v \in \mathbb{T}_d$, let $|v|$ denote its graph distance from the root. Let $A(v)$ be the event that v is ever colored red. Since the tree has no cycles, we have $P_\lambda(A(v)) = P_\lambda(A_n)$ for any $v \in \mathbb{T}_d$ with $|v| = n$, with A_n the event that red reaches a distance n on a fixed path as in Lemma 5.1.2. Linearity of expectation and the bound from Lemma 5.1.2 gives

$$ER_n = E \sum_{|v|=n} \mathbf{1}_{A(v)} = \sum_{|v|=n} P_\lambda(A(v)) \leq C_\lambda \frac{d^n (4p(1-p))^n}{n^{3/2}}. \tag{5.1.1}$$

Observe that $\lambda_c(d)$ is the smallest solution of

$$4p(1-p)d = \frac{4d\lambda}{(1+\lambda)^2} = 1.$$

It is straightforward to verify that $4dp(1-p) \leq 1$ for $\lambda \leq \lambda_c(d)$, and in this case ER_n is summable, and thus $ER < \infty$.

To prove that $P_\lambda(A) > 0$ for $\lambda > \lambda_c(d)$, observe that the lower bound in Lemma 5.1.2 ensures that for some fixed, large N we have $d^N P_\lambda(A_N) > 1$. Thus, the expected number of sites at distance N that are ever colored red is strictly greater than 1. When first occupied by red, the distance from each of these sites to the nearest blue particle is at least one. Since the tree has no cycles, the survival probability of chase-escape is monotonic on a tree; both respect to λ and the initial distance blue starts from red. This means that moving the chasing blue particles to distance 1 from each of red site at distance N will result in fewer surviving red particles at distance $2N$. Thus, the number of sites colored red at distances $N, 2N, \ldots$ dominates a Galton-Watson process with mean $d^N P_\lambda(A_N) > 1$. This expression for the mean comes from linearity of expectation applied to the d^N sites at distance N from the root. This is supercritical, and thus $P_\lambda(R = \infty) > 0$. $\square$

This proof uses the set up from Lemma 2.3.1, which considers phase transition on the path to show that $P_\lambda(A) = 0$ for $\lambda < \lambda_c$. This is done by restricting the tree to a single ray starting at the root, which is graph isomorphic to the path, and showing that the number of colored sites is summable. The proof notes that we are able to find this path because the tree is acyclic. That is to say that the path is **vertex self-avoiding**, so no node along the path is visited twice. We isolate vertex self-avoiding paths in the Erdős-Rényi graph to parallel 5.1.1 and 5.1.2. To show that coexistence does not occur on the Erdős-Rényi network for $\lambda < \lambda_c$ we follow the method of approximating the number of colored sites on the path as finite.

Proving $P_\lambda(A) > 0$ for $\lambda > \lambda_c$ requires a tree approximation that describes the number of red particles in a given generation. Authors show that for large enough generations on the tree, the Galton-Watson process is supercritical, resulting in coexistence. In order to prove the original conjecture, a similar process would investigate a tree approximation of $G(n, \mu/n)$. Attempts to couple the Erdős-Rényi branching process with a tree drove much of early research; however, the complications of collisions make coupling impossible and forced a reevaluation of the conjecture. Thus, Kordzakhia's results provide us with a lower bound for $\lambda_c(G(n, \mu/n))$ rather than true equality.

Before moving on, we formally establish the indicator function used in the last proof in Equation 5.1.1. The **indicator function** of an event e is defined as

$$\mathbf{1}_e = \begin{cases} 1 & \text{if } e \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

The indicator function is extremely useful in simplifying notation. In Equation 5.1.1 the indicator function signifies the occurrences of $A(v)$, the event that v is ever colored red. We say $\mathbf{1}_{A(v)}$ rather than $P(A(v)) = 1$ to limit the consideration of the event $A(v)$ in the sum. This is used heavily in our results.

In order to prove the lower bound, we need an asymptotically precise estimate for the probability that red can reach an arbitrary site on the path. For that we use the following Lemma:

Lemma 5.1.2 (**[DJT 2018]** [9]). *Let $A_n = A_n(\lambda)$ be the event that site n is ever colored red in the chase-escape model on $\mathbb{Z}$, and set $a_n = \frac{[4p(1-p)]^n}{n^{3/2}}$, where $p = \lambda/(\lambda + 1)$.*

i. For some $c > 0$ *and all* $n \geq 1$, $P_\lambda(A_n) \geq c\, a_n$

ii. If $\lambda < 1$, *then for some* $C_\lambda > 0$ *and all* $n \geq 1$, $P_\lambda(A_n) \leq C_\lambda\, a_n$;

Proof. Let (S_k) be the nearest-neighbor p-biased random walk as in (2.3.5). Note that the event A_n is equivalent to the event that S_k remains strictly positive for the first $2n$ steps, i.e.,

$$P_\lambda(A_n) = P_\lambda(S_k \geq 1, k \leq 2n | S_0 = 1) = \sum_{a=0}^{n} P_\lambda(S_k \geq 1, k \leq 2n; S_{2n} = 2a+1 | S_0 = 1).$$

Since S_{2n} must have the same parity as S_0, we only considered the cases where S_{2n} is odd. For any random walk path of length $2n$ from 1 to $2a+1$ that does not hit zero, there must be $(n+a)$ steps to the right and $(n-a)$ steps to the left. Using the reflection principle (see [8, Theorem 4.3.2]), the total number of such paths is

$$\binom{2n}{n+a} - \binom{2n}{n+a+1}.$$

We then have

$$\begin{aligned} P_\lambda(A_n) &= p^{2n} + \sum_{a=0}^{n-1} \left[\binom{2n}{n+a} - \binom{2n}{n+a+1} \right] p^{n+a}(1-p)^{n-a} \\ &= p^{2n} + p^n(1-p)^n \sum_{a=0}^{n-1} \binom{2n}{n+a} \frac{2a+1}{n+a+1} \left(\frac{p}{1-p} \right)^a \qquad (5.1.2) \\ &\leq p^{2n} + \frac{p^n(1-p)^n}{n+1} \binom{2n}{n} \sum_{a=0}^{\infty} (2a+1) \left(\frac{p}{1-p} \right)^a. \end{aligned}$$

When $\lambda < 1$, we have $p < \frac{1}{2} < 1-p$, the summation above is finite. Moreover, $\frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, which is known to be of order $\frac{4^n}{n^{3/2}}$ for large n. Putting $C_\lambda = \sum_{a=0}^{\infty}(2a+1)\left(\frac{p}{1-p}\right)^a = \frac{\lambda+1}{(\lambda-1)^2}$, we have the desired upper bound. The lower bound is obtained by looking at the $a = 0$ term in (5.1.2) and using the asymptotic behavior of Catalan numbers. □

The techniques and conclusions of the results shown substantiate the preliminary material necessary for rigorously setting $\lambda_c(\mathbb{T}_\mu)$ as a lower bound for $\lambda_c(G(n, \mu/n))$. Next we consider the background material to show an upper bound, λ^+, such that coexistence occurs for $\lambda > \lambda^+$ on $G(n, \mu/n)$. This result is more mathematically technical though partially non-rigorous.

5.2 Upper Bound

Though the results of [CNSW 2000] [4] are heuristic-based, they are cited frequently. This paper was motivated by work on the resilience of random networks to random or targeted deletion of nodes, particularly pertaining to transmission of the internet. The percolation criteria defined in this paper predict that random networks should be robust against the removal of nodes. We take advantage of the approach in removing vertices from the network in an order that depends on their degree.

Once the high degree nodes are deleted, we use the **configuration model** to generate a graph with a giant component contiguous to the Erdős-Rényi graph. First we specify the fraction of vertices in the network having degree k with a degree distribution p_k. The configuration model takes a degree sequence, a set of degree values k_i such that $|k_i| = n$ for vertices $i = 1, \ldots, n$ from p_k. We can think of it as giving each vertex i in the graph k_i stubs sticking out of it, determining the number of edges the node i can hold. Pairs of nodes and their associated stubs are then chosen at random to be connected, assuring that all stubs are connected by an edge.

Note that our degree sequence k_i are i.i.d Poisson λ random variables taken from p_k. The **Poisson cloning model** then takes a copy of each vertex $i \in k_i$ to make a new set $\hat{k}_i$. Then we can generate a uniform random perfect matching on the set of copies. An edge $\{v, w\}$ is in the cloning model if $\hat{v}$ is matched to $\hat{w}$ in the random perfect matching. With the results of the configuration model and the cloning process, all that is left to show is that there is still a giant component on the new graph with deleted nodes and edges. In order to do this we consider the Molloy-Reed condition.

The Molloy-Reed condition states that a giant component emerges so long as

$$\sum_k k(k-2)p_k > 0 \tag{5.2.1}$$

where k are edges that arrive at a vertex of degree k. Since the sum increases monotonically as k increases, the giant component emerges if and only if this sum is positive.

The rest of the upper bound proof relies on considering the worst case scenarios on these deleted nodes and edges to conclude that a substantial number of colored nodes exist on the configuration model's giant component, satisfying our definition of coexistence in finite space. The first of which is the Chernoff bound, a more precise bound than the fundamental Markov or Chebyshev inequality.

Theorem 5.2.1. *Chernoff Bound. Suppose* $0 \leq X_i \leq 1$. *Then for all* $\epsilon > 0$,

$$P(X \leq (1-\epsilon)\mu) \leq \text{Exp}\left(-\frac{\epsilon^2}{2}\mu\right), \quad \textit{and} \quad P(X \geq (1+\epsilon)\mu) \leq \text{Exp}\left(-\frac{\epsilon^2}{2+\epsilon}\mu\right).$$

If μ *falls within a range such that* $\mu_L \leq \mu \leq \mu_H$, *then*

$$P(X \leq (1-\epsilon)\mu_L) \leq \text{Exp}\left(-\frac{\epsilon^2}{2}\mu_L\right), \quad \textit{and} \quad P(X \geq (1+\epsilon)\mu_H) \leq \text{Exp}\left(-\frac{\epsilon^2}{2+\epsilon}\mu_H\right).$$

The other circumstances of worst case bounds are detailed in Lemmas 6.4.2 and 6.4.3. Lemma 6.4.2 follows from our definition of the degree sequence. Lemma 6.4.3 makes bounds based on facts of the Poisson random variable.

6
Results

6.1 Informal Discussion of Results

Before getting into the results, a brief overview of the chapter is provided. The results can be broken into two components, a proof of a lower and upper bound on the critical threshold resulting in phase transitions on $G(n, \mu/n)$. The lower bound is given by what happens on $\mathbb{T}_\mu$; the non-rigorous upper bound is computed as $4\mu^2$. Detailed in the previous chapter, the lower bound proof restricts chase-escape to a vertex self-avoiding path formed from 1 to a site k and shows that extinction occurs because the number of colored sites on the graph remains finite as the size of the graph grows for $\lambda \leq \lambda_c(\mathbb{T}_u)$.

The upper bound proof is broken into two theorems. The first, Theorem 6.3.3 relies on results from Callaway et al. [4], which are heuristic-based. This theorem explicates our defined bound of $4\mu^2$, showing that there exists a giant component of what we call open sites for large enough μ. First we allocate passage times to the edges of the site. Passage times dictate how long it will take a site to be colored based on their underlying exponential distribution. We say a site is *open* if the slowest red beats the fastest blue to a site. This results in a cluster of open sites. This definition dictates that once any open site is colored red in chase-escape, the entire cluster becomes red colored. To visualize this consider the counter-situation, that an open site v becomes colored red but the open cluster containing v does not entirely become red. That would mean that blue sites stop the spread of red to open sites; however this violates the criteria of being open. This proof is completed using Callaway's site percolation criteria and the configuration model to show that there exists a giant cluster of open sites for $\lambda > 4\mu^2$ and large enough μ.

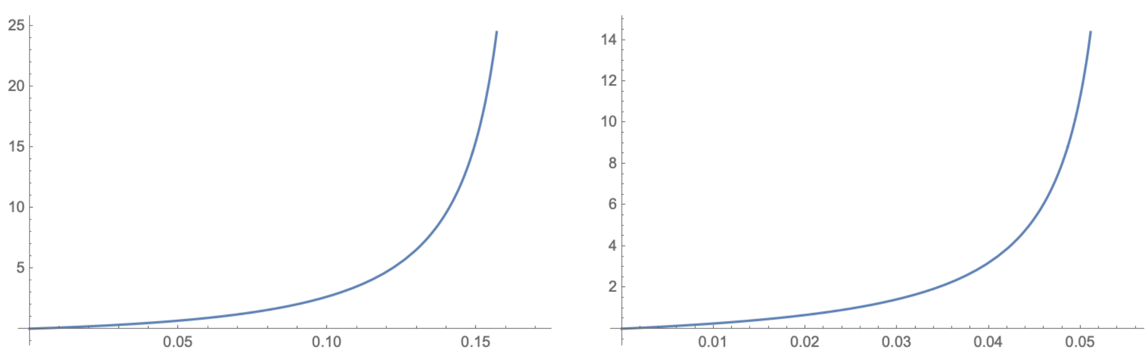
The rigorous upper bound result is Theorem 6.4.1, which is a statement of existence and does not provide an explicit form for the upper threshold. This theorem proves that there exists λ^+ allowing for coexistence on $G(n, \mu/n)$. First we consider open sites with the same definition as in the previous theorem. Next we delete the high degree vertices and control the number of edges deleted. Then we control the resulting number of vertices in the cloned set with degree i. Finally, we impose worst case scenarios to the Molloy-Reed condition, showing that there is a giant component of open sites with positive probability in this case, concluding our proof.

6.2 Lower Bound

Theorem 6.2.1. *For $\mu > 1$ and $\lambda \le \lambda_c(\mathbb{T}_\mu) = 2\mu - 1 - 2\sqrt{\mu^2 - \mu}$, strong extinction occurs on $G(n, \mu/n)$. In particular,* $\limsup_{n\to\infty} E|\mathcal{R}| \le F(\lambda, \mu)$ *where*

$$F(\lambda,\mu) = \begin{cases} \frac{\lambda+1}{(\lambda-1)^2}\sum_{k=1}^{\infty} k^{-3/2} & \text{if } \lambda = \lambda_c \\ \frac{\lambda+1}{(\lambda-1)^2}\left[\frac{4\lambda\mu}{(1+\lambda)^2 - 4\lambda\mu}\right] & \text{if } \lambda < \lambda_c. \end{cases}$$

Before getting in to the proof, let us look at a graphical representation of this bound.



(a) $\mu = 2$

(b) $\mu = 5$

Figure 6.2.1. The bound resulting for $\mu = 2$ and $\mu = 5$ on $F(\lambda, \mu)$. For $\mu = 2$ we have $\lambda_c(\mathbb{T}_2) \approx 0.17157$. For $\mu = 5$ we have $\lambda_c(\mathbb{T}_5) \approx 0.05572$. Here F is plotted with μ on the x-axis and λ on the y-axis.

Proof. Let $A_k = A_k(\lambda)$ be the event that k is ever colored red in the chase-escape model on $\mathbb{Z}_1$. By Lemma 5.1.2 we have that for $\lambda \leq \lambda_c$,

$$\mathrm{P}_\lambda(A_k) \leq C_\lambda \left(\frac{4\lambda}{(1+\lambda)^2} \right) k^{-3/2} \tag{6.2.1}$$

where $C_\lambda = \frac{\lambda+1}{(\lambda-1)^2}$. Let $G \sim G(n, \mu/n)$ and $\mathcal{R} = \mathcal{R}(G_n)$. Let Γ_k be the set of all vertex self avoiding length-k paths starting at 1. Such paths are uniquely identified by the order of the k verticies, giving $|\Gamma_k| = \mathrm{P}_{n,k} = \frac{n!}{(n-k)!}$. We say that red *survives* on a path $\gamma \in \Gamma_k$ if, for chase-escape restricted only to γ, the terminal vertex of γ is colored red. For any vertex $v \in \mathcal{R}$ it is required that there is a path red survives on where v is the terminal point,

$$\mathcal{R} \subseteq \{1\} \cup \bigcup_{k=1}^{\infty} \bigcup_{\gamma \in \Gamma_k} \{\gamma \colon \gamma \subseteq G), \gamma \subseteq A(\gamma)\} \tag{6.2.2}$$

where $A(\gamma) = \{\gamma \text{ is a vertex self-avoiding path}, \gamma_1 = 1, \text{ red survives on } \gamma\}$. Looking at the cardinality of these sets

$$|\mathcal{R}| \leq 1 + \sum_{k=1}^{\infty} \sum_{\gamma \in \Gamma_k} \mathbf{1}\{A(\gamma)\} \cdot \mathbf{1}\{\gamma \subseteq G\}. \tag{6.2.3}$$

Taking the expected value of 6.2.3 gives

$$\mathrm{E}|\mathcal{R}| \leq 1 + \mathrm{E} \sum_{k=1}^{\infty} \sum_{\gamma \in \Gamma_k} \mathbf{1}\{A(\gamma)\} \cdot \mathbf{1}\{\gamma \subseteq G\} = 1 + \sum_{k=1}^{\infty} |\Gamma_k| \mathrm{E}[\mathbf{1}\{A(\gamma)\}\mathbf{1}\{\gamma \subseteq G\}]. \tag{6.2.4}$$

Note that $\mathbf{1}\{A(\gamma)\}$ and $\mathbf{1}\{\gamma \subseteq G\}$ are conditionally independent, because $\gamma \subseteq G$ depends on the structure of G while $A(\gamma)$ on the passage times from red to blue. Conditioning gives

$$\mathrm{E}[\mathbf{1}\{A(\gamma)\}\mathbf{1}\{\gamma \subseteq G\}] = \mathrm{E}[\mathbf{1}\{A(\gamma) \mid \gamma \subseteq G\}] P(\gamma \subseteq G).$$

Thus $\mathrm{E}[\mathbf{1}\{A(\gamma) \mid \gamma \subseteq G\}] = P(A(\gamma)) = P(A_k)$. Thus,

$$\sum_{k=1}^{\infty} |\Gamma_k| \sum_{k=1}^{\infty} |\Gamma_k| \mathrm{E}[\mathbf{1}\{A(\gamma)\}\mathbf{1}\{\gamma \subseteq G\}] = \sum_{k=1}^{n} \frac{n!}{(n-k)!} P(A_k) P(\gamma \subseteq G). \tag{6.2.5}$$

The probability that a given length-k path $\gamma \in G$ is given by the probability of including edges between all k vertices in the path, so

$$P(\gamma \in G) = \left(\frac{\mu}{n} \right)^k.$$

We use Equation 6.2.1 to evaluate $P(A_k)$, which gives

$$\sum_{k=1}^{n} \frac{n!}{(n-k)!} P(A_k) P(\gamma \subseteq G) \leq C_\lambda \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left[\frac{4\lambda}{(1+\lambda)^2}^k \right] \left(\frac{1}{k^{3/2}} \right) \left(\frac{\mu}{n} \right)^k. \tag{6.2.6}$$

Plugging in C_λ,

$$\mathrm{E}|\mathcal{R}| \leq 1 + \frac{\lambda+1}{(\lambda-1)^2} \sum_{k=1}^{n} \frac{n!}{n^k (n-k)!} \left[\frac{4\lambda\mu}{(1+\lambda)^2} \right]^k k^{-3/2}. \tag{6.2.7}$$

Expanding,

$$\frac{n!}{n^k (n-k)!} = \frac{(n)(n-1)\cdots(n-k+1)}{n^k},$$

where the numerator has k terms, all less than or equal to n. Thus $n! \leq n^k (n-k)!$, and $\frac{n!}{n^k (n-k)!} \leq 1$. Thus

$$\mathrm{E}|\mathcal{R}| \leq 1 + \frac{\lambda+1}{(\lambda-1)^2} \sum_{k=1}^{n} \left[\frac{4\lambda\mu}{(1+\lambda)^2} \right]^k k^{-3/2}.$$

By Lemma 5.1.2, if $\lambda = \lambda_c$ we have that $\frac{4\lambda\mu}{(1+\lambda)^2} = 1$, so

$$\mathrm{E}|\mathcal{R}| \leq 1 + C_\lambda \sum_{k=1}^{n} k^{-3/2} < 3\, C_\lambda$$

being that $\sum_{k=1}^{n} k^{-3/2} \leq \int_1^\infty x^{-3/2} dx < 3$. If $\lambda < \lambda_c$, we bound the $k^{-3/2}$ by 1 and write

$$\mathrm{E}|\mathcal{R}| \leq 1 + C_\lambda \sum_{k=1}^{n} \left[\frac{4\lambda\mu}{(1+\lambda)^2} \right]^k = 1 + \frac{\lambda+1}{(\lambda-1)^2} \left[\frac{\frac{4\lambda\mu}{(1+\lambda)^2}}{1 - \left(\frac{4\lambda\mu}{(1+\lambda)^2} \right)} \right]. = \frac{\lambda+1}{(\lambda-1)^2} \left[\frac{4\lambda\mu}{(1+\lambda)^2 - 4\lambda\mu} \right] \tag{6.2.8}$$

Since (6.2.8) does not depend on n, $\limsup_{n\to\infty} \mathrm{E}|\mathcal{R}(G_n)|$ is bounded. Therefore strong extinction occurs. □

6.3 Heuristic-Based Upper Bound

We can also give a upper bound that uses an often-cited result from [CNSW 2000] [4]. First, we need two lemmas.

Lemma 6.3.1. *Let $X_1, \ldots, X_k$ be i.i.d.* Exp(λ) *random variables, and $Y_1, \ldots, Y_k$ be i.i.d.* Exp(1) *random variables. Let*

$$q_k = P(\max_{1\leq i\leq k} X_k < \min_{1\leq i\leq k} Y_k).$$

It holds that q_k is a decreasing function of k for any $\epsilon > 0$, there exists k_0 such that for all $k \geq k_0$ and $\lambda > k^2$ we haev $q_k > 1 - \epsilon$.

Proof. That q_k is decreasing in k is easily seen by coupling to the result with $k-1$ random variables. Using the fact that $\max X_k$ has distribution function $F(t) = (1 - e^{-\lambda t})^k$ and that $\min Y_K \sim \text{Exp}(k)$ because $\min(\text{Exp}(\lambda_1), \dots, \text{Exp}(\lambda_k)) \sim \text{Exp}(\lambda_1 + \cdots + \lambda_k)$ we can write

$$q_k = \int_0^\infty (1 - e^{-\lambda t})^k k e^{-kt} dt. \tag{6.3.1}$$

Thus, for $\lambda \geq m^2$ we have

$$q_m \geq \int_0^\infty (1 - e^{-m^2 t})^m m e^{-mt} dt \tag{6.3.2}$$

$$= \frac{\Gamma(m+1)\Gamma(1+\frac{1}{m})}{\Gamma(m+1+\frac{1}{k})}. \tag{6.3.3}$$

Here $\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt$ is the usual Γ function. Since Γ is continuous, we have the limit goes to 1 as $m \to \infty$. □

Lemma 6.3.2. *Let $D \sim \text{Poi}(\mu)$ and $\mu_k = P(D = k) = e^{-\mu}\frac{\mu^k}{k!}$. There exist $c, C > 0$ such that*

$$\sum_{k=0}^{2\mu} k(k-1)\mu_k \geq \mu^2 - C\mu^4 e^{-c\mu}$$

for all large μ.

Proof. Observe that

$$\sum_{k=0}^{2\mu} k(k-1)\mu_k = \sum_{k=0}^{\infty} k(k-1)\mu_k - \sum_{k>2\mu} k(k-1)\mu_k \tag{6.3.4}$$

$$= E[D(D-1)] - E[D(D-1)\mathbf{1}\{D > 2\mu\}]. \tag{6.3.5}$$

The Cauchy-Schwarz inequality lets us bound

$$E[D(D-1)\mathbf{1}\{D > 2\mu\}] \leq E[D^2(D-1)^2]P(D > 2\mu) \tag{6.3.6}$$

$$\leq E[D^4]P(D > 2\mu). \tag{6.3.7}$$

A standard large deviation estimate is that $P(D > 2\mu) \leq e^{-c\mu}$ for some $c > 0$. Moreover, since $ED^4 \leq C\mu^4$ for some $C > 0$ and $E[D(D-1)] = \mu^2$ we obtain the claimed inequality. □

Theorem 6.3.3. *Assuming the derivations in [CNSW 2000] [4] hold, for all large enough μ if $\lambda \geq 4\mu^2$, then coexistence occurs on $G(n, \mu/n)$.*

Proof. The spread of red and blue can be formalized by assigning passage times to edges. More precisely, we replace each edge with two directed edges with opposite orientations. To each directed edge $e = (u, v)$ we assign independent passage times $t_e^R \sim \text{Exp}(\lambda)$ and $t_e^B \sim \text{Exp}(1)$. If red occupies u, then it will occupy v after $t_{(u,v)}^R$ time units. Similarly, if blue occupies u, and v is red, then it will occupy v after $t_{(u,v)}^B$ time units. For each $v \in \mathcal{V}$ define:

$$T^R = \max\{t_{(u,v)}^R \colon (u, v) \in \mathcal{E}\} \tag{6.3.8}$$

$$T^B = \min\{t_{(v,u)}^B \colon (v, u) \in \mathcal{E}\}. \tag{6.3.9}$$

Declare v *open* if $T^R < T^B$. By construction, neighboring open sites form a cluster such that if an open site, say v, becomes red, then all sites in that component will become red. It then suffices to prove that for $\lambda > \sqrt{2\mu}$ there is a giant component of open sites. If this holds, then with positive probability 1 belongs to the component and thus coexistence occurs.

The set of open sites can be realized as a degree-dependent *site percolation* model on G. Supposing that $\deg(v) = k$, the probability that v is open is $q_k := P(T^R < T^B \mid \deg(v) = k)$ from 6.3.1. Callaway et al. provided an argument in [4] (see also the discussion in [19, Section 8.1]) that whenever

$$\sum_{k=0}^{\infty} k(k-1)\mu_k q_k > \mu \tag{6.3.10}$$

the set of open sites in the graph generated from the configuration model with degree sequence $d(v) \sim D$ contains a giant component with high probability. Since the configuration model with degree sequence $d(v) \sim D$ is contiguous to $G(n, \mu/n)$ (see [13, Theorem 1.1]), it follows that $G(n, \mu/n)$ contains a giant component of open sites whenever (6.3.10) is satisfied. Using Lemma 6.3.1 and Lemma 6.3.2, we have for $\lambda \geq 4\mu^2$ the left side of (6.3.10) is lower bounded by

$$\sum_{k=0}^{2\mu} k(k-1)e^{-\mu}\mu_k q_{2\mu} \geq q_{2\mu}(\mu^2 - C\mu^4 e^{-c\mu}) \to \mu^2. \tag{6.3.11}$$

Since $\mu^2 > \mu$, this guarantees that there is a giant component of open vertices for μ large and $\lambda \geq 4\mu^2$ which completes the argument. □

6.4 Rigorous Upper Bound

Theorem 6.4.1. *Given $\mu > 1$, there exists $\lambda^+ > 0$ such that coexistence occurs for all $\lambda \geq \lambda^+$.*

Before getting to the proof we provide two lemmas.

Lemma 6.4.2. *Let $\mu_i = e^{-\mu}\left(\frac{\mu^i}{i!}\right) = P(Poi(\mu) = i)$. There exists $c > 0$ such that for m sufficiently large it holds that*

$$\sum_{i=0}^{m} \mu_i i(i-2) > \mu^2 - \mu - e^{-cm}.$$

Proof. The facts that $\sum_{i=0}^{m} \mu_i i(i-2) = \mathrm{E}(X(X-2)) = \mathrm{E}X^2 - 2\mathrm{E}X = \mu^2 - \mu$ and $\mu_i \leq e^{-c'm \log m}$ for some $c' > 0$ and $i > m$ with m sufficiently large are both standard. It follows that $\mu_i i(i-2) \leq e^{-c''m}$ for a possibly smaller $c'' > 0$. Summing gives $\sum_{i>m} \mu_i i(i-2) \leq e^{-cm}$ for some $c > 0$ and all sufficiently large m. □

Lemma 6.4.3. *Let $\bar{\mu}_i = \frac{1}{n}\#\{d_j \colon d_j = i, j \leq n\}$. Let $A_i = \{(1-\epsilon)\mu_i < \bar{\mu}_i < (1+\epsilon)\mu_i\}$. For any $\epsilon > 0$ and $m > 0$ it holds that*

$$\lim_{n\to\infty} P\left(\bigcap_{i\leq m} A_i\right) \to 1.$$

Proof. Since $\mu_i \sim \frac{1}{n}\mathrm{Binomial}(n, \mu_i)$, a Chernoff bound (5.2.1) ensures that $P(A_i^C) \leq e^{-c_i n}$ for some $c_i > 0$ and n sufficiently large. Let $c = \min_{i\leq m} c_i$. A union bound ensures that

$$1 - P(\cup_{i\leq m} A_i) = P(\cup_{i\in I} A_i^C) \leq (m+1)e^{-cn} \to 0,$$

which gives the desired statement. □

Proof of 6.4.1. Given a graph G, let $\hat{G}_1$ be the largest component in the subgraph of G that results from keeping all vertices with degree less than or equal to m and for which $T^R < T^B$ for passage times from chase-escape on G assigned to the edge set. As in 6.3.3, if an open site becomes red in chase-escape, then the entire component containing that site will eventually be occupied by red. Thus, it suffices to prove that $P(\hat{G}_1 > \delta|G|) \to 1$ for $G \sim G(n, \mu/n)$ and some $\delta > 0$.

Let $\mathcal{H} = \{G \colon |\hat{G}_1| > \delta|G|\}$. A direct consequence of the main theorem of [13] is that for $G \sim G(n, \mu/n)$ and G' generated from the configuration model with degree sequence $d_i \sim \text{Poi}(\mu)$ we have

$$P(G \in \mathcal{H}) \to 1 \text{ if and only if } P(G' \in \mathcal{H}) \to 1.$$

Thus, it suffices to show that $|\hat{G}_1'| > \delta n$ with probability tending to 1.

After deleting the non-open vertices we obtain a new degree sequence $\hat{d}_1, \dots, \hat{d}_n$. Let $\hat{\mu}_i = n^{-1}|\{j \le n \colon \hat{d}_j = i\}|$ be the proportion of vertices with degree i in $\hat{G}$ with the convention that deleted vertices are given degree 0. The idea of of the proof from here is to apply the Molloy-Reed condition that so long as

$$\sum_{i=0}^{\infty} \hat{\mu}_i i(i-2) > 0 \tag{6.4.1}$$

then there is a giant component with probability tending to 1 [17]. We establish that (6.4.1) holds with high probability by providing deterministic worst case bounds on $\hat{\mu}_i$.

Let $\mu_i = e^{-\mu}\mu^k/k!$ be the point probabilities for the Poisson distribution with mean μ. Using Lemma 6.4.2, choose m large enough and $0 < \epsilon < 1/4$ small enough so that

$$\sum_{i>m} \mu_i i(i-2) < e^{-cm} \text{ and } (1-\epsilon)(\mu^2 - \mu) - 5e^{-cm} > 0. \tag{6.4.2}$$

Since $i \le i(i-2)$ for all $i \ge 2$, the first condition also implies that $\sum_{i>m} \mu_i i < e^{-cm}$. Let $\bar{\mu}_i = n^{-1}|\{j \le n \colon d_j = i\}|$. Using Lemma 6.4.3 we have with probability tending to 1 that

$$\sum_{i=0}^{m} \bar{\mu}_i i(i-2) \ge \sum_{i=0}^{m} (1-\epsilon)\mu_i i(i-2) \ge (1-\epsilon)(\mu^2 - \mu) - e^{-cm}$$

for all n large enough. By changing vertex i with $d_i > m$ to $\hat{d}_i = 0$, this removes at most

$$\sum_{i>m} n\bar{\mu}_i i \le \sum_{i>m} n(1+\epsilon)\mu_i i \le (1+\epsilon)e^{-cm} n$$

edges from $\hat{G}$. It minimizes the quantity at (6.4.1) if these edge removals result in $(1+\epsilon)e^{-cm}n$ more degree 1 vertices, and that many less degree m-vertices. Let $q = P(D \le m, T^R < T^B)$. As

$q \to 1$ as $m, \lambda \uparrow \infty$, we can choose $m, \lambda^+ > 0$ so that for all $\lambda > \lambda^+$ we have $P(T^R < T^B, d(v) \leq m) \geq 1 - e^{-cm}$. We are then guaranteed to have a subgraph on $N = \text{Binomial}(n, q)$ vertices with the $\hat{\mu}_i$ satisfying

$$\sum_{i=0}^{N} \hat{\mu}_i i(i-2) \geq \sum_{i=0}^{m} (1-\epsilon)\mu_i i(i-2) - 4(1+\epsilon)e^{-cm} \tag{6.4.3}$$

$$\geq (1-\epsilon)(\mu^2 - \mu) - 5e^{-cm} > 0. \tag{6.4.4}$$

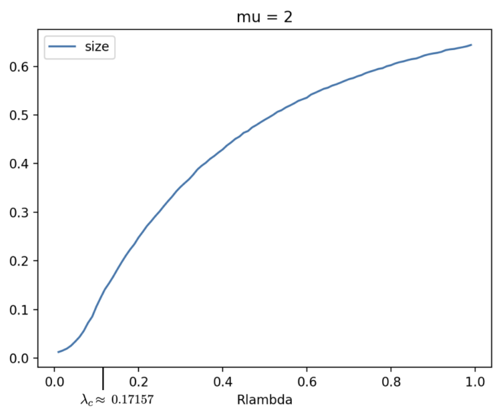
Thus, (6.4.1) is satisfied and there is a giant component of vertices that red will occupy so long as 1 belongs to that component. As this occurs with positive probability, we have coexistence occurs for $\lambda \geq \lambda^+$. □

6.5 Simulations

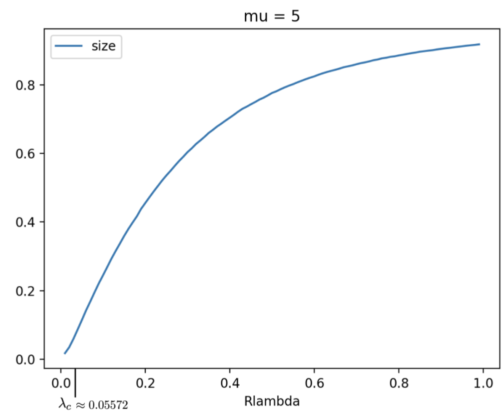
With the initial conjecture in mind, we look to test the speed of red against the fraction of uncolored sites on the Erdős-Rényi graph. To that end we modify code written by Nicole Eikmeier. The code relies on several packages:

- Epidemics on Networks (EoN) and Epidemics on Networks (chase-escape) are Python modules that were created to study infectious processes in networks. Its features include simulation, analytic approximation, and analysis of epidemics.
- NetworkX is a Python library for analysis of graphs and networks. It provides the data structures for complex graphs and allows for easy generation of random networks.
- Defaultdict is used to set all nodes empty before setting the initial conditions. This subclass should be used whenever all element's value should start with a default.
- Random, Numpy, Pandas, and Matplotlib are standard packages for mathematics and visualization.

The simple chase-escape function defined constructs $G(n, \mu/n)$ and instantiates the initial configuration with the number of colored sites. The passage rates defining red and blue behavior is encoded using NetworkX. The do_plot option can be turned on to see the number of nodes as a function of time for a set value of red speed (blue speed should always be inputted as 1). What is more useful for testing these results is seeing the number of nodes as a function of varying red speeds. The results of those plots with $\mu = 2$ and $\mu = 5$ are shown in Figure 6.5.1, where size is the number of colored nodes as a proportion of the total number of nodes.



(a) $\mu = 2$



(b) $\mu = 5$

Figure 6.5.1. Simulation Results

The expected results of these simulations would show a sharp contrast in size for values $\lambda < \lambda_c$ and $\lambda > \lambda_c$. All things running perfectly under the (now proven false) assumption that $\lambda_c(G(n, \mu/n)) = \lambda_c(\mathbb{T}_\mu)$ we anticipate a discontinuity at λ_c, indicating phase transition at the critical value. Of course the results do not fall completely in line with our expectations. Though there are hints of an inflection point around the critical speed, there are some clear issues with the model in place.

To start, this simulation likely experiences a finite size effect. We would like to run simulations on an Erdős-Rényi graph with $n > 100,000$, which is not feasible with available computing power. Moreover, facets of this code exacerbate existing strains on time and processing. For one, every iteration of the simulation, which tests a different red passage time, generates a new graph to run the dynamics on. This is an expensive task, and one we really do not want to be performing; alternatively we would generate a single graph and reinstantiate the initial conditions on each iteration.

Appendices

Appendix A

Simulation Code

```
import EoN
import EoN_chase_escape as CE
import networkx as nx
import numpy as np
import random
import pandas as pd
from collections import defaultdict
import matplotlib.pyplot as plt


# Nicole Eikmeier June 2019
# Updated Emma Bernstein November 2019



   #N is number of nodes
   #must input number for mu, we used 2,5
   #G, Gc, M ensures initial conditions, M=1 means one blue-
      red edge

N = 5000
G = nx.fast_gnp_random_graph(N,mu/(N-1))
Gc = max(nx.connected_component_subgraphs(G), key=len)
M=1
for i in range(M):
    e=(i, N+1)
    Gc.add_edge(*e)
```

```
def simple_chase_escape(G, Rgamma, Bgamma, rho, init_red,
    init_blue, do_plot=True):
    #parameters:
        # G defines the network on which the spread will occur
        # Rgamma is the rate parameter for the Exponential
           distribution for nodes turning Red
        # Bgamma is the rate parameter for the Exponential
           distribution for nodes turning Blue
        # rho is the rate parameter for the Exponential
           distribution for nodes dying
        # init_red is the set of nodes which are initially red
        # init_blue is the set of nodes which are initially
           blue
    ER_edge_attribute_dict = {edge: np.random.exponential(
        scale = Rgamma) for edge in G.edges()}
    RB_edge_attribute_dict = {edge: np.random.exponential(
        scale = Bgamma) for edge in G.edges()}
    nx.set_edge_attributes(G, values= RB_edge_attribute_dict,
         name='red2blue_weight')
    nx.set_edge_attributes(G, values= ER_edge_attribute_dict,
         name='empty2red_weight')

    node_attribute_dict = {node: np.random.exponential(scale
        = rho) for node in G.nodes()}
    nx.set_node_attributes(G, values=node_attribute_dict,
        name='reddeath_weight')

    #H defines the process of changing from Red to Death (i.e
        . Red dying), not used in this project
    H = nx.DiGraph()
    H.add_node('E')
    H.add_edge('R', 'D', rate = rho, weight_label='
        reddeath_weight')

    #J defines the process of changing from Empty to Red
    #  "              "                    Red to Blue
    J = nx.DiGraph()
    J.add_edge(('R', 'E'), ('R', 'R'), rate = 1.0,
        weight_label='empty2red_weight')
    J.add_edge(('B', 'R'), ('B', 'B'), rate = 1.0,
        weight_label='red2blue_weight')



    IC = defaultdict(lambda: 'E') #set the status of every
        node to be empty
```

```
    for node in init_red: #Define some number of nodes to
       start as red
        IC[node] = 'R'
    for node in init_blue: #Define sum set of nodes to start
       as blue
        IC[node] = 'B'

    return_statuses = ('E', 'R', 'B', 'D')

    t, E, R, B,D = EoN.Gillespie_simple_contagion(G, H, J, IC
       , return_statuses,tmax = float('Inf'))
    plt.clf()

    num = 18.3*N
    if do_plot:
    # log scale:
      # plt.semilogy(t[0:num], E[0:num], 'y', label = 'Empty
         ')
      # plt.semilogy(t[0:num], R[0:num], 'r', label = 'Red')
      # plt.semilogy(t[0:num], B[0:num], 'b', label = 'Blue')
      # plt.semilogy(t[0:num], D[0:num], 'k', label = 'Dead')

    #real scale:
        plt.plot(t[0:num], R[0:num], 'r', label = 'Red')
        plt.plot(t[0:num], B[0:num], 'b', label = 'Blue')
        plt.plot(t[0:num], D[0:num], 'k', label = 'Dead')
        plt.xticks([])
        plt.xlabel('time')
        plt.ylabel('Number of Nodes')
        plt.title('')
        plt.legend()
        plt.show()

    return t,E,R,B,D

if __name__ == "__main__":
    values = []
    R1 = range(M)
    B1 = [N+1+i for i in range(M)]
    for i in range(500):
        for Rlambda in range(17,18,1):    #speed of red (
           first, last+1, step size)
            Rlambda/=100
            t,E,R,B,D = CE.simple_chase_escape(Gc, Rlambda,
               1.0, 0, R1, B1, do_plot=False)
```

```
            #if B[-1]/N >= 0.1:  #check for over 10%
               occupation
            values.append({'Rlambda': Rlambda, 'size': B[-1]/
               N})

    values = pd.DataFrame(values)
    means = values.groupby(['Rlambda']).mean()
    means.plot()
    plt.show()
```

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