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Square Peg Problem in 2-Dimensional Lattice

Nathan M. Matsubara
Bard College, nm2834@bard.edu

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by
Nathan Matsubara

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Abstract

The Square Peg Problem, also known as the inscribed square problem poses a question: Does every simple closed curve contain all four points of a square? This project introduces a new approach in proving the square peg problem in 2-dimensional lattice.

To accomplish the result, this research first defines the simple closed curve on 2-dimensional lattice. Then we identify the existence of inscribed half-squares, which are the set of three points of a square, in a lattice simple closed curve. Then we finally add a last point to form a half-square into a square to examine whether all four points of a square exist in a lattice simple closed curve. A sage program was used to find all missing corners of all inscribed half-squares. This has enabled us to look at the pattern of sets of all missing corners in specific shapes like rectangles.

By the end, we were able to conjecture that there exist missing corners in the interior and the exterior of the lattice simple closed curve unless the shape is a square. It is obvious that the square has an inscribed square. Hence if we could prove that the set of all missing corners is connected, we could give a new proof of the square peg problem in 2-dimensional lattice.

Contents

Acknowledgments

This senior project would not have been possible without the tremendous support of my advisor, Ethan Bloch. The topic of this paper was his idea. Although I was lacking in many aspects, his enthusiasm and attention to details have kept my work on track to the end.

I also wish to thank my family for their support throughout my study.

1

Introduction

In this project, we discuss a new way of trying to prove the lattice version of the square peg problem, also known as the inscribed square problem. The problem of whether every Jordan curve has four points which form a square has not been solved in general. A survey of what is known about the problem is in [2]. The solution of the problem for every polygon is in [3]. The lattice square peg problem was solved in [4], but the proof in [4] depends upon [3], which is not a discrete proof. Hence this project attempts to write a purely discrete proof of the lattice square peg problem.

First, we introduce some basic definitions in understanding the simple closed curve on 2-dimensional lattice. Then we discuss half-squares, which are the set of three points in a square, inscribed in a lattice simple closed curve. Last, we look at the cloud, which is the set of missing corners of all inscribed half-squares. By looking at different clouds, We were able to find clear patterns of how clouds are formed in rectangular simple closed curves.

We were not able to reach to the solution of the lattice square peg problem in this project. However we could speculate that missing corners exist in the interior and the exterior of the non-square lattice simple closed curve. Because it is clear that a square has an inscribed square, if we could prove that there are missing corners in the interior and the exterior, and the set of all missing corners is connected, we would be able to solve the lattice square peg problem.

2

Lattice Simple Closed Curves

2.1 Non Degenerate Lattice Simple Closed Curves

First, let us introduce definitions in order to describe a non-degenerate lattice simple closed curve on 2-dimensional lattice.

Definition 2.1.1. The **2-dimensional lattice**, denoted by $\mathbb{Z}^2$, consists of points in $\mathbb{R}^2$ whose coordinates are only integers. $\triangle$

In 2-dimensional lattice, there are different ways to describe two neighboring points.

Definition 2.1.2. Consider two different types.

1. Two points in $\mathbb{Z}^2$ are **4-adjacent** if the distance between them is exactly 1 in either the vertical or horizontal direction.

2. Two points in $\mathbb{Z}^2$ are **8-adjacent** if the distance between them is exactly 1 or $\sqrt{2}$.

$\triangle$

Let us define a simple closed curve formed by 4-adjacent points in 2-dimensional lattice.

Definition 2.1.3. A **non-degenerate lattice simple closed curve (NDLSCC)** in $\mathbb{Z}^2$ is a finite set $C \subseteq \mathbb{Z}^2$ such that every point in C has exactly two 4-adjacent points in C. $\triangle$

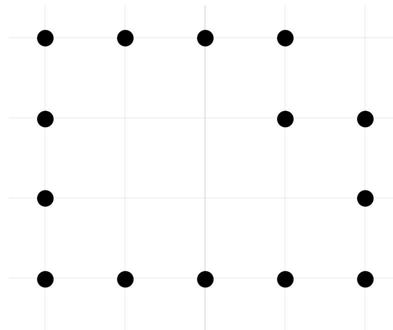

Figure 2.1.1. Example of 4-adjacent non-degenerate lattice simple closed curve (NDLSCC)

In Figure 2.1.1, the NDLSCC consists of the black points.

Figure 2.1.2 is not a NDLSCC. Note that some points have three 4-adjacent points.

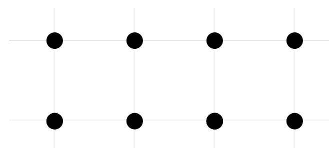

Figure 2.1.2. This is not a non-degenerate lattice simple closed curve

Now we define a path formed by 4-adjacent points in 2-dimensional lattice.

Definition 2.1.4. Let P be a finite set. A **non-degenerate lattice path (NDLP)** in $\mathbb{Z}^2$ is a finite set $P \subseteq \mathbb{Z}^2$ such that every point in P has exactly two 4-adjacent points in P except for two points in P, called **endpoints**, that have exactly one 4-adjacent point in P each. $\triangle$

We also define a path formed by 8-adjacent points in 2-dimensional lattice.

Definition 2.1.5. Let P be a finite set. A **8-adjacent non-degenerate lattice path (8-NDLP)** in $\mathbb{Z}^2$ is a set $P \subseteq \mathbb{Z}^2$ such that every point in P has exactly two 8-adjacent points in P except for two points in P, called **endpoints**, that have exactly one 8-adjacent point in P each. $\triangle$

Figure 2.1.3 is an example of non-degenerate lattice path. Notice that removing a point (or 4-adjacent points) from a NDLSCC gives NDLP because it creates two endpoints that have exactly one 4-adjacent point. Also note that Figure 2.1.4 is not a non-degenerate lattice path.

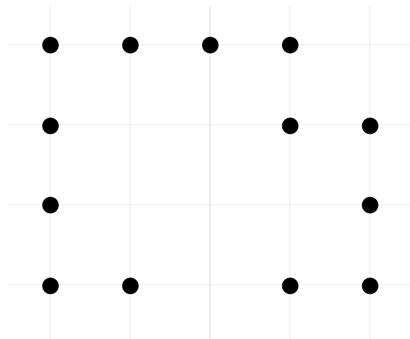

Figure 2.1.3. Example of non-degenerate lattice path (1)

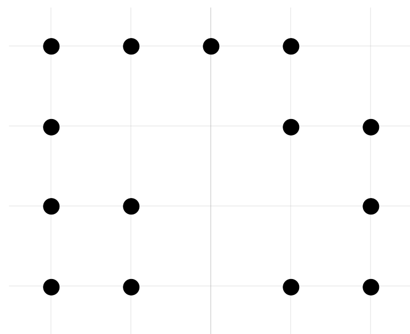

Figure 2.1.4. This is not a non-degenerate lattice path

Figure 2.1.5 is also an example of non-degenerate lattice path. However, this non-degenerate lattice path cannot be formed by removing a point (or 4-adjacent points) from a NDLSCC. This is because if we add a point anywhere on this NDLP, it would not be non-degenerate.

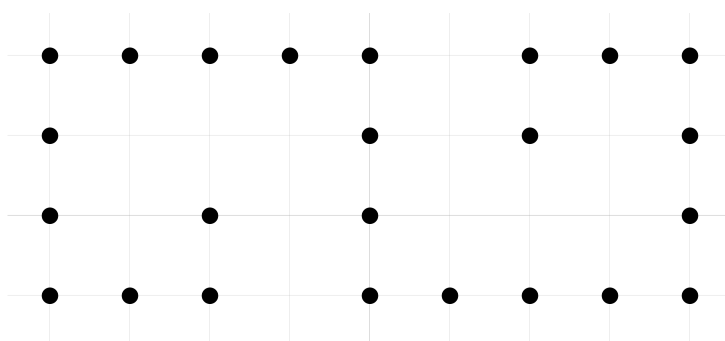

Figure 2.1.5. Example of non-degenerate lattice path (2)

Consider a corner, which is a turning point, in NDLSCC.

Definition 2.1.6. A **corner** of a NDLSCC is a point which has its two 4-adjacent points in both horizontal and vertical direction. △

In Figure 2.1.6, corners are indicated by each hexagon surrounding each point. Points that are not surrounded by hexagon is not a corner.

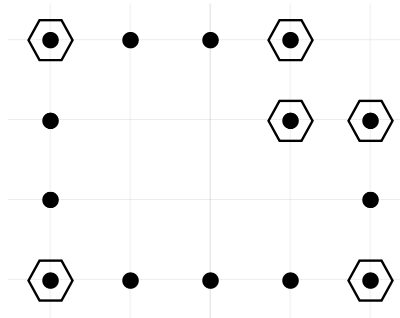

Figure 2.1.6. Example of the corner of the NDLSCC

We introduce the associate path, which is crucial in describing possible points in an inscribed square.

Definition 2.1.7. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $a \in C$, which we call a **starting point**. Let R_a be the rotation of the plane around a by $\frac{\pi}{2}$ counterclockwise. The **non-degenerate associate path for** a, denoted AP_a, is defined by $AP_a = \{R_a(b) \mid \text{for all } b \in C \text{ such that } b \neq a\}$. $\triangle$

In Figure 2.1.7 the NDLSCC consists of the black points. The starting point, a is denoted by a square around the black point. Then AP_a consists of the empty circle. The endpoints of AP_a are 4-adjacent to the starting point. (Note that the black points are inside the empty circles when they overlap.)

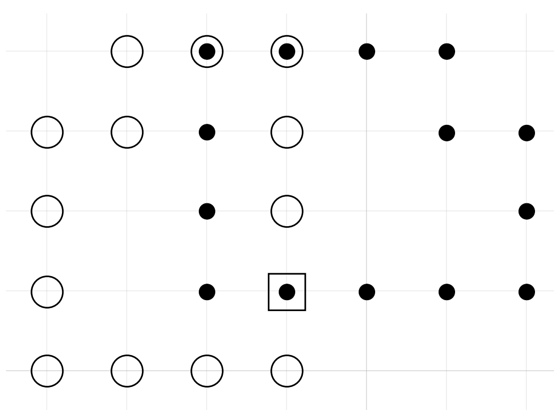

Figure 2.1.7. Example of non-degenerate associate path from starting point

2.2 Lattice Jordan Curve Theorem

In this section, we introduce Jordan Curve Theorem into 2-dimensional lattice. First we will define what component is, so that we can discuss the lattice Jordan curve theorem.

Definition 2.2.1. Let $A \subseteq \mathbb{Z}^2$.

1. The set A is **connected** if for any $x, y \in A$, there is a NDLP in A with endpoints x, y.

2. A **component** of A is a connected subset of A that is not the proper subset of another connected subset of A. △

The following theorem is the modified version of the well-known Jordan Curve Theorem in 2-dimensional lattice. The proof of this theorem is provided by [1].

Theorem 2.2.2 (Lattice Jordan Curve Theorem)**.** *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Then $\mathbb{Z}^2 - C$ has exactly two distinct components, one of which is bounded, and is called the interior of C, and one of which is unbounded, and is called the exterior of C. Let P be a 8-NDLP with one endpoint in the interior of C and the other endpoint in the exterior of C. Then $C \cap P \neq \emptyset$*

Now that we can distinguish the interior and the exterior of NDLSCC, we are going to show that the endpoints of the associate path are in different components.

Lemma 2.2.3. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $x \in C$. Suppose x is not a corner. The endpoints of AP_x are 4-adjacent to x, one endpoint in the interior of C and the other endpoint in the exterior of C.*

Proof. Let two 4-adjacent points of x in C be p, q. Because x is not a corner, then p and q are either two horizontal or two vertical 4-adjacent points of x. Let $z = R_x(p)$ and $t = R_x(q)$. Then $z, t \in AP_x$. Because $x \notin AP_x$, then z and t are endpoints of AP_x. Note that $z, t \notin C$. Hence z, t must either be in the interior or the exterior of C. Because $z = R_x(p)$ and $t = R_x(q)$, we have z and t are either two horizontal or two vertical 4-adjacent points of x. By using a standard fact about a point in polygon, one endpoint is in the interior of C and the other endpoint is in the exterior of C. □

Now we can prove that there always exists a point of intersection between NDLSCC and its non-degenerate associate path.

Lemma 2.2.4. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $x \in C$. Suppose x is not a corner. Let AP_x be the non-degenerate associate path for x. Then $C \cap AP_x \neq \emptyset$.*

Proof. Let the two 4-adjacent points of x in C be p, q. Because x is not a corner, then p, x, q are colinear. Let $z = R_x(p)$ and $t = R_x(q)$. Then $z, t \in AP_x$. By Definition 2.1.7, we have $x \notin AP_x$. Because p, x, q are colinear and $x \notin AP_x$, then z and t are endpoints of AP_x. By Lemma 2.2.3, we know that one endpoint is in the interior of C and the other endpoint is in the exterior of C. Because z and t are in different components of C and because AP_x is a NDLP, by Theorem 2.2.2, we have $C \cap AP_x \neq \emptyset$. $\square$

3
Inscribed Half-Square

3.1 Heads of Inscribed Half-Squares

In this section, we introduce half-squares inscribed in NDLSCC. We use the following notations. Let $x, y, z \in \mathbb{Z}^2$. Let $d(x, y)$ denote the distance between x and y. Let $\overline{xy} \perp \overline{yz}$ denote that xy and yz are perpendicular.

Definition 3.1.1. Let $a, b, c \in \mathbb{Z}^2$.

1. A **half-square** in $\mathbb{Z}^2$ is a set of 3 points $\{a, b, c\} \subseteq \mathbb{Z}^2$ such that $d(a, b) = d(b, c)$ and $\overline{ab} \perp \overline{bc}$.

2. The **head** of the half-square $\{a, b, c\}$ is b, and the **feet** of the half-square $\{a, b, c\}$ are a and c. △

In Figure 3.1.1, the half-square consists of black and empty points. The black point is the head of this half-square and remaining two empty points are the feet of this half-square.

If all the points of half-square is in NDLSCC, then the half-square is inscribed.

Definition 3.1.2. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. An **inscribed half-square (IHS)** in C is a set of 3 points $\{a, b, c\} \subseteq C$ such that $\{a, b, c\}$ is a half-square. △

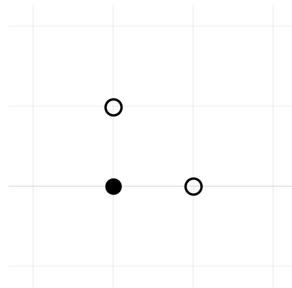

Figure 3.1.1. Example of a half-square

An example of IHS is shown in Figure 3.1.2. Note that this particular IHS is one of many IHS in this NDLSCC.

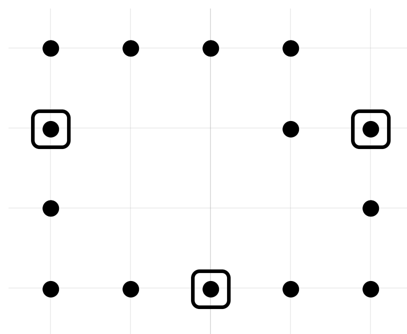

Figure 3.1.2. Example of an inscribed half-square

Now we can state that every point is the head of an IHS.

Theorem 3.1.3. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Every point in C is the head of an inscribed half-square in C.*

Proof. Let $p \in C$. Consider two cases.

1. Suppose p is the corner of C. Then p has two 4-adjacent points s and t in horizontal and vertical directions, respectively, by Definition 2.1.6. Then $\{s, p, t\}$ is an IHS in C with p its head.

2. Suppose p is not a corner of C. Let AP_p be the non-degenerate associate path for p. Then there exists $q \in AP_p \cap C$ by Lemma 2.2.4. By Definition 2.1.7 we know that $AP_p = \{R_p(b) \mid b \in C \text{ such that } b \neq p\}$. Hence there exists some $r \in C - \{p\}$ such that $q = R_p(r)$. Then $d(q, p) = d(p, r)$ and $\overline{qp} \perp \overline{pr}$. Hence $\{q, p, r\}$ is an IHS in C with p its head. $\square$

3.2 Feet of Inscribed Half-Squares

Now that it is clear that every point in NDLSCC is the head of an inscribed half-square in NDLSCC, we would also like to consider if every point in NDLSCC is the foot of an inscribed half-square in NDLSCC. First, let us redefine the associate path for this specific case.

Definition 3.2.1. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $a \in C$, which we call a **starting point**. Let S_a be the combined transformation consisting of the rotation of the plane around a by $\frac{\pi}{4}$ counterclockwise and the scaling of the plane around a by a factor of $\sqrt{2}$. The **8-adjacent non-degenerate associate path for** a, denoted 8-AP_a, is defined by 8-$AP_a = \{S_a(b) \mid$ for all $b \in C$ such that $b \neq a\}$. △

In Figure 3.2.1, a NDLSCC consists of the black points. The starting point, a is denoted by a square around the black point. Then 8-AP_a consists of the empty circles. The endpoints of 8-AP_a are 8-adjacent to the starting point. (Note that empty circles are strictly 8-adjacent to each other and the black points are inside the empty circles when they intersect.)

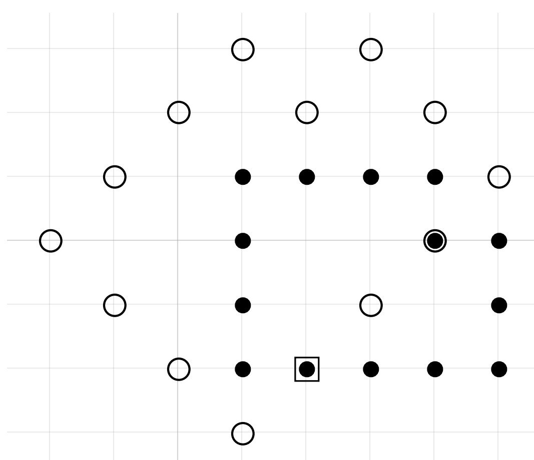

Figure 3.2.1. Example of left non-degenerate associate path from starting point

We introduce a lemma, which is similar to Lemma 2.2.3

Lemma 3.2.2. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $x \in C$. Suppose x is not a corner. The endpoints of 8-AP_x are 8-adjacent to x, one endpoint in the interior of C and the other endpoint in the exterior of C.*

Proof. Let two 8-adjacent points of x in C be p, q. Because x is not a corner, then p, x, q are colinear. Let $z = S_x(p)$ and $t = S_x(q)$. Then $z, t \in 8\text{-}AP_x$. Because $x \notin 8\text{-}AP_x$, then z and t are endpoints of $8\text{-}AP_x$. Note that $z, t \notin C$. Hence z, t must either be in the interior or the exterior of C. Because $z = S_x(p)$ and $t = S_x(q)$, we have z and t are two 8-adjacent points of x in the same direction. By using a standard fact about a point in polygon, one endpoint is in the interior of C and the other endpoint is in the exterior of C □

From here the procedure is same as Lemma 2.2.4.

Lemma 3.2.3. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $x \in C$. Suppose x is not a corner. Let $8\text{-}AP_x$ be the 8-adjacent non-degenerate associate path for x. Then $C \cap 8\text{-}AP_x \neq \emptyset$.*

Proof. Let the two 4-adjacent points of x in C be p, q. Let $z = S_x(p)$ and $t = S_x(q)$. Then $z, t \in 8\text{-}AP_x$.

By Definition 3.2.1, we have $x \notin 8\text{-}AP_x$. Because p, x and x, q are 4-adjacent and $x \notin 8\text{-}AP_x$, then z and t are endpoints of $8\text{-}AP_x$. By Lemma 3.2.2, we know that one endpoint is in the interior of C and the other endpoint is in the exterior of C. Hence z and t are in different components of C. Because $8\text{-}AP_x$ is a 8-adjacent NDLP and NDLSCC is 4-adjacent, by Theorem 2.2.2, we have $C \cap 8\text{-}AP_x \neq \emptyset$. □

Now we can prove that every point that is not a corner in NDLSCC is a foot of an inscribed half-square. We think we can prove the following theorem with every point including corners, but we were not able to do it yet.

Theorem 3.2.4. *Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Every point that is not a corner in C is a foot of an inscribed half-square in C.*

Proof. Let $p \in C$. Suppose p is not a corner. Let $8\text{-}AP_p$ be the 8-adjacent non-degenerate associate path for p. Then there exists $q \in 8\text{-}AP_p \cap C$ by Lemma 3.2.3. By Definition 3.2.1 we know that $8\text{-}AP_p = \{S_p(b) \mid b \in C \text{ such that } b \neq p\}$. Hence there exists some $r \in C - \{p\}$ such that $q = S_p(r)$. Then $d(p,r) = d(r,q)$ and $\overline{pr} \perp \overline{rq}$. Hence $\{p, r, q\}$ is an IHS in C with p and q as its feet. $\square$

3.3 Inscribed half-rectangle

Now we will examine half-rectangles in NDLSCC.

Definition 3.3.1. Let $a, b, c \in \mathbb{Z}^2$.

1. A 2×1 **half-rectangle** in $\mathbb{Z}^2$ is a set of 3 points $\{a, b, c\} \subseteq \mathbb{Z}^2$ such that $2d(a, b) = d(b, c)$ and $\overline{ab} \perp \overline{bc}$.

2. A 1×2 **half-rectangle** in $\mathbb{Z}^2$ is a set of 3 points $\{a, b, c\} \subseteq \mathbb{Z}^2$ such that $d(a, b) = 2d(b, c)$ and $\overline{ab} \perp \overline{bc}$.

3. The **head** of the half-rectangle $\{a, b, c\}$ is b, and the **feet** of the half-rectangle $\{a, b, c\}$ are a and c. $\triangle$

If all the points of half-rectangle is in NDLSCC, then the half-rectangle is inscribed.

Definition 3.3.2. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. An **inscribed half-rectangle (IHR)** in C is a set of 3 points $\{a, b, c\} \subseteq C$ such that $\{a, b, c\}$ is a half-rectangle. $\triangle$

Figure 3.3.1 and Figure 3.3.2 are the examples of IHR. 2×1 IHR and 1×2 IHR have the same physical properties, but we need to consider both to find every possible IHR in NDLSCC.

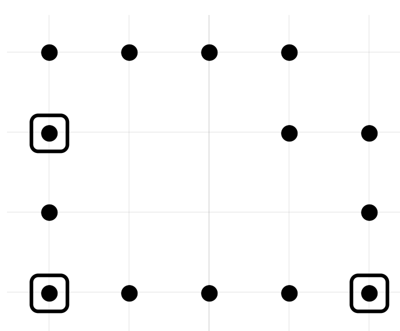

Figure 3.3.1. Example of a 2×1 IHR

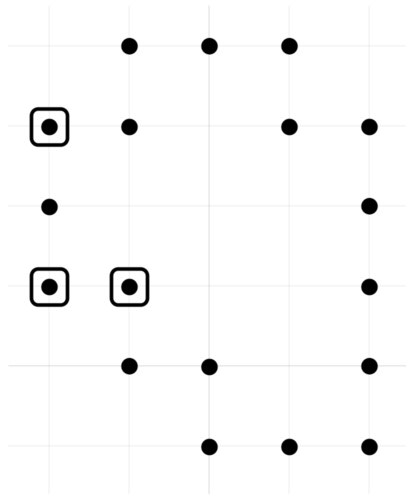

Figure 3.3.2. Example of a 1×2 IHR

Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Is every point in C the head of an inscribed half-rectangle in C? We disprove this by giving a counterexample.

Counterexample.

Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $a \in C$, which we call a **starting point**. Let T_a be the combined transformation consisting of the rotation of the plane around a by $\frac{\pi}{2}$ and the enlargement of the plane around a by a factor of 2. Let U_a be the combined transformation consisting of the rotation of the plane around a by $\frac{\pi}{2}$ and the enlargement of the plane around a by a factor of $\frac{1}{2}$. Then 1×2 **associate path for** a, denoted TAP_a, is defined by $TAP_a = \{T_a(b) \mid$ for all $b \in C$ such that $b \neq a\}$ and 2×1 **associate path for** a, denoted UAP_a, is defined by $UAP_a = \{U_a(b) \mid \text{for all } b \in C \text{ such that } b \neq a\}$. Because of the combined transformation, both TAP_a and UAP_a are not NDLP. Hence every point in C is not the head of a 1×2 or 2×1 inscribed half-rectangle in C.

In Figure 3.3.3, let black points be the NDLSCC. Suppose the black point with square around it is the head of an inscribed half-rectangle. Then larger empty circle is the every possible foot of a 1×2 half-rectangle and smaller empty circle is the every possible foot of a 2×1 half-rectangle. Because every empty circle do not overlap with any of the black points, there is no IHR.

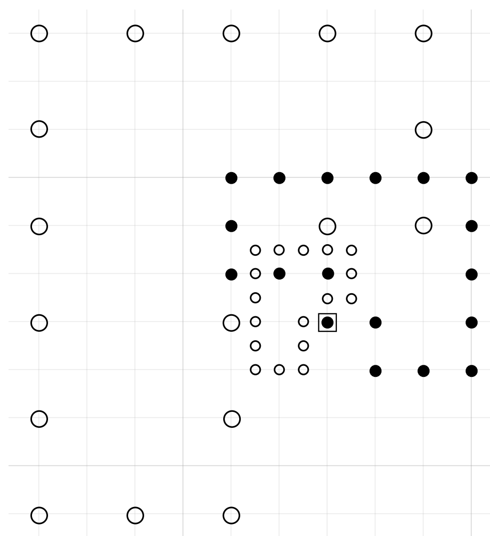

Figure 3.3.3. Countexample of an IHR

4

Cloud

4.1 Missing Corners

Now that we clearly see that every point in C is the head of an IHS, we should consider the missing point to form an IHS into a square.

Definition 4.1.1. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $\{a, b, c\} \subseteq C$ such that $\{a, b, c\}$ is a half-square. The **missing corner** in $\mathbb{Z}^2$ is a point $d \in \mathbb{Z}^2$ such that $d \neq b$ and $d(a, b) = d(b, c) = d(c, d) = d(a, d)$. △

Figure 4.1.1 shows the example of a missing corner in NDLSCC. Empty circle represents a missing corner found by 3 other points with empty squares, which is IHS.

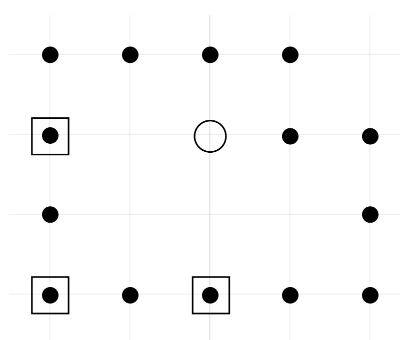

Figure 4.1.1. Example of a missing corner

We define cloud of NDLSCC as the set of missing corners.

Definition 4.1.2. Let $C \subseteq \mathbb{Z}^2$ be a NDLSCC. The **cloud** of C is a set of all missing corners of all possible inscribed half-squares in C. $\triangle$

In Figure 4.1.2, black points form NDLSCC and empty circles are the cloud of NDLSCC.

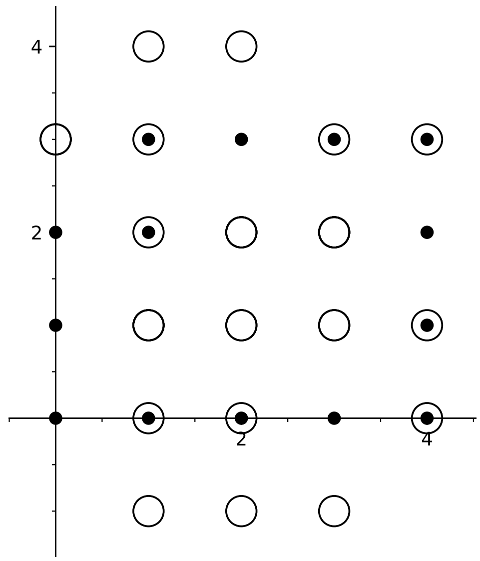

Figure 4.1.2. Example of a cloud

4.2 Cloud for Rectangle

In this section, let us consider the cloud of a rectangular NDLSCC. It is a specific case as it demonstrates clear pattern. First, we define integer interval to keep expressions simple.

Definition 4.2.1. Let $a, b \in \mathbb{N}$. The **integer interval** from a to b is denoted as $[[a, b]] = \{x \in \mathbb{Z} \mid a \leq x \leq b\} = \{a, a+1, ..., b\}$. $\triangle$

Let us define a rectangular NDLSCC.

Definition 4.2.2. Let $a, b \in \mathbb{N}$ such that $a, b \geq 3$. Suppose $a > b$. Let

$$B(a, b) = [[0, a]] \times \{0\},$$

$$T(a, b) = [[0, a]] \times \{b\},$$

$$L(a, b) = \{0\} \times [[1, b-1]],$$

and

$$R(a,b) = \{a\} \times [[1, b-1]].$$

Let $Rect(a,b)$ be NDLSCC that is the $a \times b$ rectangle given by

$$Rect(a,b) = B(a,b) \cup T(a,b) \cup L(a,b) \cup R(a,b).$$

△

We will look at three different types of rectangles. First, consider $Rect(a,b)$ with $a \geq 2b$ and $b \geq 3$.

In Figure 4.2.1, empty circles are the cloud of $Rect(15,5)$.

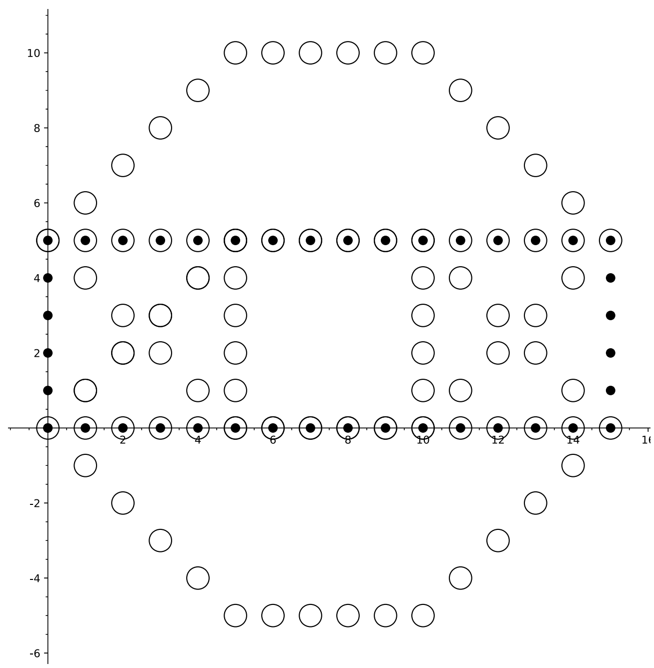

Figure 4.2.1. The cloud of $Rect(a,b)$ when $a \geq 2b$

We separate the cloud of a rectangle into beam, arm, body, bridge and cross.

Definition 4.2.3. Let $a, b \in \mathbb{N}$. Suppose $b \geq 3$, and suppose $a \geq 2b$.

1. Let the **beam** of $Rect(a,b)$ be

$$Be(a,b) = Be_1(a,b) \cup Be_2(a,b)$$

where

$$Be_1(a,b) = \{(x,-b) \mid x \in [[b, a-b]]\}$$

and

$$Be_2(a,b) = \{(x,2b) \mid x \in [[b, a-b]]\}.$$

2. Let the **arm** of $Rect(a,b)$ be

$$Ar(a,b) = Ar_1(a,b) \cup Ar_2(a,b) \cup Ar_3(a,b) \cup Ar_4(a,b)$$

where

$$Ar_1(a,b) = \{(x,b+x) \mid x \in [[1, b-1]]\},$$

$$Ar_2(a,b) = \{(x,-x) \mid x \in [[1, b-1]]\},$$

$$Ar_3(a,b) = \{((a-b)+x, 2b-x) \mid x \in [[1, b-1]]\}$$

and

$$Ar_4(a,b) = \{((a-b)+x, -b+x) \mid x \in [[1, b-1]]\}.$$

3. Let the **body** of $Rect(a,b)$ be

$$Bd(a,b) = Bd_1(a,b) \cup Bd_2(a,b)$$

where

$$Bd_1(a,b) = \{(x,0) \mid x \in [[0,a]]\}$$

and

$$Bd_2(a,b) = \{(x,b) \mid x \in [[0,a]]\}.$$

4. Let the **bridge** of $Rect(a,b)$ be

$$Br(a,b) = Br_1(a,b) \cup Br_2(a,b)$$

where

$$Br_1(a,b) = \{(b,x) \mid x \in [[1,b-1]]\}$$

and

$$Br_2(a,b) = \{(a-b,x) \mid x \in [[1,b-1]]\}.$$

5. Let the **cross** of $Rect(a,b)$ be

$$Cr(a,b) = Cr_1(a,b) \cup Cr_2(a,b) \cup Cr_3(a,b) \cup Cr_4(a,b)$$

where

$$Cr_1(a,b) = \{(x,x) \mid x \in [[1,b-1]]\},$$

$$Cr_2(a,b) = \{(x,b-x) \mid x \in [[1,b-1]]\},$$

$$Cr_3(a,b) = \{((a-b)+x,x) \mid x \in [[1,b-1]]\}$$

and

$$Cr_4(a,b) = \{((a-b)+x,b-x) \mid x \in [[1,b-1]]\}.$$

△

In Figure 4.2.2, empty circles are the beams of $Rect(15,5)$.

In Figure 4.2.3, empty circles are the arms of $Rect(15,5)$.

In Figure 4.2.4, empty circles are the bodies of $Rect(15,5)$.

In Figure 4.2.5, empty circles are the bridges of $Rect(15,5)$. Note that when $a = 2b$, two bridges overlap.

In Figure 4.2.6, empty circles are the crosses of $Rect(15,5)$.

Now we show that all the missing corners given by a rectangle is the union of beam, arm, body, bridge and cross.

Theorem 4.2.4. *Let $a,b \in \mathbb{N}$. Suppose $b \geq 3$, and suppose $a \geq 2b$. Then the cloud of $Rect(a,b)$ is $Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$.*

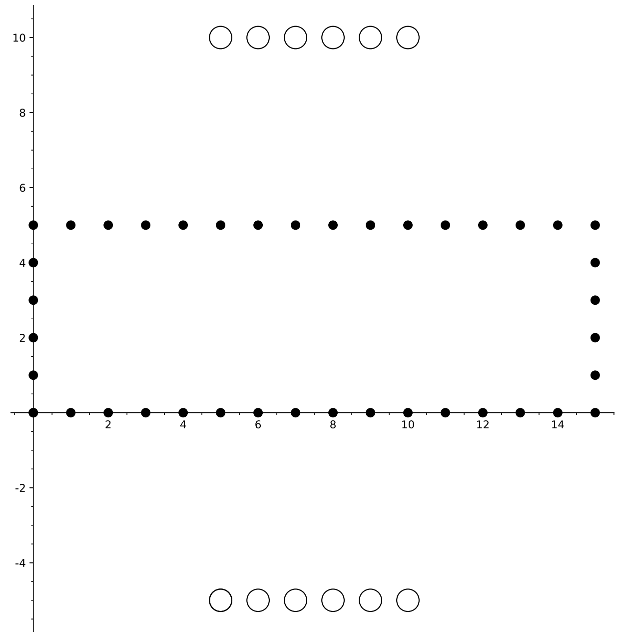

Figure 4.2.2. Example of beam when $a \geq 2b$

Proof. Let C be the cloud of $Rect(a,b)$. First we are going to show that $C \subseteq Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$.

By Theorem 3.1.3, we know that every point in $Rect(a,b)$ is the head of an IHS. Let $p \in Rect(a,b)$. Consider different cases for all possible IHS with p as head in $Rect(a,b)$.

1. Suppose $p = (0,0)$, $p = (0,b)$, $p = (a,0)$ or $p = (a,b)$. First, let us consider when $p = (0,0)$. All possible feet of IHS with p as head are $(0,i)$ and $(i,0)$ for each $i \in [[1,b]]$. Then for each $i \in [[1,b]]$, missing corner is (i,i). Hence the set of all missing corners is

$$\{(i,i) \mid i \in [[1,b]]\} = Cr_1(a,b) \cup \{(b,b)\} \subseteq Cr_1(a,b) \cup Bd_2(a,b).$$

By symmetry, we see that sets of all missing corners from other p in this case are

$$\{(i,b-i) \mid i \in [[1,b]]\} = Cr_2(a,b) \cup \{(b,0)\} \subseteq Cr_2(a,b) \cup Bd_1(a,b),$$

$$\{((a-b)+i,b-i) \mid i \in [[0,b-1]]\} = Cr_4(a,b) \cup \{(a-b,b)\} \subseteq Cr_4(a,b) \cup Bd_2(a,b)$$

and

$$\{((a-b)+i,i) \mid i \in [[0,b-1]]\} = Cr_3(a,b) \cup \{(a-b,0)\} \subseteq Cr_3(a,b) \cup Bd_1(a,b).$$

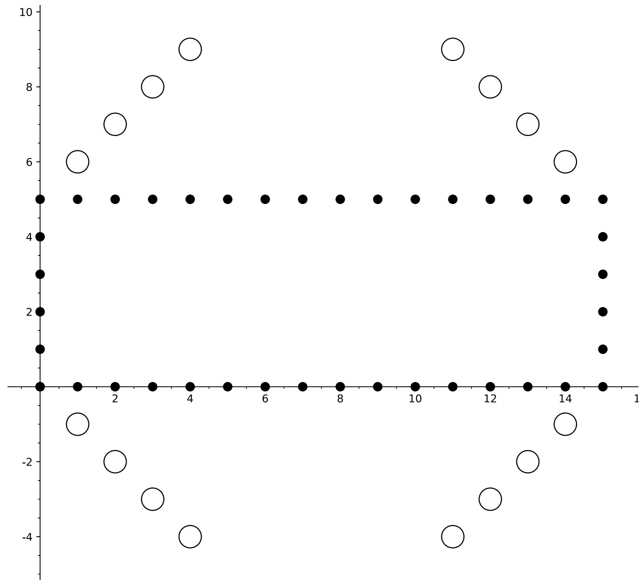

Figure 4.2.3. Example of arm when $a \geq 2b$

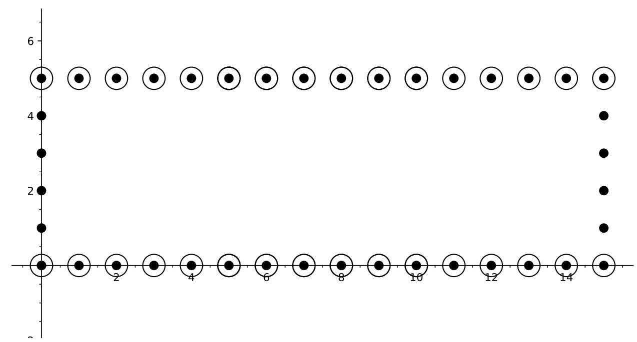

Figure 4.2.4. Example of body when $a \geq 2b$

2. Suppose $p \in [[1, b-1]] \times \{0\}$, $p \in [[1, b-1]] \times \{b\}$, $p \in [[(a-b)+1, a-1]] \times \{0\}$ or $p \in [[(a-b)+1, a-1]] \times \{b\}$. Let us consider when $p = (1, 0) \in [[1, b-1]] \times \{0\}$. All possible feet of IHS with p as head are $(k+b, 0)$ and (k, b) for each $k \in [[1, b]]$. Then for each $k \in [[1, b]]$, missing corner is $(k+b, b)$. Hence the set of all missing corners is

$$\{(k+b, b) \mid k \in [[1, b-1]]\} \subseteq Bd_2(a, b).$$

By symmetry, we see that sets of all missing corners from other p in this case are

$$\{(k+b, 0) \mid k \in [[1, b-1]]\} \subseteq Bd_1(a, b),$$

$$\{(k-b, b) \mid k \in [[(a-b)+1, a-1]]\} \subseteq Bd_2(a, b)$$

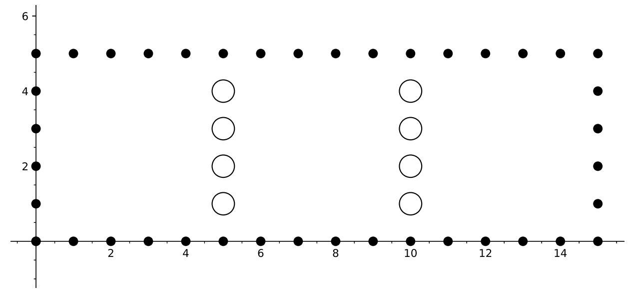

Figure 4.2.5. Example of bridge when $a \geq 2b$

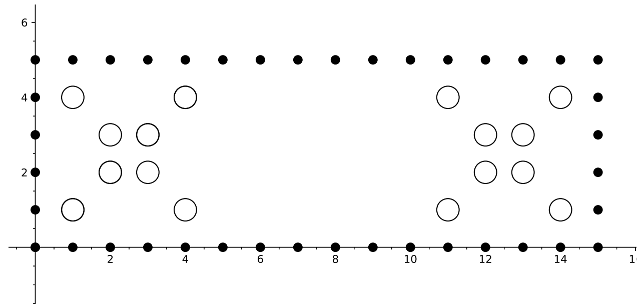

Figure 4.2.6. Example of cross when $a \geq 2b$

and

$$\{(k-b,0) \mid k \in [[(a-b)+1, a-1]]\} \subseteq Bd_1(a,b).$$

3. Suppose $p=(b,0)$, $p=(b,b)$, $p=(a-b,0)$ or $p=(a-b,b)$. Let us consider when $p=(b,0)$. The possible IHS with p as head are following sub-cases.

   (a) The feet of IHS with p as head are $(2b,0)$ and (b,b). Hence the missing corner is $(2b,b) \in Bd_2(a,b)$.

   (b) The feet of IHS with p as head are $(0,i)$ and $(b+i,b)$ for each $i \in [[0,b]]$. Then for each $i \in [[0,b]]$, missing corner is $(i,b+i)$. Hence the set of all missing corners is

   $$\{(i,b+i) \mid i \in [[0,b]]\} = Ar_1(a,b) \cup \{(0,b),(b,2b)\} \subseteq Ar_1(a,b) \cup Bd_2(a,b) \cup Be_2(a,b).$$

   By symmetry, missing corners from other p from sub-case (a) are $(2b,0) \in Bd_1(a,b)$, $(a-2b,b) \in Bd_2(a,b)$ and $(a-2b,0) \in Bd_1(a,b)$. We see that sets of all missing corners from other p in sub-case (b) are

$$\{(i,-i) \mid i \in [[0,b]]\} = Ar_2(a,b) \cup \{(0,0),(b,-b)\} \subseteq Ar_2(a,b) \cup Bd_1(a,b) \cup Be_1(a,b),$$

$\{((a-b)+i, 2b-i) \mid i \in [[0,b]]\} = Ar_3(a,b) \cup \{(a,b),(a-b,2b)\} \subseteq Ar_3(a,b) \cup Bd_1(a,b) \cup Be_2(a,b)$

and

$\{((a-b)+i, -b+i) \mid i \in [[0,b]]\} = Ar_4(a,b) \cup \{(a,0),(a-b,-b)\} \subseteq Ar_4(a,b) \cup Bd_2(a,b) \cup Be_1(a,b).$

4. Suppose $p \in [[b+1,(a-b)-1]] \times \{0\}$ or $p \in [[b+1,(a-b)-1]]$. Let us consider when $p = (b+1,0) \in [[b+1,(a-b)-1]] \times \{0\}$. The possible IHS with p as head are following sub-cases.

   (a) The feet of IHS with p as head are $(k+b,0)$ and (k,b) for each $k \in [[b+1,(a-b)-1]]$. Then for each $k \in [[b+1,(a-b)-1]]$, missing corner is $(k+b,b)$. Hence the set of all missing corners is

   $$\{(k+b,b) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Bd_2(a,b).$$

   (b) The feet of IHS with p as head are $(k-b,0)$ and (k,b) for each $k \in [[b+1,(a-b)-1]]$. Then for each $k \in [[b+1,(a-b)-1]]$, missing corner is $(k-b,b)$. Hence the set of all missing corners is

   $$\{(k-b,b) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Bd_2(a,b).$$

   (c) The feet of IHS with p as head are $(k-b,b)$ and $(k+b,b)$ for each $k \in [[b+1,(a-b)-1]]$. Then for each $k \in [[b+1,(a-b)-1]]$, missing corner is $(k,2b)$. Hence the set of all missing corners is

   $$\{(k,2b) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Be_2(a,b).$$

By symmetry, we see that sets of all missing corners from $p \in [[b+1,(a-b)-1]]$ are

$$\{(k+b,0) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Bd_1(a,b),$$

$$\{(k-b,0) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Bd_1(a,b),$$

and

$$\{(k,-b) \mid k \in [[b+1,(a-b)-1]]\} \subseteq Be_1(a,b).$$

5. Suppose when $p \in \{0\} \times [[1, b-1]]$ or $p \in \{b\} \times [[1, b-1]]$. Let us consider when $p = (0, 1) \in \{0\} \times [[1, b-1]]$. All possible feet of IHS with p as head are (k, b) and $(b-k, 0)$ for each $k \in [[1, b-1]]$. Then for each $k \in [[1, b-1]]$, missing corner is $(b, b-k)$. Hence the set of all missing corners is

$$\{(b, b-k) \mid k \in [[1, b-1]]\} = Br_1(a, b).$$

By symmetry, we see that set of all missing corners from $p \in \{b\} \times [[1, b-1]]$ is

$$\{(a-b, b-k) \mid k \in [[1, b-1]]\} = Br_2(a, b).$$

Given all the cases, the set of all missing corners, which is the cloud of $Rect(a, b)$, is a subset of $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$. Hence $C \subseteq Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$.

Now we are going to show that $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b) \subseteq C$ by showing that each set in the union is a subset of C.

First, we claim that $Be(a, b) \subseteq C$. The set $Be(a, b)$ is given by

$$Be(a, b) = \{(x, -b) \mid x \in [[b, a-b]]\} \cup \{(x, 2b) \mid x \in [[b, a-b]]\}.$$

Let $q \in Be(a, b)$. Then $q \in \{(x, -b) \mid x \in [[b, a-b]]\}$ or $q \in \{(x, 2b) \mid x \in [[b, a-b]]\}$. We consider cases.

1. Suppose $q \in \{(x, -b) \mid x \in [[b, a-b]]\}$. Consider sub-cases.

   (a) Suppose $q = (b, -b)$ or $q = ((a-b), -b)$. By Case 3(b) of the previous part of the proof, we have $q \in \{(i, -i) \mid i \in [[0, b]]\} \cup \{((a-b)+i, -b+i) \mid i \in [[0, b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, -b) \mid x \in [[b+1, (a-b)-1]]\}$. By Case 4(c) of the previous part of the proof, we have $q \in \{(k, -b) \mid k \in [[b+1, (a-b)-1]]\} \subseteq C$.

2. Suppose $q \in \{(x, 2b) \mid x \in [[b, a-b]]\}$. Consider sub-cases.

   (a) Suppose $q = (b, 2b)$ or $q = ((a-b), 2b)$. By Case 3(b) of the previous part of the proof, we have $q \in \{(i, b+i) \mid i \in [[0, b]]\} \cup \{((a-b)+i, 2b-i) \mid i \in [[0, b]]\} \subseteq C$.

(b) Suppose $q \in \{(x, 2b) \mid x \in [[b+1, (a-b)-1]]\}$. By Case 4(c) of the previous part of the proof, we have $q \in \{(k, 2b) \mid k \in [[b+1, (a-b)-1]]\} \subseteq C$.

The above cases show that $q \in C$. Hence $Be(a, b) \subseteq C$.

Second, we claim that $Ar(a, b) \subseteq C$. Let $q \in Ar(a, b)$. Then $q \in Ar_1(a, b)$, $q \in Ar_2(a, b)$, $q \in Ar_3(a, b)$ or $q \in Ar_4(a, b)$. We consider cases.

1. Suppose $q \in Ar_1(a, b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_1(a, b) \cup \{(0, b), (b, 2b)\} \subseteq C$.

2. Suppose $q \in Ar_2(a, b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_2(a, b) \cup \{(0, 0), (b, -b)\} \subseteq C$.

3. Suppose $q \in Ar_3(a, b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_3(a, b) \cup \{(a, b), (a-b, 2b)\} \subseteq C$.

4. Suppose $q \in Ar_4(a, b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_4(a, b) \cup \{(a, 0), (a-b, -b)\} \subseteq C$.

The above cases show that $q \in C$. Hence $Ar(a, b) \subseteq C$, which is the subset of C.

Third, We claim that $Bd(a, b) \subseteq C$. The set $Bd(a, b)$ is given by

$$Bd(a, b) = \{(x, 0) \mid x \in [[0, a]]\} \cup \{(x, b) \mid x \in [[0, a]]\}.$$

Let $q \in Bd(a, b)$. Then $q \in \{(x, 0) \mid x \in [[0, a]]\}$ or $q \in \{(x, b) \mid x \in [[0, a]]\}$. We consider cases.

1. Suppose $q \in \{(x, 0) \mid x \in [[0, a]]\}$. Consider sub-cases.

   (a) Suppose $q = (b, 0)$ or $q = (a-b, 0)$. By Case 1 of the previous part of the proof, we have $q \in \{(i, b-i) \mid i \in [[0, b]]\} \cup \{((a-b)+i, i) \mid i \in [[0, b-1]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, 0) \mid x \in [[b+1, 2b-1]]\}$ or $q \in \{(x, 0) \mid x \in [[(a-2b)+1, (a-b)+1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k+b, 0) \mid k \in [[1, b-1]]\} \cup \{(k-b, 0) \mid k \in [[(a-b)+1, a-1]]\} \subseteq C$.

(c) Suppose $q = (2b, 0)$ or $(a - 2b, 0)$. By Case 3(a) of the previous part of the proof, we have $q \in \{(2b, 0), (a - 2b, 0)\} \subseteq C$.

(d) Suppose $q = (0, 0)$ or $(a, 0)$. By Case 3(b) of the previous part of the proof, we have $q \in \{(i, -i) \mid i \in [[0, b]]\} \cup \{((a - b) + i, b + i) \mid i \in [[0, b]]\} \subseteq C$.

(e) Suppose $q \in \{(x, 0) \mid x \in [[2b + 1, a - 1]]\}$ or $q \in \{(x, 0) \mid x \in [[1, (a - 2b) - 1]]\}$. By Case 4 of the previous part of the proof, we have $q \in \{(k + b, 0) \mid i \in [[b + 1, (a - b) - 1]]\} \cup q \in \{(k - b, 0) \mid i \in [[b + 1, (a - b) - 1]]\} \subseteq C$.

2. Suppose $q \in \{(x, b) \mid x \in [[0, a]]\}$.

(a) Suppose $q = (b, b)$ or $q = (a - b, b)$. By Case 1 of the previous part of the proof, we have $q \in \{(i, i) \mid i \in [[0, b]]\} \cup \{((a - b) + i, -b - i) \mid i \in [[0, b - 1]]\} \subseteq C$.

(b) Suppose $q \in \{(x, b) \mid x \in [[b+1, 2b-1]]\}$ or $q \in \{(x, b) \mid x \in [[(a-2b)+1, (a-b)+1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k+b, b) \mid k \in [[1, b-1]]\} \cup \{(k - b, b) \mid k \in [[(a - b) + 1, a - 1]]\} \subseteq C$.

(c) Suppose $q = (2b, b)$ or $(a - 2b, b)$. By Case 3(a) of the previous part of the proof, we have $q \in \{(2b, b), (a - 2b, b)\} \subseteq C$.

(d) Suppose $q = (0, b)$ or (a, b). By Case 3(b) of the previous part of the proof, we have $q \in \{(i, b + i) \mid i \in [[0, b]]\} \cup \{((a - b) + i, 2b - i) \mid i \in [[0, b]]\} \subseteq C$.

(e) Suppose $q \in \{(x, b) \mid x \in [[2b + 1, a - 1]]\}$ or $q \in \{(x, b) \mid x \in [[1, (a - 2b) - 1]]\}$. By Case 4 of the previous part of the proof, we have $q \in \{(k + b, b) \mid i \in [[b + 1, (a - b) - 1]]\} \cup q \in \{(k - b, b) \mid i \in [[b + 1, (a - b) - 1]]\} \subseteq C$.

The above cases show that $q \in C$. Hence $Bd(a, b) \subseteq C$.

Fourth, We claim that $Br(a, b) \subseteq C$. Let $q \in Br(a, b)$. Then $q \in Br_1(a, b)$ or $q \in Br_2(a, b)$. We consider cases.

1. Suppose $q \in Br_1(a, b)$. By Case 5 of the previous part of the proof, we have $q \in Br_1(a, b) \subseteq C$.

2. Suppose $q \in Br_2(a, b)$. By Case 5 of the previous part of the proof, we have $q \in Br_2(a, b) \subseteq C$.

The above cases show that $q \in C$. Hence $Br(a, b) \subseteq C$.

Last, we claim that $Cr(a, b) \subseteq C$. Let $q \in Cr(a, b)$. we have $q \in Cr_1(a, b)$, $q \in Cr_2(a, b)$, $q \in Cr_3(a, b)$ or $q \in Cr_4(a, b)$. We consider cases.

1. Suppose $q \in Cr_1(a, b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_1(a, b) \cup \{(b, b)\} \subseteq C$.

2. Suppose $q \in Cr_2(a, b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_2(a, b) \cup \{(b, 0)\} \subseteq C$.

3. Suppose $q \in Cr_3(a, b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_3(a, b) \cup \{(a - b, 0)\} \subseteq C$.

4. Suppose $q \in Cr_4(a, b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_4(a, b) \cup \{(a - b, b)\} \subseteq C$.

The above cases show that $q \in C$. Hence $Cr(a, b) \subseteq C$.

Because each set is a subset of C, $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b) \subseteq C$.

Since we have proved that $C \subseteq Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$ and $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b) \subseteq C$, we can conclude that the cloud of $Rect(a, b)$ is $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$. □

This time, we consider a rectangle $Rect(a, b)$ with $b < a < 2b$ and $b \geq 3$.

In Figure 4.2.7, empty circles are the cloud of $Rect(10, 7)$.

We still name parts of cloud with same term, but some of them will be defined differently.

Definition 4.2.5. Let $a, b \in \mathbb{N}$. Suppose $b \geq 3$, and suppose $b < a < 2b$. We define $Br(a, b)$ and $Cr(a, b)$ as we did in Definition 4.2.3.

1. Let the **beam** of $Rect(a, b)$ be

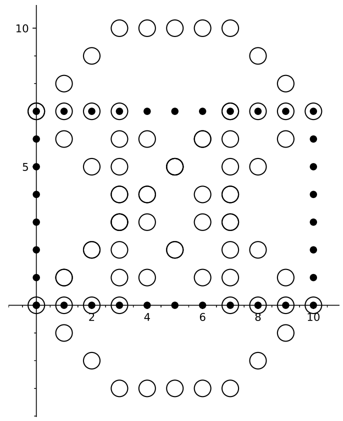

Figure 4.2.7. The cloud of $Rect(a,b)$ when $b < a < 2b$

$$Be(a,b) = Be_1(a,b) \cup Be_2(a,b)$$

where

$$Be_1(a,b) = \{(x, -(a-b)) \mid x \in [[a-b, b]]\}$$

and

$$Be_2(a,b) = \{(x, (a-b)+b) \mid x \in [[a-b, b]]\}.$$

2. Let the **arm** of $Rect(a,b)$ be

$$Ar(a,b) = Ar_1(a,b) \cup Ar_2(a,b) \cup Ar_3(a,b) \cup Ar_4(a,b)$$

where

$$Ar_1(a,b) = \{(x, b+x) \mid x \in [[(a-b)-1, b+1]]\},$$

$$Ar_2(a,b) = \{(x, -x) \mid x \in [[(a-b)-1, b+1]]\},$$

$$Ar_3(a,b) = \{((a-b)+x, 2b-x) \mid x \in [[(a-b)-1, b+1]]\}$$

and

$$Ar_4(a,b) = \{((a-b)+x, -b+x) \mid x \in [[(a-b)-1, b+1]]\}.$$

3. Let the **body** of $Rect(a,b)$ be

$$Bd(a,b) = Bd_1(a,b) \cup Bd_2(a,b) \cup Bd_3(a,b) \cup Bd_4(a,b)$$

where

$$Bd_1(a,b) = \{(x,0) \mid x \in [[0, a-b]]\},$$

$$Bd_2(a,b) = \{(x,0) \mid x \in [[b, a]]\},$$

$$Bd_3(a,b) = \{(x,b) \mid x \in [[0, a-b]]\}$$

and

$$Bd_4(a,b) = \{(x,b) \mid x \in [[b, a]]\}.$$

△

In Figure 4.2.8, empty circles are the beams of $Rect(10,7)$.

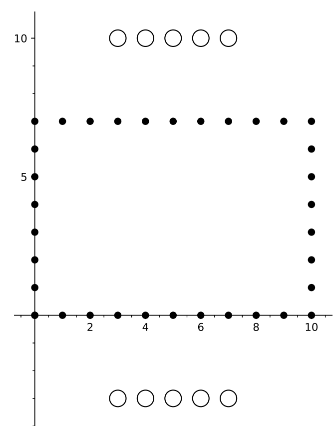

Figure 4.2.8. Example of beam when $b < a < 2b$

In Figure 4.2.9, empty circles are the arms of $Rect(10,7)$.

In Figure 4.2.10, empty circles are the bodies of $Rect(10,7)$.

In Figure 4.2.11, empty circles are the bridges of $Rect(10,7)$.

In Figure 4.2.12, empty circles are the crosses of $Rect(10,7)$.

The following proof follows the same procedure as Theorem 4.2.4.

Theorem 4.2.6. *Let $a, b \in \mathbb{N}$. Suppose $b \geq 3$, and suppose $b < a < 2b$. Then the cloud of $Rect(a,b)$ is $Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$.*

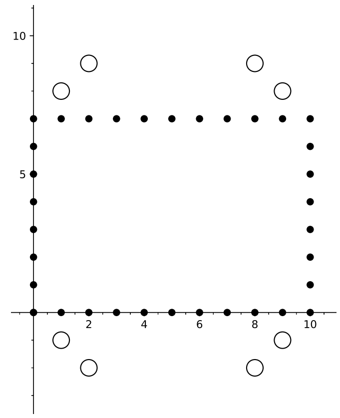

Figure 4.2.9. Example of arm when $b < a < 2b$

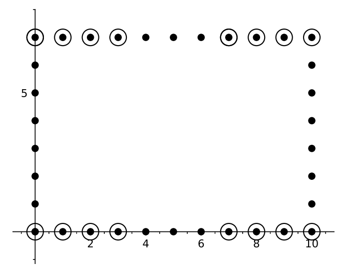

Figure 4.2.10. Example of body when $b < a < 2b$

Proof. Let C be the cloud of $Rect(a,b)$. First we are going to show that $C \subseteq Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$. Let $p \in Rect(a,b)$. Consider different cases for all possible IHS with p as head in $Rect(a,b)$.

1. Suppose $p = (0,0)$, $p = (0,b)$, $p = (a,0)$ or $p = (a,b)$. First, let us consider when $p = (0,0)$. All possible feet of IHS with p as head are $(0,i)$ and $(i,0)$ for each $i \in [[1,b]]$. Then for each $i \in [[1,b]]$, missing corner is (i,i). Hence the set of all missing corners is

$$\{(i,i) \mid i \in [[1,b]]\} = Cr_1(a,b) \cup \{(b,b)\} \subseteq Cr_1(a,b) \cup Bd_2(a,b).$$

By symmetry, we see that sets of all missing corners from other p in this case are

$$\{(i,b-i) \mid i \in [[1,b]]\} = Cr_2(a,b) \cup \{(b,0)\} \subseteq Cr_2(a,b) \cup Bd_1(a,b),$$

$$\{((a-b)+i,b-i) \mid i \in [[0,b-1]]\} = Cr_4(a,b) \cup \{(a-b,b)\} \subseteq Cr_4(a,b) \cup Bd_2(a,b)$$

and

$$\{((a-b)+i,i) \mid i \in [[0,b-1]]\} = Cr_3(a,b) \cup \{(a-b,0)\} \subseteq Cr_3(a,b) \cup Bd_1(a,b).$$

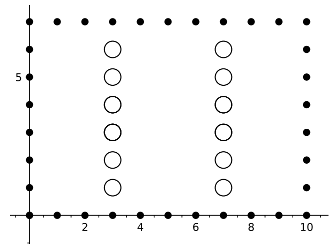

Figure 4.2.11. Example of bridge when $b < a < 2b$

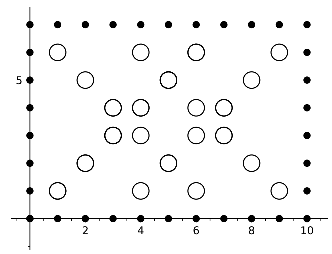

Figure 4.2.12. Example of cross when $b < a < 2b$

2. Suppose $p \in [[1,(a-b)-1]] \times \{0\}$, $p \in [[1,(a-b)-1]] \times \{b\}$, $p \in [[b+1,a-1]] \times \{0\}$ or $p \in [[b+1,a-1]] \times \{b\}$. Let us consider when $p = (1,0) \in [[1,(a-b)-1]] \times \{0\}$. All possible feet of IHS with p as head are $(k+b,0)$ and (k,b) for each $k \in [[1,(a-b)-1]]$. Then for each $k \in [[1,(a-b)-1]]$, missing corner is $(k+b,b)$. Hence the set of all missing corners is

$$\{(k+b,b) \mid k \in [[1,(a-b)-1]]\} \subseteq Bd_4(a,b).$$

By symmetry, we see that sets of all missing corners from other p in this case are

$$\{(k+b,0) \mid k \in [[1,(a-b)-1]]\} \subseteq Bd_2(a,b),$$

$$\{(k-b,b) \mid k \in [[b+1,a-1]]\} \subseteq Bd_3(a,b)$$

and

$$\{(k-b,0) \mid k \in [[b+1,a-1]]\} \subseteq Bd_1(a,b).$$

3. Suppose $p = (b,0)$, $p = (b,b)$, $p = (a-b,0)$ or $p = (a-b,b)$. Let us consider when $p = (a-b,0)$. The possible IHS with p as head are following sub-cases.

   (a) The feet of IHS with p as head are $(a,0)$ and $(a-b,b)$. Hence the missing corner is $(a,b) \in Bd_2(a,b)$.

(b) The feet of IHS with p as head are $(0,i)$ and $(b+i,b)$ for each $i \in [[0,a-b]]$. Then for each $i \in [[0,a-b]]$, missing corner is $(i,b+i)$. Hence the set of all missing corners is

$$\{(i,b+i) \mid i \in [[0,a-b]]\} = Ar_1(a,b) \cup \{(0,b),(b,a)\} \subseteq Ar_1(a,b) \cup Bd_3(a,b) \cup Be_1(a,b).$$

By symmetry, missing corners from other p from sub-case (a) are $(0,0) \in Bd_1(a,b)$, $(0,b) \in Bd_2(a,b)$ and $(a,0) \in Bd_1(a,b)$. We see that sets of all missing corners from other p in sub-case (b) are

$$\{(i,-i) \mid i \in [[0,a-b]]\} = Ar_2(a,b) \cup \{(0,0),(a-b,-a+b)\} \subseteq Ar_2(a,b) \cup Bd_1(a,b) \cup Be_1(a,b),$$

$$\{(b+i,a-i) \mid i \in [[0,a-b]]\} = Ar_3(a,b) \cup \{(b,a),(a,b)\} \subseteq Ar_3(a,b) \cup Bd_4(a,b) \cup Be_2(a,b)$$

and

$$\{((b+i,-(a-b)+i) \mid i \in [[0,a-b]]\} = Ar_4(a,b) \cup \{(a,0),(b,-a+b)\} \subseteq Ar_4(a,b) \cup Bd_2(a,b) \cup Be_1(a,b).$$

4. Suppose $p \in [[(a-b)+1,b-1]] \times \{0\}$ or $p \in [[(a-b)+1,b-1]]$. Let us consider when $p=(b+1,0) \in [[(a-b)+1,b-1]] \times \{0\}$. All possible feet of IHS with p as head are $(k-b,b)$ and $(k+b,b)$ for each $k \in [[(a-b)+1,b-1]]$. Then for each $k \in [[(a-b)+1,b-1]]$, missing corner is $(k,2b)$. Hence the set of all missing corners is

$$\{(k,a) \mid k \in [[(a-b)+1,b-1]]\} \subseteq Be_2(a,b).$$

By symmetry, we see that sets of all missing corners from $p \in [[(a-b)+1,b-1]]$ is

$$\{(k,-(a-b)) \mid k \in [[(a-b)+1,b-1]]\} \subseteq Be_1(a,b).$$

5. Suppose when $p \in \{0\} \times [[1,b-1]]$ or $p \in \{b\} \times [[1,b-1]]$. Let us consider when $p=(0,1) \in \{0\} \times [[1,b-1]]$. All possible feet of IHS with p as head are (k,b) and $(b-k,0)$ for each $k \in [[1,b-1]]$. Then for each $k \in [[1,b-1]]$, missing corner is $(b,b-k)$. Hence the set of all missing corners is

$$\{(b,b-k) \mid k \in [[1,b-1]]\} = Br_1(a,b).$$

By symmetry, we see that set of all missing corners from $p \in \{b\} \times [[1, b-1]]$ is

$$\{(a-b, b-k) \mid k \in [[1, b-1]]\} = Br_2(a, b).$$

Given all the cases, the set of all missing corners, which is the cloud of $Rect(a, b)$, is a subset of $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$. Hence $C \subseteq Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$.

Now we are going to show that $Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b) \subseteq C$ by showing that each set in the union is a subset of C.

First, we claim that $Be(a, b) \subseteq C$. The set $Be(a, b)$ is given by

$$Be(a, b) = \{(x, -(a-b) \mid x \in [[a-b, b]]\} \cup \{(x, a) \mid x \in [[a-b, b]]\}.$$

Let $q \in Be(a, b)$. Then $q \in \{(x, -(a-b) \mid x \in [[a-b, b]]\}$ or $q \in \{(x, a) \mid x \in [[a-b, b]]\}$. We consider cases.

1. Suppose $q \in \{(x, -(a-b) \mid x \in [[a-b, b]]\}$. Consider sub-cases.

   (a) Suppose $q = ((a-b), -(a-b))$ or $q = (b, -(a-b))$. By Case 3(b) of the previous part of the proof, we have $q \in \{(i, -i) \mid i \in [[0, a-b]]\} \cup \{(b+i, a-i) \mid i \in [[0, a-b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, -(a-b) \mid x \in [[(a-b)+1, b-1]]\}$. By Case 4(c) of the previous part of the proof, we have $q \in \{(k, -(a-b)) \mid k \in [[(a-b)+1, b-1]]\} \subseteq C$.

2. Suppose $q \in \{(x, a) \mid x \in [[a-b, b]]\}$. Consider sub-cases.

   (a) Suppose $q = ((a-b), a)$ or $q = (b, a)$. By Case 3(b) of the previous part of the proof, we have $q \in \{(i, b+i) \mid i \in [[0, a-b]]\} \cup \{((a-b)+i, 2b-i) \mid i \in [[0, b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, a) \mid x \in [[(a-b)+1, b-1]]\}$. By Case 4(c) of the previous part of the proof, we have $q \in \{(k, a) \mid k \in [[(a-b)+1, b-1]]\} \subseteq C$.

The above cases show that $q \in C$. Hence $Be(a, b) \subseteq C$.

Second, we claim that $Ar(a, b) \subseteq C$. Let $q \in Ar(a, b)$. Then $q \in Ar_1(a, b)$, $q \in Ar_2(a, b)$, $q \in Ar_3(a, b)$ or $q \in Ar_4(a, b)$. We consider cases.

1. Suppose $q \in Ar_1(a,b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_1(a,b) \cup \{(0,b),(b,2b)\} \subseteq C$.

2. Suppose $q \in Ar_2(a,b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_2(a,b) \cup \{(0,0),(b,-b)\} \subseteq C$.

3. Suppose $q \in Ar_3(a,b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_3(a,b) \cup \{(a,b),(a-b,2b)\} \subseteq C$.

4. Suppose $q \in Ar_4(a,b)$. By Case 3(b) of the previous part of the proof, we have $q \in Ar_4(a,b) \cup \{(a,0),(a-b,-b)\} \subseteq C$.

The above cases show that $q \in C$. Hence $Ar(a,b) \subseteq C$, which is the subset of C.

Third, We claim that $Bd(a,b) \subseteq C$. The set $Bd(a,b)$ is given by

$$Bd(a,b) = \{(x,0) \mid x \in [[0,a-b]]\} \cup \{(x,0) \mid x \in [[b,a]]\} \cup \{(x,b) \mid x \in [[0,a-b]]\} \cup \{(x,b) \mid x \in [[b,a]]\}.$$

Let $q \in Bd(a,b)$. Then $q\{(x,0) \mid x \in [[0,a-b]]\}$, $q\{(x,0) \mid x \in [[b,a]]\}$, $q\{(x,b) \mid x \in [[0,a-b]]\}$ or $q \in \{(x,b) \mid x \in [[b,a]]\}$. We consider cases.

1. Suppose $q \in \{(x,0) \mid x \in [[0,a-b]]\}$. Consider sub-cases.

   (a) Suppose $q = (a-b,0)$. By Case 1 of the previous part of the proof, we have $q \in \{((a-b)+i,i) \mid i \in [[0,b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x,0) \mid x \in [[1,a-b-1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k-b,0) \mid k \in [[b+1,a-1]]\} \subseteq C$.

   (c) Suppose $q = (0,0)$. By Case 3(b) of the previous part of the proof, we have $q = (0,0) \subseteq C$.

2. Suppose $q \in \{(x,0) \mid x \in [[b,a]]\}$. Consider sub-cases.

   (a) Suppose $q = (b,0)$. By Case 1 of the previous part of the proof, we have $q \in \{(i,b-i) \mid i \in [[1,b]]\} \subseteq C$.

(b) Suppose $q \in \{(x, 0) \mid x \in [[b+1, a-b-1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k+b, 0) \mid k \in [[1, (a-b)+1]]\} \subseteq C$.

(c) Suppose $q = (a, 0)$. By Case 3(b) of the previous part of the proof, we have $q = (a, 0) \subseteq C$.

3. Suppose $q \in \{(x, b) \mid x \in [[0, a-b]]\}$. Consider sub-cases.

   (a) Suppose $q = (b, b)$. By Case 1 of the previous part of the proof, we have $q \in \{(a-b)+i, b-i) \mid i \in [[1, b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, b) \mid x \in [[b+1, a-b-1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k-b, b) \mid k \in [[b+1, a-1]]\} \subseteq C$.

   (c) Suppose $q = (0, b)$. By Case 3(b) of the previous part of the proof, we have $q = (0, b) \subseteq C$.

4. Suppose $q \in \{(x, b) \mid x \in [[b, a]]\}$. Consider sub-cases.

   (a) Suppose $q = (b, b)$. By Case 1 of the previous part of the proof, we have $q \in \{(i, i) \mid i \in [[1, b]]\} \subseteq C$.

   (b) Suppose $q \in \{(x, b) \mid x \in [[b+1, a-b-1]]\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k+b, b) \mid k \in [[1, (a-b)+1]]\} \subseteq C$.

   (c) Suppose $q = (a, b)$. By Case 3(b) of the previous part of the proof, we have $q = (a, b) \subseteq C$.

The above cases show that $q \in C$. Hence $Bd(a, b) \subseteq C$.

Fourth, We claim that $Br(a, b) \subseteq C$. Let $q \in Br(a, b)$. Then $q \in Br_1(a, b)$ or $q \in Br_2(a, b)$. We consider cases.

1. Suppose $q \in Br_1(a, b)$. By Case 5 of the previous part of the proof, we have $q \in Br_1(a, b) \subseteq C$.

2. Suppose $q \in Br_2(a, b)$. By Case 5 of the previous part of the proof, we have $q \in Br_2(a, b) \subseteq C$.

The above cases show that $q \in C$. Hence $Br(a,b) \subseteq C$.

Last, we claim that $Cr(a,b) \subseteq C$. Let $q \in Cr(a,b)$. Then $q \in Cr_1(a,b)$, $q \in Cr_2(a,b)$, $q \in Cr_3(a,b)$ or $q \in Cr_4(a,b)$. We consider cases.

1. Suppose $q \in Cr_1(a,b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_1(a,b) \cup \{(b,b)\} \subseteq C$.

2. Suppose $q \in Cr_2(a,b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_2(a,b) \cup \{(b,0)\} \subseteq C$.

3. Suppose $q \in Cr_3(a,b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_3(a,b) \cup \{(a-b,0)\} \subseteq C$.

4. Suppose $q \in Cr_4(a,b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_4(a,b) \cup \{(a-b,b)\} \subseteq C$.

The above cases show that $q \in C$. Hence $Cr(a,b) \subseteq C$.

Because each set is a subset of C, $Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b) \subseteq C$.

Since we have proved that $C \subseteq Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$ and $Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b) \subseteq C$, we can conclude that the cloud of $Rect(a,b)$ is $Be(a,b) \cup Ar(a,b) \cup Bd(a,b) \cup Br(a,b) \cup Cr(a,b)$. $\square$

Note that $Rec(a,a)$ would be a square. Now we consider $Rect(a,a)$ with $a \geq 3$.

In Figure 4.2.13, empty circles are the cloud of $Rect(10,10)$.

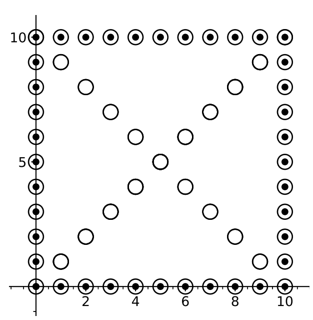

Figure 4.2.13. The cloud of $Rect(a,a)$

Note that cross is defined differently.

Definition 4.2.7. Let $a \in \mathbb{N}$. Suppose $a \geq 3$.

1. Let the **cross** of $Rec(a, a)$ be

$$Cr(a, a) = Cr_1(a, b) \cup Cr_2(a, b)$$

where

$$Cr_1(a, a) = \{(x, x) \mid x \in [[1, b - 1]]\},$$

and

$$Cr_2(a, a) = \{(x, b - x) \mid x \in [[1, b - 1]]\}.$$

△

In Figure 4.2.14, empty circles are the crosses of $Rect(10, 10)$.

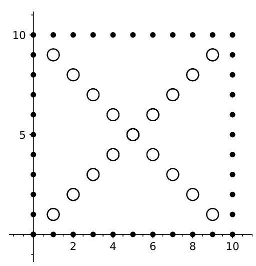

Figure 4.2.14. Example of cross for $Rec(a, a)$

Again, the following proof follows the same procedure as Theorem 4.2.4.

Theorem 4.2.8. *Let $a \in \mathbb{N}$. Suppose $a \geq 3$. Then the cloud of $Rec(a, a)$ is $Rec(a, a) \cup Cr(a, a)$.*

Proof. Let C be the cloud of $Rect(a, a)$ for $a > 2$ such that $a \in \mathbb{N}$. First we are going to show that $C \subseteq Be(a, b) \cup Ar(a, b) \cup Bd(a, b) \cup Br(a, b) \cup Cr(a, b)$.

By Theorem 3.1.3, we know that every point in $Rect(a, b)$ is the head of an IHS. Let $p \in Rect(a, b)$. Consider different cases for all possible IHS with p as head in $Rect(a, b)$.

1. Suppose $p = (0, 0)$, $p = (0, a)$, $p = (a, 0)$ or $p = (a, a)$. First, let us consider when $p = (0, 0)$. All possible feet of IHS with p as head are $(0, i)$ and $(i, 0)$ for each $i \in [[0, a]]$. Then for

each $i \in [[0,a]]$, missing corner is (i,i). Hence the set of all missing corners is

$$\{(i,i) \mid i \in [[0,a]]\} = Cr_1(a,a) \cup \{(a,a),(0,0)\} \subseteq Cr_1(a,a) \cup Rec(a,a).$$

By symmetry, we see that sets of all missing corners from other p in this case are

$$\{(i,a-i) \mid i \in [[0,a]]\} = Cr_2(a,a) \cup \{(a,0),(0,a\} \subseteq Cr_2(a,a) \cup Rec(a,a),$$

2. Suppose $p \in [[1,a-1]] \times \{0\}$, $p \in [[1,a-1]] \times \{b\}$, $p \in \{0\} \times in[[1,a-1]]$ or $p \in \{a\} \times in[[1,a-1]]$. Let us consider when $p = (1,0) \in [[1,a-1]] \times \{0\}$. All possible feet of IHS with p as head are $(0,a-k)$ and (k,a) for each $k \in [[1,a-1]]$. Then for each $k \in [[1,a-1]]$, missing corner is $(a-k,a)$. Hence the set of all missing corner is

   $$\{(a-k,a) \mid k \in [[1,a-1]]\} \subseteq T(a,a)$$

   . By symmetry, we see that sets of all missing corners from other p in this case are

   $$\{(a,k) \mid k \in [[1,a-1]]\} = R(a,a)$$

   ,

   $$\{(k,0) \mid k \in [[1,a-1]]\} \subseteq B(a,a)$$

   , and

   $$\{(0,a-k) \mid k \in [[1,a-1]]\} = L(a,a).$$

Given all the cases, the set of all missing corners, which is the cloud of $Rect(a,a)$, is a subset of $Cr(a,a) \cup Rect(a,a)$. Hence $C \subseteq Cr(a,a) \cup Rect(a,a)$.

Now we are going to show that $Cr(a,a) \cup Rect(a,a) \subseteq C$ by showing that each set in the union is a subset of C.

First, we claim that $Cr(a,a) \subseteq C$. Let $q \in Cr(a,a)$. Then $q \in Cr_1(a,a)$, $q \in Cr_a(a,a)$. We consider cases.

1. Suppose $q \in Cr_1(a,a)$. By Case 1 of the previous part of the proof, we have $q \in Cr_1(a,a) \cup \{(a,a),(0,0)\} \subseteq C$.

2. Suppose $q \in Cr_2(a,b)$. By Case 1 of the previous part of the proof, we have $q \in Cr_2(a,b) \cup \{(a,0),(0,a)\} \subseteq C$.

The above cases show that $q \in C$. Hence $Cr(a,b) \subseteq C$.

Now, we claim that $Rect(a,a) \subseteq C$. Let $q \in Rect(a,a)$. Then $q \in [[0,a]] \times \{0\}$, $q \in [[0,a]] \times \{b\}$, $q \in \{0\} \times in[[1,a-1]]$ or $q \in \{a\} \times \in [[1,a-1]]$ We consider cases.

1. Suppose $q \in [[0,a]] \times \{0\}$. Consider sub-cases.
   (a) Suppose $q \in (0,0)$ or $(a,0))$. By Case 1 of the previous part of the proof, we have $q \in \{(i,i) \mid i \in [[0,a]]\} \cup \{(i,a-i) \mid i \in [[0,a]]\} \subseteq C$.
   (b) Suppose $q \in [[1,a-1]] \times \{0\}$. By Case 2 of the previous part of the proof, we have $q \in \{(k,0) \mid k \in [[1,a-1]]\} \subseteq C$.
2. Suppose $q \in [[0,a]] \times \{a\}$. Consider sub-cases.
   (a) Suppose $q \in (0,a)$ or $(a,a))$. By Case 1 of the previous part of the proof, we have $q \in \{(i,i) \mid i \in [[0,a]]\} \cup \{(i,a-i) \mid i \in [[0,a]]\} \subseteq C$.
   (b) Suppose $q \in [[1,a-1]] \times \{a\}$. By Case 2 of the previous part of the proof, we have $q \in \{(a-k,a) \mid k \in [[1,a-1]]\} \subseteq C$.
3. Suppose $q \in \{0\} \times in[[1,a-1]]$. By Case 2 of the previous part of the proof, we have $q \in L(a,a) \subseteq C$.
4. Suppose $q \in \{a\} \times \in [[1,a-1]]$. By Case 2 of the previous part of the proof, we have $q \in R(a,a) \subseteq C$

The above cases show that $q \in C$. Hence $Rect(a,a) \subseteq C$.

Because each set is a subset of C, we have $Cr(a,a) \cup Rect(a,a) \subseteq C$.

Since we have proved that $C \subseteq Cr(a,a) \cup Rect(a,a)$ and $Cr(a,a) \cup Rect(a,a) \subseteq C$, we can conclude that the cloud of $Rect(a,a)$ is $Cr(a,a) \cup Rect(a,a)$. $\square$

4.3 Internal and external corners

Now we should examine whether missing corners exist in both interior and exterior of NDLSCC. First, we should consider corners of NDLSCC because it is clear that any IHS with a corner of NDLSCC as its head forms a missing corner.

Definition 4.3.1. Let $S \subseteq \mathbb{Z}^2$ be a NDLSCC. Let p be a corner of S. Then we form the IHS with p as head and its two 4-adjacent points as feet.

1. If the IHS with p as head has its missing corner in the interor of S, then p is an **internal corner**.

2. If the IHS with p as head has its missing corner in the exterior of S, then p is an **external corner**.

$\triangle$

To examine whether missing corners exist in both interior and exterior of a NDLSCC. We want to know the existence of internal and external corners.

We think the following is true. Let $S \subseteq \mathbb{Z}^2$ be a NDLSCC.

1. There exists a corner in S.

2. There exists an internal corner in S.

3. Suppose S is not $Rect(a, b)$ for some $a, b \in \mathbb{N}$. There exists an external corner in S.

Therefore we think the following is true. Let $S \subseteq \mathbb{Z}^2$ be a NDLSCC. There exists a missing corner in the interior of S.

Also we think the following is true. Let $S \subseteq \mathbb{Z}^2$ be a NDLSCC. Let $a, b \in \mathbb{N}$. Suppose S is not $Sq(a, b)$. There exists a missing corner in the exterior of S. However, we have not proved these statements.

Appendix A
Sage Code

A.1 Cloud

This program finds all the IHSs by using each point as a head and plots all the missing corners deduced from the IHSs.

This function reverses order of element in the list.

```
def reverse(lst):
    lis = []
    for i in range(len(lst)-1,-1,-1):
        lis.append(lst[i])
    return lis
```

This function lists coordinates of the rectangle.

```
def rectangle(x,y):
    lis = []
    for i in range (0, x + 1):
        lis.append((i, 0))
    for k in range (1, y + 1):
        lis.append((x, k))
```

```
    revlis = []
    for j in range (0, y):
        revlis.append((0, j))
    for h in range (0, x):
        revlis.append((h, y))
    newlis = reverse(revlis)
    reclis = lis + newlis
    return reclis
```

This function plots a point.

```
def diskbrown((x1,y1),r):
    cp = circle((x1,y1), r, fill=True, thickness=1, edgecolor='black', facecolor='black')
    return cp
```

This function rotates a vector clockwise.

```
def rotateclockw((x1,y1)):
    v = vector([y1, -x1])
    return v
```

This function returns the list of missing corner of a square if there exist an IHS with given point as a head.

```
def missingcornercenter((x1,y1), lis):
    newlis = [vector(x) for x in lis if x != (x1, y1)]
    miscornerlis = []
    w1 = vector((x1,y1))
    for w2 in newlis:

        zz = w2 - w1
```

```
        w3 = w1 + rotateclockw(zz)


        if w3 in newlis:
            w4 = w2 + rotateclockw(zz)
            miscornerlis.append(w4)


    return miscornerlis
```

This function returns the list of missing corner of a square if there exist an IHS with given point as a foot.

```
def missingcornerfoot((x1,y1), lis):
    newlis = [vector(x) for x in lis if x != (x1, y1)]
    miscornerlis = []
    w1 = vector((x1,y1))
    for w2 in newlis:


        zz = w2 - w1


        w3 = w2 + rotateclockw(zz)


        if w3 in newlis:
            w4 = w1 + rotateclockw(zz)
            miscornerlis.append(w4)


    return miscornerlis
```

The fuctions returns the list of missing corners of a square for each point on the curve by using given point as a head.

```
def allmissingcornercenter(lis):
    allmiscornerlis = []
    for x in lis:
        allmiscornerlis.append(missingcornercenter(x, lis))
    #show(allmiscornerlis)

    result = []
    for list in allmiscornerlis:
        for item in list:
            result.append(item)

    return result
```

The fuctions returns the list of missing corners of a square for each point on the curve by using given point as a foot.

```
def allmissingcornerfoot(lis):
    allmiscornerlis = []
    for x in lis:
        allmiscornerlis.append(missingcornerfoot(x, lis))

    result = []
    for list in allmiscornerlis:
        for item in list:
            result.append(item)

    return result
```

This is the interactive controls. We now come to the main part of the Sage code.

```
@interact
def _(
    vv = input_box(default = initlis, label="List of Vertices of Polygon"),
    switch = ("Show original", False),
    u = slider(3, 30, 1, default=20, label='bound for graph'),
    rx = slider(3, 30, 1, default=8, label='Length'),
    ry = slider(3, 30, 1, default=5, label='Height'),
    selec = selector(['Head', 'Foot'], label = 'Orientation')
    ):

    rad = sqrt(u)

    reclis = rectangle(rx,ry)

    newlis = allmissingcornercenter(reclis)
    #show("Corners = ", newlis)

    footlis = allmissingcornerfoot(reclis)

    circs = [circle(newlis[k], rad/15, fill = False, thickness = 1.0, edgecolor =
    "black", facecolor = "white") for k in range(0, len(newlis))]
    circssum = sum(circs)

    circs1 = [circle(footlis[k], rad/15, fill = False, thickness = 1.0, edgecolor =
    "black", facecolor = "white") for k in range(0, len(footlis))]
    circssum1 = sum(circs1)
```

```
#hs = [diskbrown(vv[k], rad/40) for k in range(len(vv))]
hs = [diskbrown(reclis[k], rad/40) for k in range(len(reclis))]

same = duplicates(newlis)
#show("Overlap = ", same)

#show('There are ', len(vv), 'points')

#show(circssum + sum(hs), figsize=12, xmin=-u, xmax=u, ymin=-u, ymax=u)

if selec == 'Head':
    show(circssum + sum(hs), figsize=12, xmin=-1, xmax=rx+1, ymin=-ry-1,
    ymax=(2*ry)+1)

if selec == 'Foot':
    show(circssum1 + sum(hs), figsize=12, xmin=-1, xmax=rx+1, ymin=-ry-1,
    ymax=(2*ry)+1)

if switch == True:
    show(sum(hs))
```

Bibliography

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