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Determinantal Conditions on Integer Splines

Kathryn Elizabeth Blaine
Bard College

Determinantal Conditions on Integer Splines

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Kate Blaine

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Abstract

In this project, we work with integer splines on graphs with positive integer edge labels. We focus on graphs that are (m, n)-cycles for some natural numbers m, n, specifically the diamond graph, which consists of two triangles joined at an edge. We extend previous research on integer splines over the diamond graph. In particular, we prove that a set of splines on the diamond graph forms a basis if and only if it satisfies a certain determinantal criterion.

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Dedication

For my parents, who have stood by me through the laughter and tears.

Acknowledgments

I owe many thanks to my adviser, Lauren Rose, for providing me with inspiration and guidance every step of the way. Her support and that of the other math department professors has been incredible. My parents' unfailing love and kindness during all four years was more than I could have asked for. Lastly, I thank my friend, Maya Schwartz, and my boulder, Nick Bader, for keeping me sane during the final months.

1

Introduction

Splines are used in multiple branches of mathematics. Additionally, before computers were developed for engineering purposes, splines were used to create smooth and flexible curves out of wood. They were used to build many structures, for example, ships and instruments.

Mathematically, splines are piecewise polynomial functions connected at various degrees of smoothness. Splines can be represented using Cartesian coordinates, or by graphs with labeled vertices and edges. In this project, we explore generalized integer splines, which are represented by edge-labeled graphs. We will be working with splines over the diamond graph, which can be described as two triangles connected by an edge. A diamond graph is pictured in Figure 1.0.1.

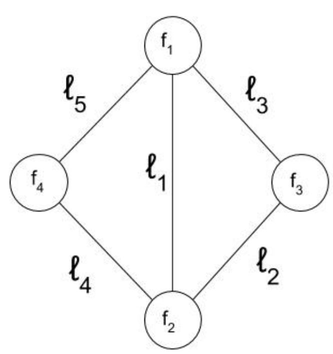

Figure 1.0.1. $F = (f_1, f_2, f_3, f_4)$ is a spline over the diamond graph.

We will study the module of splines over the ring $\mathbb{Z}$. Unlike vector spaces over fields, modules are not necessarily guaranteed to have a basis. Modules that do have a basis are known as free modules. In fact, for any edge-labeled graph, the module of splines over that graph has a basis, and is thus a free module.

The main part of my project focuses on determinantal conditions of splines and spline bases. Previously, Emmet Mahdavi made a conjecture that a set of splines over the diamond graph forms a module basis if and only if its determinant is equal to a certain value based on the integer edge labels. Mahdavi was unable to complete the proof without imposing special conditions on the edge labels, namely, that they are relatively prime. While Mahdavi was able to prove his result about bases for relatively prime edge labels, we will show that the result is true for the general case using properties of the greatest common divisor.

Moving forward, the next step of this research would be to explore determinantal conditions on more complex graphs, for example, an (m, n)-cycle graph.

In Chapter 2, we introduce the reader to basic concepts of number theory and graph theory. In Chapter 3, we introduce several definitions necessary for understanding splines and we present past results which will be useful for later proofs. In Chapter 4, we present the main results of the project, namely, the expansion of Mahdavi's results to the general case, and some smaller results for splines over (m, n)-cycle graphs.

2
Preliminaries

Before we can introduce the definition of a generalized integer spline and detail previous results on the topic, we must first provide some background on number theory. In this chapter, we will introduce the reader to some basic concepts required to understand later chapters. Additionally, we will define terminology and notation.

2.1 Elementary Number Theory

First, we introduce modular arithmetic, a crucial aspect of the study of splines. Most of the following results can be found in [1] and [2].

Definition 2.1.1 ([1], Chapter 4, Section 4.1)**.** Let m be a positive integer. If a and b are integers, we say that a is **congruent** to b modulo m if $m \mid (a-b)$. We denote this by $a \equiv b \bmod m$ if a is congruent to b modulo m. △

Example 2.1.2. Let $a = 7$, $b = 12$, and $m = 5$. We know that $7 \equiv 12 \bmod 5$ because $5|(7-12)$. Also, note that $k \equiv k \bmod j$ for any $j, k \in \mathbb{Z}$ because $k - k = 0$ and hence, $j|(k-k)$. ♢

Now we define the greatest common divisor and least common multiple of two integers.

Definition 2.1.3 ([1], Chapter 1, Section 1.5)**.** The **greatest common divisor** (*gcd*) of two integers a and b, which are not both 0, is the largest integer that divides both a and b. The *gcd* of a and b is denoted by (a, b). △

Definition 2.1.4 ([2], Chapter 2, Section 2.3)**.** The **least common multiple** (*lcm*) of two integers a and b which are not both 0, denoted by $[a, b]$, is the smallest positive integer m that is divisible by both a and b. △

We can also define the greatest common divisor and least common multiple of multiple integers.

Definition 2.1.5 ([1], Chapter 3, Section 3.3)**.** Let $a_1, a_2, \ldots, a_n$ be integers which are not all zero. The *gcd* of these integers is the largest integer that is a divisor of all the integers in the set. The *gcd* of $a_1, a_2, \ldots, a_n$ is denoted by $(a_1, a_2, \ldots, a_n)$. △

Definition 2.1.6. Let $a_1, a_2, \ldots, a_n$ be integers which are not all zero. The *lcm* of these integers is the smallest positive integer that is divisible by all the integers in the set. The *lcm* of $a_1, a_2, \ldots, a_n$ is denoted by $[a_1, a_2, \ldots, a_n]$. △

In the following two chapters, we will require the use of some properties of greatest common divisors and least common multiples. We establish those properties here. Most of the following results can be found in [1] and [2]. We provide proofs here to aid the reader.

Lemma 2.1.7. *Let $a, b, c \in \mathbb{Z}$ such that not all of a, b, c are zero. Then $(a, b, c) = (a, (b, c))$.*

Proof. Let $x = (a, b, c)$ and let $y = (a, (b, c))$. Then $x|a$, $x|b$, and $x|c$. Since $x|b$ and $x|c$, we can deduce that $x|(b, c)$. Adding the fact that $x|a$, we deduce that $x|(a, (b, c))$. Hence $x|y$. Since $y = (a, (b, c))$, we know that $y|a$ and $y|(b, c)$. From $y|(b, c)$, it follows that $y|b$ and $y|c$. Then $y|(a, b, c)$. Hence $y|x$. Since $x|y$ and $y|x$ and $x, y \geq 0$, we conclude that $x = y$. Hence $(a, b, c) = (a, (b, c))$. □

Lemma 2.1.8. *Let $a, b, c \in \mathbb{Z}$ such that not all of a, b, c are zero. Then $(ca, cb) = c(a, b)$.*

Proof. Let $x = (a, b)$. Then $x|a$ and $x|b$. Hence $a = jx$ and $b = kx$ for some $j, k \in \mathbb{Z}$. Equivalently, $cjx = ca$ and $ckx = cb$. By associativity and commutativity, we can write these equations as $j(cx) = ca$ and $k(cx) = cb$. Hence $cx|ca$ and $cx|cb$. We deduce that $cx|(ca, cb)$. Hence $c(a, b)|(ca, cb)$.

Now, let $y = (ca, cb)$. Then $y|ca$ and $y|cb$. Equivalently, we know $my = ca$ and $ny = cb$ for some $m, n \in \mathbb{Z}$. We also know that $c|ca$ and $c|cb$, so it follows that $c|(ca, cb)$, and hence $c|y$. So, we know that $cy_0 = y$ for some $y_0 \in \mathbb{Z}$. By substitution, then $mcy_0 = ca$ and $ncy_0 = cb$. By cancellation, then $my_0 = a$ and $ny_0 = b$. We deduce from this that $y_0|a$ and $y_0|b$, and it follows that $y_0|(a, b)$. Hence $y_0z = (a, b)$ for some $z \in \mathbb{Z}$. Multiplying both sides of this equation by c, we get $cy_0z = c(a, b)$. Equivalently, we have $yz = c(a, b)$. Then we know that $y|c(a, b)$. Hence $(ca, cb)|c(a, b)$. Hence, we conclude that $c(a, b) = (ca, cb)$. □

We can extend these properties for greatest common divisors of multiple integers.

Lemma 2.1.9. ([1], Chapter 3, Section 3.3). *Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ such that not all of $a_1, a_2, \ldots, a_n$ are zero. Then $(a_1, a_2, \ldots, a_{n-1}, a_n) = (a_1, a_2, \ldots, (a_{n-1}, a_n))$.*

Proof. Let $x = (a_1, a_2, \ldots, a_{n-1}, a_n)$ and let $y = (a_1, a_2, \ldots, (a_{n-1}, a_n))$. Let $i \in \{1, \ldots, n\}$. Then $x|a_i$. Since $x|a_{n-1}$ and $x|a_n$, we can deduce that $x|(a_{n-1}, a_n)$. Because $x|a_i$ for all $i \in \{1, \ldots, n\}$, we deduce that $x|(a_1, a_2 \ldots, (a_{n-1}, a_n))$. Hence $x|y$. Let $j \in \{1, \ldots, n-2\}$. Since $y = (a_1, a_2, \ldots, (a_{n-1}, a_n))$, we know that $y|a_j$ and $y|(a_{n-1}, a_n)$. From $y|(a_{n-1}, a_n)$, it follows that $y|a_{n-1}$ and $y|a_n$. Hence $y|a_i$ for all $i \in \{1, \ldots, n\}$. So $y|(a_1, a_2, \ldots, a_{n-1}, a_n)$. Hence $y|x$. Since $x|y$ and $y|x$, we conclude that $x = y$. Hence $(a_1, a_2, \ldots, a_{n-1}, a_n) = (a_1, a_2, \ldots, (a_{n-1}, a_n))$. □

The following example demonstrates this property.

Example 2.1.10. Suppose we have $a_1 = 4, a_2 = 24, a_3 = 6, a_4 = 18$. Then $(4, 24, 6, 18) = 2$ and $(4, 24, (6, 18)) = (4, 24, 6) = 2$. ◊

Lemma 2.1.11. *Let $a_1, a_2, \ldots, a_n, c \in \mathbb{Z}$ such that not all of $a_1, a_2, \ldots, a_n, c$ are zero. Then $(ca_1, ca_2, \ldots, ca_n) = c(a_1, a_2, \ldots, a_n)$.*

Proof. We will show by induction that $c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$ for all $n \in \mathbb{N}$ such that $n \geq 2$.

Base Case: We know by Lemma 2.1.8 that $c(a_1, a_2) = (ca_1, ca_2)$.

Inductive Step: Let $k \in \mathbb{N}$ such that $k \geq 2$. Suppose $c(a_1, a_2, \ldots, a_k) = (ca_1, ca_2, \ldots, ca_k)$. By Lemma 2.1.9, we know that $c(a_1, a_2, \ldots, a_k, a_{k+1}) = c((a_1, a_2, \ldots, a_k), a_{k+1})$. From the base case, we know that $c((a_1, a_2, \ldots, a_k), a_{k+1}) = (c(a_1, a_2, \ldots, a_k), ca_{k+1})$. By hypothesis, we know $(c(a_1, a_2, \ldots, a_k), ca_{k+1}) = ((ca_1, ca_2, \ldots, ca_k), ca_{k+1})$. Using Lemma 2.1.9, we know that $((ca_1, ca_2, \ldots, ca_k), ca_{k+1}) = (ca_1, ca_2, \ldots, ca_k, ca_{k+1})$. Hence, we conclude that $c(a_1, a_2, \ldots, a_{k+1}) = (ca_1, ca_2, \ldots, ca_{k+1})$. □

The following example demonstrates this property.

Example 2.1.12. Suppose we have $a_1 = 4, a_2 = 24, a_3 = 6, a_4 = 18$. Then $(4, 24, 6, 18) = 2$ and $2(2, 12, 3, 9) = 2 \cdot 1 = 2$. ◊

Here, we define notation which will be used in the next proof.

Definition 2.1.13. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Let

$$\hat{a}_1 = a_2 a_3 \cdots a_n$$

$$\hat{a}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$$

$$\hat{a}_n = a_1 \cdots a_{n-1}$$

for all i such that $1 < i < n$. △

The following theorem is an extension of a common number theory result. It will be useful for proofs in later chapters. A proof of this result can be found in [4], but we provide an alternate method here.

Theorem 2.1.14. *Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. Then* $[a_1, a_2, \ldots, a_n] = \dfrac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)}$.

Proof. Let $x = [a_1, a_2, \ldots, a_n]$. Then $a_1|x, a_2|x, \ldots, a_n|x$. We can rewrite $a_1|x$ as $a_1\hat{a}_1|x\hat{a}_1$. Similarly, we can rewrite $a_i|x$ as $a_i\hat{a}_i|x\hat{a}_i$ for all $i \in \{1, \ldots, n\}$. This simplifies to $a_1 a_2 \cdots a_n|x\hat{a}_i$ for all $i \in \{1, \ldots, n\}$. We deduce that

$$a_1 a_2 \cdots a_n|(x\hat{a}_1, x\hat{a}_2, \ldots, x\hat{a}_n).$$

Then

$$a_1 a_2 \cdots a_n | x(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n).$$

It follows that $k(a_1 a_2 \cdots a_n) = x(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$ for some $k \in \mathbb{Z}$. Then $k\dfrac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)} = x$. Since $\hat{a}_i | a_1 a_2 \cdots a_n$ for all $i \in \{1, \ldots, n\}$, then $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) | a_1 a_2 \cdots a_n$. It follows that $\dfrac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)} \in \mathbb{Z}$.

Hence,

$$\frac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)} | x.$$

Let $z = \dfrac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)}$. We know that $z = a_i \dfrac{\hat{a}_i}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)}$ for all $i \in \{1, \ldots, n\}$. Since $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) | \hat{a}_i$, then we know that $\dfrac{\hat{a}_i}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)} \in \mathbb{Z}$. Hence $a_i | z$ for all $i \in \{1, \ldots, n\}$. We deduce from this that $[a_1, a_2, \ldots, a_n] | z$. Hence $x | z$. Since $z | x$ and $x | z$, then $x = z$. Hence, we conclude that $[a_1, a_2, \ldots, a_n] = \dfrac{a_1 a_2 \cdots a_n}{(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)}$. □

The following example demonstrates this property.

Example 2.1.15. Suppose we have $a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 7$. Then $[2, 3, 7, 5] = 210$ and $\dfrac{2 \cdot 3 \cdot 5 \cdot 7}{(\hat{2}, \hat{3}, \hat{5}, \hat{7})} = \dfrac{210}{(105, 70, 42, 30)} = \dfrac{210}{1} = 210.$ ♢

2.2 Graph Theory

In this section, we introduce and illustrate some graph theory definitions which will be helpful for the following chapter. First, we define an edge-labeled graph, a concept which will be necessary for our definition of a generalized integer spline.

Definition 2.2.1 ([3], Definition 2.1)**.** Let G be a graph with k edges ordered $e_1, e_2, \ldots, e_k$ and n vertices ordered $v_1, \ldots, v_n$. Let l_i be a positive integer label on edge e_i and let $L = \{l_1, \ldots, l_k\}$ be the set of edge labels. Then (G, L) is called an **edge-labeled graph**. △

Next, we define various types of edge-labeled graphs that will be used throughout the paper.

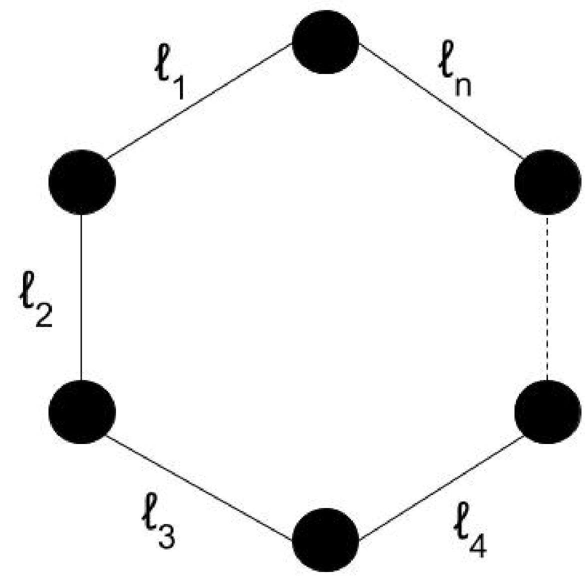

Figure 2.2.1. An edge-labeled n-cycle graph.

Definition 2.2.2. We define an ***n*-cycle** with edge labels $L = (l_1, l_2, \ldots, l_n)$ to be the graph shown in Figure 2.2.1. A given n-cycle for $n \geq 3$ will be denoted by C_n, and a given edge-labeled n-cycle for $n \geq 3$ will be denoted by (C_n, L).

△

Definition 2.2.3. We define a **diamond graph** with edge labels $L = (l_1, l_2, l_3, l_4, l_5)$ to be the graph shown in Figure 2.2.2. A diamond graph will be denoted by D, and an edge-labeled diamond graph will be denoted by (D, L).

△

Definition 2.2.4. We define an **(m, n)-cycle** with edge labels $L = (l_1, l_2, \ldots, l_{n+m-1})$ to be the graph shown in Figure 2.2.3. A given (m, n)-cycle for $m, n \geq 3$ will be denoted by $C_{(m,n)}$, and a given edge-labeled (m, n)-cycle for $m, n \geq 3$ will be denoted by $(C_{(m,n)}, L)$.

△

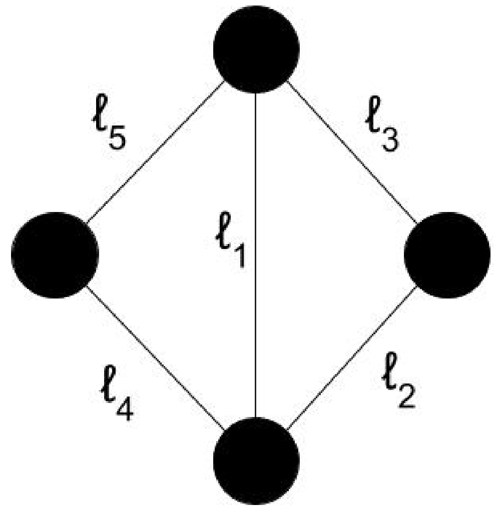

Figure 2.2.2. An edge-labeled diamond graph.

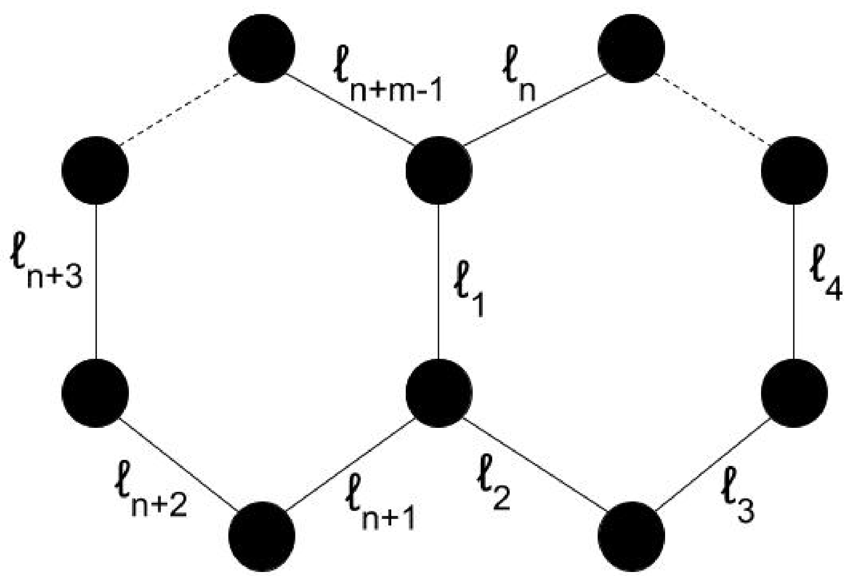

Figure 2.2.3. An edge-labeled (m, n)-cycle graph.

3

Past Results on Generalized Integer Splines

In this chapter, we discuss previous results found for splines over n-cycle graphs and the diamond graph. In Section 3.1, we introduce the definition of a generalized integer spline and a $\mathbb{Z}$-module. In Section 3.2, we discuss flow-up classes and define a minimal element of a flow-up class. In Section 3.3 and Section 3.4, we describe results previously found for splines over the diamond graph, which will be useful for proofs in the following chapter.

3.1 Introduction to Splines

To begin, we define a generalized spline over the integers.

Definition 3.1.1 ([3], Definition 2.2)**.** A **generalized spline** on the edge-labeled graph (G, L) is a vertex labeling $(f_1, \ldots, f_n) \in \mathbb{Z}^n$ satisfying the following: if two vertices are connected by an edge e_i then the labels on the two vertices are equivalent modulo the label on the edge. We denote the set of all splines on (G, L) by $S(G, L)$.

△

To illustrate the defining properties of a spline, we provide an example of a spline over a 4-cycle edge-labeled graph.

Example 3.1.2. Fix the edge labels on (C_4, L) where $L = (4, 9, 7, 2)$. Let $f = (7, 3, 12, 5)$.

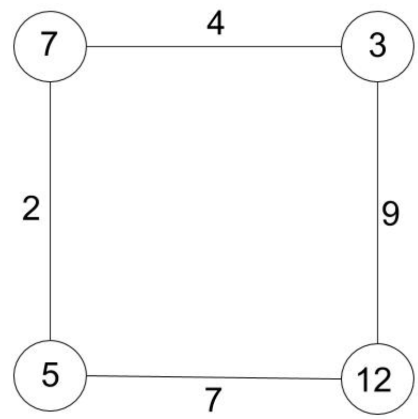

Figure 3.1.1. A spline over a 4-cycle graph.

This spline is pictured in Figure 3.1.1. We can verify that f is a spline by checking that it satisfies the required conditions.

$$7 \equiv 3 \bmod 4$$

$$3 \equiv 12 \bmod 9$$

$$12 \equiv 5 \bmod 7$$

$$5 \equiv 7 \bmod 2$$

Hence $f \in S(C_4, L)$. $\Diamond$

We now provide the definition of a module and state that, for any edge-labeled graph (G, L), the set of all splines $S(G, L)$ forms a $\mathbb{Z}$-module.

Definition 3.1.3. ([7], Chapter 5, Section 1). A **module** over a ring R is a set M together with a binary operation, and an operation of R on M, satisfying the following properties:

1. M is an abelian group under addition.

2. For all $a \in R$ and all $f, g \in M$, $a(f + g) = af + ag$.

3. For all $a, b \in R$ and all $f \in M$, $(a + b)f = af + bf$.

4. For all $a, b \in R$ and all $f \in M$, $(ab)f = a(bf)$.

5. If 1 is the multiplicative identity in R, then $1f = f$ for all $f \in M$.

$\triangle$

To show that $S(G, L)$ forms a $\mathbb{Z}$-module for any edge-labeled graph (G, L), we defer to the following theorem, a proof of which can be found in [4] and [5].

Theorem 3.1.4. *Fix the edge labels on (G, L) where G is any graph with m vertices and $L = (l_1, l_2, \ldots, l_n)$. Then $S(G, L)$ is a subgroup of $\mathbb{Z}^m$, and hence a $\mathbb{Z}$-module.*

It is important to note that scalars in a vector space come from a field F, while scalars in a module come from a ring R. Additionally, while all vector spaces have a basis, it is not necessarily the case that all modules have a basis.

Definition 3.1.5 ([5], Definition 3.2.3)**.** An R-module M is a **free module** if it has a basis. $\triangle$

In order to show that $S(G, L)$ is a free module for any edge-labeled graph (G, L), the following theorem, included without proof, is necessary.

Theorem 3.1.6 ([8], Theorem 6.1)**.** *Let F be a free module over a principle ideal domain R and let G be a submodule of F. Then G is a free R-module.*

We know that $\mathbb{Z}$ is a principle ideal domain and $\mathbb{Z}^m$ is a free module over $\mathbb{Z}$. From Theorem 3.1.4, we know for any edge-labeled graph with m vertices that $S(G, L)$ is a submodule of $\mathbb{Z}^m$. Hence $S(G, L)$ is a free $\mathbb{Z}$-module, and we conclude that $S(G, L)$ has a basis.

3.2 Flow-Up Classes

Before we begin exploring previous work, we must provide a few additional definitions which will be helpful in determining criteria for a module basis. In this section, we define a flow-up class and a minimal element of a flow-up class.

Definition 3.2.1. ([3], Definition 2.3). Fix a graph (G, L) with n vertices. Let $k \in [0, n-1]$. The **flow-up class** $\mathcal{F}_k$ is defined by $\mathcal{F}_k = \{F \in S(G, L) \mid F \text{ has exactly } k \text{ leading zeroes}\}$. $\triangle$

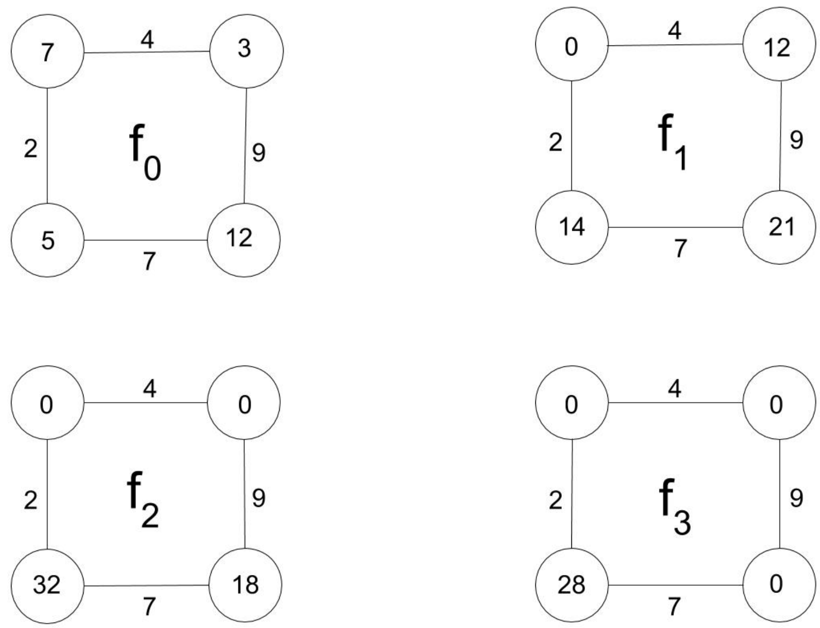

Figure 3.2.1. An element of each flow-up class over a 4-cycle graph.

The following example shows elements of each flow-up class on a 4-cycle edge-labeled graph.

Example 3.2.2. Fix the edge labels on (C_4, L) where $L = (4, 9, 7, 2)$. The following splines are elements of the flow-up classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, and are illustrated in Figure 3.2.1.

$$f_0 = (7, 3, 12, 5) \in \mathcal{F}_0$$

$$f_1 = (0, 12, 21, 14) \in \mathcal{F}_1$$

$$f_2 = (0, 0, 18, 32) \in \mathcal{F}_2$$

$$f_3 = (0, 0, 0, 28) \in \mathcal{F}_3$$

It is easy to verify that f_0, f_1, f_2, f_3 satisfy the defining properties of $S(C_4, L)$. $\Diamond$

In order to define a minimal element of a flow-up class, we must first define and establish notation for the leading term of a spline.

Definition 3.2.3 ([6], Definition 2.2.5)**.** . Let (G, L) be an edge-labeled graph with n vertices and let $f \in S(G, L) - \{\mathbf{0}\}$ be a spline. Then the **leading term** of f is the term f_i with the smallest i such that $f_j = 0$ for all $j < i$. We will denote the leading term of a spline f as f_L. △

Example 3.2.4. Fix the edge labels on (C_4, L) where $L = (4, 9, 7, 2)$. Let $f = (7, 3, 12, 5)$. It is easy to verify that f satisfies the defining properties of $S(C_4, L)$. Hence $f_L = 7$. Note that if $g = (g_1, \ldots, g_n) \in \mathcal{F}_i$, then $g_L = g_{i+1}$. ♢

We are now ready to define the minimal element of a flow-up class, which is any element of a flow-up class with the smallest positive leading term.

Definition 3.2.5. Fix the graph (G, L). Let $\mathcal{F}_k \in S(G, L)$ be a flow-up class. Let $b_k \in \mathcal{F}_k$. Then b_k is a **minimal element** of $\mathcal{F}_k$ if $(b_k)_L \in \mathbb{N}$ and $(b_k)_L \leq (p_k)_L$ for all $p_k \in \mathcal{F}_k$ such that $(p_k)_L \in \mathbb{N}$. △

Next, we provide an example of minimal elements of each flow-up class over a 4-cycle edge-labeled graph.

Example 3.2.6. Fix the edge labels on (C_4, L) where $L = (4, 9, 7, 2)$. The following splines are minimal elements of the flow-up classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, and are illustrated in Figure 3.2.2.

$$b_0 = (1, 1, 1, 1) \in \mathcal{F}_0, \quad (b_0)_L = 1$$

$$b_1 = (0, 4, 13, 6) \in \mathcal{F}_1, \quad (b_1)_L = 4$$

$$b_2 = (0, 0, 9, 16) \in \mathcal{F}_2, \quad (b_2)_L = 9$$

$$b_3 = (0, 0, 0, 14) \in \mathcal{F}_3, \quad (b_3)_L = 14$$

Since $(1, 1, 1, 1)$ is the trivial spline, then any spline $f \in \mathcal{F}_0$ where $f_L = 1$ is a minimal element. For any $g \in \mathcal{F}_1$, we know $g_L \equiv 0 \bmod 4$. The smallest number that satisfies this requirement and still allows a vertex labeling such that a spline is created is 4. For any $h \in \mathcal{F}_2$, we know $h_L \equiv 0 \bmod 9$. The smallest number that satisfies this requirement and still allows a vertex labeling such that a spline is created is 9. For any $j \in \mathcal{F}_3$, we know $j_L \equiv 0 \bmod 7$ and $j_L \equiv 0 \bmod 2$.

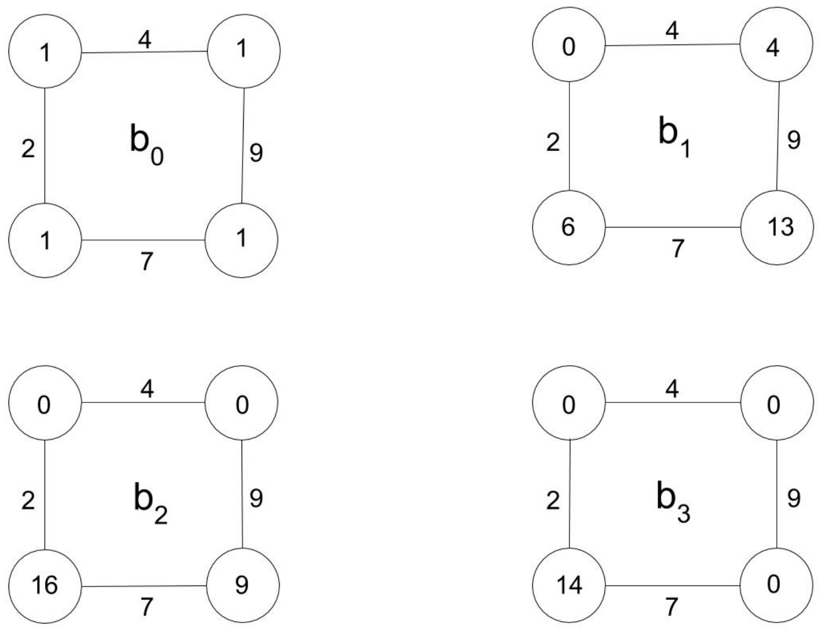

Figure 3.2.2. A minimal element of each flow-up class over a 4-cycle graph.

The smallest that number that satisfies this requirement is 14. Hence b_0, b_1, b_2, b_3 are minimal elements of their respective flow-up classes. $\Diamond$

3.3 Bases of Splines over the Diamond Graph

Now that we have defined some basic concepts necessary for understanding how splines form a basis, we can continue on to discussing previous work. In his senior project at Bard, Emmet Mahdavi studied splines over the diamond graph. We will include some of his results in this section. We begin with an example of a spline over the diamond graph.

Example 3.3.1. Fix the edge labels on (D, L) where $L = (2, 7, 3, 4, 5)$. Let $f = (3, 5, 12, 33)$. We can verify that f is a spline by checking that it satisfies the required conditions:

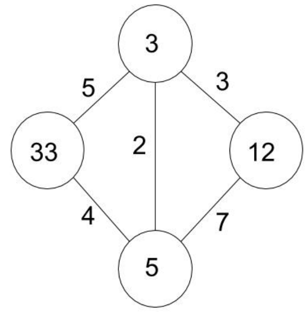

Figure 3.3.1. A spline over the diamond graph.

$$3 \equiv 5 \bmod 2$$

$$5 \equiv 12 \bmod 7$$

$$12 \equiv 3 \bmod 3$$

$$5 \equiv 33 \bmod 4$$

$$33 \equiv 3 \bmod 5$$

Hence $f \in S(D, L)$. This spline is shown in Figure 3.3.1.

◊

Before we can discuss previous results found on properties of flow-up class elements, it is first necessary to establish that the flow-up classes exist on any edge-labeled diamond graph (D, L). The following theorem, included here without proof, shows this to be true.

Theorem 3.3.2. ([5], Lemma 4.1.1, Lemma 4.1.2). *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. The flow-up classes $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ in $S(D, L)$ are non-empty.*

Note that we exclude $\mathcal{F}_0$ from this theorem because the existence of the trivial spline implies the existence of $\mathcal{F}_0$ on any edge-labeled graph.

The following theorem provides general descriptions for the leading terms of any element of a flow-up class, specifically any minimal element. This result is included without proof, but a proof may be found in [5].

Theorem 3.3.3. ([5], Lemma 4.1.4, Lemma 4.1.5, Lemma 4.1.6). *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. Let $x = (0, f_1, f_2, f_3)$ be in the flow-up class $\mathcal{F}_1$ on (D, L), let $y = (0, 0, g_2, g_3)$ be in the flow-up class $\mathcal{F}_2$ on (D, L), and let $h = (0, 0, 0, h_3)$ be in the flow-up class $\mathcal{F}_3$ on (D, L).*

1. *The leading element f_1 of x is a multiple of $[l_1, (l_2, l_3), (l_4, l_5)]$ and $f_1 = [l_1, (l_2, l_3), (l_4, l_5)]$ is the smallest positive value such that x is a spline.*

2. *The leading element g_2 of y is a multiple of $[l_2, l_3]$ and $g_2 = [l_2, l_3]$ is the smallest positive value such that y is a spline.*

3. *The leading element h_3 of z is a multiple of $[l_4, l_5]$ and $h_3 = [l_4, l_5]$ is the smallest positive value such that z is a spline.*

Corollary 3.3.4. *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. Let $b_0 \in \mathcal{F}_0$, let $b_1 \in \mathcal{F}_1$, let $b_2 \in \mathcal{F}_2$, and let $b_3 \in \mathcal{F}_3$. If $(b_0)_L = 1$, $(b_1)_L = [l_1, (l_2, l_3), (l_4, l_5)]$, $(b_2)_L = [l_2, l_3]$, and $(b_3)_L = [l_4, l_5]$, then b_0, b_1, b_2, b_3 are minimal elements of their respective flow-up classes.*

To demonstrate the properties of the leading terms of an element of a flow-up class, we provide the following example.

Example 3.3.5. Fix the edge labels on (D, L) where $L = (2, 7, 3, 4, 5)$. We already found an element of $\mathcal{F}_0$ in Example 3.3.1. By Theorem 3.3.3, the leading term of a spline in $\mathcal{F}_1$ is a multiple of $[2, (7, 3), (4, 5)] = 2$. Let $f_1 = (0, 6, 27, 30)$. It is easy to verify that f_1 satisfies the required conditions. Hence $f_1 \in \mathcal{F}_1$. Next, by Theorem 3.3.3, the leading term of a spline in $\mathcal{F}_2$ is a multiple of $[7, 3] = 21$. Let $f_2 = (0, 0, 21, 20)$. It is easy to verify that f_2 satisfies the required conditions. Hence $f_2 \in \mathcal{F}_2$. Finally, by Theorem 3.3.3, the leading term of a spline in $\mathcal{F}_3$ is a multiple of $[4, 5] = 20$. Let $f_3 = (0, 0, 0, 40)$. Hence $f_3 \in \mathcal{F}_3$. ♢

Now that we have some information regarding what form a minimal element takes, we can show that a minimal element from each of the flow-up classes forms a basis.

Lemma 3.3.6. ([5], Theorem 4.1.7). *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. Let b_0, b_1, b_2, b_3 be minimal elements of the corresponding flow-up classes in $S(D, L)$. Then $\{b_0, b_1, b_2, b_3\}$ are a basis for $S(D, L)$.*

Proof. Let $b_0 = (f_0, f_1, f_2, f_3), b_1 = (0, g_1, g_2, g_3), b_2 = (0, 0, h_2, h_3)$ and $b_3 = (0, 0, 0, j_3)$. Forming a column matrix from these splines, we have

$$M = \begin{bmatrix} f_0 & 0 & 0 & 0 \\ f_1 & g_1 & 0 & 0 \\ f_2 & g_2 & h_2 & 0 \\ f_3 & g_3 & h_3 & j_3 \end{bmatrix}.$$

Since the determinant of M is $f_0 g_1 h_2 j_3 \neq 0$, we know that b_0, b_1, b_2, b_3 are linearly independent.

Let $X = (x_0, x_1, x_2, x_3) \in S(D, L)$. Let $X' = X - x_0 b_0$. Since $b_0 = (1, 1, 1, 1)$, then $X' = \begin{pmatrix} 0 \\ x_1 - x_0 \\ x_2 - x_0 \\ x_3 - x_0 \end{pmatrix}$. Because X' is a linear combination of splines and $S(D, L)$ is a module, then $X' \in S(D, L)$. Since X' is a spline, we know that $x_1 - x_0 = a_1 g_1$ for some $a_1 \in \mathbb{Z}$.

Now suppose $X'' = X' - a_1 b_1 = \begin{pmatrix} 0 \\ a_1 g_1 \\ x_2 - x_0 \\ x_3 - x_0 \end{pmatrix} - a_1 \begin{pmatrix} 0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_2 - x_0 - a_1 g_2 \\ x_3 - x_0 - a_1 g_3 \end{pmatrix}$. Then $X'' \in S(D, L)$. Since X'' is a spline, we know that $x_2 - x_0 - a_1 g_2 = a_2 h_2$ for some $a_2 \in \mathbb{Z}$.

Next suppose $X''' = X'' - a_2 b_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3 - x_0 - a_1 g_3 - a_2 h_3 \end{pmatrix}$. Then $X''' \in S(D, L)$. Since X''' is a spline, we know that $x_3 - x_0 - a_1 g_3 - a_2 h_3 = a_3 j_3$ for some $a_3 \in \mathbb{Z}$. We know that $X''' - a_3 b_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $X = x_1 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$. Since we can write $X = (x_1, x_2, x_2, x_3)$ as a linear combination of b_0, b_1, b_2, b_3, we deduce that $X \in \text{span}(S(D, L))$. Hence $\{b_0, b_1, b_2, b_3\}$ form a module basis for $S(D, L)$. □

3.4 Determinants of Splines over the Diamond Graph

Our goal is to prove that four splines in $S(D, L)$ where $L = (l_1, l_2, l_3, l_4, l_5)$ form a module basis for $S(D, L)$ if and only if their determinant is equal to a certain value, namely $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5))}$. For the remainder of this chapter, we provide results from

Mahdavi's paper that will be necessary for our main proof, and we include proofs of the stated results for the reader's convenience.

First, we define determinant notation.

Definition 3.4.1. Let M be a square matrix. Then $\det(M) = |M|$. △

Lemma 3.4.2. ([5], Corollary 4.2.2). *Fix the edge labels on* (D, L) *where* $L = (l_1, l_2, l_3, l_4, l_5)$. *Let* $Q = \frac{l_1 l_2 l_3 l_4 l_5}{((l_2,l_3)(l_4,l_5), l_1(l_2,l_3,l_4,l_5))}$. *Let* b_0, b_1, b_2, b_3 *be minimal elements of the corresponding flow-up classes in* $S(D, L)$. *Then* $\begin{vmatrix} b_0 & b_1 & b_2 & b_3 \end{vmatrix} = Q$.

Proof. We know $b_0 = (1, 1, 1, 1)$. Let $b_1 = (0, g_1, g_2, g_3), b_2 = (0, 0, h_2, h_3)$ and $b_3 = (0, 0, 0, j_3)$. From Theorem 3.3.3, we know $g_1 = [l_1, (l_2, l_3), (l_4, l_5)]$, $h_2 = [l_2, l_3]$, and $j_3 = [l_4, l_5]$. Since $M = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}$ is a lower triangular matrix, then $|M| = [l_1, (l_2, l_3), (l_4, l_5)][l_2, l_3][l_4, l_5]$. By Theorem 2.1.14, then

$$\begin{aligned} |M| &= \frac{l_1(l_2, l_3)(l_4, l_5)}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3), l_1(l_4, l_5))} \frac{l_2 l_3}{(l_2, l_3)} \frac{l_4 l_5}{(l_4, l_5)} \\ &= \frac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), (l_1(l_2, l_3), l_1(l_4, l_5)))} \\ &= \frac{l_1 l_2 l_3 l_4 l_5}{(l_1((l_2, l_3), (l_4, l_5)), (l_2, l_3)(l_4, l_5))} \\ &= \frac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5))} \\ &= Q. \end{aligned}$$

□

Our next goal is to show that two bases for the set of splines over an edge-labeled diamond graph have the same determinant. In order to prove this, we need to first show that the determinant of one basis divides the determinant of four arbitrary splines.

Lemma 3.4.3. ([5], Lemma 4.2.7). *Fix the edge labels on* (D, L) *where* $L = (l_1, l_2, l_3, l_4, l_5)$. *If* $\{W, X, Y, Z\}$ *is a basis of* $S(D, L)$ *and* $\{A, B, C, D\} \in S(D, L)$ *then* $\begin{vmatrix} W & X & Y & Z \end{vmatrix}$ *divides* $\begin{vmatrix} A & B & C & D \end{vmatrix}$.

Proof. Since $A, B, C, D \in S(D, L)$, then each of A, B, C and D can be written as a linear combination of W, X, Y, Z as follows:

$$A = a_1W + a_2X + a_3Y + a_4Z \qquad a_1, a_2, a_3, a_4 \in \mathbb{Z}$$

$$B = b_1W + b_2X + b_3Y + b_4Z \qquad b_1, b_2, b_3, b_4 \in \mathbb{Z}$$

$$C = c_1W + c_2X + c_3Y + c_4Z \qquad c_1, c_2, c_3, c_4 \in \mathbb{Z}$$

$$D = d_1W + d_2X + d_3Y + d_4Z \qquad d_1, d_2, d_3, d_4 \in \mathbb{Z}$$

We can write each of the four splines in matrix form. If $A = (A_1, A_2, A_3, A_4)$, then

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} a_1w_1 + a_2x_1 + a_3y_1 + a_4z_1 \\ a_1w_2 + a_2x_2 + a_3y_2 + a_4z_2 \\ a_1w_3 + a_2x_3 + a_3y_3 + a_4z_3 \\ a_1w_4 + a_2x_4 + a_3y_4 + a_4z_4 \end{bmatrix} = \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}.$$

Using the same process for B, C, and D, we can write

$$\begin{bmatrix} A & B & C & D \end{bmatrix} = \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$$

Since $|AB| = |A| \cdot |B|$, we know that

$$\begin{vmatrix} A & B & C & D \end{vmatrix} = \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$= \begin{vmatrix} W & X & Y & Z \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Since all the entries of $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$ are in $\mathbb{Z}$, it follows that $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ is in $\mathbb{Z}$. Hence, $\begin{vmatrix} W & X & Y & Z \end{vmatrix}$ divides $\begin{vmatrix} A & B & C & D \end{vmatrix}$. □

From Lemma 3.4.3, we can now deduce that any two bases have the same determinant, up to sign.

Lemma 3.4.4. ([5], Lemma 4.2.8). *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. If $\{W, X, Y, Z\}$ and $\{A, B, C, D\}$ are bases of $S(D, L)$ then $\begin{vmatrix} W & X & Y & Z \end{vmatrix} = \pm \begin{vmatrix} A & B & C & D \end{vmatrix}$.*

Proof. Since $A, B, C, D \in S(D, L)$ and $\{W, X, Y, Z\}$ is a basis, then we know by Lemma 3.4.3 that $\begin{vmatrix} W & X & Y & Z \end{vmatrix}$ divides $\begin{vmatrix} A & B & C & D \end{vmatrix}$. Similarly, because $W, X, Y, Z \in S(D, L)$ and $\{A, B, C, D\}$ is a basis, then we know that $\begin{vmatrix} A & B & C & D \end{vmatrix}$ divides $\begin{vmatrix} W & X & Y & Z \end{vmatrix}$. Hence $\begin{vmatrix} W & X & Y & Z \end{vmatrix} = \pm \begin{vmatrix} A & B & C & D \end{vmatrix}$. □

Lemma 3.4.5. ([5], Lemma 4.2.5). *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. Let $Q = \frac{l_1 l_2 l_3 l_4 l_5}{((l_2,l_3)(l_4,l_5), l_1(l_2,l_3,l_4,l_5))}$. Let $W, X, Y, Z, H \in S(D, L)$. If $\begin{vmatrix} W & X & Y & Z \end{vmatrix} = \pm Q$ then $QH \in$ span$(\{W, X, Y, Z\})$.*

Proof. Let $W = (w_1, w_2, w_3, w_4), X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4)$, and $Z = (z_1, z_2, z_3, z_4)$. Let the column matrix formed by these splines be

$$M = \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix}.$$

Suppose $\begin{vmatrix} M \end{vmatrix} = \pm Q$. Let $H = (h_1, h_2, h_3, h_4)$. In order to show that $QH \in \text{span}(\{W, X, Y, Z\})$, we must show that there exist $a, b, c, d \in \mathbb{Z}$ such that $QH = aW + bX + cY + dZ$. In other words, we want to find an integer solution to the following equation:

$$\begin{bmatrix} Qh_1 \\ Qh_2 \\ Qh_3 \\ Qh_4 \end{bmatrix} = M \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Using Cramer's rule over $\mathbb{Q}$, we can solve for a, b, c, and d in $\mathbb{Q}$. Beginning with a, we have

$$a = \frac{\begin{vmatrix} Qh_1 & x_1 & y_1 & z_1 \\ Qh_2 & x_2 & y_2 & z_2 \\ Qh_3 & x_3 & y_3 & z_3 \\ Qh_4 & x_4 & y_4 & z_4 \end{vmatrix}}{\begin{vmatrix} M \end{vmatrix}} = \frac{Q \begin{vmatrix} h_1 & x_1 & y_1 & z_1 \\ h_2 & x_2 & y_2 & z_2 \\ h_3 & x_3 & y_3 & z_3 \\ h_4 & x_4 & y_4 & z_4 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} h_1 & x_1 & y_1 & z_1 \\ h_2 & x_2 & y_2 & z_2 \\ h_3 & x_3 & y_3 & z_3 \\ h_4 & x_4 & y_4 & z_4 \end{vmatrix}.$$

Using the same process to solve for b, c, and d, we have

$$b = \pm \begin{vmatrix} w_1 & h_1 & y_1 & z_1 \\ w_2 & h_2 & y_2 & z_2 \\ w_3 & h_3 & y_3 & z_3 \\ w_4 & h_4 & y_4 & z_4 \end{vmatrix},$$

$$c = \pm \begin{vmatrix} w_1 & x_1 & h_1 & z_1 \\ w_2 & x_2 & h_2 & z_2 \\ w_3 & x_3 & h_3 & z_3 \\ w_4 & x_4 & h_4 & z_4 \end{vmatrix}, \text{ and}$$

$$d = \pm \begin{vmatrix} w_1 & x_1 & y_1 & h_1 \\ w_2 & x_2 & y_2 & h_2 \\ w_3 & x_3 & y_3 & h_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix}.$$

Because all the entries in the matrices above are integers, we know that their determinants are integers. Hence $a, b, c, d \in \mathbb{Z}$. Hence, we can conclude that $QH \in \text{span}(\{W, X, Y, Z\})$ over $\mathbb{Z}$. □

The following lemma from Mahdavi's paper is crucial to our final proof. Mahdavi was only able to prove this result for a specific case, namely, the case in which all the edge labels are relatively prime. We provide his original lemma here and expand his result to the general case in the next chapter.

Lemma 3.4.6. ([5], Lemma 4.2.4). *Fix the edge labels on (D, L) such that* $L = (l_1, l_2, l_3, l_4, l_5)$. *Let* $(l_2, l_3, l_4, l_5) = (l_1, l_2) = (l_1, l_3) = (l_1, l_4) = (l_1, l_5) = 1$. *Let* $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5)}$. *Let* $W, X, Y, Z \in S(D, L)$. *Then* $Q \mid \begin{vmatrix} W & X & Y & Z \end{vmatrix}$.

Proof. Since $W, X, Y, Z \in S(D, L)$, then $l_2 \mid (w_2 - w_3)$, $l_2 \mid (x_2 - x_3)$, $l_2 \mid (y_2 - y_3)$, $l_2 \mid (z_2 - z_3)$. Let $M = \begin{vmatrix} W & X & Y & Z \end{vmatrix}$. Then

$$M = \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} = \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 - w_3 & x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} = l_2 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix}$$

for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}$. We also know that $l_3 \mid (w_3 - w_1)$, $l_3 \mid (x_3 - x_1)$, $l_3 \mid (y_3 - y_1)$, and $l_3 \mid (z_3 - z_1)$. Then

$$M = l_2 l_3 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix}$$

for some $b_1, b_2, b_3, b_4 \in \mathbb{Z}$. Similarly, we know that $l_5 \mid (w_4 - w_1)$, $l_5 \mid (x_4 - x_1)$, $l_5 \mid (y_4 - y_1)$, and $l_5 \mid (z_4 - z_1)$. Then

$$M = l_2 l_3 l_5 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{Z}$.

Hence, we know that $l_2 l_3 l_5 \mid M$. Using the same procedure, we can prove that $l_2 l_3 l_4 \mid M$, $l_2 l_4 l_5 \mid M$, and $l_3 l_4 l_5 \mid M$. Since these four products divide M, it follows that their least common multiple divides M. Then $[l_2 l_3 l_4, l_2 l_3 l_5, l_2 l_4 l_5, l_3 l_4 l_5] \mid M$. By Theorem 2.1.2, we know that $[l_2 l_3 l_4, l_2 l_3 l_5, l_2 l_4 l_5, l_3 l_4 l_5] = \dfrac{(l_2 l_3 l_4 l_5)^3}{(\widehat{l_2 l_3 l_4}, \widehat{l_2 l_3 l_5}, \widehat{l_2 l_4 l_5}, \widehat{l_3 l_4 l_5})} = \dfrac{l_2 l_3 l_4 l_5}{(l_2, l_3, l_4, l_5)}$. Hence $\frac{l_2 l_3 l_4 l_5}{(l_2, l_3, l_4, l_5)} \mid M$. We know that $(l_2, l_3, l_4, l_5) = 1$ by hypothesis, so $l_2 l_3 l_4 l_5 \mid M$.

Using the matrix method once again, we can show that $l_1 \mid M$. Hence $[l_1, l_2 l_3 l_4 l_5] \mid M$. Since we assumed that l_1 is pairwise coprime with l_2, l_3, l_4, and l_5, then $[l_1, l_2 l_3 l_4 l_5] = l_1 l_2 l_3 l_4 l_5$. Hence $l_1 l_2 l_3 l_4 l_5 \mid M$. By hypothesis, we know that $((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5)) = ((l_2, l_3)(l_4, l_5), l_1) = 1$. Hence $Q = l_1 l_2 l_3 l_4 l_5$. Hence $Q \mid M$. □

To demonstrate this result, we provide the following example.

Example 3.4.7. Fix the edge labels on (D, L) where $L = (3, 4, 7, 5, 11)$. Let $f_0, f_1, f_2, f_3 \in S(D, L)$ where

$$f_0 = (2, 5, 37, 35),$$

$$f_1 = (0, 3, 35, 33),$$

$$f_2 = (0, 0, 28, 55), \text{ and}$$

$$f_3 = (0, 0, 0, 55).$$

It is easy to verify that f_0, f_1, f_2, f_3 satisfy the required conditions. Putting these splines into matrix form and taking the determinant, we have

$$M = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 37 & 35 & 28 & 0 \\ 35 & 33 & 55 & 55 \end{vmatrix} = 2 \cdot 3 \cdot 28 \cdot 55 = 9240.$$

For this particular edge-labeling, we have

$$Q = \frac{3 \cdot 4 \cdot 7 \cdot 5 \cdot 11}{((4,7)(5,11), 3(4,7,5,11))} = 3 \cdot 4 \cdot 7 \cdot 5 \cdot 11 = 4620.$$

Since $9240 = 2 \cdot 4620$, then $Q|M$. ◊

4

Determinantal Conditions on Splines

In this chapter, we present the main results of the project. In Section 4.1, we expand Mahdavi's results to the general case and show that four splines over a diamond graph (D, L) with $L = (l_1, l_2, l_3, l_4, l_5)$ form a module basis for $S(D, L)$ if and only if their determinant is equal to $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5))}$. In Section 4.2, we further expand some of Mahdavi's results to splines over (m, n)-cycle graphs.

4.1 Splines over the Diamond Graph

We begin by expanding Lemma 3.4.6 to a more general case, where the conditions on the edge labels are less restrictive.

Lemma 4.1.1. *Fix the edge labels on (D, L) such that* $L = (l_1, l_2, l_3, l_4, l_5)$. *Let* $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5))}$. *Let* $W, X, Y, Z \in S(D, L)$. *If* $(l_2, l_3) = 1$ *and* $(l_4, l_5) = 1$, *then* $Q \mid \begin{vmatrix} W & X & Y & Z \end{vmatrix}$.

Proof. Suppose $(l_2, l_3) = 1$ and $(l_4, l_5) = 1$. Then $((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5)) = (1, l_1) = 1$. Hence $Q = l_1 l_2 l_3 l_4 l_5$. Let $M = \begin{vmatrix} W & X & Y & Z \end{vmatrix}$. We have already shown using properties of determinants that $l_2 l_3 l_5 | M$, $l_2 l_3 l_4 | M$, $l_2 l_4 l_5 | M$, and $l_3 l_4 l_5 | M$. Using the same procedure, we can prove that any combination of three edge labels, except for $l_1 l_2 l_3$ and $l_1 l_4 l_5$, divides M. In addition, using the same procedure, it is easy to show that any combination of two edge labels divides M.

We will now show that $l_1l_2l_3$ and $l_1l_4l_5$ divide M on the condition that $(l_2,l_3)=(l_4,l_5)=1$ is true. Since any combination of two edge labels divides M, then we know that $l_1l_2|M$, $l_1l_3|M$, and $l_2l_3|M$. Then it follows that $[l_1l_2,l_1l_3,l_2l_3]|M$. By Theorem 2.1.14, we know that $[l_1l_2,l_1l_3,l_2l_3]=\dfrac{l_1l_2l_1l_3l_2l_3}{(l_1l_2l_1l_3,l_1l_2l_2l_3,l_1l_3l_2l_3)}=\dfrac{l_1l_2l_1l_3l_2l_3}{l_1l_2l_3(l_1,l_2,l_3)}=\dfrac{l_1l_2l_3}{(l_1,l_2,l_3)}$. Because $(l_2,l_3)=1$, then $\dfrac{l_1l_2l_3}{(l_1,l_2,l_3)}=l_1l_2l_3$. Hence $l_1l_2l_3|M$.

Similarly, we know that $l_1l_4|M$, $l_1l_5|M$, and $l_4l_5|M$. Then it follows that $[l_1l_4,l_1l_5,l_4l_5]|M$. By Theorem 2.1.14, we know that $[l_1l_4,l_1l_5,l_4l_5]=\dfrac{l_1l_4l_1l_5l_4l_5}{(l_1l_4l_1l_5,l_1l_4l_4l_5,l_1l_5l_4l_5)}=\dfrac{l_1l_4l_1l_5l_4l_5}{l_1l_4l_5(l_1,l_4,l_5)}=\dfrac{l_1l_4l_5}{(l_1,l_4,l_5)}$. Because $(l_4,l_5)=1$, then $\dfrac{l_1l_4l_5}{(l_1,l_4,l_5)}=l_1l_4l_5$. Hence $l_1l_4l_5|M$.

We have already shown that if $l_2l_3l_5 \mid M$, $l_2l_3l_4 \mid M$, $l_2l_4l_5 \mid M$, and $l_3l_4l_5 \mid M$, then $\dfrac{l_2l_3l_4l_5}{(l_2,l_3,l_4,l_5)} \mid M$. By hypothesis, then $(l_2,l_3,l_4,l_5)=1$. Then $l_2l_3l_4l_5 \mid M$, or equivalently $\hat{l}_1 \mid M$. Similarly, it can be shown that $\hat{l}_2 \mid M$, $\hat{l}_3 \mid M$, $\hat{l}_4 \mid M$, and $\hat{l}_5 \mid M$. Since all of these divide M, it follows that their least common multiple divides M. Then $[\hat{l}_1,\hat{l}_2,\hat{l}_3,\hat{l}_4,\hat{l}_5] \mid M$. By Theorem 2.1.14, we know that $[\hat{l}_1,\hat{l}_2,\hat{l}_3,\hat{l}_4,\hat{l}_5]=\dfrac{l_1l_2l_3l_4l_5}{(l_1,l_2,l_3,l_4,l_5)}$. Then $\dfrac{l_1l_2l_3l_4l_5}{(l_1,l_2,l_3,l_4,l_5)} \mid M$. Since $(l_2,l_3)=1$ and $(l_4,l_5)=1$, then $(l_1,l_2,l_3,l_4,l_5)=1$. Hence $l_1l_2l_3l_4l_5 \mid M$. Hence $Q \mid M$. □

The following two lemmas are proved without any restrictions on the edge labels, and they will be essential for our final proof.

Lemma 4.1.2. *Fix the edge labels on (D,L) such that $L=(l_1,l_2,l_3,l_4,l_5)$. Let $W,X,Y,Z\in S(D,L)$. Let $M=\begin{vmatrix}W & X & Y & Z\end{vmatrix}$. Then $\dfrac{l_1l_2l_3l_4}{(l_1,l_2,l_3)}|M$, $\dfrac{l_1l_2l_3l_5}{(l_1,l_2,l_3)}|M$, $\dfrac{l_1l_3l_4l_5}{(l_1,l_4,l_5)}|M$, and $\dfrac{l_1l_2l_4l_5}{(l_1,l_4,l_5)}|M$.*

Proof. In Lemma 4.1.1, we showed that $\dfrac{l_2l_3l_4l_5}{(l_2,l_3,l_4,l_5)}|M$. We will now show that $\dfrac{l_1l_2l_3l_4}{(l_1,l_2,l_3)}|M$. Let $X=\dfrac{l_1l_2l_3}{(l_1,l_2,l_3)}$. We showed in Lemma 4.1.1 that $l_1l_2l_4|M$, $l_1l_3l_4|M$, $l_2l_3l_4|M$, and $X|M$. It follows that $[l_1l_2l_4, l_1l_3l_4, l_2l_3l_4, X]|M$. By Theorem 2.1.14, we can rewrite this as follows:

$$
\begin{aligned}
[l_1l_2l_3, l_1l_3l_4, l_2l_3l_4, X] &= \frac{l_1l_2l_4 \cdot l_1l_3l_4 \cdot l_2l_3l_4 \cdot X}{(l_1l_3l_4 \cdot l_1l_2l_4 \cdot l_2l_3l_4, l_1l_2l_4 \cdot l_1l_3l_4 \cdot X, l_1l_2l_4 \cdot l_2l_3l_4 \cdot X, l_1l_3l_4 \cdot l_2l_3l_4 \cdot X)} \\
&= \frac{l_1l_2l_3l_4 \cdot X}{(l_1l_2l_3l_4, l_1 \cdot X, l_2 \cdot X, l_3 \cdot X)} \\
&= \frac{l_1l_2l_3l_4 \cdot X}{(l_1l_2l_3l_4, X(l_1, l_2, l_3))} \\
&= \frac{l_1l_2l_3l_4 \cdot X}{(l_1l_2l_3l_4, \frac{l_1l_2l_3}{(l_1,l_2,l_3)} \cdot (l_1, l_2, l_3))} \\
&= \frac{l_1l_2l_3l_4 \cdot X}{(l_1l_2l_3l_4, l_1l_2l_3)} \\
&= \frac{l_4 \cdot X}{(l_4, 1)} \\
&= l_4 \cdot X \\
&= \frac{l_1l_2l_3l_4}{(l_1, l_2, l_3)}.
\end{aligned}
$$

Hence, we can conclude that $\frac{l_1l_2l_3l_4}{(l_1, l_2, l_3)}|M$. Using a similar method, we can show that $\frac{l_1l_2l_3l_5}{(l_1, l_2, l_3)}|M$.

Now, let $Y = \frac{l_1l_4l_5}{(l_1, l_4, l_5)}$. We showed in Lemma 4.1.1 that $l_1l_3l_4|M$, $l_1l_3l_5|M$, $l_3l_4l_5|M$, and $Y|M$. It follows that $[l_1l_3l_4, l_1l_3l_5, l_3l_4l_5, Y]|M$. By Theorem 2.1.14, we can rewrite this as follows:

$$
\begin{aligned}
[l_1l_3l_4, l_1l_3l_5, l_3l_4l_5, Y] &= \frac{l_1l_3l_4 \cdot l_1l_3l_5 \cdot l_3l_4l_5 \cdot Y}{(l_1l_3l_4 \cdot l_1l_3l_5 \cdot l_3l_4l_5, l_1l_3l_4 \cdot l_1l_3l_5 \cdot Y, l_1l_3l_4 \cdot l_3l_4l_5 \cdot Y, l_1l_3l_5 \cdot l_3l_4l_5 \cdot Y)} \\
&= \frac{l_1l_3l_4l_5 \cdot Y}{(l_1l_3l_4l_5, l_1 \cdot Y, l_4 \cdot Y, l_5 \cdot Y)} \\
&= \frac{l_1l_3l_4l_5 \cdot Y}{(l_1l_3l_4l_5, Y(l_1, l_4, l_5))} \\
&= \frac{l_1l_3l_4l_5 \cdot Y}{(l_1l_3l_4l_5, \frac{l_1l_4l_5}{(l_1,l_4,l_5)} \cdot (l_1, l_4, l_5))} \\
&= \frac{l_1l_3l_4l_5 \cdot Y}{(l_1l_3l_4l_5, l_1l_4l_5)} \\
&= \frac{l_3 \cdot Y}{(l_3, 1)} \\
&= l_3 \cdot Y \\
&= \frac{l_1l_3l_4l_5}{(l_1, l_4, l_5)}.
\end{aligned}
$$

Hence, we can conclude that $\frac{l_1l_3l_4l_5}{(l_1, l_4, l_5)}|M$. Using a similar method, we can show that $\frac{l_1l_2l_4l_5}{(l_1, l_4, l_5)}|M$. □

We will demonstrate this result using an example.

Example 4.1.3. Fix the edge labels on (D, L) where $L = (2, 3, 6, 5, 10)$. Notice that $(l_2, l_3) = (3, 6) = 3$ and $(l_4, l_5) = (5, 10) = 5$, so L does not meet the required conditions imposed by Lemma 4.1.1. Let $f_0, f_1, f_2, f_3 \in S(D, L)$ where

$$f_0 = (1, 1, 1, 1),$$

$$f_1 = (0, 30, 18, 50),$$

$$f_2 = (0, 0, 30, 50), \text{ and}$$

$$f_3 = (0, 0, 0, 50).$$

It is easy to verify that f_0, f_1, f_2, f_3 satisfy the required conditions. Putting these splines into matrix form and taking the determinant, we have

$$M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 30 & 0 & 0 \\ 1 & 18 & 30 & 0 \\ 1 & 50 & 50 & 50 \end{vmatrix} = 1 \cdot 30 \cdot 30 \cdot 50 = 45000.$$

For this particular edge-labeling, we have

$$\frac{l_1 l_2 l_3 l_4}{(l_1, l_2, l_3)} = \frac{2 \cdot 3 \cdot 6 \cdot 5}{(2, 3, 6)} = 180 = \frac{M}{250}$$
$$\frac{l_1 l_2 l_3 l_5}{(l_1, l_2, l_3)} = \frac{2 \cdot 3 \cdot 6 \cdot 10}{(2, 3, 6)} = 360 = \frac{M}{125}$$
$$\frac{l_1 l_3 l_4 l_5}{(l_1, l_4, l_5)} = \frac{2 \cdot 6 \cdot 5 \cdot 10}{(2, 5, 10)} = 600 = \frac{M}{75}$$
$$\frac{l_1 l_2 l_4 l_5}{(l_1, l_4, l_5)} = \frac{2 \cdot 3 \cdot 5 \cdot 10}{(2, 5, 10)} = 300 = \frac{M}{150}$$

Hence $\frac{l_1 l_2 l_3 l_4}{(l_1, l_2, l_3)} | M$, $\frac{l_1 l_2 l_3 l_5}{(l_1, l_2, l_3)} | M$, $\frac{l_1 l_3 l_4 l_5}{(l_1, l_4, l_5)} | M$, and $\frac{l_1 l_2 l_4 l_5}{(l_1, l_4, l_5)} | M$. ◊

Lemma 4.1.4. *Fix the edge labels on (D, L) such that $L = (l_1, l_2, l_3, l_4, l_5)$. Let $P = \frac{l_1 l_2 l_3 l_4 l_5}{(l_1(l_2, l_3, l_4, l_5), l_2(l_1, l_4, l_5), l_3(l_1, l_4, l_5), l_4(l_1, l_2, l_3), l_5(l_1, l_2, l_3))}$. Let $W, X, Y, Z \in S(D, L)$. Let $M = \begin{vmatrix} W & X & Y & Z \end{vmatrix}$. Then $P \mid M$.*

Proof. Without using any constraints on the edge labels, we proved in Lemma 4.1.1 that $\frac{l_2 l_3 l_4 l_5}{(l_2, l_3, l_4, l_5)} | M$, and we proved in Lemma 4.1.2 that $\frac{l_1 l_3 l_4 l_5}{(l_1, l_4, l_5)} | M$, $\frac{l_1 l_2 l_4 l_5}{(l_1, l_4, l_5)} | M$, $\frac{l_1 l_2 l_3 l_5}{(l_1, l_2, l_3)} | M$, and $\frac{l_1l_2l_3l_4}{(l_1,l_2,l_3)}|M$. We will denote these fractions as $L_1 = \frac{l_2l_3l_4l_5}{(l_2,l_3,l_4,l_5)}, L_2 = \frac{l_1l_3l_4l_5}{(l_1,l_4,l_5)}$, $L_3 = \frac{l_1l_2l_4l_5}{(l_1,l_4,l_5)}, L_4 = \frac{l_1l_2l_3l_5}{(l_1,l_2,l_3)}$, and $L_5 = \frac{l_1l_2l_3l_4}{(l_1,l_2,l_3)}$. Since $L_i \mid M$ for all $i \in \{1,2,3,4,5\}$, then $[L_1,L_2,L_3,L_4,L_5] \mid M$. Let $x = [L_1,L_2,L_3,L_4,L_5]$. Then $L_1 \mid x, L_2 \mid x, L_3 \mid x, L_4 \mid x$, and $L_5 \mid x$.

We can rewrite $L_1 \mid x$ as $L_1l_1(l_2,l_3,l_4,l_5) \mid xl_1(l_2,l_3,l_4,l_5)$, which simplifies to $l_1l_2l_3l_4l_5 \mid xl_1(l_2,l_3,l_4,l_5)$. Similarly, we can rewrite

$$L_2 \mid x \text{ as } L_2l_2(l_1,l_4,l_5) \mid xl_2(l_1,l_4,l_5),$$

$$L_3 \mid x \text{ as } L_3l_3(l_1,l_4,l_5) \mid xl_3(l_1,l_4,l_5),$$

$$L_4 \mid x \text{ as } L_4l_4(l_1,l_2,l_3) \mid xl_4(l_1,l_2,l_3), \text{ and}$$

$$L_5 \mid x \text{ as } L_5l_5(l_1,l_2,l_3) \mid xl_5(l_1,l_2,l_3).$$

These simplify to

$$l_1l_2l_3l_4l_5 \mid xl_2(l_1,l_4,l_5),$$

$$l_1l_2l_3l_4l_5 \mid xl_3(l_1,l_4,l_5),$$

$$l_1l_2l_3l_4l_5 \mid xl_4(l_1,l_2,l_3), \text{ and}$$

$$l_1l_2l_3l_4l_5 \mid xl_5(l_1,l_2,l_3).$$

We deduce that

$$l_1l_2l_3l_4l_5 \mid (xl_1(l_2,l_3,l_4,l_5), xl_2(l_1,l_4,l_5), xl_3(l_1,l_4,l_5), xl_4(l_1,l_2,l_3), xl_5(l_1,l_2,l_3)).$$

Then

$$l_1l_2l_3l_4l_5 \mid x(l_1(l_2,l_3,l_4,l_5), l_2(l_1,l_4,l_5), l_3(l_1,l_4,l_5), l_4(l_1,l_2,l_3), l_5(l_1,l_2,l_3)).$$

It follows that $k \cdot l_1l_2l_3l_4l_5 = x(l_1(l_2,l_3,l_4,l_5), l_2(l_1,l_4,l_5), l_3(l_1,l_4,l_5), l_4(l_1,l_2,l_3), l_5(l_1,l_2,l_3))$ for some $k \in \mathbb{Z}$. Then $k\frac{l_1l_2l_3l_4l_5}{(l_1(l_2,l_3,l_4,l_5), l_2(l_1,l_4,l_5), l_3(l_1,l_4,l_5), l_4(l_1,l_2,l_3), l_5(l_1,l_2,l_3))} = x$. It is easy to show that $l_1(l_2,l_3,l_4,l_5) \mid l_1l_2l_3l_4l_5$, $l_2(l_1,l_4,l_5) \mid l_1l_2l_3l_4l_5$, $l_3(l_1,l_4,l_5) \mid l_1l_2l_3l_4l_5$, $l_4(l_1,l_2,l_3) \mid l_1l_2l_3l_4l_5$, and $l_5(l_1,l_2,l_3) \mid l_1l_2l_3l_4l_5$.

Hence $(l_1(l_2,l_3,l_4,l_5), l_2(l_1,l_4,l_5), l_3(l_1,l_4,l_5), l_4(l_1,l_2,l_3), l_5(l_1,l_2,l_3))|l_1l_2l_3l_4l_5$.

Hence $\dfrac{l_1l_2l_3l_4l_5}{(l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))} \in \mathbb{Z}$. Then, we can conclude that

$$\frac{l_1l_2l_3l_4l_5}{(l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))} \mid x.$$

Since $x \mid M$, then $\dfrac{l_1l_2l_3l_4l_5}{(l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))} \mid M$. □

Now we are able to prove with no restrictions on the edge labels that Q divides the determinant of four arbitrary splines in $S(D,L)$.

Theorem 4.1.5. *Fix the edge labels on (D,L) where $L=(l_1,l_2,l_3,l_4,l_5)$. Let $Q = \dfrac{l_1l_2l_3l_4l_5}{((l_2,l_3)(l_4,l_5),l_1(l_2,l_3,l_4,l_5))}$. Let $W,X,Y,Z \in S(D,L)$. Let $M = \begin{vmatrix} W & X & Y & Z \end{vmatrix}$. Then $Q \mid \begin{vmatrix} W & X & Y & Z \end{vmatrix}$.*

Proof. Let $P = \dfrac{l_1l_2l_3l_4l_5}{(l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))}$. Let $X = l_1l_2l_3l_4l_5$. Let $A = (l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))$, and let $B = ((l_2,l_3)(l_4,l_5),l_1(l_2,l_3,l_4,l_5))$. Then $P = \dfrac{X}{A}$ and $Q = \dfrac{X}{B}$. Using properties of the greatest common denominator, we can expand

$$\begin{aligned}
A &= (l_1(l_2,l_3,l_4,l_5),l_2(l_1,l_4,l_5),l_3(l_1,l_4,l_5),l_4(l_1,l_2,l_3),l_5(l_1,l_2,l_3))\\
&= (l_1l_2,l_1l_3,l_1l_4,l_1l_5,l_2l_1,l_2l_4,l_2l_5,l_3l_1,l_3l_4,l_3l_5,l_4l_1,l_4l_2,l_4l_3,l_5l_1,l_5l_2,l_5l_3)\\
&= (l_1l_2,l_1l_3,l_1l_4,l_1l_5,l_2l_4,l_2l_5,l_3l_4,l_3l_5)\\
&= (l_1(l_2,l_3,l_4,l_5),l_2(l_4,l_5),l_3(l_4,l_5))\\
&= (l_1(l_2,l_3,l_4,l_5),(l_2,l_3)(l_4,l_5))\\
&= B
\end{aligned}$$

Hence $A = B$. Then $P = \dfrac{X}{A} = \dfrac{X}{B} = Q$. By Lemma 4.1.4, we know that $P|M$. Hence $Q|M$. □

Finally, we show that four splines form a module basis for $S(D,L)$ if and only if their determinant is equal to Q.

Theorem 4.1.6. *Fix the edge labels on (D, L) where $L = (l_1, l_2, l_3, l_4, l_5)$. Let $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{((l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5))}$. Let $W, X, Y, Z \in S(D, L)$, and let $M = \begin{vmatrix} W & X & Y & Z \end{vmatrix}$. Then $\{W, X, Y, Z\}$ forms a module basis for $S(D, L)$ if and only if $M = \pm Q$.*

Proof. First, suppose that $\{W, X, Y, Z\}$ forms a module basis for $S(D, L)$. Let b_0, b_1, b_2, b_3 be minimal elements of their respective flow-up classes. By Lemma 3.3.6, we know that b_0, b_1, b_2, b_3 form a module basis for $S(D, L)$. Also, by Lemma 3.4.2, we know that $\begin{vmatrix} b_0 & b_1 & b_2 & b_3 \end{vmatrix} = \pm Q$. Since $\{b_0, b_1, b_2, b_3\}$ and $\{W, X, Y, Z\}$ are bases, then by Lemma 3.4.4, we know that $\begin{vmatrix} b_0 & b_1 & b_2 & b_3 \end{vmatrix} = M$. Hence $M = \pm Q$.

Now, suppose that $M = \pm Q$. Since $M \neq 0$, then $\{W, X, Y, Z\}$ are linearly independent. Let $H \in S(D, L)$. By Lemma 3.4.5, we know that $QH \in \text{span}(\{W, X, Y, Z\})$. In order to show that $\{W, X, Y, Z\}$ spans $S(D, L)$, we must show that H can be written as a linear combination of $W, X, Y,$ and Z. Since $QH \in \text{span}(\{W, X, Y, Z\})$, then there exist $a, b, c, d \in \mathbb{Z}$ such that $QH = aW + bX + cY + dZ$. We know that $M = \pm Q$. Then, by the properties of determinants,

$$\begin{aligned} \pm aQ &= a \begin{vmatrix} W & X & Y & Z \end{vmatrix} \\ &= \begin{vmatrix} aW & X & Y & Z \end{vmatrix} \\ &= \begin{vmatrix} aW + bX + cY + dZ & X & Y & Z \end{vmatrix} \\ &= \begin{vmatrix} QH & X & Y & Z \end{vmatrix} \\ &= Q \begin{vmatrix} H & X & Y & Z \end{vmatrix}. \end{aligned}$$

Hence, $a = \pm \begin{vmatrix} H & X & Y & Z \end{vmatrix}$. Using the same procedure, we can show that

$$\begin{aligned} b &= \pm \begin{vmatrix} W & H & Y & Z \end{vmatrix} \\ c &= \pm \begin{vmatrix} W & X & H & Z \end{vmatrix} \\ d &= \pm \begin{vmatrix} W & X & Y & H \end{vmatrix}. \end{aligned}$$

By Theorem 4.1.5, we know that $Q \mid \begin{vmatrix} H & X & Y & Z \end{vmatrix}$. Hence, we deduce that $k_1 Q = \begin{vmatrix} H & X & Y & Z \end{vmatrix} = a$ for some $k_1 \in \mathbb{Z}$. Similarly, we can show that there exists

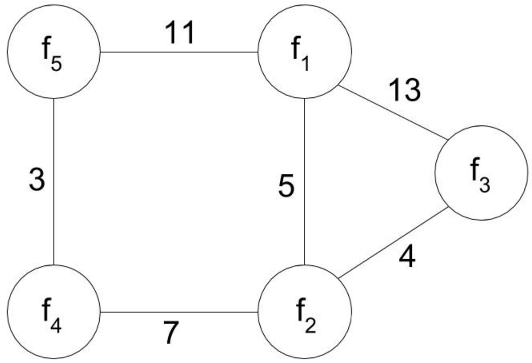

Figure 4.2.1. An edge-labeled (4,3)-cycle graph.

$k_2, k_3, k_4 \in \mathbb{Z}$ such that $k_2Q = b$, $k_3Q = c$, and $k_4Q = d$. Then

$$\begin{aligned} QH &= aW + bX + cY + dZ \\ &= k_1QW + k_2QX + k_3QY + k_4QZ \\ &= Q(k_1W + k_2X + k_3Y + k_4Z). \end{aligned}$$

We conclude that $H = k_1W + k_2X + k_3Y + k_4Z$. Since H is a linear combination of $W, X, Y,$ and Z, then $\{W, X, Y, Z\}$ span $S(D, L)$. Hence, we conclude that $\{W, X, Y, Z\}$ is a module basis for $S(D, L)$. □

4.2 Splines over (m,n)-cycles

In this section, we explore properties of splines over (m, n)-cycle graphs, and we expand some of Mahdavi's results from splines over the diamond graph to splines over (m, n)-cycles. To begin, we provide an example of elements of each flow-up class over a $(4, 3)$-cycle edge-labeled graph.

Example 4.2.1. Fix the edge labels on $(C_{(4,3)}, L)$ where $L = (5, 4, 13, 7, 3, 11)$. This graph is pictured in Figure 4.2.1. The following splines are elements of the flow-up classes $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$.

$$f_0 = (2, 7, 15, 14, 35) \in \mathcal{F}_0$$

$$f_1 = (0, 5, 65, 12, 33) \in \mathcal{F}_1$$

$$f_2 = (0, 0, 52, 14, 11) \in \mathcal{F}_2$$

$$f_3 = (0, 0, 0, 7, 22) \in \mathcal{F}_3$$

$$f_4 = (0, 0, 0, 0, 33) \in \mathcal{F}_4$$

It is easy to verify that f_0, f_1, f_2, f_3, f_4 satisfy the defining properties of a spline. $\Diamond$

Some of Mahdavi's work has already been expanded by Sangam and Zhang, as shown in the following theorem, included without proof.

Theorem 4.2.2. *Fix the edge labels on an $(C_{(m,n)}, L)$ where $L = \{l_1, l_2, \ldots, l_n, l_{n+1}, \ldots l_{m+n-1}\}$ and $m, n \geq 3$ are integers. Then the flow-up classes exist on $S(C_{(m,n)}, L)$ and the set of minimal elements of each flow-up class $B = \{b_0, \ldots, b_{m+n-3}\}$, where b_k is a minimal element of $\mathcal{F}_k$ for some $0 \leq k \leq m+n-3$, forms a module basis over $S(C_{(m,n)}, L)$. In addition, the leading term $(b_k)_L$ of each basis element b_k can be described as follows:*

$$(b_k)_L = \begin{cases} 1 & \text{if } k = 0 \\ [l_1, (l_2, \ldots, l_n), (l_{n+1}, \ldots, l_{m+n-1})] & \text{if } k = 1 \\ [l_k, (l_{k+1}, \ldots, l_n)] & \text{if } 2 \leq k \leq n-1 \\ [l_{k+1}, (l_{k+2}, \ldots, l_{m+n-1})] & \text{if } n \leq k \leq m+n-3 \end{cases}$$

We now state Mahdavi's conjecture for the determinantal criterion for splines over an (m, n)-cycle graph.

Conjecture 4.2.3 ([5], Conjecture 5.1.3)**.** *Fix the edge labels on $(C_{(m,n)}, L)$ such that $L = (l_1, l_2, \ldots, l_n, l_{n+1}, \ldots, l_{n+m-1})$. Let $X_1, X_2, \ldots, X_{n+m-2} \in S(C_{(m,n)}, L)$. Let*

$$\begin{aligned} Q &= \frac{\textit{product of edge labels}}{((\textit{edges of cycle 1})(\textit{edges of cycle 2}), \textit{center edge}(\textit{outer edges}))} \\ &= \frac{l_1 l_2 \cdots l_{n+m-1}}{((l_2, l_3, \ldots, l_n)(l_{n+1}, \ldots, l_{n+m-1}), l_1(l_2, \ldots, l_{n+m-1}))}. \end{aligned}$$

Then $\{X_1, X_2, \ldots, X_{n+m-2}\}$ is a basis for $S(C_{(m,n)}, L)$ if and only if $\begin{vmatrix} X_1 & X_2 & \cdots & X_{n+m-2} \end{vmatrix} = \pm Q$.

Some of Mahdavi's results, presented in Chapter 3, can be expanded from splines over the diamond graph to splines over a more general (m,n)-cycle graph. Here, we state the results that we have proved so far for any (m,n)-cycle graph.

Lemma 4.2.4. *Fix the edge labels on $(C_{(m,n)}, L)$ where $L = (l_1, \ldots, l_{n+m-1})$. If $\{B_1, \ldots, B_{n+m-2}\}$ is a basis of $S(C_{(m,n)}, L)$ and $X_1, \ldots, X_{n+m-2} \in S(C_{(m,n)}, L)$, then $\begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix}$ divides $\begin{vmatrix} X_1 & \cdots & X_{n+m-2} \end{vmatrix}$.*

Proof. Since $X_1, \ldots, X_{n+m-2} \in S(C_{(m,n)}, L)$, then each of $X_1, \ldots, X_{n+m-2}$ can be written as a linear combination of $B_1, \ldots, B_{n+m-2}$ as follows. For all $i \in \{1, \ldots, n+m-2\}$, $X_i = a_{i1}B_1 + a_{i2}B_2 + \ldots + a_{i(n+m-2)}B_{n+m-2}$ where $a_{i1}, \ldots, a_{i(n+m-2)} \in \mathbb{Z}$.

We can write each of the splines in matrix form. Let $j \in \{1, \ldots, n+m-2\}$. If $X_j = (X_{j1}, X_{j2}, \ldots, X_{j(n+m-2)})$, then

$$X_j = \begin{bmatrix} X_{j1} \\ X_{j2} \\ \vdots \\ X_{j(n+m-2)} \end{bmatrix} = \begin{bmatrix} a_{j1}b_{11} + a_{j2}b_{21} + \ldots + a_{j(n+m-2)}b_{(n+m-2)1} \\ a_{j1}b_{12} + a_{j2}b_{22} + \ldots + a_{j(n+m-2)}b_{(n+m-2)2} \\ \vdots \\ a_{j1}b_{1(n+m-2)} + a_{j2}b_{2(n+m-2)} + \ldots + a_{j(n+m-2)}b_{(n+m-2)(n+m-2)} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{(n+m-2)1} \\ b_{12} & b_{22} & \cdots & b_{(n+m-2)2} \\ \vdots & & & \\ b_{1(n+m-2)} & b_{2(n+m-2)} & \cdots & b_{(n+m-2)(n+m-2)} \end{bmatrix} \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{j(n+m-2)} \end{bmatrix}.$$

Using the same process for the remaining splines, we can write $\begin{bmatrix} X_1 & X_2 & \cdots & X_{n+m-2} \end{bmatrix} =$

$$\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{(n+m-2)1} \\ b_{12} & b_{22} & \cdots & b_{(n+m-2)2} \\ \vdots & & & \\ b_{1(n+m-2)} & b_{2(n+m-2)} & \cdots & b_{(n+m-2)(n+m-2)} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{(n+m-2)1} \\ a_{12} & a_{22} & \cdots & a_{(n+m-2)2} \\ \vdots & & & \\ a_{1(n+m-2)} & a_{2(n+m-2)} & \cdots & a_{(n+m-2)(n+m-2)} \end{bmatrix}.$$

Since $\left|AB\right| = \left|A\right| \cdot \left|B\right|$, we know that $\begin{vmatrix} X_1 & X_2 & \cdots & X_{n+m-2} \end{vmatrix} =$

$$\begin{vmatrix} b_{11} & b_{21} & \cdots & b_{(n+m-2)1} \\ b_{12} & b_{22} & \cdots & b_{(n+m-2)2} \\ \vdots & & & \\ b_{1(n+m-2)} & b_{2(n+m-2)} & \cdots & b_{(n+m-2)(n+m-2)} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{(n+m-2)1} \\ a_{12} & a_{22} & \cdots & a_{(n+m-2)2} \\ \vdots & & & \\ a_{1(n+m-2)} & a_{2(n+m-2)} & \cdots & a_{(n+m-2)(n+m-2)} \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & B_2 & \cdots & B_{n+m-2} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{(n+m-2)1} \\ a_{12} & a_{22} & \cdots & a_{(n+m-2)2} \\ \vdots & & & \\ a_{1(n+m-2)} & a_{2(n+m-2)} & \cdots & a_{(n+m-2)(n+m-2)} \end{vmatrix}$$

Since all the entries of $\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{(n+m-2)1} \\ a_{12} & a_{22} & \cdots & a_{(n+m-2)2} \\ \vdots & & & \\ a_{1(n+m-2)} & a_{2(n+m-2)} & \cdots & a_{(n+m-2)(n+m-2)} \end{bmatrix}$ are in $\mathbb{Z}$, it follows that $\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{(n+m-2)1} \\ a_{12} & a_{22} & \cdots & a_{(n+m-2)2} \\ \vdots & & & \\ a_{1(n+m-2)} & a_{2(n+m-2)} & \cdots & a_{(n+m-2)(n+m-2)} \end{vmatrix}$ is in $\mathbb{Z}$. Hence $\begin{vmatrix} B_1 & B_2 & \cdots & B_{n+m-2} \end{vmatrix}$ divides $\begin{vmatrix} X_1 & X_2 & \cdots & X_{n+m-2} \end{vmatrix}$ □

Lemma 4.2.5. *Fix the edge labels on $(C_{(m,n)}, L)$ where $L = (l_1, \ldots, l_{n+m-1})$. If $\{A_1, \ldots, A_{n+m-2}\}$ and $\{B_1, \ldots, B_{n+m-2}\}$ are bases of $S(C_{(m,n)}, L)$, then $\begin{vmatrix} A_1 & \cdots & A_{n+m-2} \end{vmatrix} = \pm \begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix}$.*

Proof. Since $A_1, \ldots, A_{n+m-2} \in S(C_{(m,n)}, L)$ and $\{B_1, \ldots, B_{n+m-2}\}$ is a basis, then we know by Lemma 4.2.4 that $\begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix}$ divides $\begin{vmatrix} A_1 & \cdots & A_{n+m-2} \end{vmatrix}$. Similarly, because $B_1, \ldots, B_{n+m-2} \in S(C_{(m,n)}, L)$, and $\{A_1, \ldots, A_{n+m-2}\}$ is a basis, then we know that $\begin{vmatrix} A_1 & \cdots & A_{n+m-2} \end{vmatrix}$ divides $\begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix}$. Hence $\begin{vmatrix} A_1 & \cdots & A_{n+m-2} \end{vmatrix} = \pm \begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix}$.

□

5

Future Work

Although we have not yet proved all of the results from Chapter 3 carry over to splines over (m,n)-cycle graphs, we believe that Mahdavi's conjecture regarding the determinantal criterion is correct. Future work may include showing that the remaining Chapter 3 lemmas carry over to splines over (m,n)-cycle graphs, which will be required for a proof of Mahdavi's conjecture. We would ultimately hope to show that a set of splines over an edge-labeled (m,n)-cycle graph forms a basis if and only if its determinant is equal to

$$Q = \frac{\text{product of edge labels}}{((\text{edges of cycle 1})(\text{edges of cycle 2}), \text{center edge}(\text{outer edges}))}$$
$$= \frac{l_1 l_2 \cdots l_{n+m-1}}{((l_2, l_3, \ldots, l_n)(l_{n+1}, \ldots, l_{n+m-1}), l_1(l_2, \ldots, l_{n+m-1}))}.$$

Here we state some conjectures on the expansion of Mahdavi's results.

Conjecture 5.0.1. *Fix the edge labels on* $(C_{(m,n)}, L)$ *where* $L = (l_1, \ldots, l_{n+m-1})$. *Let* $Q = \frac{l_1 l_2 \cdots l_{n+m-1}}{((l_2, \ldots, l_n)(l_{n+1}, \ldots, l_{n+m-1}), l_1(l_2, \ldots, l_{n+m-1})}$. *Let* $H \in S(C_{(m,n)}, L)$, *and let* $\{B_1, \ldots, B_{n+m-2}\} \in S(C_{(m,n)}, L)$. *If* $\begin{vmatrix} B_1 & \cdots & B_{n+m-2} \end{vmatrix} = \pm Q$, *then* $QH \in span(\{B_1, \ldots, B_{n+m-2}\})$.

Conjecture 5.0.2. *Fix the edge labels on* $(C_{(m,n)}, L)$ *where* $L = (l_1, \ldots, l_{n+m-1})$. *Let* $Q = \frac{l_1 l_2 \cdots l_{n+m-1}}{((l_2, \ldots, l_n)(l_{n+1}, \ldots, l_{n+m-1}), l_1(l_2, \ldots, l_{n+m-1})}$, *and let* $\{X_1, \ldots, X_{n+m-2}\} \in S(C_{(m,n)}, L)$. *Then* $Q \mid \begin{vmatrix} X_1 & \cdots & X_{n+m-2} \end{vmatrix}$.

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