Spring 2016

An Exploration of Chaos in Electrical Circuits

Ryan Ann O'Connell
Bard College

An Exploration of Chaotic Behavior in Electrical Circuits

Ryan O'Connell

A Senior Project presented for the degree of

Bachelor of Arts

Physics Division of Science, Mathematics, and Computing

Bard College

Annandale-on-Hudson, NY

May 2016

An Exploration of Chaotic Behavior in Electrical Circuits

Ryan O'Connell

Abstract

This senior project investigates chaotic behavior and controlled chaos in electrical circuits. It primarily focuses on two types of circuits: one with a varicap diode and one with a operational amplifier based Chua diode. These are the non-linear elements of the circuit that produce chaotic behavior in voltage outputs. Attempts to get the circuits to function correctly were unsuccessful, but the project includes a theoretical investigation of the chaotic behavior of the circuits through bifurcations and period doubling, as well as ways this behavior could be theoretically controlled.

to Kim for being strong for me.

Acknowledgements

I want to thank the Physics department at Bard College for their patience, instruction, and support throughout my time here. I would specifically like to thank my senior project adviser Matthew Deady for his guidance, encouragement, and motivation. I would also like to thank James Belk for providing a punch-line at the end of every class and for strengthening my interest in chaos theory.

Lastly, I am very grateful to have such a caring and loving support system of friends and family, both at school and home, that keep me motivated, focused, and happy. I sincerely don't know where I would be without all of you and your love.

Contents

Chapter 1

Introduction

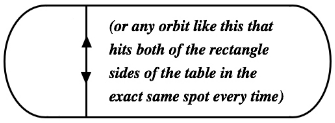
1.1 What is Chaos?

What does the word "chaos" bring to mind. The average person would probably say it is synonymous with dysfunction, disorder, or confusion. Ask a physicist or mathematician and you would get a much more complex answer. Chaos, in the mathematical sense, is a type of behavior that is exhibited by particular dynamical systems which have a sensitive dependence on initial conditions. [20] This sensitive dependence has been referred to as the butterfly effect. [13] This is the idea that a butterfly in China can flap its wings and through a series of events that happened because of that flap, Texas experiences a hurricane. Basically, it is a dramatized expression of a simple notion: small changes produce large-scale results in interconnecting and complex environments.

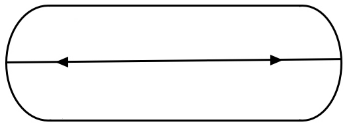
To better understand this, let's look at the simple example of the rounded edge pool table:

Figure 1.1: rounded-edge pool table

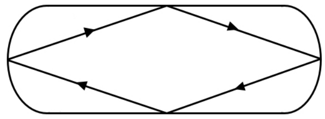
For this example, let's neglect friction so that the pool ball will continue its motion forever only being forced to change its direction when it hits the edge of the pool table. The trajectory of the pool ball is the orbit in this case. The following orbits are periodic, meaning that the orbit will repeat itself in time:



(a) Period-2 orbits



(b) Period-2 orbit



(c) Period-4 orbit

These periodic orbits are stable and repeat themselves. However, if I was to hit the pool ball from a slightly different angle or at a slightly different spot on the edge of the table, I would get a very different looking orbit:

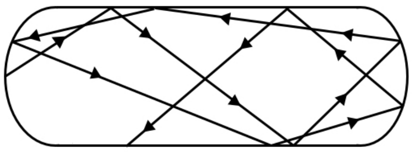


Figure 1.3: Chaotic orbit

And even shifting the starting angle of trajectory slightly from theh starting angle used in Figure 1.3 would produce something very different than the above orbit. This behavior is chaotic.

We could describe this behavior in a quasi-mathematical way: whenever the ball hits a straight wall it bounces off in such a direction that the angle between the ball's trajectory and the table on both sides of the collision are the same, like so:

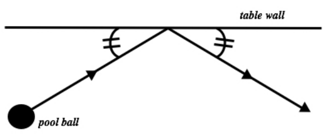


Figure 1.4: Straight wall angle of deflection

When the ball hits a rounded edge of the pool table, a similar rule in applied to the trajectory which determines the next point of the orbit. We could think of this as an iterated piecewise function, one that takes into account which edge of the pool table the ball hits and the angle at which its hits it, and provides the next part of the orbit. We could then continuously take the output of our hypothetical piecewise function and put it back into our hypothetical piecewise function to get the next point of our orbit, and continue to do this forever. This iterated function would produce chaotic behavior. Not all iterated functions do. Some, like f(x) = 2x, just produce a series where each element is twice the value of the one before it. But in special cases, all of which have some nonlinear element to them, like the rounded edge pool table, the iterated function can produce chaos — deterministic, but unpredictable (because of initial condition sensitivity).

1.2 Attractors, Strange Attractors, and Fractals

This next, very famous, example may come as a surprise to someone who has never seen it. Let's consider the following iterated formula: We have three points equidistant and the same angle apart from each other:

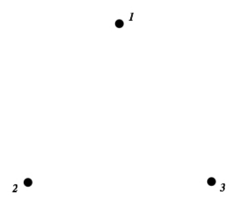
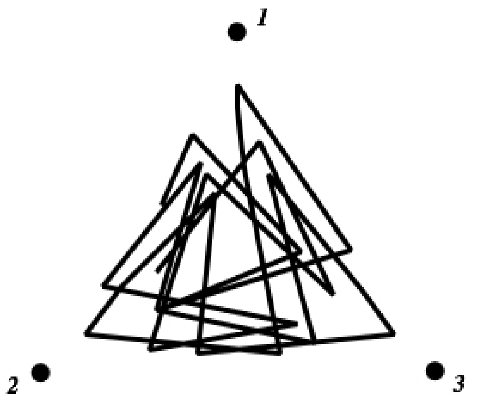


Figure 1.5: Vertex points of equilateral triangle

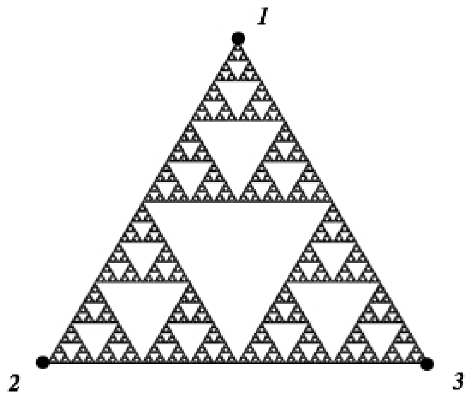
We choose any point within the triangle these three points connect to make to be our starting point. At random, we will choose an integer from 1 to 3 and draw a line from our starting point, halfway to the corresponding point. From there, we iterate through this formula some large number of times. This process will create something that looks disorderly or chaotic like Figure 1.6a.

If we repeat this process, but for an extremely high number of iterations (something like 100,000) and only plotted the endpoints of our lines, we would get a completely different image resembling that of Figure 1.6b.

This image is known as Sierpinski's Triangle and it is one of the simplest fractals. Fractals are attractors of chaotic systems. Attractors are groups of points (or a single



(a) Twenty iterations of the formula given



(b) Sierpenski Triangle [21]

point) that the a system's orbit tends to evolve towards. Strange attractors are a particular type of attractor which have fractal structures, like Sierpinski's Triangle, and demonstrate infinite self-similarity, meaning that each piece looks like the whole. It is impossible to predict where on a strange attractor a system will be at any given time.[20]

1.3 Lorenz Attractor

Chaotic behavior, as well as strange attractors and fractals can be observed in iterated function systems, as we have just observed, but they can also be the product of a system of differentiable equations. The first system to be proven chaotic rigorously was the Lorenz equations, a set of three differential equations that describe the behavior of all the convention cycles which dictate the central component in the forming of weather patterns.[13] These are the Lorenz equations:

$$\frac{dx}{dt} = \alpha(y - x) \tag{1.1}$$
$$\frac{dy}{dt} = x(\rho - z) - y \tag{1.2}$$
$$\frac{dz}{dt} = xy - \beta z \tag{1.3}$$

Simple enough right? Unfortunately for those of us who would like to accurately know the weather next Saturday, this system of equations exhibits chaotic behavior. An orbit of this system mapped out over time while plotting x, y, and z against each other will look like this:

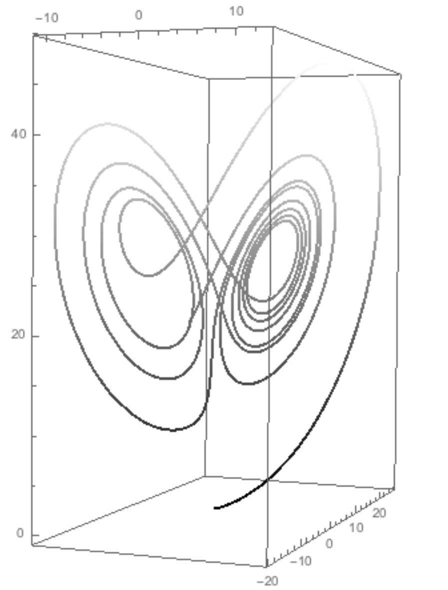


Figure 1.7: The Lorenz Attractor

This is Lorenz Attractor. [23] It is a strange attractor and it is self-similar and is sensitive to initial conditions. The orbit is completely deterministic, however, it is impossible to predict which part of the attractor the orbit will be on at any given time. If you started simultaneous orbits of this system with initial conditions one-millionth of a decimal place different from each other, their orbits would start off identical, but eventually would show slight differences that would ultimately result in completely different orbits. This is a main reason why weather predictions are very difficult to make with any degree of accuracy past a few days.

1.4 The Logistic Equations and Bifurcation Diagrams

As previously shown, chaos is not easily defined and has many components to identifying and explaining it. All of the examples I have just presented evolve over time, but time doesn't have to be the parameter that is evolving in the system for chaos to be observed. For instance, the logistic equations,

$$x_{n+1} = \lambda x_n(1 - x_n)$$

display chaotic behavior by varying the λ-value from 0 to 4. Let's take a look as some of the possible orbits the logistic equations can produce. Below is the orbits of twenty iterations with different λ-values all starting with the initial value of 0.5:

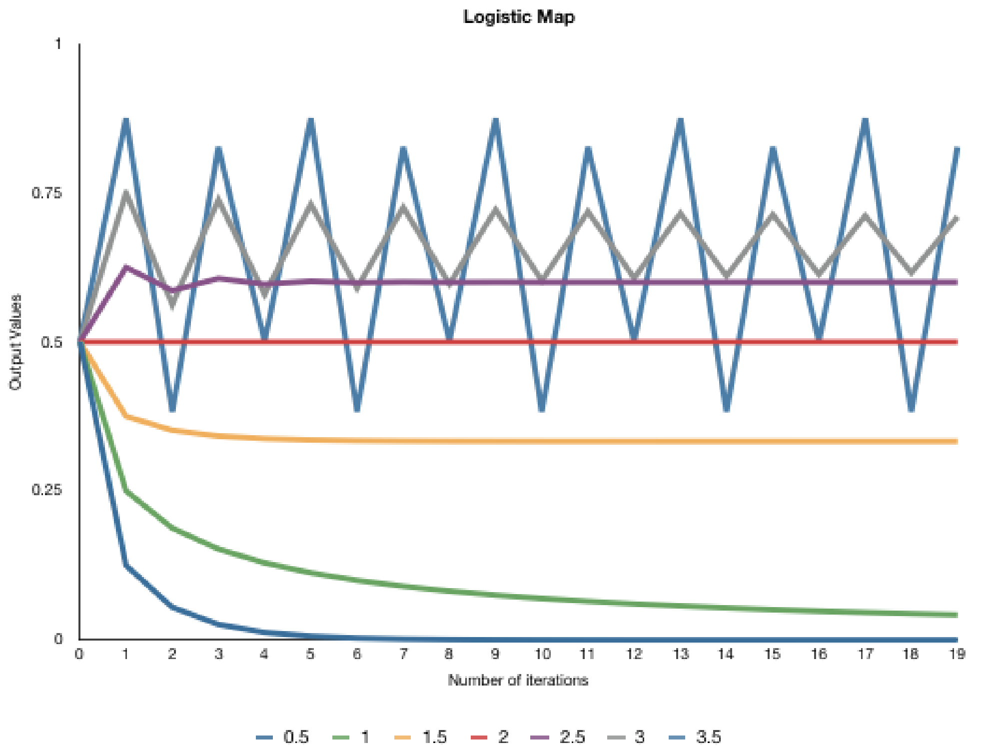


Figure 1.8: Orbits with inital condition at 0.5 with varying λ values

For $\lambda = 0.5$ and $\lambda = 1$, the orbit goes to zero, but for λ-values 1.5, 2, and 2.5, the orbit gravitates towards a particular value and stays on that point once it hits it. This is a fixed point attractor. For $\lambda = 3$, the orbit seems to be gravitating towards two points and once it hits them its oscillates between those two points. This orbit is periodic with period-2. For $\lambda = 3.5$, the orbit appears to just be oscillating essentially randomly. [20] There is a more useful way to visualize this data, which will also allow us to see where these differences in attractors and period points come from. The scatter-plot of the λ-values against the x_n values outputted by the logistic equations is called a bifurcation diagram, as shown in Figure 1.9

This bifurcation diagram shows the second 100 of 200 iterations (the first 100 were thrown away to insure that the orbits had reached their attractors, if they had one).

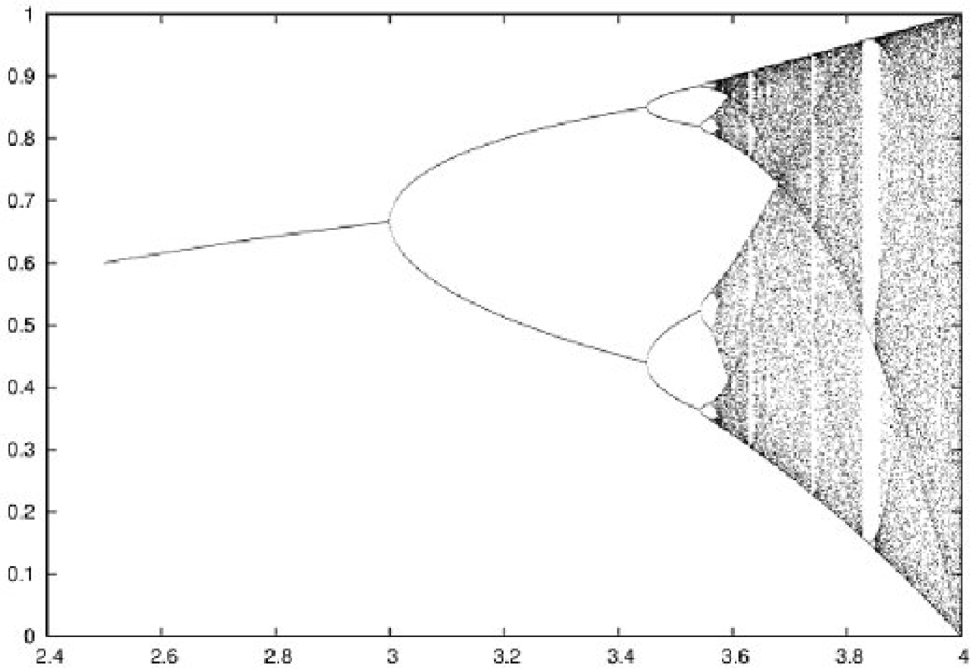


Figure 1.9: Bifurcation Diagram of the Logistic Map where x-axis is the λ-value and y-axis is x_n [22]

From $\lambda = 1$ to $\lambda \approx 3$ there is one periodic point that the orbit converges to, but past that point, there is now two period points the orbit will converge to. This is the first period-doubling. Each fork in the diagram represents another period doubling, meaning that the number of periodic points doubles. We see a few of these period-doubling forks before the noise-filled region starts. This noise is chaos — completely deterministic but random-looking and unpredictable — the orbit could land on essentially any output within the region. There are bands of breaks in the chaos where we, once again, see the orbits converge on some number of period points. If we were to zoom in on one of these bands we would see a period-doubling route to chaos, identical to how the bifurcations looks as a whole, meaning that this diagram is also self-similar. If we were to zoom-in infinitely, we would observe the same structure at every level of the zoom. This is one of the ways we know that the behavior of the orbits in the noise region, are not random, but are chaotic.

1.5 Chaos in Physical Systems

As we have already observed to a certain extent, chaos can be observed in many physical systems. The logistic map has been used to model the stability of ecosystems. A system as simple as a double pendulum or a 3-body system are chaotic systems. While the individual parts of the systems are simple and easy to predict, when these parts interact with other simple parts, the sensitivity on initial conditions is amplified and the system becomes chaotic. For instance, if you took a break shot in pool, the balls will hit into each other in such a way that, even if you tried your hardest to hit them in the exact same way, you would never be able to replicate. Think of it like a 15-body problem — extremely dependent on initial conditions. Perhaps not as obviously as these physical systems have, particular electrical circuits also exhibit chaotic behavior. This project is an investigation into the different ways we can observe this behavior in two of these particular circuits. The two circuits I have chosen to focus my exploration on are the varicap diode circuit and the Chua circuit. the diodes in both cases provide the non-linearity needed to exhibit chaotic behavior. The varicap diode circuit is much simpler in its structure as it is essentially just a RLD circuit (that at times acts like an RLC circuit, but that will be explained later). The Chua circuit is a bit more complicated with an operational amplifier based diode and a few more reactive elements. Equipped with the chaos theory vocabulary to describe the behavior of these chaotic circuits, let's take a looks at the varicap diode circuit.

Chapter 2

Varicap Diode Circiut

2.1 How It Works

First, let's look at the behavior of the varicap diode circuit. The diode's name comes from the combination of the words "variable" and "capacitance" because its capacitance is voltage-dependent. There were three major papers published describing the behavior of this diode during 1981 and 1982, which I have drawn most of my understanding of the circuit from. The circuit containing the varicap diode that exhibits period doubling and chaotic behavior looks like this: where

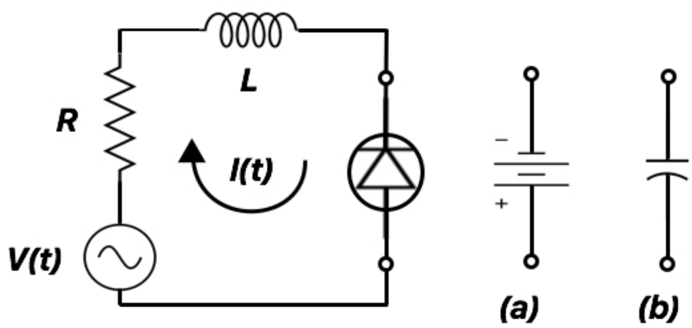


Figure 2.1: Varicap Diode Circuit

the circled element is the varicap diode and the elements labeled (b) and (c) are in reference to its behavior under specific conditions. The varicap diode exploits the voltage-variable property of the junction capacitance, which every diode has to some extent but in most cases tries to minimize. In the varicap diode, the depletion region between the P and N parts of the diode is large, making the diode essentially a capacitor in reverse bias. This is where (b) can be switched in for the varicap diode in the above circuit. In forward bias, the diode conducts and could be thought to be replaced by an emf with voltage V_F. This is where (a) can be switched in for the varicap diode in the above circuit. However, the diode will not conduct until Vf is reached and will stay at V_f for the remaining time the circuit is in forward bias. Until then, the diode does not conduct but instead has a fixed capacitance. When the current through the diode passes through zero, the diode does not immediately stop conducting. It takes some reverse-recovery time to stop conducting completely.

2.2 Voltage-Dependent Capacitance as a Means of Chaotic Behavior

There are two ways to exhibit the chaotic behavior of the circuit above. The first of which was first discovered by Paul S. Linsay in 1981 [1] and later expanded upon by James Testa, Jose Perez and Carson Jeffries the following year [3]. Linsay's experimentation was done in response to Feigenbaum's theory of non-linear systems that exhibit period doubling, which states that their behavior, and specifically their bifurcation and period doubling patterns, follow universal laws independent of the equations that describe their particular behavior. The goal of Linsay's experiment was to show that this theory holds for physical systems, and in particular, for a driven anharmonic oscillator. Feigenbaum's theory states that given some system described by

$$\frac{dx_i}{dt} = F_i(x_1, x_2, ..., x_N)$$

for $i = 1, 2, ..., N$ and given that $x_i(t)$ are periodic with period $T_n = 2^n T_0$ at $\lambda = \lambda_n$, the recurrence relation,

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} = \delta$$

is satisfied, where δ is the Feigenbaum convergence rate, which, for quadratic extrema, is 4.669. For an electrical circuit like this one, the $x_i(t)$ is the current flowing through the system and the λ is the amplitude of the drive voltage at a particular subharmonic, n. Linsay chose to observe this circuit's chaotic behavior by analyzing its spectral peaks at different drive voltages, which would be directly influenced by the diode's capacitance dependence on said drive voltages. This dependence is described by this equation:

$$C(V) = \frac{C_0}{(1 + \frac{V}{\phi^{\gamma}})}$$

where $C_0 = 81.8\rho\text{F}$, $\gamma = 0.44$, and $\phi = 0.6\text{V}$ determined by measured values of capacity. A spectrum analyzer was connected to the circuit between the inductor and the resistor, see Figure 2.2, to measure the subharmonics at specific frequencies

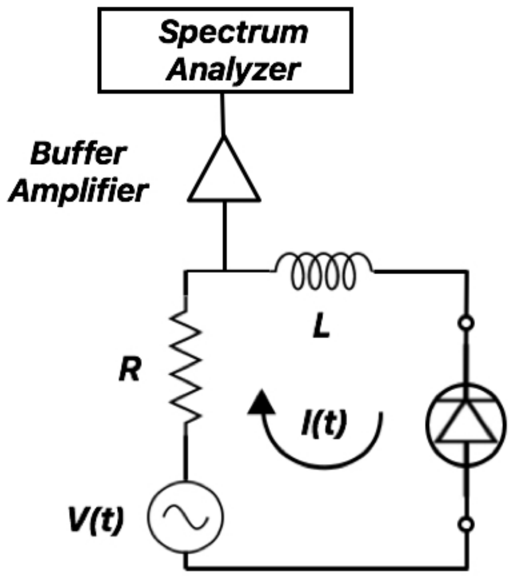


Figure 2.2: Varicap Diode Circuit with Spectrum Analyzer

which would occur as the drive voltage was increased. At very low drive voltages, the circuit acted like a regular RLC circuit in series, where the resonance frequency was found to be 1.78MHz. As the drive voltages increased, spectral peaks were observed, the first being at the frequency one-half of the resonance frequency. Three more subharmonic frequencies were observed as the drive voltage increased: at one-fourth, one-eighth, and one-sixteenth the resonance frequency. The calculated value of δ, the convergence rate in Linsay's experiment, for which he was able to compute two given the data collected, is displayed in Figure 2.3. The Feigenbaum convergence rate for quadratic extremum is very close to the calculated values of δ in this experiment, therefore giving experimental evidence of a physical system exhibiting Feigenbaum's theory of nonlinear systems that exhibit period doubling.

n	Subharmonic	$\Delta V_{threshold}$ (V)	δ_n
1	$f_1 / 2$		
		3.2 ± 0.02	
2	$f_1 / 4$		4.4 ± 0.1
		0.72 ± 0.02	
3	$f_1 / 8$		4.5 ± 0.6
		0.16 ± 0.02	
4	$f_1 / 16$		

Figure 2.3: Table used to compare the difference in $V_{threshold}$ for each subharmonic to calculate the convergence rate

The period-doubling of spectral peaks is displayed in this spectrum, where the numbers at the top of the peaks corresponds to the order of appearance and the y-axis is the amplitude in decibels:

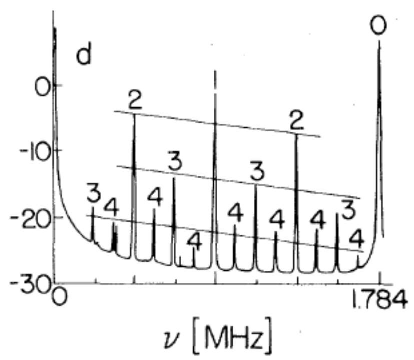
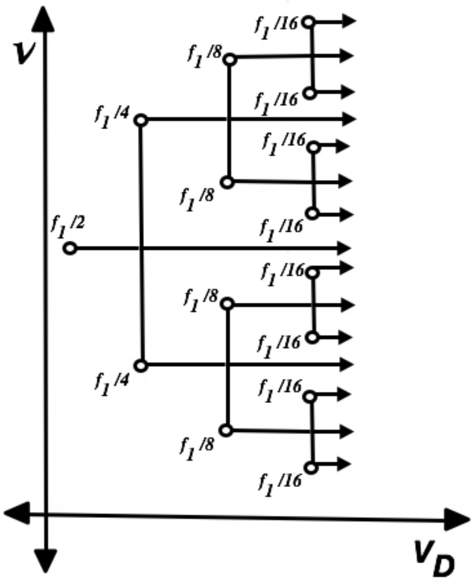


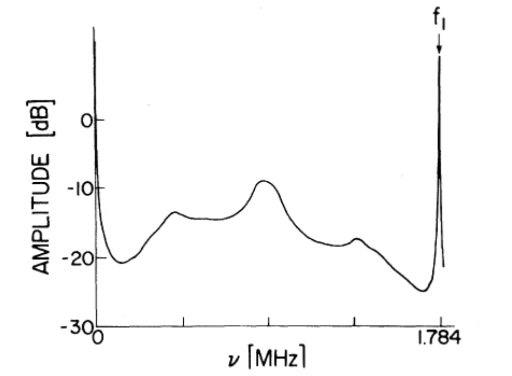
Figure 2.4: Period-Doubling Spectral Peaks [1]

When translated into a bifurcation diagram, this spectral peaks appear in the following fashion as shown in Figure 2.5a. (It is important to note that this does not look like a traditional bifurcation diagram since the peaks remain as new peaks appear)

Increasing the drive voltage past this point, produced chaotic behavior as exhibited in Figure 2.5b. Replacement of the varicap diode with a capacitor in parallel



(a) Bifurcation Diagram of Spectral Peaks of Varicap Diode Circuit



(b) The full onset of chaos [1]

with a 1N4154 diode which had the same resonant frequency as the varicap diode set up did but lacked the period-doubling behavior, proves, according to Linsay, that the chaotic nature of the circuit is due to the nonlinear capacity of the varicap diode as described by these three equations:

$$V_c = \frac{Q}{CV_c} = (1 + \frac{V_c}{\phi})^{\gamma} \frac{Q}{C_0}$$
$$L\frac{dI}{dt} = V_0 \sin{(2\pi f_1 t)} V_c - RI$$
$$\frac{dQ}{dt} = I$$

whose solutions, according to Linsay, support the data.

2.3 Reverse-Recovery Time as a Means of Chaotic Behavior

As I had mentioned previously, there is a secondary path to proving the varicap circuit presented exhibits chaotic behavior. Instead of observing the voltage-dependent capacitance, one could prove that the large reverse-recovery time of the varicap diode is the property of this diode that produces the chaotic behavior. Rollins and Hunt actually speculated that the large reverse-recovery time is, in fact, the *only* reason for the chaotic behavior, [2] not the voltage-dependent capacitance that Linsay had suspected, but never actually proved, to be the reason for the period-doubling. Rollins and Hunt give a step by step procedure on how to observe chaos this way. They claim that the finite forward bias voltage and reverse-recovery time are the two attributes of the varicap, that when combined, cause the circuit to exhibit chaotic behavior. As described previously, the varicap diode takes some time τ_{RR}, the reverse-recovery time, to stop conducting when bias is switched from forward to reverse. They described the reverse recovery time using the equation below:

$$\tau_{RR} = \tau_m[1 - e^{\frac{-|I_m|}{I_c}}]$$

where I_{max} is the most recent maximum forward current and τ_m and I_c are parameters distinct to each diode.

For $V(t) = V_0 \cos \omega t$, these are the general solutions for both states of the diode:

When the diode is conducting:

$$I(t; A) = (\frac{V_0}{Z_a} \cos{(\omega t - \theta_a)} + Ae^{\frac{-Rt}{L}} + \frac{V_f}{R} \tag{2.1}$$

$$V_d(t) = -V_f \tag{2.2}$$

where $(Z_a)^2 = R^2 + \omega^2 L^2$, $\theta_a = \arctan(\omega L/R)$, and A is a constant to be determined by the boundary conditions.

When the diode is not conducting:

$$I(t; B, \Phi) = (\frac{V_0}{Z_b} \cos{(\omega t - \theta_b)} + Be^{\frac{-2Rt}{L}} \cos{(\omega_b t + \Phi)} \tag{2.3}$$

$$V_d(t; B, \Phi) = V_0 \cos \omega t - I(t; B, \Phi)R - L\frac{dI(t; B, \Phi)}{dt} \tag{2.4}$$

where $(Z_b)^2 = R^2 + (\frac{L}{\omega})^2(\omega^2 - {\omega_0}^2)^2$, $\theta_b = arctan[L(\omega^2 - \omega_0^2)/R\omega]$, $\omega_0^2 = \frac{1}{LC}$,

$\omega_b^2 = \omega_0^2 - (\frac{R}{2L})^2$, and B and Φ are constants to be determined by the boundary conditions.

Given an alternating current, the diode will go through different cycles of specific behavior. An example of three of these cycles that Rollins and Hunt found using the system modeled above, is given in Figure 2.6.

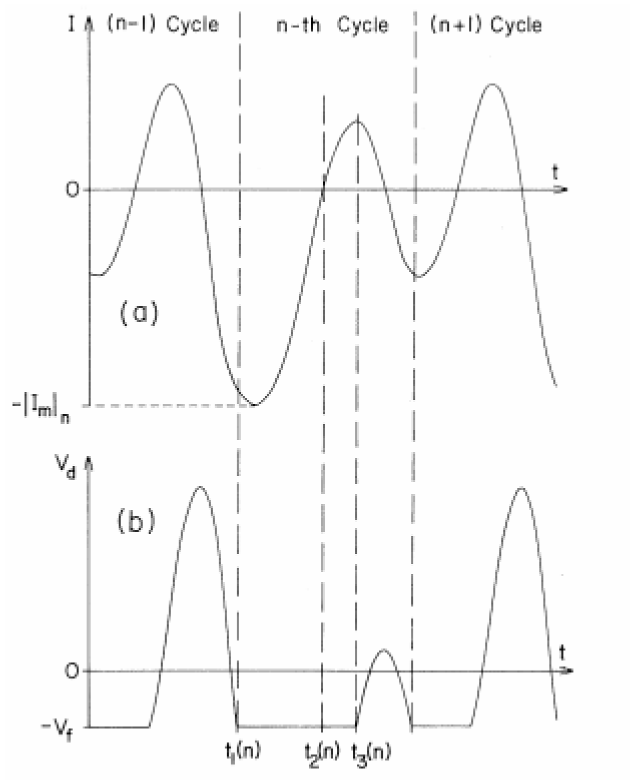


Figure 2.6: Three cycles of the varicap circuit. The top graph depicts the current across the diode as a function of time, and the bottom graph depicts the voltage across the diode also as a function of time. $t_i(n)$'s refer to key times when something about the diode's behavior has changed. [2]

In this diagram, $t_1(n)$ depicts the beginning of the time that the diode will be conducting, so the diode will be in forward bias, and the start of the n-th cycle. $t_2(n)$ is the time at which the current passes through zero. Notice that the diode is still conducting, making this the start of τ_{RR}. The reverse-recovery time is dictated in this case by I_m, which is the lowest or highest point of I during the cycle being considered. One would assume, correctly, then that $t_3(n) = t_2(n) + \tau_{RR}(n)$.

Now for the interesting part. The value of $t_3(n)$ is determined on a 2-case basis. Case 1: If current at $t_3(n)$ is passing through the diode in the reverse direction, then the diode will stop conducting and it is not until the voltage across the diode is back at -V_f, that the next cycle begins, as depicted in Figure 2.6. Case 2: If current at $t_3(n)$ is passing through the diode in the forward direction, then the diode does not stop conducting and the next cycle starts with $t_1(n+1) = t_3(n)$. This is where the nonlinearity of the circuit that is needed to produce the chaotic behavior comes from. Since the previously stated system of equations 2.1-2.4 fully describe the behavior of the system, each new $|I_{max}|_n$ can be determined from the previous cycle's one, given a set of initial conditions, the trajectory of the voltage across the diode is deterministic, but slight changes in the initial conditions will greatly influence the outcome once a different case of $t_3(n)$ is satisfied. This is sensitive dependence on initial conditions stemming from a nonlinear element of the system, which is our principle definition of chaos. This is how Rollins and Hunt identified the central component in the production of chaotic behavior in the varicap diode circuit.

In a later paper by Mariz de Mores and Anlage a more detailed depiction of how the p-n junction in the diode effects the nonlinearities that the reverse-recovery time depends on through the different stages of the circuit. [17] They, unlike Rollins and Hunt, do not make the assertion of the reverse-time constant being the principle creator of chaotic behavior, but rather, chose to not make a distinction between effects of the nonlinear capacitance and reverse-recovery time of the diode, as they can be similar, as they both take the history of the circuit into account when determining the trajectory of the system.

This concludes my discussion of the varicap diode circuit. Now that we are familiar with the way in which chaotic behavior can be seen in circuit systems, let's take a look at the, slightly more complex, Chua Circuit and the many ways it (and its variations) exhibits chaotic behavior.

Chapter 3

Chua's Circuit

3.1 The Classical Chua's Circuit

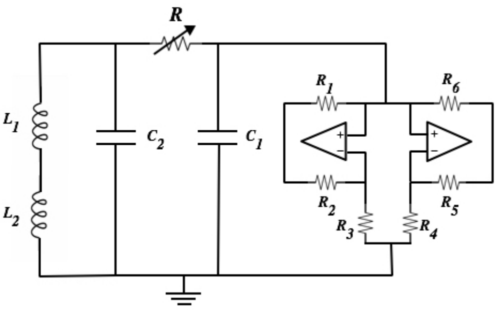
The second simple circuit exhibiting chaotic behavior that I have explored is called Chua's circuit. One of the aspects of Chua's circuit that makes it so attractive to work with is that it does not require the purchase of a special element as its nonlinear component, but instead can be made using simple inductors, resistors, capacitors, and a couple of operational amplifiers as shown in Figure 3.1a with the right-most section of the circuit being referred to as Chua's diode, a nonlinear resistor. The original Chua circuit only has one inductor, but I attempted to use two in series to get the correct inductance, as I did not have an inductor of the value needed. Inductors in series add their values, so this should not have effected the circuit at all. The equations that govern this circuit are as follows, all of which can be easily obtained with the exception of $i_d(v_1)$ which is a bit more complicated as it is the current through the diode as a function of the voltage across C_1:

$$\begin{aligned} C_1\frac{dv_1}{dt} &= \frac{v_2 - v_1}{R} - i_d(v_1) \\ C_2\frac{dv_2}{dt} &= \frac{(v_1 - v_2)}{R} - i_l \\ L\frac{di_L}{dt} &= -v_2 \end{aligned}$$

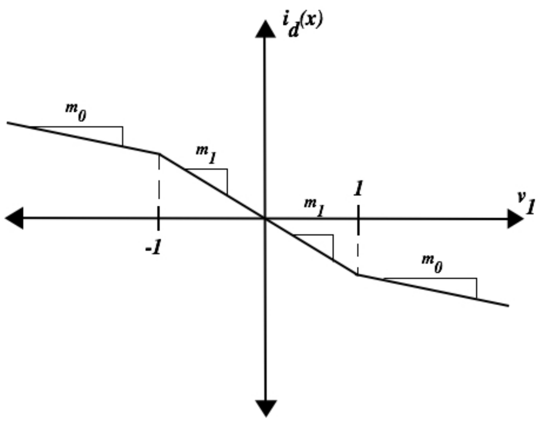
Value	Component	Schematic Label
10nF	Capacitor	C_1
100nF	Capacitor	C_2
10mH	Inductor	L_1
10mH	Inductor	L_2
1.5kΩ	Potentiometer	R
220Ω	Resistor	R_1
220Ω	Resistor	R_2
2.2kΩ	Resistor	R_3
22kΩ	Resistor	R_4
22kΩ	Resistor	R_5
3.3kΩ	Resistor	R_6

Table 3.1: Element Values for the Chua Circuit

Here, v_1 and v_2 refer to the voltage across the corresponding capacitor and i_L is the current across the inductor. The m variables in $i_d(v_1)$ correspond to the slopes of the line created by the relationship between the voltage across the first capacitor and the current through the diode as Figure 3.1b displays. This piecewise function, $i_d(v_1)$ is what makes this circuit non-linear and is responsible for its chaotic behavior.



(a) Chua's Circuit Schematic



(b) The current across the Chua diode as a function of the voltage across C_1

With the guidance of [19], I attempted to construct this current with the element values described in Table 3.1. I encountered various set-backs along the way including a power board that was exhibiting 6V of noise. Ultimately, while trying to measure the resonant frequency of the diode-excluded circuit, my adviser and I determined that the values of some of the elements I was using were incorrect. Unfortunately, my error was caught too late in the allotted project time to reconcile, so I will be giving a theoretical analysis of the chaotic behavior of the Chua Circuit using some Mathematica modeling and research I have done of previously published projects.

Modeling of the circuit equations with the help of [21], gave the following results:

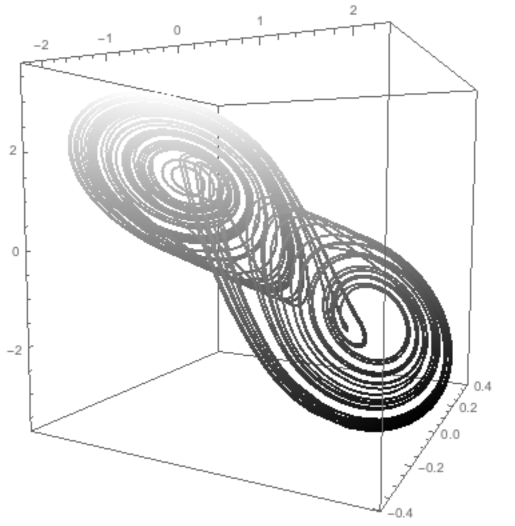


Figure 3.2: Chua Circuit's Equations plotted in the $(v_1(t),\, v_2(t),\, i_L(t))$ plane as t goes from 0 to 100

This is known as the Double Scroll strange attractor. The above orbit along the Double Scroll attractor does not systematically move from one scroll to the other in any kind of predicable fashion, but rather one that is unpredictable and seemingly random, but deterministic. Other initial conditions would give a completely different orbit, but one that still has the general shape. Adjusting parameters, such as the inductance and the resistance of R will give you different sections of the double scroll and periods before that of the chaotic attractor, respectively.

3.2 The Double Scroll

One of the first to publish on the appearance of the Double Scroll attractor in an electrical circuit was T. Matsumoto, who observed the behavior of a slightly different circuit than the one outlined above. [4] Leon O. Chua was actually the one to discover this attractor, but had to be rushed to the hospital for an emergency surgery after his discovery and left this work to Matsumoto to be published. This circuit had the inductor (or inductors, in my case) switched with the second capacitor, like so:

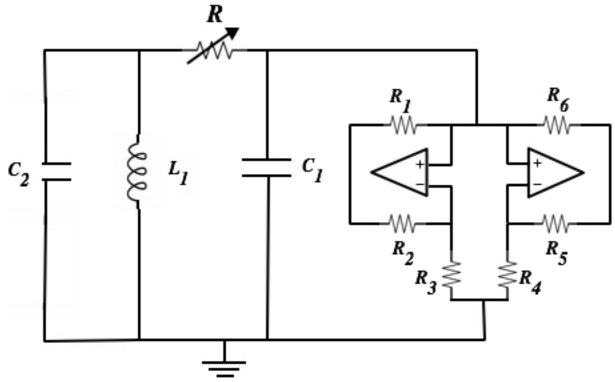


Figure 3.3: Matsumoto's Chua Circuit

The equations that govern this circuit are as follows:

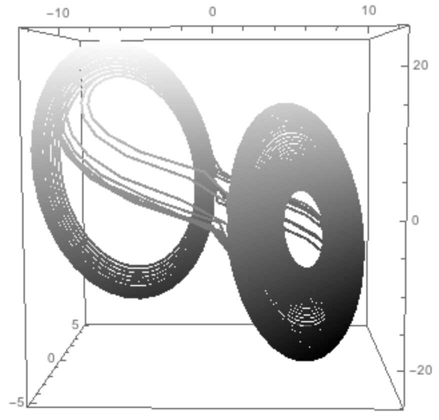
$$C_1\frac{dv_1}{dt} = R(v_2 - v_1) - i_d(v_1) \tag{3.1}$$

$$C_2\frac{dv_2}{dt} = R(v_1 - v_2) + i_l \tag{3.2}$$

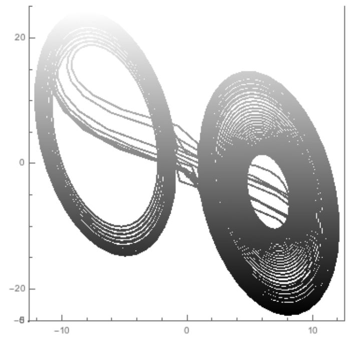
$$L\frac{di_L}{dt} = -v_2 \tag{3.3}$$

The difference in structure does lend itself to a slightly different looking attractor for this circuit as compared to the common construction of the Chua circuit, displayed at the beginning of the chapter, which I will present later. Both attractors are types of Double Scroll attractors. The attractor made by Matsumoto's circuit is modeled in Figures 3.4a - 3.4d.

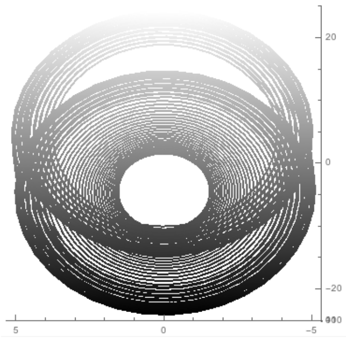
This attractor was proven to be different from the other strange attractors known at the time (Lorenz Attractor and Rossler's Attractor) by comparing the signs of the the eigenvalues on the origin to those not on the origin in each case. [4] And so, this was the first of its own class of strange attractors, later coined by Chua as the Double Scroll family of attractors. [5]



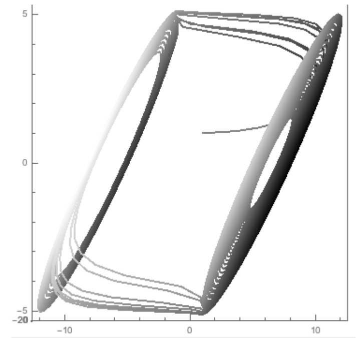
(a) 3-D view of Matsumoto's Chua Circuit Attractor observed after a time lapse of 300 seconds



(b) Left view of attractor where x-axis is the voltage drop across the first capacitor and y-axis is the current across the inductor.



(c) Right view of attractor where x-axis is the voltage drop across the second capacitor and y-axis is the current across the inductor.



(d) Right view of attractor where x-axis is the voltage drop across the first capacitor and y-axis is the voltage drop across the second capacitor.

3.3 Variations of Chua's Circuit

A slightly more canonical version of Chua's Circuit is the Chua Oscillator. [11] This circuit is more canonical in the sense that it exhibits a wider range of chaotic phenomena than the classical Chua Circuit, particularly in the routes it takes to display this chaos, as we will investigate.

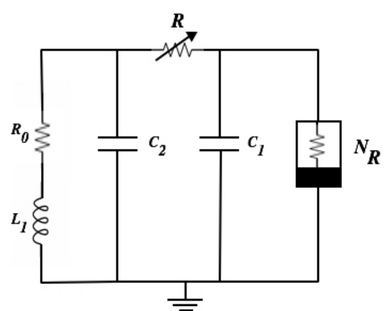


Figure 3.5: Chua Oscillator Schematic Circuit as described by [12]

Value	Schematic Label
5.75nF	C_1
21.32nF	C_2
12mH	L_1
1.5kΩ	R
30.86Ω	R_1
Chua Diode	N_R

Table 3.2: Element values for the Chua Oscillator

A period-doubling route to chaos was observed in Chua's Oscillator by varying the R_0 value $1.5k\Omega < R < 1.6k\Omega$. My attempts to computer model this period doubling using the values given did not produce the same results as Chua [12], most likely due to the fact that these results were obtained experimentally, allowing for different circumstances to influence the results, so I will provide the observations made by Chua as well as my failed code in Appendix A.

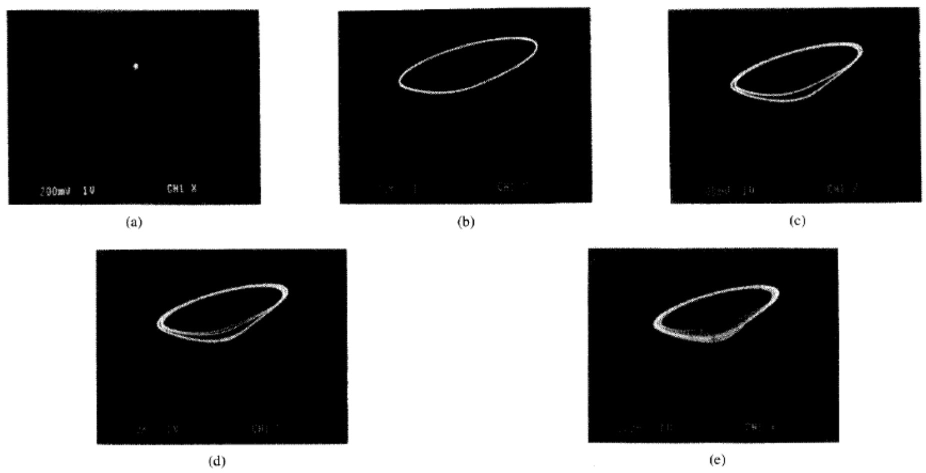


Figure 3.6: Period Doubling route to chaos observed by decreasing the R value from $1.6k\Omega$ to $1.5k\Omega$ [12]

Looking at Figure 3.6: in (a) there is an equillibrium point that loses stability and from it emerges a stable limit cycle as seen in (b) as R is decreased. Further decreasing of R allows for this limit cycle to lose its stability and transform into a limit cycle with twice the period as depicted in (c). This happens once again to give us (d), which is a limit cycle with period-4. This process continues until $R = 1.503k\Omega$ at which time chaos is observed as shown in (e). This period-doubling route to chaos is just one of the routes presented by Chua, that the Chua Oscillator can take to chaos.

Also described in [12] is the type-1 intermittency route to chaos. This route to chaos is observed in systems that appear to be periodic in their trajectory but experience chaotic bursts. This route to chaos was observed for $R = 1.501k\Omega$ in the Chua Oscillator, where a stable period-3 limit cycle was observed, as shown in Figure 3.7.

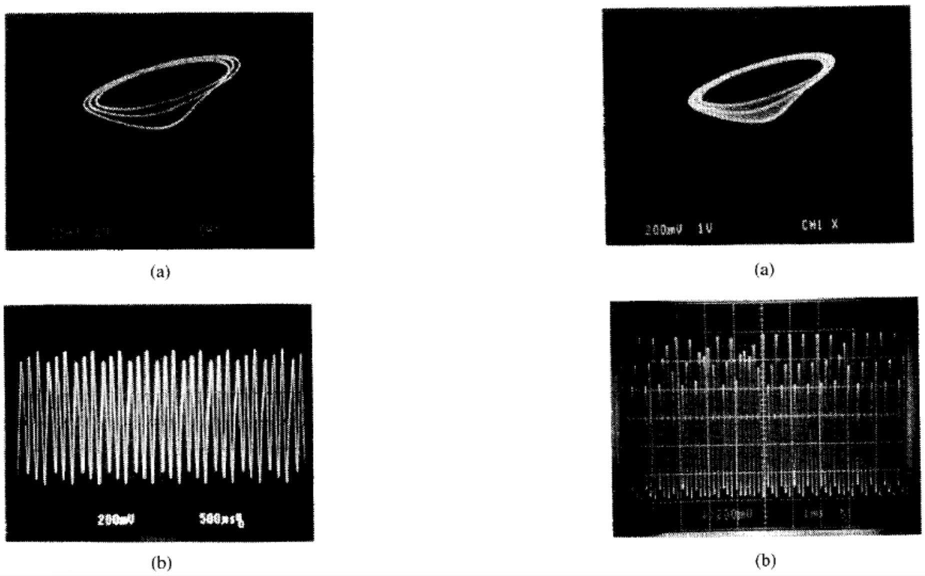


Figure 3.7: Type-1 intermittency route to chaos observed by increasing the R value from $1.501k\Omega$ to $1.502k\Omega$ [12]

Here we see phase portraits of a stable period-3 limit cycle in the left (a) and an intermittency near the period-3 cycle. The (b)'s are both time waveforms of v_1 that show a period-3 limit cycle and intermittent chaos near that period-3 limit cycle, from left to right.

Chua goes on to explain that the period-doubling into chaos is not just observed at these values of R, but that there are windows of chaos that revert back to being periodic, and then revert back to chaos, similarly to the way the logisitc map does. This is the another way he proves this circuit to be chaotic.

Lastly, the Chua Oscillator's torus breakdown route to chaos is explained, this time, by varying the values of C_1 and fixing R. For specified element values and a specified C_1, a double-torus attractor with a trajectory jumping from one torus to the other is observed. As the C_1 values is decreased, the two consolidate into a single torus attractor. Reducing the value of C_1 further, Chua observed a "period-adding sequence of periodic windows of consecutively decreasing periods" [12] until the torus deforms and a double-scroll attractor is revealed after devolving from period-16 to period-4. The bifurcation diagram of this torus breakdown as a route to chaos is presented here:

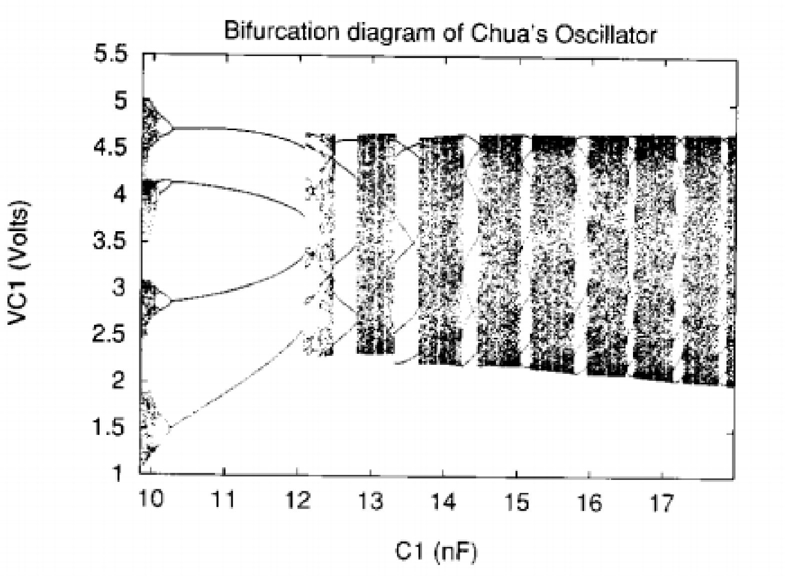


Figure 3.8: The Bifurcation Diagram of the torus breakdown route to chaos of the Chua Oscillator [12].

The Chua Oscillator, as well as the other members of the Chua family have been extremely useful in understanding chaos in physical systems as they exhibit a wide variety of routes to chaos. The Chua family of circuits has also been thoroughly analyzed since it has been in existence for nearly three decades, and analyses have been consistent in their theoretical, experimental, numerical findings, which is not the case for most other physical systems rigorously proven to exhibit chaos.

Many modifications have been made to the Chua Circuit to obtain desired behavior. Chua published a paper describing a canonical realization of the Chua family of circuits [8], which gives examples of other members of the Chua family, like the torus circuit [6] and the double hook circuit [7] and explains how the specific behaviors of one cannot be replicated in the others. He does this by identifying the eigenvalue patterns, which determine the conjugacy of the system to another, that can be produced by Chua's circuit and then states reasons for which particular eigenvalue patterns can not be replicated by the other members of the Chua family. Ultimately, he describes a circuit of his own creation that is general enough to, given the proper element values and choosing the right parameters, display all of the behaviors of the circuits in the Chua family. This canonical circuit is pictured here:

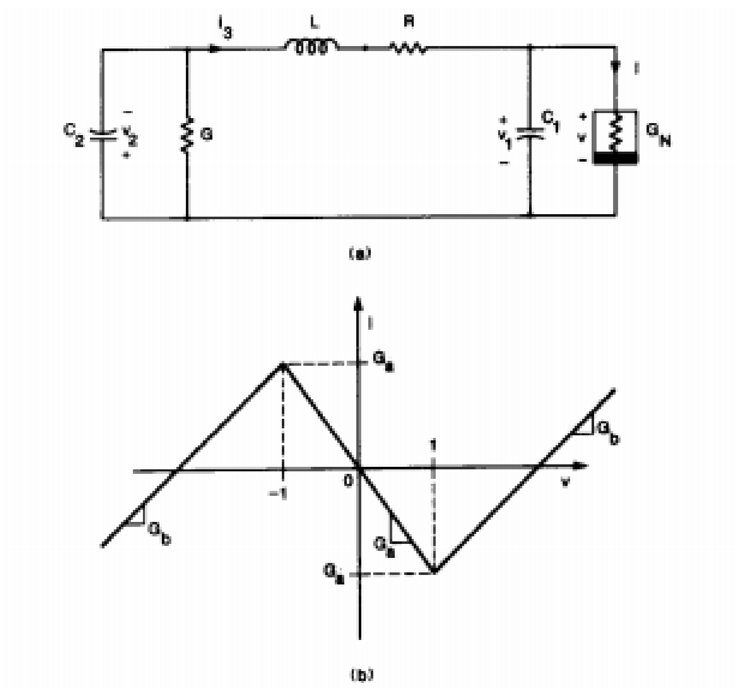


Figure 3.9: The Canonical Chua Circuit. (a) The schematic of the circuit (b) the current across the non-linear resistor as a function of voltage across C_1 (referred to as the v-i characteristic) [8].

where the G variables are like our m variables from previous cases. As one can see, the are only six elements in the circuit, and only one of them is nonlinear making for an exceedingly simple circuit for such a plethora of chaotic behavior to be observed from. Chua goes on to prove that this circuit can realize any eigenvalue pattern in the class of three-region symmetric, with respect to the origin, piecewise-linear continuous vector fields which has come to define a circuit as belonging to the Chua family. [8]

3.4 Controlling Chaos

Chaotic systems naturally exhibit a variety of different behaviors in a single orbit. Being able to control when these naturally occurring behaviors are exhibited is very desirable since only small changes can produce large effects, increasing efficiency. However, it is also for these reasons, that it controlling chaos would be a difficult task. It is also sometimes undesirable to have a system be chaotic when, for whatever reason, you do not want sensitive dependence on initial conditions. This has been the motivation behind recent work on controlling chaos in systems like the Chua Circuit.

This is the line of thinking that argues for the control of chaotic systems: Analyzing a system by its performance, let's call that P, we would see that P depends on the time average of the function of the system state, $f(x)$. This function of the system state would be a function of the time dependent orbit of the system, x. For chaotic systems, x would be the chaotic orbit on the attractor. These attractors have periodic points on them, some stable, some unstable. Each of the unstable period points on the attractor has its own time dependent orbit, x_i and therefore will have its own performance P_i. Since P is the weighted average of all the P_i's, there must exist P_i's that are higher than P (and some that are lower).

Theoretically, giving a system initial conditions that will project its motion directly on an unstable periodic point with a large P_i would do the trick. However, this becomes virtually impossible when dealing with chaotic systems because of their sensitive dependence on initial conditions. It is also important to note that if you land only near an unstable periodic point and not on it, the system's orbit could be repelled by the unstable periodic point. This means that you must kick the system back towards the unstable period point. This can be done using a few different techniques including placing the orbit near the stable manifold of the desired unstable periodic point, using a continuous time-delay feedback system, and the pole-placement technique. [18]

All techniques of controlling chaos can be broken down into two categories: feedback and non-feedback, feedback techniques being the more versatile and widely used method. Non-feedback control techniques require small period parametric perturbations which continuously kick the system back to the desired fixed point. This is a quicker fix for chaotic behavior, but without the use of feedback, this becomes difficult to globalize. There has been successful non-feedback control techniques performed on the Chua Circuit, and other circuits exhibiting chaotic behavior. [10] [9] [15] However, since the methods of feedback control methods appear to be more common and non-feedback methods seem to be better when high-speed control is needed (not the case here), I will focus primarily on describing and explaining some of the ways feedback control methods have been use to control the chaotic behavior of the Chua Circuit. The first method we will explore is a type of linear feedback control method used on the Chua Circuit by Hwang, Hsieh and Lin. [14] The paper cited proposed the control method of adding a voltage source in series with the inductor in Chua's circuit. Our new controlled Chua Circuit will behave according to the following system of equations:

$$C_1 \frac{dv_1}{dt} = \frac{v_2 - v_1}{R} - i_d(v_1) \tag{3.4}$$

$$C_2 \frac{dv_2}{dt} = \frac{(v_1 - v_2)}{R} - i_l \tag{3.5}$$

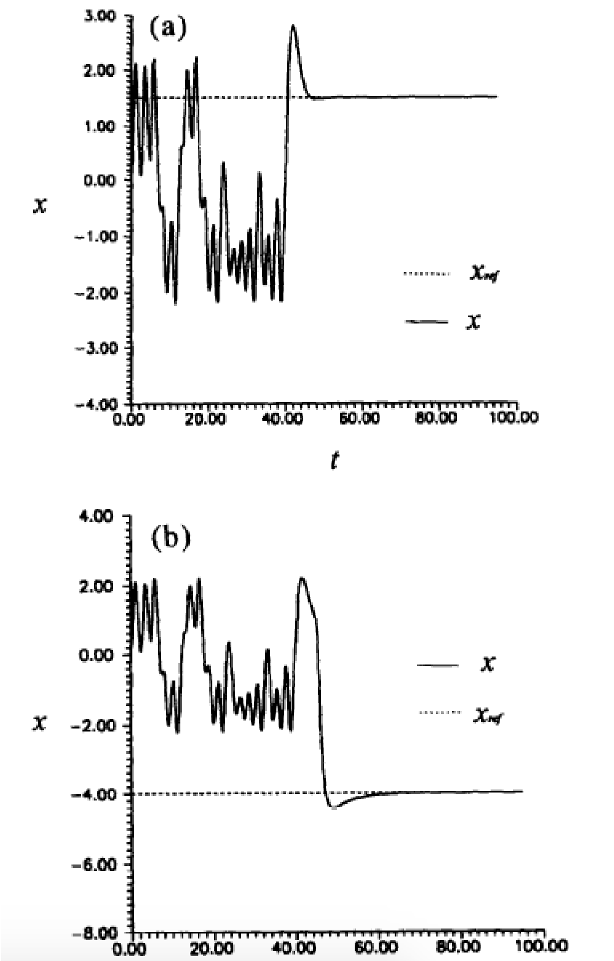
$$L \frac{di_L}{dt} = -v_2 + u \tag{3.6}$$

Our control addition, u has two parts: the first, mimics the second equation and in doing so extends the equilibrium points of the system, and the second, cancels out the non-control part of the equation and adds in what has the same form as a restoring force, damping the behavior of the system around an equilibrium point. The following is the, as previously described, equation of u:

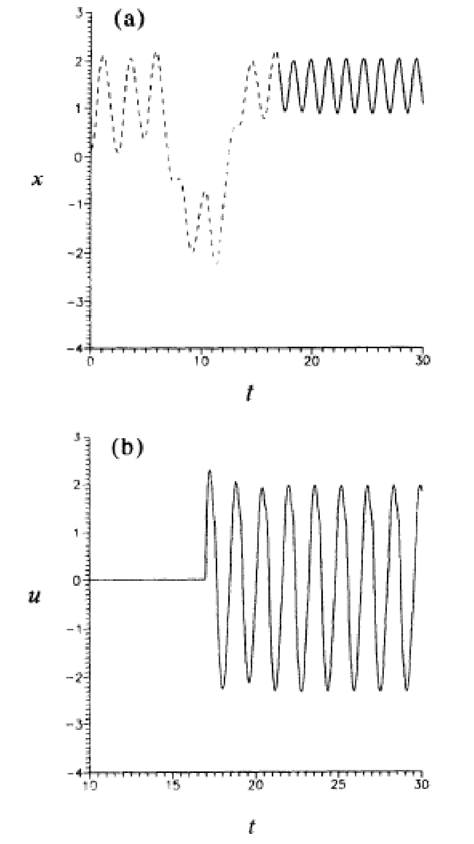
$$u = k(x - y + z) + \beta y + k_p(x_{ref} - x)$$

where x_{ref} is your desired reference x-coordinate.

Hwang, Hsieh and Lin observed the control of Chua's Circuit's chaotic behavior from Figure 3.10a. The restoring nature of the control addition is apparent in Figure 3.10a (b) as the orbit takes an oscillatory path to the desired fixed point.



(a) [a] and [b] both depict the controlling of chaos to a fixed point, just to different x_{ref} points [14].



(b) [a] Uses the time responses of the controlled system to depict the controlling of chaos to a limit cycle of period-2 and [b] illustrates u as a function of time. [14].

After establishing a control method for bringing a chaotic circuit to a desired fixed point, Hwang, Hsieh and Lin tested their method on bringing the Chua Circuit to a desired limit cycle. This was also successful according to Figure 3.10b

The method of feedback control not only works for fixed points and limit cycles alike, but it also has a very small overshoot, meaning it does not usually oscillate very much before settling on a point or cycle, making for a more efficient method of control.

In a later paper by Tzuyin Wu and Min-Sin Chen, using a backstepper control design, a nonlinear feedback control method for Chua's circuit is presented which is proven to be a "global" control, meaning that any initial values of the system will allow you to control the output to any periodic point. They make the case that since chaotic systems are nonlinear, it is necessary to use a nonlinear feedback system to globalize the control. The controlled system set up is in the same in terms of equations 3.4-3.6, but a smooth approximation is used to describe the current across the diode as a function of the voltage across the first capacitor that is a follows:

$$f(x) = \frac{2x^3 - x}{7}$$

.

The function of u for this control method is much more complicated than the previous one and is depicted in Figure 3.11.

$$\begin{aligned} u = {} & a(x)x + b(x)y + c(x)z + d(x)y^2 + ef(x) + gf(x)f'(x) + p^2 f(x)f'(x)^2 \\ & + p^2 f^2(x) f''(x) + \frac{1}{8p} k_x k_y k_z x_r, \end{aligned}$$

where

$$\begin{aligned} a(x) &= 1 - \frac{1}{2}(k_x + k_y + k_z) - \frac{1}{8p} k_x k_y k_z + pf'(x), \\ b(x) &= q - p - 1 + \frac{1}{2}(k_x + k_y + k_z) - \frac{1}{4}(k_x k_y + k_y k_z + k_z k_x) \\ & \quad - \left(1 - \frac{1}{2}(k_x + k_y + k_z)\right) pf'(x) - p^2 f'(x)^2 - 2p^2 f(x) f''(x), \\ c(x) &= 1 - \frac{1}{2}(k_x + k_y + k_z) + pf'(x), \qquad d(x) = p^2 f''(x), \\ e &= p + \frac{1}{4}(k_x k_y + k_y k_z + k_z k_x), \qquad g = -\frac{p}{2}(k_x + k_y + k_z), \end{aligned}$$

Figure 3.11: The control non-linear function, u. [16]

Wu and Chen go on to rigorously prove the stability and globalization of this equation of u.

All of the articles I was able to find on controlling chaos were theoretical and numerical, but not experimental. For future research, I would be interested in how one would go about implementing one of the many control methods used on Chua's circuit in an experimental capacity.

Chapter 4

Conclusion

After first establishing a language for which to talk about the chaotic behavior exhibited by the circuits explored in my project, we were able to consolidate almost three decades of progress in chaotic behavior in electrical circuits with a high degree of detail and comprehension. The evolution of the understanding of each circuit has been outlined and samples from each realm of research pertaining to the circuts has been presented.

To summarize, first we looked at the varicap diode circuit. It was first speculated that the principle nonlinearity that lead to the a period-doubling route to chaos as well as other chaotic behavior was the voltage-dependent capacitance. It was later, more mathematically rigorously explained that the large reverse-recovery time constant of the circuit had a hand in providing the nonlinearity.

Next, we explored the extensively modeled and understood world of Chua's Circuit. We talked about how its different parameters effect its behavior and the variations of it that belong to the Chua family and their behaviors. The Chua Oscillator, a member of the Chua family of circuits, displays three different route to chaos that were outlined: period-doubling, intermittency, and torus breakdown. A canonical version of Chua's Circuit was also present which incorporated every type of behavior observed by members of the Chua family into one circuit with only one nonlinear element. Lastly, we discussed ways in which one can control the chaotic behavior exhibited by the Chua circuit, or any other chaotic system, and the reasons why this may be desirable. Linear and non-linear feedback control methods were discussed as well as the advantages and disadvantages of using linear versus nonlinear and feedback versus non-feedback control methods.

My motivation for exploring chaotic circuits was mainly due to their applications and observational nature, so my failure to produce an actual circuit to model the chaotic behavior was definitely disappointing for me. However, I was still able to explore the varicap and Chua Circuit behavior from multiple perspectives, through prior written articles and Mathematica simulation and modeling, which has definitely been rewarding.

Appendix A

Mathematica Code

A.1 Lorenz Attractor

```
In[3]:= Leq = {
          x'[t] == -3 (x[t] - y[t]),
          y'[t] == -x[t] z[t] + n x[t] - y[t],
          z'[t] == x[t] y[t] - z[t]
         };

In[4]:= P = ParametricNDSolveValue[
          {
           Leq, x[0] == 0, z[0] == 0, y[0] == 1
          },
          Function[
           Evaluate[
            {
             x[#], y[#], z[#]
            }
           ]
          ],
          {t, 0, 15},
          n];

In[5]:= ParametricPlot3D[
         P [28] [t],
         {
          t, 0, 15
         },
         ColorFunction → GrayLevel
        ]
```

A.2 Double Scroll Attractor

```
In[24]:= m0 = -1.143;
         m1 = -0.714;
         c1 = 15.6;
         c2 = 1;
         c3 = 28;
         f[x_] := m1 x + (m0 - m1) / 2 (Abs[x + 1] - Abs[x - 1]);

In[84]:= p = NDSolve[
            {x'[t] == c1 (y[t] - x[t] - f[x[t]]),
             y'[t] == c2 (x[t] - y[t] + z[t]),
             z'[t] == -c3 y[t] ,
             x[0] == 0.7, y[0] == 0, z[0] == 0},
            {x, y, z},
            {t, 1000}
           ];

In[94]:= ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. p], {t, 0, 100},
          PlotRange → All, Mesh → None, BoxRatios → {1, 1, 1}, ColorFunction -> GrayLevel]
```

A.3 Matsumoto's Double Scroll Attractor

```
m0 = -4;
m1 = -0.1;
c1 = {1/10.0};
c2 = {1/0.5};
G = 0.7;
L = {1/7};
f[x_] := m1 x + (m0 - m1) / 2 (Abs[x + 1] - Abs[x - 1]);

p = NDSolve[
    {c1 x'[t] == G (y[t] - x[t]) - f[x[t]],
     c2 y'[t] == G (x[t] - y[t]) + z[t],
     L z'[t] == - y[t] ,
     x[0] == 1, y[0] == 1, z[0] == 1},
    {x, y, z},
    {t, 1000}
   ];


ds = ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. p], {t, 0, 300},
  PlotRange → All, Mesh → None, BoxRatios → {1, 1, 1}, ColorFunction -> GrayLevel]
```

A.4 Chua Oscillator, Failed

```
In[276]:= m0 = -0.879 * 10^(-3);
          m1 = -0.4124 * 10^(-3);
          c1 = 5.75 * 10^(-6);
          c2 = 21.32 * 10^(-6);
          L = 12 * 10^(-3);
          R0 = 30.86;
          R = 1.558 (10^3);
          f[x_] := m1 x + (m0 - m1) / 2 (Abs[x + 1] - Abs[x - 1]);

In[284]:= p = NDSolve[
             {c1 x'[t] == (1/R) (y[t] - x[t]) - f[x[t]],
              c2 y'[t] == (1/R) (x[t] - y[t]) + z[t],
              L z'[t] == - (y[t] + R0 z[t]),
              x[0] == 0.5, y[0] == 0, z[0] == 0},
             {x, y, z},
             {t, 1000}
            ];


In[285]:= ds = ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. p], {t, 0, 1000},
            PlotRange → All, Mesh → None, BoxRatios → {1, 1, 1}, ColorFunction -> GrayLevel]
```

Bibliography

[1] Paul S. Linsay. “Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator”. In: *Phys. Rev. Lett.* 47.19 (1981), pp. 1349–1352.

[2] R.W. Rollins and E.R. Hunt. “Exactly Solvable Model of a Physical System Exhibiting Universal Chaotic Behavior”. In: *IEEE Trans. Circuits Syst.* 49.18 (1982), pp. 1295–1298.

[3] James Testa, Jose Perez, and Carson Jeffries. “Evidence for Universal Chaotic Behavior of a Driven Nonlinear Oscillator”. In: *Phys. Rev. Lett.* 48.11 (1982), pp. 714–717.

[4] T. Matsumoto. “A chaotic attractor from Chua’s circuit”. In: *IEEE Trans. Circuits Syst.* 31.12 (1984), pp. 1055–1058.

[5] Leon O. Chua, Motomasa Komuro, and Takashi Matsumoto. “The double scroll family”. In: *IEEE Trans. Circuits Syst.* CAS-33 (1986), pp. 1073–1118.

[6] T. Matsumoto, L.O. Chua, and M. Tokunaga. “Chaos via torus breakdown”. In: *IEEE Trans. Circuits Syst.* CAS-34 (1987), pp. 240–253.

[7] P. Bartissol and Leon O. Chua. “The double hook”. In: *IEEE Trans. Circuits Syst.* 35 (1988), pp. 1512–1522.

[8] Leon O. Chua and Gui-Nian Lin. “Canonical realization of Chua’s circuit family”. In: *IEEE Trans. Circuits Syst.* 37.7 (1990), pp. 885–902.

[9] R. Lima and M. Pettini. “Suppression of chaos by resonant parametric perturbations”. In: *Phys. Rev.* 41 (1990), pp. 726–733.

[10] Y. Braiman and I. Goldhirsch. “Taming chaotic dynamics with weak periodic perturbations”. In: *Phys. Rev. Lett.* 66 (1991), pp. 2545–2548.

[11] R.N. Madan. “Observing and Learning Chaotic Phenomena”. In: *Proc. 35th Midwest Symp. Circuits Syst.* 736-745 (1992).

[12] Leon O. Chua, Chai Wah Wu, and Anshan Huang. “A Universal Circuit for Studying and Generating Chaos - Part I. Routes to chaos”. In: *IEEE Trans. Circuits Syst. 1* 40.10 (1993), pp. 732–744.

[13] Robert C. Hilborn. *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers.* New York: Oxford University Press, 1994.

[14] Chi-Chuan Hwang, Jin-Yuan Hsieh, and Rong-syh Lin. “A linear continuous feedback control of Chua’s circuit”. In: *Chaos, Solitons and Fractals* 8 (1997), pp. 1507–1515.

[15] S. Rajasekar, K. Murali, and M. Lakshmana. “Control of chaos by nonfeedback methods in a simple electric circuit system and FitzHugh-Nagumo Equation”. In: *Chaos, Solitons, and Fractals* 9 (1997), pp. 1545–1558.

[16] Tzuyin Wu and Min-Shin Chen. “Chaos control of the modified Chua’s circuit system”. In: *Physica D: Nonlinear Phenomena* 164.1-2 (2002), pp. 53–58.

[17] Renato Mariz de Moraes and Steven M. Anlage. “Unified model and reverse recovery nonlinearities of the driven diode resonator”. In: *Physical Review E* 68.2 (2003).

[18] Edward Ott. *Controlling chaos.* 2006. URL: `http://www.scholarpedia.org/article/Controlling_chaos`.

[19] Gaurav Gandhi and Tamas Roska Bharathwaj Muthuswamy. “Chua’s Circuit for High School Students”. In: *International Journal of Bifurcation and Chaos* (2007).

[20] Geoff Boeing. *Chaos Theory and the Logistic Map.* 2015. URL: `http://geoffboeing.com/2015/03/chaos-theory-logistic-map/`.

[21] Valentin Siderskiy. *Chua's circuit diagrams, equations, simlulations and how to build*. 2016. URL: http://Chuacircuits.com.

[22] *Population Growth*. URL: http://www.pspwp.pwp.blueyonder.co.uk/starting-points/simulations-chaos/pop-growth.html.

[23] *Sensitivty of Lorenz Equations*. URL: https://www.wolfram.com/mathematica/new-in-9/parametric-differential-equations/sensitivity-of-the-lorenz-equations.html.