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Welfare versus Stability in "Stabilizing an Unstable Economy": A Minskyan Growth Model

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by
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Abstract

The project focuses on Minsky's financial fragility hypothesis incorporated in a growth model and investigates whether an inherently unstable economy can be stabilized by a big and proactive government. Using dynamical systems theory and expanding a supply-driven growth model developed by Lin, Day and Tse (1992), the project explores how different government spending programs and financing paths can affect the growth, as well as the stability of the economy. The results and implications of the model are analyzed, using analytical and numerical methods of bifurcation, to examine the dependence of 'optimal' government intervention on the economic environment.

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Dedication

To my family for all the support. To my grandfathers, who are my role models.

Acknowledgments

First, I would like to thank Professor Papadimitriou for all the support and guidance during my college years. Everything would have been different without him. Also, I want to thank my senior project advisor Maria Belk for devoting so much effort and time and without whom this project would not have been possible. I will always be grateful to my professors John Cullinan, Greg Landweber, Matthew Deady, Arthur Gibbons and Garry Hagberg. Their engagement and enthusiasm in their courses, shaped my academic life. Last but not least, I would like to express my appreciation to the Levy Economics Institute of Bard College, for the internship position, and particularly to Taun Toy for being a great "out of class" professor.

1
Introduction

The housing bubble and the financial crisis that followed in 2008 triggered the controversy over the causes of crises and their impact on real economy. The debate about the role of government and whether austerity measures or spending programs could stimulate the economy was fueled once again. As Hyman Minsky suggested, 'if a theory is to explain an event, the event must be possible within the theory' (Minsky, 1984, p.16). However, in standard economic theory and neoclassical synthesis, financial crisis and serious fluctuations of output and employment are anomalies; the theory offers no explanation of these phenomena.

On the contrary, Minsky's Financial Instability Hypothesis (1984), explains instability as a result of the normal functioning of a capitalist economy. According to Minsky, financial crises should be viewed as systemic rather than accidental events. As he pointed out,

> 'A capitalist economy with sophisticated financial institutions is capable of a number of modes of behavior and the mode that actually rules at any time depends upon institutional relations, the structure of financial linkages, and the history of the economy.' (Minsky, 1984, p.92)

Based on Minsky's framework and its interpretation in a supply-driven real growth model developed by Lin, Day and Tse (1992), this project addresses the critical question of whether government intervention can stabilize an unstable economy.

1.1 Hyman Minsky

1.1.1 Introduction to Hyman Misnky

In his analysis of economic development, Minsky followed Schumpeter's integration of financial entrepreneurship and financial evolution. Schumpeter focused on endogenous factors that change the equilibrium of the system, 'so that the new one cannot be reached from the old one by infinitesimal steps' (Schumpeter 1934 [1949], p.64). Economic development is viewed as the result of innovation, as new combinations of materials and forces are performed in the economy.

The basic core of Minsky's thought is centered around (i) the interpretation of Keynes's theory focusing on the role of financial markets, the endogeneity and non-neutrality of money and the systematic uncertainty surrounding the decisions made by the agents involved, (ii) the financial instability hypothesis and (iii) the thesis that discretionary economic policies can smooth cyclical instability, creating ceilings and floors and constraining the dynamic behavior of the economy.

Market mechanisms often seem structurally flawed because they fail to provide full employment, fair inequalities and economic stability. Historical data on long periods of economic growth provides evidence that the private sector was never able to attain full employment, so stimulating the private sector through government spending and incentives is flawed.

Minsky argued that the Great Depression represented a failure of the small-government economic model, while the New Deal promoted a 'Big Government/ Big Bank' highly successful model for financial capitalism. However, new regulations in a society that is highly dynamic lead to regulatory arbitrage, and the creativity of economic agents generates new economic structures and so a need to change regulations. Profit-seeking agents learn how a regulatory structure operates and because regulation means that some perceived profit opportunities are not open to exploitation, they have incentives to change their behavior to evade or avoid the constraints. This implies that over time the consequences of the structure of intervention changes. According to Bellfiore, Halevi and Passarella (2010) Minsky's theory suggests that a perfect economic policy, good for all possible times and places, does not exist.

The twin pillars of Minsky's recommended policy structure are 'Big Government', a fiscal authority that engages in large spending and taxing programs, and a 'Big Bank', a lender of last resort. The key role of big government operates through the 'Kalecki mechanism', which Minsky has adopted as a cornerstone of his analysis of fiscal policy: economic downturns cause reduced profit flows. Reduced profits weaken the ability of firms to service existing debt commitments. Consequently, asset quality deteriorates and the price of capital assets declines relative to the price of current output. This, in turn, reduces investment and magnifies the downturn. According to Kalecki, government deficits support business profits. Minsky was conscious that big-government capitalism, while solving important problems, also creates new ones. Although fiscal discretion can fine-tune the economy, it can also create inequality, inflationary pressure and long-term unemployment through unproductive government intervention that focuses solely on fine-tuning.

The issue of fiscal policy is not the nominal amount of expenditure; what matters most is the structural content of the intervention. Not all the forms of government intervention are consistent with stabilizing the economy. Minsky criticized traditional Keynesian deficit spending, claiming that defense and private consumption does not contribute to a more efficient productive structure. 'These policies have reduced in time the degree of robustness of the financial structures, have shortened units time-horizon, and have slowed down productivity growth'(Bellofiore, Halevi and Passarella, 2010). Minsky advocated of a proactive form of government that takes initiatives to direct economy toward more stable and more fair forms of capitalism, rather than a passive government that just reacts to economic problems through spending, taxing and manipulations of financial conditions.

1.1.2 Profits and Deficits in Minsky's Theory

As Eisner (1978) indicates, it is expectations about future profit flows that motivate investment. In a modern capitalist economy, investment is always an exchange of money now, which is used to pay for the creation of capital assets, for future money that is expected to be forthcoming as the capital assets are employed. The economic theory that begins with the formulations of Keynes, Kalecki and Minsky makes financial relations and the financing of activity critical determinants of the system's behavior (Minsky, 1992).

A systemic shortfall of profits below what had been anticipated, by the time financial engagements were undertaken, diminishes the flow of the new financing for investment. This will tend to lower investment and lower gross profits, leading to a potential downturn in economic activity.

Profits are determined by macroeconomic relations, particularly by the structure of demand and the availability of financing. In a small laissez-faire capitalist economy, with balanced trade, total profits equals investment. In a more sophisticated economy, the relation between profits and investment can be expanded into

Profits = Investment + Government Deficit − Trade Deficit − Savings out of Wage + Consumption out of Profits[1].

In big-government capitalism, monetary and fiscal discretion can fine-tune the economy. Government intervention in the form of fiscal actions and built-in stabilizers can sustain realized and expected profits, if government expenditures and taxes are as big as

[1] See Minsky, 1992

investment and when the government deficit moves quickly and strongly in the opposite direction. On the monetary side, the central bank can limit the spread of the crisis by the timely refinancing of the economy, in order to prevent the bankruptcy of banks and financial intermediaries.

1.1.3 Financial Instability Hypothesis

A strong form of orthodox economic theory finds what happens in financial markets to be of little or no importance in determining what happens in the economy. In sharp contrast, Minsky's Financial Instability Hypothesis focuses on business debt and profits as an explanation of crises since this debt is an essential characteristic of a capitalist economy. The theories of investment and endogenous money, which are the building blocks of Minsky's broader ideas about macroeconomics, are key to his central concern; the financial instability that drives fluctuations in the economy as a whole.

In a capitalist economy, prolonged economic growth generates an upward fundamental instability, due to the tendency to transform euphoria into a speculative investment boom (Minsky, 1984). As a result, 'stability is destabilizing' and market mechanisms tend to promote inflationary pressure and financial fragility as the economy tends toward full employment.

According to Minsky (1984) the longer the boom continues the more the liabilities of firms must be increased to finance investment, i.e. the demands on current cash flows to finance debt payments increases substantially. This increased 'financial fragility sows the seeds of the next downturn'(ibid), placing financial instability in an inherently dynamic and cyclical context, which is interpreted into hedge, speculative and Ponzi finance.

In a hedge financial structure, the prospective income cash flows provided from their own economic activity are greater than cash payment contractual commitments for every period. So the debt volume tends to fall from one period to another. Hedge-financing faces an economic risk, but is safe relative to financial risk.

In the case of speculative financial structure, cash flows allow the servicing of the interest on debt, but no longer suffice to cover, in every period, the part of the principal that must be paid back.

Finally, in the Ponzi financial structure, interest payments alone exceed the cash flow earnings for a significant number of periods, so that units must either liquidate part of their assets or raise new funds (refinancing) and increase debt.

In such a context, a small, unexpected rise in the short-term interest rates or fall in asset prices leads to a financial crisis.

1.2 Mathematical Modeling

Every model is at best an idealization of a real phenomenon. The goal of the model is to capture some feature of the physical process. The question to ask about a model is whether the behavior it exhibits is because of its simplifications or if it captures the behavior despite the simplifications.

Financial institutions, markets and instruments are historically highly fluid and adaptable to economic conditions. The financial response to changes of economic environment takes the form of transformation and innovation of financial institutions and financial instruments. Economists' knowledge of the real world takes the form of simplified mathematical models, in which the relevant phenomena are represented by an unchanging set of quantitative variables. Mathematical models have the great conceptual advantage that they can be easily manipulated to generate predictions by changing the parameters of the model. Usually, mathematical models are validated by econometric techniques when at least some observed data can be shown to exhibit at least some of the correlations predicted by the model. However, the available data may be compatible with mathematical models that have contradictory counterfactual implications.[2].

When Minsky's early followers tried to formulate his ideas mathematically, they were unable to represent the shift in the average risk of financial positions as a result of the gradual loosening of financial constraints, which is characteristic of all capitalist booms, and plays an important role in supporting and extending boom conditions. In addition, quantitative variables that represented the risk tolerance of investors could not be validated by statistical data, due to the key role played by unobservable variables. In Minsky's discourse, the shift toward more exposed financial positions is not only a psychological phenomenon based on the increasing optimism of investors, but it also involves strong competitive pressures on individual investors to follow the group norms that are themselves shifting.

Modern capitalism is a complex and dynamic system and its essential features cannot be captured in a static framework. 'Since historically cyclical fluctuations took place against the background of a rising trend output, a theory of the cycle ought to be built around a dynamic theory of economic development rather than in a static framework' (Kaldor 1951 b, p.841)

Developments in the mathematical theory of dynamical stability and bifurcation theory have influenced social sciences and particularly the field of economics. The considerable new insight gained by the applications following from nonlinear dynamics led to the revision of past claims in theory. Mathematical chaos made possible the presence of new dynamical features in social systems that theoreticians had never addressed before.

The modeling toolbox has been advanced by the addition of bifurcation analysis and nonlinear dynamic systems. Nonlinear models can come closer to reflecting dialectical

[2]See 'The Logic of Counterfactuals in Causal Inference', Judea Pearl, 2000

insights. These tools have been the main route through which economists have tried to formalize the insights of Minsky's financial fragility hypothesis.

As Kregel points out though, the possibility of instability as a result of nonlinear dynamics is not the essential issue. The key is the substance of the model's structure that generates nonlinearities and instability. Thus, despite our attempt to build a body of rigorous evidence (a robust mathematical model) to support Minsky's ideas, we should always keep in mind that mathematical models are tools to clarify our understanding of simplified, imaginary systems that we hope represent a complex economic reality. Therefore the insights the mathematical models offer should not be exaggerated or their implications recklessly extrapolated beyond the natural limits of the simplified system on which they are based. Following Minsky's dialectic investigation of economic issues, any conclusion should be derived only after a strong dialogue between dialectical and mathematical perspectives and methods.

The project is organized as follows. Chapter 2 presents the mathematical toolbox which is used to construct the model and to investigate its implications. It also introduces the economic theory on which the project is based. In Chapter 3, the growth model developed by Lin, Day and Tse (1992), is modified and expanded in order to incorporate the public sector and expectation dynamics that could formalize mathematically the Financial Instability Hypothesis. Finally, Chapter 4 analyzes the results and implications of the model and contains some concluding remarks[3].

[3]The reader should keep in mind that the conclusions and the graphs presented in this project are for specific parameter values. However, the graphs represent the general picture and are not biased towards validating the conclusions. Analytical results, as well as results for other parameter values are shown in the appendices.

2

Basic Framework

2.1 Mathematical Background

2.1.1 Dynamical systems

The theory of dynamical systems is a major mathematical discipline, whose concepts and methods have stimulated research in many sciences. A **dynamical system**[1] consists of a set of variables that describe its state and a law that describes the evolution of the state variables with time (how the state of the system in the next moment of time depends on the input and its state in the previous moment of time).

The evolution of a dynamical system means a change in the state of the system with time $t \in T$, where T is a number set. There are two types of dynamical systems: those with continuous (real) time $T = \mathbb{R}^1$, and those with discrete (integer) time $T = \mathbb{Z}$. Systems of the first type are called **continuous-time** dynamical systems, while those of the second type are called **discrete-time** dynamical systems. Discrete-time systems appear naturally in economics when the state of a system at a certain moment of time t completely determines its state after a year.

[1] The definitions and theorems presented in this chapter can be found in a standard dynamical system textbook, such as *Nonlinear Dynamics and Chaos*, by S. Strogatz.

In this project, we will only be concerned with discrete time dynamical systems. In particular, we will be concerned with dynamical systems where we iterate functions of the form: $f \colon \mathbb{R} \longrightarrow \mathbb{R}$. Given a function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ and an initial value x_0, consider the sequence of iterates of x_0 under the function f

$$x_0, \quad f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \quad \dots$$

We will use the notation f^n to denote the n-fold composition of a function f with itself, e.g. $f^2(x_0) = f(f(x_0))$. Let $x_n = f^n(x_0)$. Then the iterates of x_0 under f can be written as the solution of the first-order difference equation.

$$x_{n+1} = f(x_n). \tag{2.1.1}$$

Definition 2.1.1. A continuous function whose domain space and range space are the same is called a **map**. Let x be a point and let f be a map. The **orbit** of x under f is the set of points $\{x, f(x), f^2(x), \dots\}$. The starting point x for the orbit is called the **initial value** of the orbit. $\triangle$

Definition 2.1.2. The point $\overline{x}$ is called **fixed point** for f if $f(\overline{x}) = \overline{x}$. $\triangle$

Definition 2.1.3. A fixed point $\overline{x}$ is said to be **stable** if, for any $\epsilon > 0$, there is a $\delta > 0$, such that, for every x_0 for which $|x_o - \overline{x}| < \delta$, the iterates of x_0 satisfy the inequality $|f^n(x_0) - \overline{x}| < \epsilon$ for all $n \geq 0$. The fixed point $\overline{x}$ is said to be **unstable** if it is not stable. $\triangle$

An informal interpretation of Definition 2.1.3 that clarifies the stability of different types of fixed points is as follows:

Let f be a map on $\mathbb{R}$ and let $x \in \mathbb{R}$ such that $f(x) = x$. If all points sufficiently close to x are attracted to x, then x is called an **attracting fixed point** (stable). If all points sufficiently close to x are repelled from x, then x is called a **repelling fixed point**(unstable).

Theorem 2.1.4. *Let f be a map on $\mathbb{R}$, and assume that p is a fixed point of f.*

1. *If $|f'(x)| < 1$, then x is an attracting fixed point.*

2. *If $|f'(x)| > 1$, then x is a repelling fixed point.*

Definition 2.1.5. A point x^* is called a **periodic point of minimal period n** if $f^n(x^*) = x^*$ and n is the least such positive integer. The set of all iterates of a periodic point is called a periodic orbit. △

The study of changes in the qualitative structure of a difference equation as parameters are varied is called **bifurcation theory**. At a given parameter value, a difference equation is said to have a **stable orbit structure** if the qualitative structure of the system does not change for sufficiently small variations of the parameters. A parameter value for which the system does not have stable orbit structure is called a **bifurcation value**, and the equation is said to be at a **bifurcation point**.

Definition 2.1.6. An **attractor** is a set to which all neighboring trajectories converge. △

A useful graphical method for depicting some of the important dynamical features in Equation 2.1.1 consists of drawing curves on the $(c, x)-$plane, where the curves depict the attractor for each of the parameter c. The resulting picture is called **bifurcation diagram**.

In numerical approximations, bifurcation diagrams can be generated by the following algorithm:

1. Choose a value of c, starting with $c = c_0$.

2. Choose x at random within the domain of the function

3. Calculate the orbit of $f_{c_0}(x)$

4. Ignore the first N iterates and plot the orbit beginning at iterate $N + 1$

5. Increment c by ϵ and begin the procedure again

The points that are plotted will approximate the fixed or periodic points. The resulting figure is a computer-generated bifurcation diagram. As, N increases and ϵ decreases, the computer-generated picture will more adequately represent the actual bifurcation diagram.

Discrete dynamical systems can exhibit periodic orbits or aperiodic for certain parameter values. To characterize a behavior 'chaotic', a system should show sensitive dependence on initial conditions. This means that two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. A practical implication is that long-term predictions becomes impossible in a system like this, where small uncertainties are amplified enormously fast, in the sense that neighboring orbits separate exponentially fast and lead to completely different results.

Definition 2.1.7. Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions. △

Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates. Suppose that $|\delta_n| \approx |\delta_0| e^{n\lambda}$. Then, by taking logarithms and noting that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$, we obtain

$$\lambda \approx \frac{1}{n} \ln |\frac{\delta_n}{\delta_0}|$$

$$= \frac{1}{n} \ln |\frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0}|$$

$$= \frac{1}{n} \ln (f^n)'(x_0)$$

$$= \frac{1}{n} \ln | \prod_{i=0}^{n-1} f'(x_i)|$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

Definition 2.1.8. The Lyapunov exponent for a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ for the orbit starting at x_0 is

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

△

The Lyapunov exponent is used to determine whether or not a system is chaotic. The Lyapunov exponent may be computed for a sample of points near the attractor to obtain an average Lyapunov exponent.

Theorem 2.1.9. *If the average Lyapunov exponent is positive, then the system is chaotic; if the average Lyapunov exponent is negative, then the orbit is periodic; and if the average Lyapunov exponent is zero, a bifurcation occurs.*

2.1.2 The Tent Map

As a simple introduction to nonlinear discrete dynamical systems, consider the tent map, $T\colon [0,1] \longrightarrow [0,1]$, defined by

$$T(x) = \begin{cases} ax & 0 \le x \le \dfrac{1}{2}, \\ a(1-x), & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{2.1.2}$$

The graph of T is given in Figure 2.1.1.

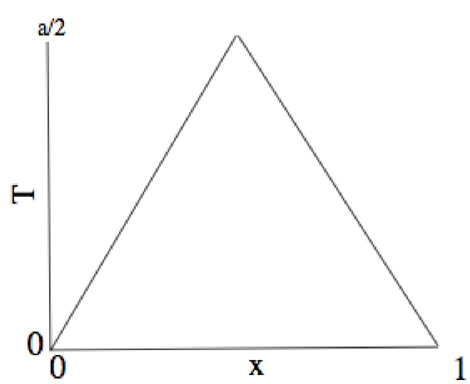

Figure 2.1.1. The Tent Map

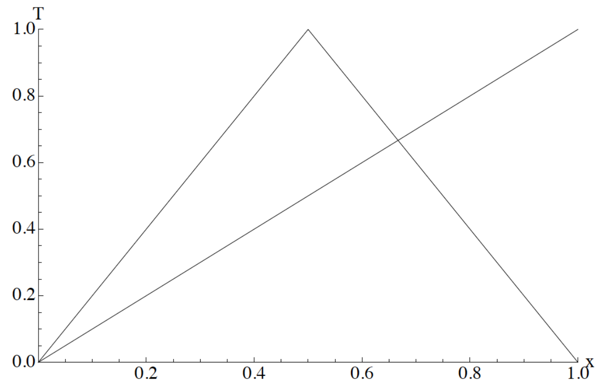

Figure 2.1.2. The intersection $T(x)$ and the diagonal $y = x$ when $a = 2$

Define the iterative map by

$$x_{n+1} = T(x_n), \tag{2.1.3}$$

where $x_n \in [0, 1]$. Despite the simple form of the tent map, the system can exhibit complex behavior and chaotic phenomena for certain parameter values.

In order to determine the points of period N for the tent map we need to solve the equation $x_{n+N} = T^N(x_n) = x_n$, for all n. Graphically, the periodic points can be found by identifying intersections of the function $T^N(x)$ with the diagonal $(y = x)$.

Consider the case where $a = 2$. The fixed points are at $x = 0$ and $x = \dfrac{2}{3}$, which are the intersections of $T(x)$ with the diagonal (see Figure 2.1.2). To determine the points of period two, we need to find the points of intersections of $T^2(x)$ with the diagonal, and so on. As in any dynamical system, the periodic points can be attracting, repelling or indifferent. The type of the periodic point can either be determined by the derivative or using the Lyapunov exponent. For example, $|T'(0)| = 2 > 1$, while $|T'(2/3)| = 2 > 1$ and therefore both fixed points are unstable. A different method to verify this result is to calculate the Lyapunov exponent for the two points. The Lyapunov exponent can be found easily since $|T'(x)| = 2$ for all values of x. Hence

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |T'(x_i)| = \ln 2 > 0,$$

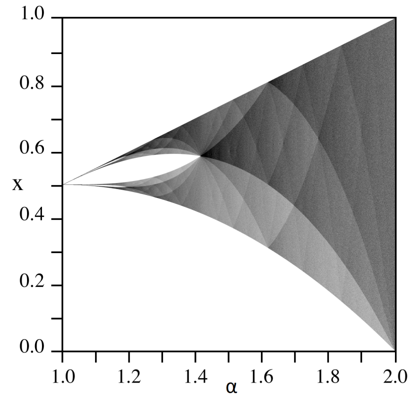

Figure 2.1.3. The Bifurcation Diagram for the Tent Map

Since $\lambda > 0$, the system is chaotic, so the two fixed points are repelling.

2.2 Basic Neoclassical Growth Model

2.2.1 The Solow's Growth Model

Neoclassical models of real growth of market-clearing type originate in Solow's seminal contribution (Solow,1956). The Solow's model represents a simple approach to supply-driven economic growth that allows for factor substitution.

This section starts with the analysis of the basic Solow's model (Solow, 1956) assuming centralized allocation. Assume that there is a social planner that chooses the allocation of resources and dictates those allocations to the households of the economy. Some features of the economy are the following:

- Time is discrete, $t \in \{0, 1, 2, \ldots\}$. Every period is a year, or generation or any arbitrary length of time.

- There is a closed economy, with no imports and exports. There are no markets and production is centralized.

- There is only one good produced with two factors of production, capital and labor. The produced good can be either consumed in the same period, or invested as capital for the next period.

- Households are endowed with one unit of labor, which they supply inelastically. The labor force with the accumulated aggregate capital stock is used to produce the one good of the economy.

- In each period, the social planner saves a constant fraction $s \in (0, 1)$ of contemporaneous output, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the households of the economy.

- Population grows with a constant rate n.

- Existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$.

Let L_t denote the number of households (and the size of the labor force) in period t, K_t the aggregate capital stock at the beginning of period t, Y_t the aggregate output, C_t the aggregate consumption, and I_t the aggregate investment, in period t. The corresponding lower-case variables represent per-capita measures: $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $i_t = I_t/L_t$, and $c_t = C_t/L_t$.

The production function is given by

$$Y_t = F(K_t, L_t) \tag{2.2.1}$$

where $F\colon \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$. We assume that F is continuous and (although not always necessary) twice differentiable.

An example of such a function is the Cobb-Douglas technology function

$$F(K_t, L_t) = K_t^{\alpha} L_t^{\alpha-1} \tag{2.2.2}$$

or equivalently in per capita terms

$$f(k_t) = k_t^{\alpha} \tag{2.2.3}$$

where α and $1-\alpha$ are the elasticities of capital and labor respectively.

Since the economy is closed, the aggregate consumption and aggregate investment cannot exceed aggregate output. The resource constraint is the following

$$C_t + I_t \leq Y_t \tag{2.2.4}$$

Consumption is, by assumption, a fixed fraction of output

$$C_t = (MPC)Y_t \tag{2.2.5}$$

where $MPC \in [0, 1]$ is the marginal propensity to consume and it indicates the portion of the income that we consume.

Equivalently, in per-capita terms

$$c_t + i_t \leq y_t \tag{2.2.6}$$

Since population grows with a constant rate n per period, the size of the labor force evolves over time as follows

$$L_t = (1+n)L_{t-1} = (1+n)^t L_0 \tag{2.2.7}$$

The existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$. The capital stock at the beginning of the next period is given by the non-depreciated part of current-period capital, plus contemporaneous investment. Therefore, the law of motion for capital is

$$K_{t+1} = (1-\delta)K_t + I_t \tag{2.2.8}$$

Equivalently, in per-capita terms

$$(1+n)k_{t+1} = (1-\delta)k_t + i_t \tag{2.2.9}$$

The fundamental equation of Solow's model states that the change in the capital stock is given by aggregate output, minus capital depreciation, minus aggregate consumption.

$$k_{t+1} - k_t = f(k_t) - c_t - (\delta + n)k_t \tag{2.2.10}$$

A steady state of the economy is defined as any level k^*, such that, if the economy starts with $k_0 = k^*$,then $k_t = k^*$ for all $t \geq 1$.That is, a steady state is any fixed point (c^*, k^*) of the system.

2.2.2 The Dynamics of Capital and Consumption

The key of the analysis of a growth model is to derive a dynamic system that characterizes the evolution of aggregate consumption and capital in the economy. We are interested to know whether the economy will converge to the steady state, if it starts away from it.

Figure 2.2.1 depicts that in the Solow's model given any initial $k_0 \in (0, \infty)$, the economy converges asymptotically to a stable steady state k^* following a monotonic transition with decreasing growth rate.

2.3 Real Growth Model with Adaptive Expectations

Early attempts to incorporate instability within the framework of neoclassical theory (Benhabib and Nishura (1985), Boldrin and Montrucchio (1986), Cass and Shell (1983) and Grandmont (1985)) had been criticized because of 'the seriously lacking in the very elements that are relevant to evaluating the dynamic behavior of the real-world economies'(D. J. Harris (1988)). Harris (ibid) continues arguing that 'the missing links relate to labor market interactions, production changes associated with both technology and organization, active price-quantity interventions firms, the role of financial variables and, last but not least, the formation of expectations.' Without a specific treatment of these complicating factors, it would be premature to make a final judgment of stability and instability as a general rule, Harris argues.

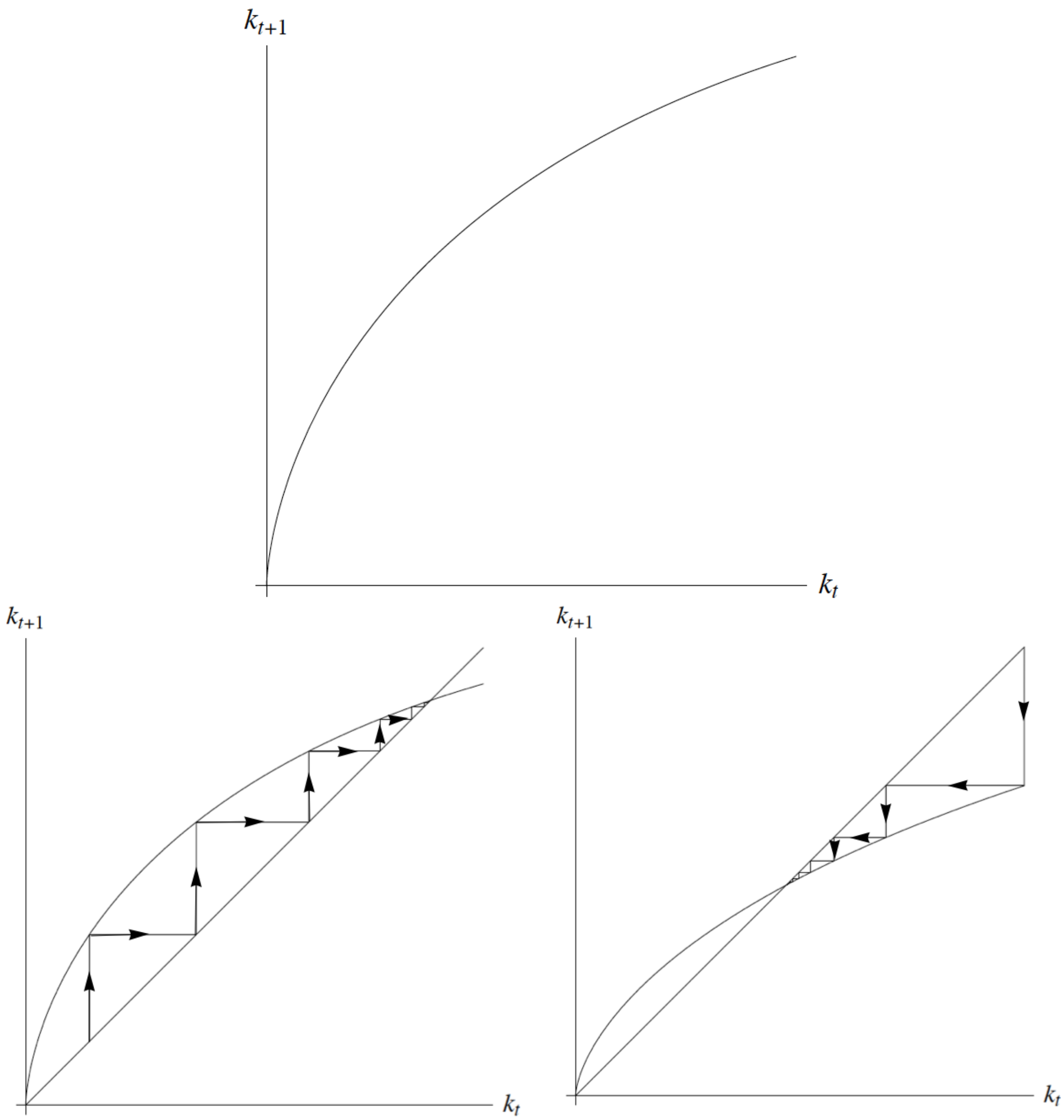

Figure 2.2.1. Transitional Dynamics of Solow's Model. The economy convergence to its steady state independently from the initial conditions. If $k_0 < \overline{k}$ ($k_0 > \overline{k}$), the growth rate is positive (negative) and its rate of change decreases over time.

In the *Real Growth Cycle with Adaptive Expectations* , a growth model developed by Lin, Tse and Day (1992), the behavior of the system can exhibit periodic cycles or non-periodic; even chaotic behavior can occur within plausible parameter ranges. The various types of behavior depend on preferences, the rate of technological advance, the labor growth rate and expectational adjustments; elements that capture important insights of real-world economies.

2.3.1 The Model

Similarly to Solow's model, the savings are invested and the capital stock that results, allowing for depreciation, constitutes the endowment of the next generation.The major difference of this model lies on each generation's plan to choose its own consumption and to endow its heir, with a substantial level of living without considering an indefinitely long future. Thus, succeeding generations decide for themselves what to do with the income generated by their inherited capital stock based on their current situation and modified expectations.

The capital accumulation identity of the economy is the same as in the Solow model.

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s_t] = \frac{1}{1+n}[(1-\delta)k_t + y_t - c_t] \tag{2.3.1}$$

The current consumption (c_t) is not a constant proportion of income, as in the neoclassical model, but it is defined by the maximization of the utility function, which is based on the generation's preference presented by

$$u(c, c^1) = v(c) + V(c^1) \tag{2.3.2}$$

where v represents the utility of current consumption, while $V(c^1)$ is the utility to the present generation from the bequest to its heir. Let

$$v(c) = \log(c) \text{ and } V(c^1) = \gamma \log(c^1), \tag{2.3.3}$$

where γ is the future weight and is equal to the reciprocal of the subjective rate of time preference i, which is the premium that a consumer places on enjoyment nearer in time over more remote enjoyment. If the future weight is large, then i is small. Also, we use logarithmic functions for v and V, because of the principle of diminishing marginal utility (every extra unit of a good gives us less utility than the previous).

We define c^1 as the sustainable consumption level for the next generation, which is determined such that capital stock is maintained the same[2], taking into account population growth. Thus

$$c^1 = y^1 - (n+\delta)k^1 \tag{2.3.4}$$

where the anticipated income y^1 for the next generation in terms of wages w^1 and dividends r^1k^1,assuming efficient markets, is

$$y^1 = w^1 + r^1k^1 \tag{2.3.5}$$

The income distribution of the model does not consider any dichotomy between workers and entrepreneurs and it does not exclude the possibility that workers might save a portion of their income to acquire assets and earn a return on them.

The anticipated and current net rates of returns, which represent the per capita purchasing power of the rate of return on capital, are respectively

$$\rho^1 = \frac{r^1-(n+\delta)}{1+n} \quad \text{and} \quad \rho = \frac{r-(n+\delta)}{1+n} \tag{2.3.6}$$

Therefore the sustainable consumption level is

$$c^1 \quad = \quad w^1 + \rho^1[y - c + (1-\delta)k] \tag{2.3.7}$$

Since in our model the economy is closed, consumption is bounded between

$$0 \leq c \leq y \tag{2.3.8}$$

[2]Alternatively, think of c^1 as a minimum standard of future consumption

In order to find the current and future consumption level we maximize our utility function (2.3.2) using equation (2.3.7) and subject to equation (2.3.8)

$$L = v(c) + V\{w^1 + \rho^1[y - c + (1-\delta)k]\} \tag{2.3.9}$$

Maximizing equation (2.3.9) with respect to c using Langrange multipliers, we get the unconstrained consumption function

$$c(y, k, r^1, w^1, \gamma) = \frac{1}{1+\gamma}[\frac{w^1}{\rho^1} + (y + (1-\delta)k)] \tag{2.3.10}$$

The production for a given generation is given by the production function

$$y = f(k) = Bk^\beta. \tag{2.3.11}$$

The real rate of interest is

$$r = f'(k) = \beta y/k, \tag{2.3.12}$$

while wages are given by

$$w = y - rk = (1-\beta)y. \tag{2.3.13}$$

Some economists, among who Kaldor, questioned the relevance of marginal productivity in a distribution theory and supported that fluctuations over short periods occur. However, they accepted that in the long run real wages rise at the same rate as productivity per worker does, which validates the assumption to set wage equal to the labor share of output.

Leaving aside the inability of a given generation to predict the future accurately, we assume that the current level of income and interest are used as proxies for future values. Then, we get the pure extrapolative forecast for interest and wages

$$r^1 = r, \quad \rho^1 = \rho, \quad w^1 = w. \tag{2.3.14}$$

Using equations (2.3.6), (2.3.12), (2.3.13) and (2.3.14), we obtain the consumption function

$$c(k) = \frac{1}{1+\gamma}[(1-\delta)k + Bk^\beta + \frac{(1+n)(1-\beta)Bk^\beta}{\beta Bk^{\beta-1} - (n-\delta)}] \tag{2.3.15}$$

which is a monotonically increasing function and becomes unbounded as it approaches an asymptote k^0 where $\rho = 0$. Since f is also monotonic but finite valued at k^0, there exists a unique k^s such that $f(k^s) = c(k^s)$.

Define

$$K^s = [0, k^s] \quad and \quad K^d = [k^s, \infty] \tag{2.3.16}$$

Then K^s is the set of k for which savings are positive, while K^d is the set where no savings occur (see Figure 2.3.1). Therefore consumption can be written in the form

$$c(k)^3 = \begin{cases} \frac{1}{1+\gamma}[(1-\delta)k + Bk^\beta + \frac{(1+n)(1-\beta)Bk^\beta}{\beta Bk^{\beta-1} - (n-\delta)}] & k \in K^s, \\ \\ Bk^\beta & k \in K^d. \end{cases} \tag{2.3.17}$$

Combining (2.3.1) and (2.3.17), we have our capital accumulation identity that describes the evolution of per capita wealth generation after generation.

$$k_{t+1} = \theta(k) = \begin{cases} \frac{1}{1+n}\frac{\gamma}{1+\gamma}[(1-\delta)k_t + Bk_t^\beta] - \frac{1}{1+\gamma}\frac{(1-\beta)Bk_t^\beta}{\beta Bk_t^{\beta-1} - (n+\delta)} & k \in K^s, \\ \\ \frac{1-\delta}{1+n}k_t & k \in K^d. \end{cases} \tag{2.3.18}$$

In such an economy different kinds of developments can occur, such as monotonic growth, dampened fluctuations, periodic cycles and chaos. An interesting aspect, which is explored in Chapter 4, is that different parameter values (β, γ, n, δ) can create either a stable or an unstable behavior. In general, as long as saving is positive ($k \in K^s$) growth

[3] $k \in K^s$ if the maximization consumption level is an inferior solution of equation (2.3.8), while $k \in K^d$, if the maximization consumption level is a boundary solution

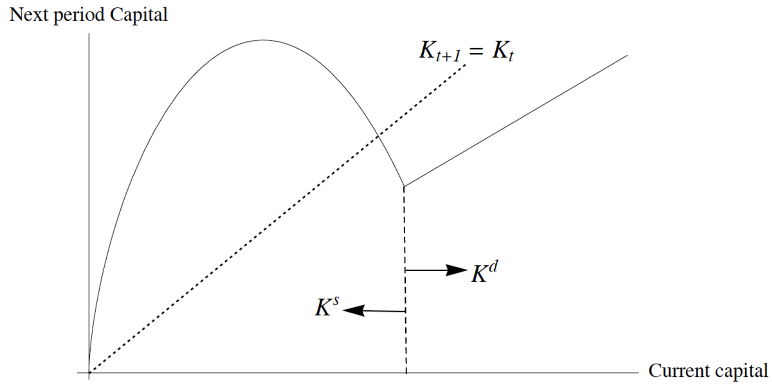

Figure 2.3.1. Phase Diagram of Capital Accumulation

may[4] occur, but when no saving takes place ($k \in K^d$) , capital depreciates at the maximum rate[5].

2.3.2 Numerical Experiment

> *'Even a theory of the simplified, essentially macroeconomic kind under consideration is of considerably more interest if its properties derive from parameter values that fall within plausible ranges.'* (Day, Lin, Tse 1992)

The values of n and δ should vary directly with respect to the length of the period considered, as well as the durability of capital. In the long-run, where t represents generation, it is intuitively clear that the value of capital depreciation and of the population growth rate should be greater than in the short run. If for example, $n_{\text{annual}} = .014$ and $\delta_{\text{annual}} = .0115$, the corresponding values on a generational basis will be $n = .414$ and $\delta = .25$. Regarding the production function, the value of β varies according to the degree of industrialization of the economy. Solow suggested a range of β for the US of .3 to .4, while for Germany a value of .67. We will explore the behavior of the system for the range $.1 \leq \beta \leq .7$. Finally,

[4] If savings $(y - c)$ are greater than the depreciated capital (δk) and the increase is substantial to cover the increase of population, then growth occur since $k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + y - c] > k_t$

[5] If savings are zero $(y = c)$, then $k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t] < k_t$ since δ, $n > 0$.

because the future weight γ is the subjective element of the model, instead of deciding the value of the parameter, we investigate the types of behavior associated with a wide range of values of γ. After all, γ is determined by institutional relations, the structure of financial linkages and the history of the economy. Values for γ as small as .1 and as large as 30 are feasible, in a generational time frame, where a small annual discount of the future compounded through say 25 years implies a very large discount for the future.

Suppose that population is constant, so $n = 0$, but there is a generational rate of technical progress with $\zeta = 2.4$. If neutral technological change takes place at a constant rate ζ, then the term $(1 + n)$ should be replaced by $(1 + n)(1 + \zeta)$ in equation (2.3.1) and correspondingly below. However, any of the results can be associated with positive values of both technical progress and population growth by setting $(1 + n^*)(1 + \zeta) = 1 + n$. Any value of actual population growth n^* and technical progress ζ that satisfy this equation produce the same result as for $n > 0$. Finally, suppose that capital depreciates at a generational rate $\delta = .25$ and the output elasticity β is equal to .67. The corresponding bifurcation diagrams[6] and the steady states of some key macroeconomic variables of the system are presented in Figure (2.3.2).

As figure 2.3.2 depicts, future weights up to 16 produce long-run stable states. After that, two-cycle generation cycles occur followed by a full range of bifurcation results of complexity. According to Lin, Day and Tse (1992) extrapolative expectations combined with a high capital share and rapid technical progress have a great potential to produce generational fluctuations of chaotic nature when the future weight is large.

As one would expect, an increase in future weight (decrease in time preference) decreases consumption as % of GDP since agents place a higher value on future consumption and

[6]The maps (graphs) are conjugate, which means that they are equivalent as far as their dynamics are concerned. This follows from the fact that all the equations are autonomous equations of k (capital accumulation). Informally, this means that if capital accumulation has a two cycle steady state, consumption, production and consumption as % of GDP will also have a two cycle steady state. Therefore the stability of all macroeconomic variables depend on the stability of consumption, since it is the variable that determines the stability of capital accumulation.

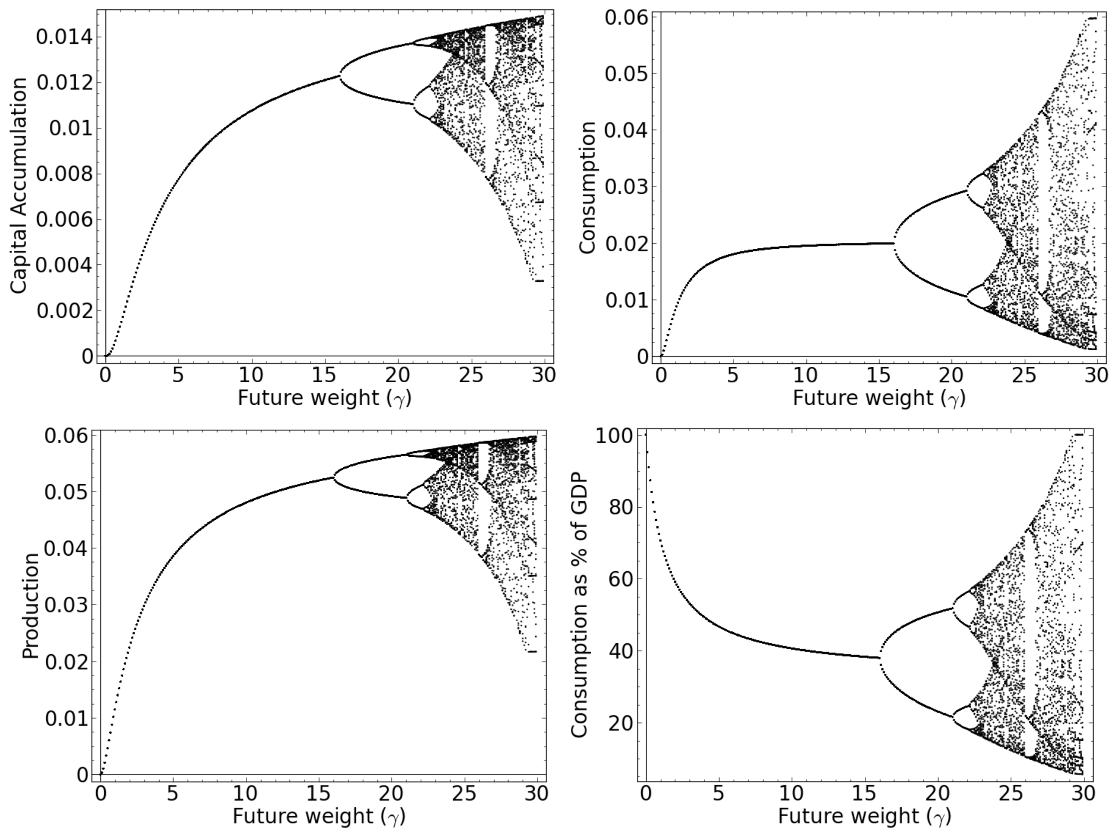

Figure 2.3.2. Bifurcations Diagrams: Basic Model ($\delta = .25$, $n = 2.4$, $\beta = .67$)

they increase savings. The increased level of investment (in a closed economy savings are equal to investment), combined with the uncertainty of future values that leads to erroneous expectations, destabilize the economy and ultimately give rise to chaotic trajectories. As Keynes first argued and then Minsky supported, the uncertain conditions under which investment decisions are made, opens the door to output fluctuations and crises.

3
Minskyan Growth Model

3.1 Mathematical Interpretation of Minsky's Ideas

> 1. *Why haven't we had a great or even a serious depression since 1946?*
> 2. *Why was 1946-66 a period of tranquil progress and why has it been followed by turbulence?*
> 3. *Are there feasible policies short of accepting a deep and long depression that will lead to a resumption of tranquil progress such as took place in the first post-World War II? (Minsky, 1984, p.16)*

To address these questions we need an economic theory which explains why our economy is sometimes stable and sometimes unstable. However, in the neoclassical synthesis of the Solow Model, instability cannot happen as a normal result of the economic process and therefore, does not qualify to formalize Minsky's ideas. In the Solow Model, a serious depression cannot occur as a result of internal operations of the economy, but only as the result of policy errors or of an exogenous shock. However, 'if a theory is to guide a policy that aims at controlling or preventing an event, the event must be possible within the theory.'(Minsky, 1984, p.16)

Minsky had never sought to build a mathematical model of rigorous evidence to support his basic positions, an effort which may had led him to question, modify, or even strengthen his perspective (Dymski and Pollin, 2010). Nor had he attempted to test whether any alternative framework may explain the patterns he describes equally well. Working from the Wall Street paradigm, Minsky (ibid) first sees a network of financial institutions and cash flows, and then a production and distribution mechanism. According to Dymski and Pollin (2010), Minsky relegates to a second order of concern questions located within the production and distribution mechanisms that are not only intrinsically vital, but also directly related to the problems of finance, instability, and crisis that are central to his vision.

Distributional questions, as reflected in the determination of wage rates and markups, have major macroeconomic implications. Among other things, they will affect realized profit flows and the expected level of profitability. Distributional change will affect profit expectations and investment of two distinct and possibly countervailing effects. The first effect is that capitalists will benefit immediately from a rising of the profits share out of national income. In addition, after a lag, capitalists also benefit from the demand side from a high aggregate wage bill, even if the relative share of national income favors wages over profits.

According to Fazzari, the sense that one gets from conversation with Hyman Minsky is that modern capitalism, in what he calls 'financially-sophisticated economies', is a complex and dynamic system. Its essential features cannot be captured in a static framework, nor can one abstract from finance still have a model that describes modern reality. The endogenous dynamics of the system have the potential to break out into instability. Wise policy intervention is essential to containing the instability. Such intervention may lead to acceptable behavior for a time, but the risk remains that the system will overcome any particular structure of institutional checks.

In Minsky's interpretation of Keynes, capitalism is inherently unstable because of the volatility of private investment. Private investment drives the capitalist economy; its volatility is therefore the root cause of fluctuations in aggregate demand and employment. Investment is more volatile than other components of aggregate demand because it is based on investors' subjective evaluation of the future, and in particular on their expectations of future cash flows from profits.

The growth model introduced by R. Day, T. Lin and W. Tse (1992) seems to reflect Minsky's ideas on a broad basis since irregularities and instability is not an artifact, but a possibility inherent in the structure of the model. In addition, it allows the agents to form expectations about the future values of wages and rates of return, a feature that captures how real world operates and as stated before, is the root of volatility in investment in a capitalist economy.

Minsky argued that economic instability, unemployment and unfair distribution are inherent of capitalism, and thus he advocated a big-government capitalism. Any mathematical formalization of Minsky's ideas should have government intervention as a cornerstone. It is for this reason that we will extend the growth model developed by Lin, Day and Tse (1992) from a model of *laissez faire* capitalism, where government plays an insignificant role in the economy, to a *big government* capitalist model, incorporating as well expectation dynamics that reflect Minsky's basic idea of the Hedge-Speculative-Ponzi finance.

3.2 Unproductive Government Spending

> *The primary source of cash for households is wages, for business firms it is gross profits and for government units it is taxes. Household wage income, business profit flow, and government tax receipts are related to the performance of the economy. (Minsky, 1984, p.21)*

We introduce government in the competitive market economy. We first assume that government spends resources without contributing to production or capital accumulation. The incorporation of unproductive government spending is grounded on welfare programs, as well as income inequalities. Households around the poverty line cannot be assumed to face any consumption-saving trade off. Another interpretation for the incorporation of an unproductive government is the case where there is no correlation between public infrastructure spending program and productivity growth. Since the goal of the project is to testify whether government can stabilize an unstable economy, the model should consider all extreme cases.

We assume that government spending is financed with proportional income taxation, at rate $\tau \in [0, 1]$. Thus the government spending absorbs a fraction τ of aggregate output

$$g_t = \tau y_t. \tag{3.2.1}$$

The new capital accumulation identity becomes

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + y_t - c_t - g_t] \tag{3.2.2}$$

The disposable income of households is now $(1-\tau)y_t$, and thus consumption is bounded between

$$0 \leq c \leq (1-\tau)y. \tag{3.2.3}$$

Although the majority of growth models assume a lump sum tax without any separation between wage and capital returns, the distribution of income in the model allows us to take a step closer to the real world economy. The distinction of the income tax rate between wage tax and tax on capital allows us to testify the validity of Minsky's assumption that a more stable consumption, possibly through a higher tax rate on wage, would eliminate the instability of the economy. Furthermore, it provides us more specific policy tools, which might be important, without perplexing the model.

Thus, we assume that the rate of return on capital is taxed wih a rate α.The after-tax real rate of interest is

$$r_\tau = f'(k)(1-\alpha) = \frac{\beta y}{k}(1-\alpha), \tag{3.2.4}$$

while the wage is taxed with a rate x and the after-tax wage is give by

$$w_\tau = (1-x)(y-rk) = (1-x)(1-\beta)y. \tag{3.2.5}$$

The disposable income for a generation in terms of wages and dividends is

$$(1-\tau)y = w_\tau + r_\tau k = (1-x)(1-\beta)y + \beta y(1-\alpha). \tag{3.2.6}$$

Therefore the relationship between the tax rate on capital and wage is

$$\alpha(x) = \frac{\tau-(1-\beta)x}{\beta} \quad \text{and} \quad x(\alpha) = \frac{\tau-\beta\alpha}{1-\beta} \tag{3.2.7}$$

or respectively between income tax and wage/rate on capital tax

$$\tau = (1-\beta)x + \beta\alpha. \tag{3.2.8}$$

Since none of the tax rates can be negative, we deduce that $\tau \in [0,1]$, $\alpha \in [0, \frac{\tau}{\beta}] \cap [0,1]$ and $x \in [0, \frac{\tau}{1-\beta}] \cap [0,1]$.

Maximizing the utility function, subject to the consumption constraint, we derive the generation's consumption function[1]

$$c(k) = \begin{cases} \dfrac{1}{1+\gamma}[(1-\delta)k + Bk^\beta(1-\tau) + \dfrac{(1+n)(1-\beta)(1-x)Bk^\beta}{\beta Bk^{\beta-1}(1-\alpha)-(n-\delta)}] & k \in K^s, \\ \\ Bk^\beta(1-\tau) & k \in K^d. \end{cases} \tag{3.2.9}$$

The corresponding capital accumulation identity is

$$k_{t+1} = \theta(k) = \begin{cases} \dfrac{1}{1+n}\dfrac{\gamma}{1+\gamma}[(1-\delta)k_t + Bk_t^\beta(1-\tau)] - \dfrac{1}{1+\gamma}\dfrac{(1-\beta)(1-x)Bk_t^\beta}{\beta Bk_t^{\beta-1}(1-\alpha)-(n+\delta)} & k \in K^s, \\ \\ \dfrac{1-\delta}{1+n}k_t & k \in K^d. \end{cases} \tag{3.2.10}$$

Bifurcation analysis of the new capital accumulation identity is more complex than that of the basic model. The existence of three parameters (α, x and γ) implies that the parameter space is defined by a higher dimensional surface instead of a curve. The introduction

[1] See Appendix A for the derivations

of tax rates in the model complexes the equations and thus analytical results about the parameter space are extremely arduous and not straightfoward. Numerical simulations and approximation techniques are necessary and are employed in Chapter 4 for the analysis of regions of stability.

Keeping the values of output elasticity, population growth and depreciation rate the same as in the numerical example in Section 2.3.2, we assume that the government imposes a 30% tax rate on wages ($x = .3$), while it taxes capital by 10% ($\alpha = .1$). It can be inferred from the graphs in Figures 3.2.2 and Figure 3.2.3 that the introduction of government and the imposition of tax rates dampens the cyclic behavior of the system, since bifurcation occur for higher values of future weight. Although output and thus private consumption decreased in absolute terms, total consumption (Private Consumption + Government Spending) as % of GDP remained relatively constant. However, we cannot have a clear view about the effect of government's incorporation in the model, neither can we advocate for a tax policy, unless we investigate the effect on consumption and growth more thoroughly.

A comparison of the levels of production and consumption between the two frameworks (Basic Model and Unproductive Government spending) is necessary, before any valid conclusion is deduced about whether the tax rate is beneficial for the society or not. Although not always accurate, the greater the output and consumption the more well off the citizens are.

3.3 Productive Government Spending

Ashauer (1998) asserted that there is a strong correlation between public capital and productivity and assumed that the public capital acts as an input to private production. Barro (1990) developed a simple model of endogenous growth in which the government uses tax revenue to finance government expenditure and this expenditure enters into the

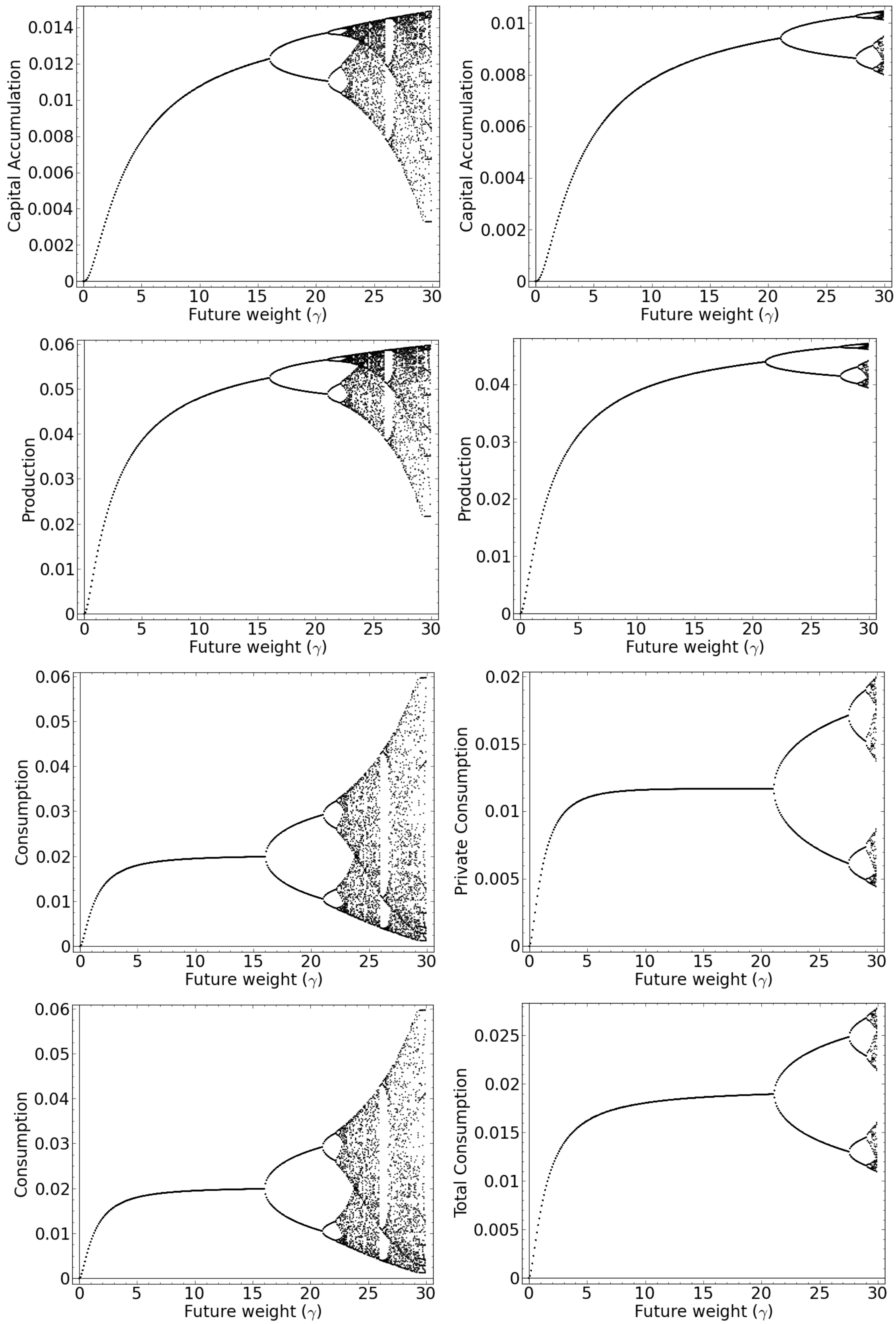

Figure 3.2.1. Bifurcation Diagrams of macroeconomic variables: Right side graphs correspond to the Basic Model ($\delta = .25$, $n = 2.4$, $x = 0$, $\alpha = 0$); left side graphs correspond to the Unproductive Government Spending framework ($\delta = .25$, $n = 2.4$, $x = .3$, $\alpha = .1$)

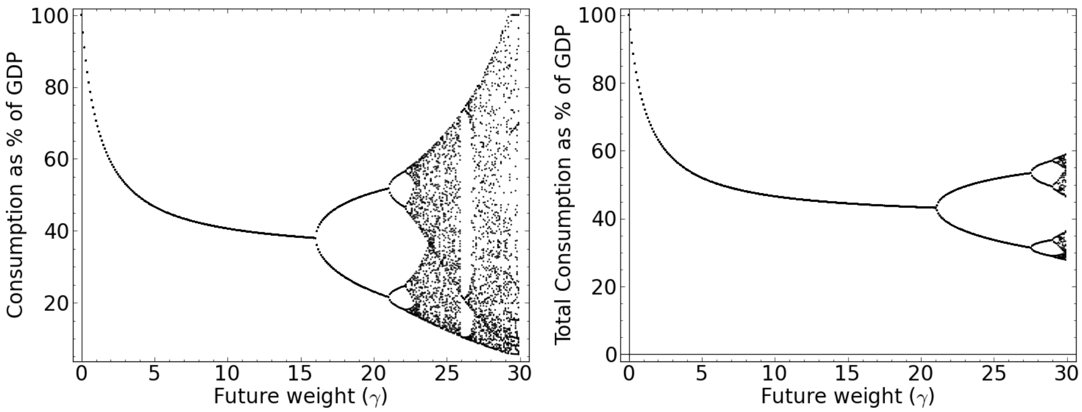

Figure 3.2.2. Bifurcation Diagrams of macroeconomic variables: Right side graphs correspond to the Basic Model ($\delta = .25$, $n = 2.4$, $x = 0$, $\alpha = 0$); left side graphs correspond to the Unproductive Government Spending framework ($\delta = .25$, $n = 2.4$, $x = .3$, $\alpha = .1$)

production function as a productive input. Barro regarded public services, a flow variable, as a productive input in private production.

Based on Aschauer's idea and following Barro's line, we now reuse our model assuming that the current flows of government spending, rather than the services of the existing stocks, generates additions to production. An alternative approach would be to allow the government to accumulate stocks, such as roads, that are likely to impact the economy through their accumulated stock, rather than their current flow[2].

We assume again that government spending is financed with proportional income taxation, at rate $\tau \in [0, 1]$. Thus the government spending absorbs a fraction τ of aggregate output

$$g_t = \tau y_t. \tag{3.3.1}$$

[2]Based on Barro's (1990) endogenous model, Futagami, Morita and Shibata (1993) developed a model that included two state variables: private and public capital stocks. The rationalization of their modeling strategy of incorporating public capital lies on the fact that many public infrastructures, such as highways, airports, and electrical and gas facilities are stock variables in nature. As they argued, in the theoretical literature on public investment, stock of public capital, instead of the flow of public services, is the productive input to private production. However, the structure of the model adopted in the project restricts the analysis of this case. Appendix A includes some preliminary research on this area.

We define $i \in (0,1]$ to be the portion of government spending that is productive. Then $(1-i)$ represents the portion that government spends without contributing to production. The separation between the productive and unproductive spending is crucial in various ways.

First, evidence from empirical studies shows that not all public capital has the same contribution to the production function. For example, over half of the public capital stock is composed of a core infrastructure (streets and highways, water supply, sewers, and publicly owned electrical and gas facilities), which might be expected to function as an input to the private production function more closely than does the total public capital stock. Thus an increase in the output elasticity of public capital should be accompanied by a decrease in i, indicating that a narrower definition of public capital is used.

A second reason for this separation is based on the real function of government spending. Once again, welfare programs and military expenses despite the fact that they raise total consumption, they have no effect on productivity.

Finally, a shift in the government budget, represented by a change in i, can be a policy tool to either raise total consumption through increased unproductive government spending, or increase the productivity of private sector through increased spending in infrastructure programs.

The production is now given by

$$y_t = Bk_t^{\beta}(ig_t)^{\zeta}. \tag{3.3.2}$$

Substituting $g_t = \tau y_t$ into (3.3.2) and solving for y_t, we infer

$$(3.3.2) \Rightarrow y_t^{1-\zeta} = Bk_t^{\beta}(i\tau)^{\zeta}$$

$$y_t = B^{\frac{1}{1-\zeta}} k_t^{\frac{\beta}{1-\zeta}} (i\tau)^{\frac{\zeta}{1-\zeta}}$$

$$y_t = B' k_t^{\beta'} (i\tau)^{\zeta'},$$

where $B' \equiv B^{\frac{1}{1-\zeta}}$, $\overline{\beta} \equiv \frac{\beta}{1-\zeta}$ and $\overline{\zeta} \equiv \frac{\zeta}{1-\zeta}$.

Since B', i, and τ are considered as constants, let $\overline{B} \equiv B'(i\tau)^{\overline{\zeta}}$. Then the production function becomes

$$y_t = \overline{B} k_t^{\overline{\beta}} \tag{3.3.3}$$

Following the exact same process and reasoning as in Section 3.2, we deduce that

Current Consumption

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{3.3.4}$$

$$= \begin{cases} \frac{1}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}} + \frac{(1+n)(1-x)(1-\overline{\beta})\overline{B}k^{\overline{\beta}}}{(1-\alpha)\overline{\beta}\,\overline{B}k^{\overline{\beta}-1} - (n+\delta)}] & k \in K^s, \\ \\ \overline{B}k^{\overline{\beta}}(1-\tau) & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \frac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \\ \frac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{3.3.5}$$

$$\begin{cases} \frac{1}{1+n}\frac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}}] - \frac{1}{1+\gamma}\frac{(1-\overline{\beta})(1-x)\overline{B}k^{\overline{\beta}}}{(1-\alpha)\overline{\beta}\,\overline{B}k^{\overline{\beta}-1} - (n+\delta)} & k \in K^s, \\ \\ \frac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

The analysis of the behavior of the system for various values of the parameters, α, x, ι and ζ is presented in Chapter 4. For now, we simply illustrate the potential impact that productive government spending can have on the economy. As an example, we use Munnell's estimated values for output elasticity of public capital ($\zeta = .15$) and private capital ($\beta = .31$) and we also assume that the tax rate on wages and capital are 20% and 10% respectively. Figure 3.3.1 displays some macroeconomic variables for the specific parameter values. Without any thorough analysis we can infer that the incorporation of public capital as a productive factor is non trivial. It affects both the growth and the stability of the economy. A detailed and formal analysis is presented in Chapter 4.

3.4 Dynamic Expectations

> *To understand the behavior of our economy it is necessary to integrate financial relations into an explanation of employment, income and prices.(Minsky, 1984, p.17)*
>
> *The trade cycles and economic growth are the resultant of particular attitudes of entrepreneurs-more precisely, of the volatility of entrepreneurial expectations. (Hicks and Harrods)*

In the final section, we attempt to incorporate Minsky's Financial Instability Hypothesis (F.I.H.), through expectation dynamics. In the basic model, we assumed that the current value of rate of return is used as a basis for the consumption-savings-bequest tradeoff. Now, we presume that the expectation dynamics may be related to the consumption decision. It could be that the expected pay-offs themselves are dynamic and exacerbate the instability of the system.

Minsky was not the only economist who stressed the importance of 'entrepreneurial' expectations. Hicks and Harrod suggested that that 'an economy is likely to grow at the rate at which its business men expect it to grow.' This proposition is very strict and it limits the power of economic analysis. Thus we will argue that economic growth and cycles are significantly affected by the attitudes of entrepreneurs and we will check the validity of this looser claim within the context of our model.

The main idea for the formalization of F.I.H. in the model is that during euphoric periods the optimism of markets is expressed as an increase in expectations for the future rate of returns on capital assets. This optimism increases investment, potentially leading to a speculative investment boom, and occurs until downturned economic activity does not validate the high expected rate of returns on capital.

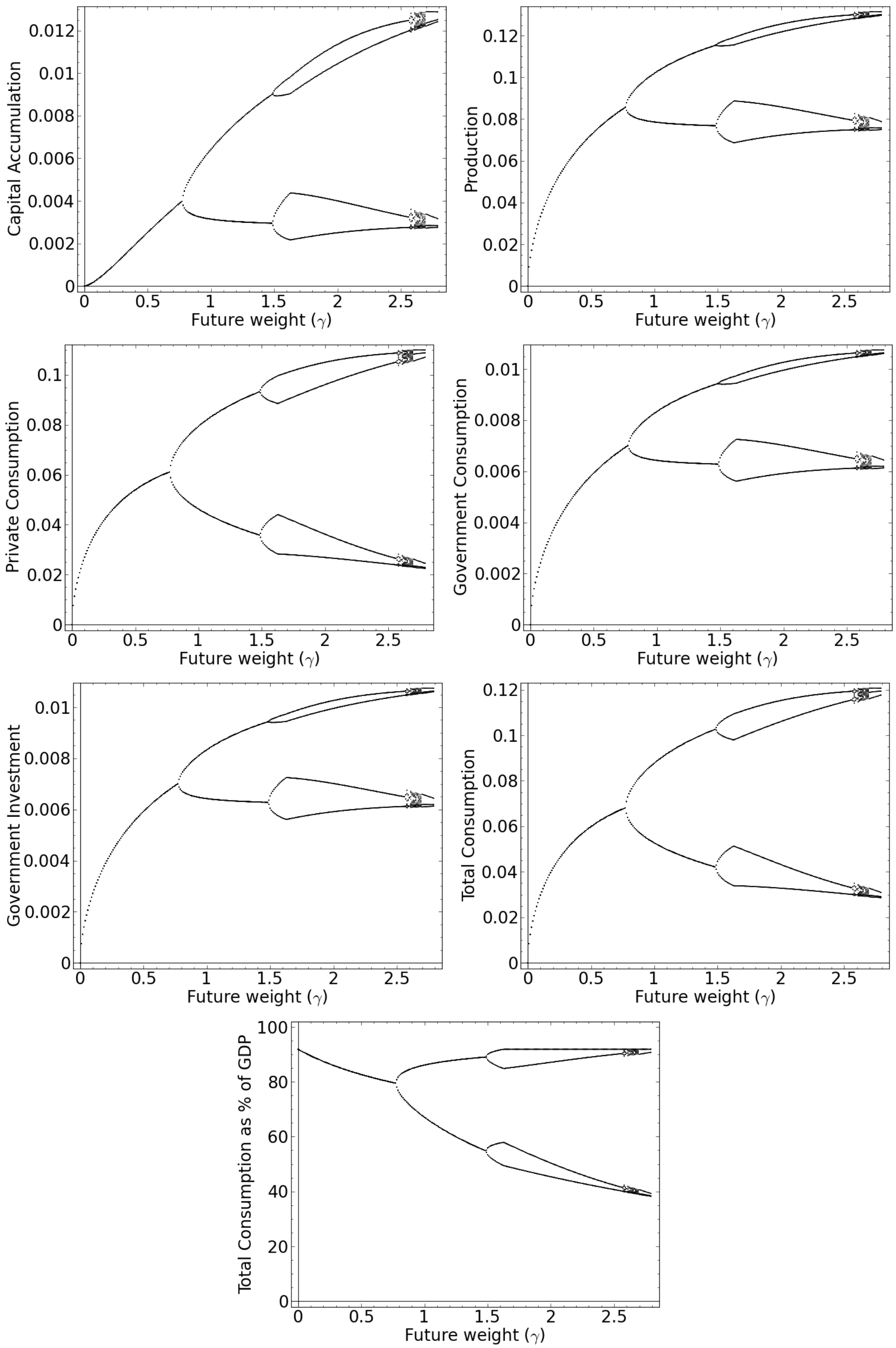

Figure 3.3.1. Bifurcation Diagrams: Productive Government Spending ($\delta = .25$, $n = 2.4$, $\beta = .31$, $\zeta = .15$). Although it cannot be seen in the graph, chaotic behavior occurs for $\gamma \approx 2.6$.

The expectation dynamics of the model will follow a simple and straightfoward rule; optimism feeds optimism and pessimism feeds pessimism. However, we will assume that the expectations for tax rates and the budget deficit will remain constant[3].

Expectation Dynamics

$$r_t^1 = \left(1 + \frac{r_t - r_{t-1}}{r_{t-1}}\right) r_t \tag{3.4.1}$$

$$\rho_t^1 = \left(1 + \frac{\rho_t - \rho_{t-1}}{\rho_t}\right) \rho_t \tag{3.4.2}$$

$$w_t^1 = \left(1 + \frac{w_t - w_{t-1}}{w_{t-1}}\right) w_t \tag{3.4.3}$$

$$\tau_t^1 = \tau_t, \tag{3.4.4}$$

which implies that $\alpha_t^1 = \alpha_t$ and $x_t^1 = x_t$.

This simple version of expectation dynamics captures the intuition of the more complex models and may better explain the impact of expectations on real economy. For example, if the real wage or the rate of return on capital increases one year, then, according to the dynamics of the model, the agents would be optimistic about the future values of the variables and they would expect that the real wage or the rate of return would further increase in the following year.

The equations of current consumption and capital accumulation, are derived[4] following the same process and reasoning as the ones in the basic model presented in Section 2.3.1.

Current Consumption

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{3.4.5}$$

$$= \begin{cases} \frac{1}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}} \\ + \frac{(1+n)(1-x)(1-\overline{\beta})(\overline{B}k^{\overline{\beta}})^2}{[(1-\alpha)\overline{\beta}\,\overline{B}(k^{\overline{\beta}-1}) - (n+\delta)]^2} \frac{((1-\alpha)\overline{\beta}\,\overline{B}k_{-1}^{\overline{\beta}-1} - (n+\delta))}{\overline{B}k_{-1}^{\overline{\beta}}}] & k \in K^s, \\ \\ \overline{B}k^{\overline{\beta}}(1-\tau) & k \in K^d. \end{cases}$$

[3]Among the interesting features and capabilities of the model developed in this project is the possibility to investigate the impact of anticipated versus unanticipated fiscal policies.

[4]See Appendix A for a thorough analysis of the derivations

where k_{-1} is the capital accumulation in the previous year.

Finally, the Capital Accumulation Identity is

$$k_{t+1} = \theta(k) = \begin{cases} \dfrac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \\ \dfrac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{3.4.6}$$

$$\begin{cases} \dfrac{1}{1+n}\dfrac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}}] \\ -\dfrac{1}{1+\gamma}\dfrac{(1-x)(1-\overline{\beta})(\overline{B}k^{\overline{\beta}})^2}{[(1-\alpha)\overline{\beta}\ \overline{B}(k^{\overline{\beta}-1}) - (n+\delta)]^2}\dfrac{((1-\alpha)\overline{\beta}\ \overline{B}k_{-1}^{\overline{\beta}-1} - (n+\delta))}{\overline{B}k_{-1}^{\overline{\beta}}} & k \in K^s, \\ \\ \dfrac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

Figure 3.4.1 and 3.4.2 show the bifurcation diagram for the capital accumulation in the Basic Model and the Productive Government Spending framework, adjusted to incorporate the expectation dynamics introduced in this section. It is clearly evident that dynamic expectations open the door to crisis and serious output fluctuations; unstable chaotic trajectories occur for all possible values of the future weight (γ). The questions that arises are if and how government intervention can mitigate this instability.

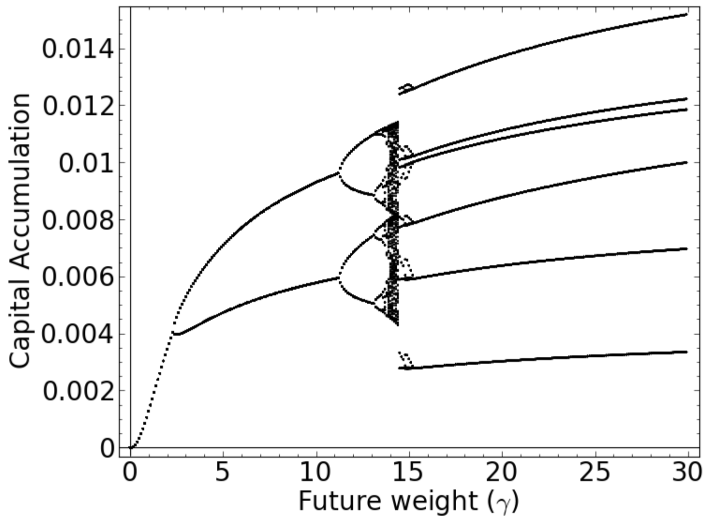

Figure 3.4.1. Bifurcation Diagram: Basic Model framework with Expectations Dynamics ($\beta = .67$, $\zeta = 0$)

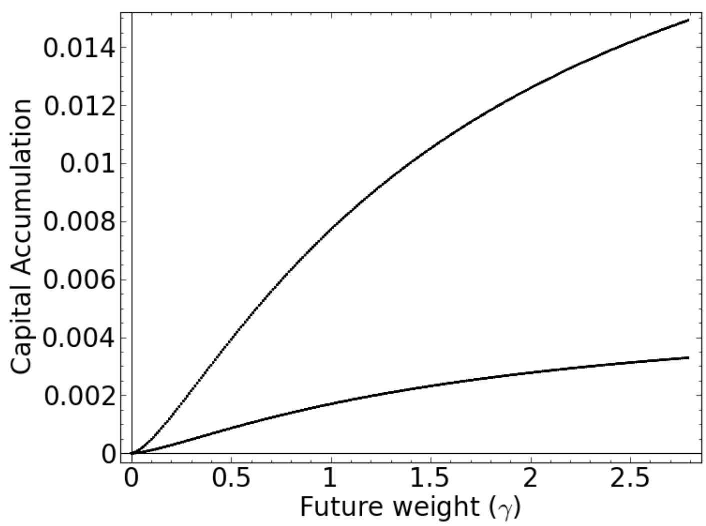

Figure 3.4.2. Bifurcation Diagram: Productive Government Spending framework with Expectation Dynamics ($\beta = .31$, $\zeta = .15$)

4
Economic Policy

The effect of government expenditure on aggregate economic activity and overall economic welfare is a subject with a long and controversial history. The emergence of large fiscal and current account deficits in the United States during the 1980s stimulated interest in the theoretical analysis of fiscal policies. A central concern in traditional discussion of this issue was whether or not and to what extent public expenditure crowds out private activity. A relative question was whether government expenditure could serve as an effective policy tool in mitigating the intensity of business cycle fluctuations.

Recent literature addresses two main issues: the real effects of changes in government expenditure policy and issues pertaining to debt and tax-financing policies. The contribution of this project lies in its equally distributed focus on the effects of government policies on economic performance and on the stability of an inherently unstable economy.

Minsky advocated for a Big Government as a policy to smooth cyclical instability. The questions that are addressed in this chapter are

- Can government intervention stabilize an unstable economy?
- What form should government spending take: productive or unproductive?

- How do different forms and different ways of financing affect economic growth, performance and stability?

- How is optimal government intervention affected by other parameters of the economy, such as the output elasticity of the private or public sector?

4.1 Literature Review

As has been stated by Atkinson and Stiglitz (1980), we should consider how taxation may discourage or encourage the long-term rate of growth of the economy. According to economic theory, an increase in the tax rates creates two opposing forces. In the long run, the negative effect comes from the fact that an increase in tax rate reduces the level of private disposable income and thereby the level of private investment. The positive effect is due to the fact that an increase in the tax rate raises the ration of public to private capital and thus the marginal productivity of private capital, assuming a strong positive relationship between the two. Assuming that government spending is unproductive and goes toward consumption, in the short run, the rise in the tax rate has a positive effect on consumption. The change in tax rate consists of two effects: the intertemporal substitution effect and the income effect. Since the growth rate is reduced by the tax change, the income effect is negative. However, the intertemporal substitution effect increases present consumption at expense of future consumption. The net effect depends on the parameters of the model and thus no valid generalized argument can be made.

While there are many potential mechanisms to raise productivity growth, most turn on boosting the rate of capital accumulation. Within this context, the role of public capital has been widely discussed during the postwar period. The debate now addresses the effects of public infrastructure investment on private sector productivity. Many question the mere existence of a relationship between public capital and productivity. A logical case can be made that public investment may well respond to changes in private economy instead of initiating them; slower growth in productivity and per capita income reduces tax revenue and thus induces the government at all levels to reduce spending on public capital projects. Others simply state that though a relationship may actually exist, it is impossible to quantify the magnitude or establish a precise level of correlation. In addition, they argue that although public infrastructure investment may contribute to the long-term competitiveness, deficit reduction and public investment are mutually exclusive strategies.

Empirical results, though, offer the possibility of a direct channel by which fiscal policy can affect national investment and national productivity growth. According to Aschauer (1993), public infrastructure (streets and highways, mass transit, and water and sewer systems) should be considered as a factor of production. He claims that a decrease in public capital accumulation may be responsible for explaining "a very substantial portion of the productivity slowdown and cross-country differences in productivity growth"(Aschauer, 1993, p.10). According to Aschauer (1998), the output elasticity of private capital is between .3 and .35. His optimistic results initiated a wave of criticism. Aschauer (ibid) was accused of omitting variables that may better explain the productivity slowdown, and he was criticized for dismissing the prospect of reverse causation. The controversy about the existence of a strong correlation between public and capital productivity was fueled once again. Arron (1990), for example, argued that "the association between private output and stocks of public capital may also be coincidental." Aschuer (1993) responded that "the lack of theoretical motivation is viewed as something of a virtue. Without any theoretical rationale, any data series becomes admissible, and the ability of researchers to find one or more data series to accomplish their goal will be constrained only by the extent of their desire to debunk a particular theory" (Aschauer, 1993, p.20).

Apart from Aschauer, other empirical studies (e.g. Munnell (1990) [1] and Eisner(1991)[2]) also support the importance of public capital in private production and find a positive relationship between public capital and marginal productivity of the private sector.

Among the more mediocre critics of infrastructure accumulation as a key determinant of productivity growth is Holtz-Eakin (1993). His argument is based on the application of different statistical approaches, [3] which yield results that a broad-based spending program for additional infrastructure is unlikely to augment economy-wide productivity growth. Increased public capital might increase output and even a one-half percentage point increase in growth can make a large difference in living standards. However, the point is that it takes decades for the effect to accumulate. In the short run, there will probably be no dramatic turnaround, and therefore, a large scale infrastructure spending program will not have any appreciable effect on productivity growth.According to Holtz-Eakin (1993), the best infrastructure program does not focus on new spending. Instead, the top priority should be to "get the prices right"(ibid) by charging user fees for infrastructure services.

The difficulty comparing different empirical results arises due to the lack of a common definition of the public capital stock across studies. In some cases, the public capital stock is limited to highways, while in others it is more inclusive, including mass transit, airports, and water and sewer systems as well. Thus, it is inappropriate to compare elasticities of different types of public capital. However, it is quite safe to claim that the estimates of the output elasticities of public capital show a fairly systematic relationship with the level of public investment. The estimates tend to be larger at the federal level than at the state level, and larger at the state level than the municipal level.

[1] Munnell estimated an output elasticity of public capital of .15, while the output elasticity of private capital is .31.

[2] Assuming a Cobb-Douglas production function and using cross section data, Eisner found that the output elasticity of highways equals .06, of water and sewer systems .12, and other public capital .01.

[3] Only the use of OLS showed a strong causal relationship between public capital and private productivity. Other methods such as LONG, FIX, GLS, IV and HNR show weak or even negative relationship.

In the theoretical models that were developed by various economists, the incorporation of the government sector concerns only the economic performance of the economy and not its stability. After all, in the majority of them the economy follows a monotonic growth path.

Buiter (1987) was among the first who tried to address the question about the choice between borrowing and tax-financing of a given level of government expenditure. His main conclusion was that as long as the returns to infrastructure investment exceed the growth rate of the economy, an increase in public investment financed through the reduction in either public or private consumption, will favorably tilt the national consumption profile toward the future.

Futagami, Morita and Shibata (1993) found that the long-run growth rate of market equilibrium is maximized when the rate of income tax equals the elasticity of output with respect to public capital in the case where this elasticity is constant.

In the analysis of Turnovsky and Pen (1991) temporary fiscal expansion can give rise to permanent effects. In a dynamic endogenous model the steady state depends upon the initial stock of assets. A temporary fiscal expansion can alter these initial conditions and thus, repeated shocks can generate random walk behavior and hysteresis.

Implications of temporary shocks though have been given little attention. This is a crucial issue, especially in the light of recent interest pertaining to hysteresis and the random walk behavior of real variables such as output. Of particular relevance is the impact of such shocks on the welfare of the representative agent in the economy.

4.2 Analysis

A good analysis should conduct analyses over the full range of plausible values of key parameters and their interactions, to access how impacts change *in response to changes in key parameters. (EC handbook for extended impact assessment)*

Using the different variations of the model developed in Chapter 3, we investigate the effects of different government budgets and financing paths on a number of key macroeconomic variables, such as the rate of capital accumulation and the level of consumption, as well as on stability.

The controversy over the contribution of public capital on economic growth dictates that our analysis should be conducted over a wide range of possible values for the output elasticities of the private and public sectors. Furthermore, since our model is a theoretical representation of a real economy, we should consider the extreme case that no correlation between public capital and productivity exists, using the unproductive government spending framework.

The goal of the project is to investigate whether a big and proactive government can stabilize an unstable economy. Thus, the analysis of the instability of the modeled economy will focus on the parameters that can be affected by government, i.e. the tax rate on wages (x), the tax rate on capital (α) and the proportion that government spends productively (i) or unproductively ($1 - i$).

The analysis of the stability of the economy for the different values of our parameters will be based on the concept of the Lyapunov exponent. In mathematics, the Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories and determines a notion of predictability-stability for a dynamical system. A positive value is taken as an indication that the system is chaotic, while nonpositive values indicate a predictability in the system. The more negative the value the more stable the system is.

In order to find the relationship between the regions of stability and the exogenous parameters of our model, (i.e. output elasticity of private sector, output elasticity of public sector, rate of depreciation and population growth rate), we employ the mathematical toolbox presented in Section 2.1.

Policy proposals though should not be based solely on the stability of the economy. In the example in section 3.2, we saw that the stability of the economy comes at the expense of its reduced dynamism. Any decision about the appropriate policy should also take into consideration the capital accumulation and the level of consumption. Before we argue in favor of any particular level of tax rates we need to see what are the levels of consumption and the output for different tax rates and spending programs.

However, the complete analysis of dynamical models with a large number of parameters is difficult. As a consequence, if analytical results are not obtainable we will resort to numerical simulations on the basis of approximation techniques or dynamic programming algorithms.

4.2.1 Basic Model

As we saw in Section 2.2, in the real growth model developed by Lin, Tse and Day (1992), the trajectories can either converge monotonically or fluctuate periodically; even non-periodic behavior can occur. Bifurcation analysis of the capital accumulation identity illustrates the general theoretical possibilities of the system for different values of the parameters.

To examine the stability of the system, the derivative of capital accumulation ($\theta'(k)$) must be evaluated at the steady state. Recall from Theorem 2.1.4 that the value of the derivative indicates the stability of a point. By taking the derivative of Equation 2.3.18,

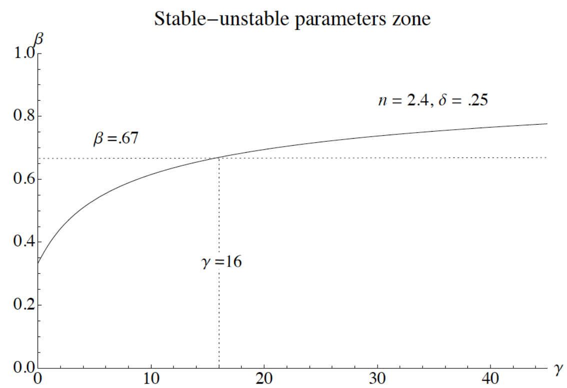

Figure 4.2.1. Bifurcation Boundaries

we find that after tedious mathematical derivations,[4]

$$\theta'(\overline{k}) = 1 - \frac{(1-\beta)\gamma^2}{(1+n)^2(1+\gamma)}[\frac{1-\beta}{\beta}(\overline{r})^2 + \frac{1+n}{\gamma}\overline{r}], \tag{4.2.1}$$

where $\overline{k}$ and $\overline{r}$ are the steady state capital-labor ratio and rate of return respectively.

The bifurcation diagram in Figure (4.2.1) illustrates the points in the parameter space for the numerical example in Section 2.3.2 that give rise to stable and unstable steady states. Points on the curve are the solutions of the equation $|\theta'(k)| = 1$. Points above the curve are attracting fixed points ($|\theta'(k)| < 1$), while points below the graph are repelling since $|\theta'(k)| > 1$. The relative position of the vertical line $\beta = .67$ with respect to the curve $|\theta'(k)| = 1$, determines the stability of the steady states. If the line is above the curve, trajectories converge to a steady state; at the intersection the first bifurcation takes place, while as long as the line lies below the curve, cycles and chaos arise.

The bifurcation diagram of capital accumulation in Figure 4.2.2 shows period-doubling bifurcations (points of period one become period two; points of period two become period four, and so on) as γ increases beyond 16. It is not clear though whether some orbits are chaotic or whether the orbits are just periodic of a very high period.

[4]See Appendix A

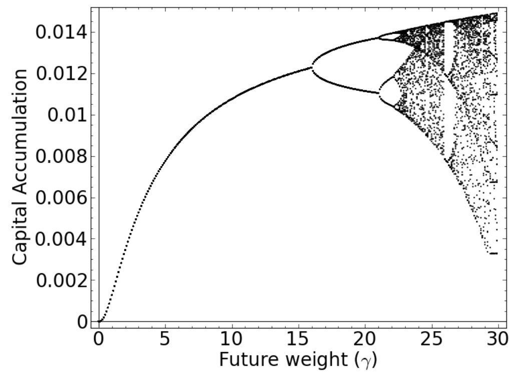

Figure 4.2.2. Capital Accumulation: Basic Model

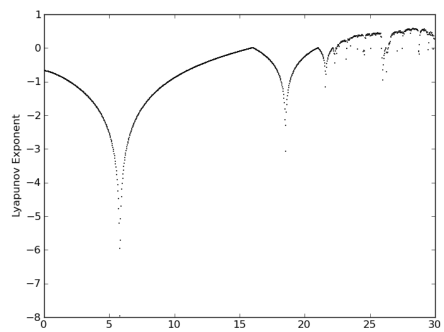

Figure 4.2.3. Lyaponov Exponent: Basic Model

A method often used to determine whether or not a system is chaotic is the Lyapunov exponent.[5] The Lyapunov exponent may be computed for a sample of points near the attractor to obtain an average Lyapunov exponent. Figure 4.2.3 plots the corresponding Lyapunov exponent computed for several values of the parameter γ. The value of the exponent approaches zero as γ approaches 16, and thus period-doubling bifurcation occurs, while it becomes positive for $\gamma \geq 22$, indicating that the economy becomes less stable for high values of future weight and becomes chaotic for values greater than 22.

[5] See Section 2.1.1

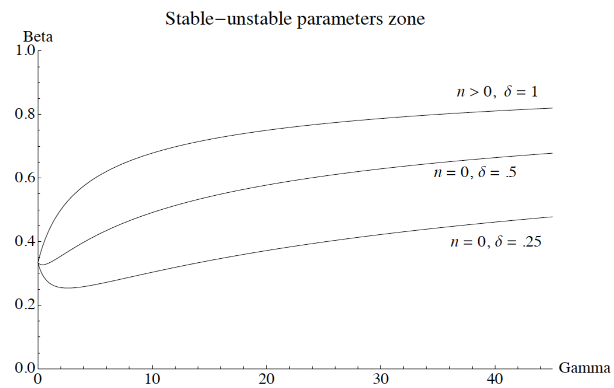

Figure 4.2.4. Stable-unstable parameter zones. Points below the line are unstable, points above the line are stable

Figure 4.2.4 shows stable/unstable regions in the parameter space for different values of n and δ. Again, points below the curve give rise to unstable steady states, while points above the curve give rise to stable ones. The upper curve ($n > 0$, $\delta = 1$) is the upper bound of the boundaries between stable and unstable states. We can deduce that instability is positively related with the value of future weight (negatively related with the time preference), as well as the population and technological growth rates, and the rate of depreciation. Any increase of these parameters shift the curve upward and increases the unstable region. However, instability is negatively related with the capital share on output; an increase in β shifts the vertical line upward and thus bifurcation occurs at a higher future weight. So, the larger the value of n, δ and γ or the smaller the value of β the more likely the system is to be unstable[6].

In general, the Basic Model implies that extrapolative expectations combined with a moderate output elasticity of capital (β) and rapid technological progress(or population growth) (n), has the potential for producing fluctuations in the economy, specifically when the rate of time preference is small (large future weight (γ)). This suggests that countries

[6]A more analytical analysis is presented in Appendix A.1.

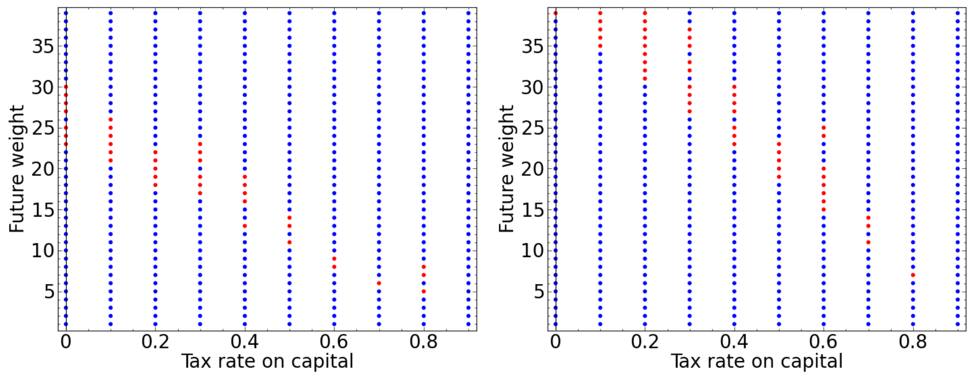

Figure 4.2.5. The Lyaponov Exponent with respect to the tax rate on capital: left graph ($x = 0$), right graph, ($x = 0.4$)

with low levels of consumption and high population growth rates can experience destabilizing forces, unless the capital share on output is substantial.

4.2.2 Unproductive Government Spending

Assuming no relationship between government spending and productivity, government intervention can focus on two targets. One is to stabilize the economy through a mixture of tax rates. The second is to increase the level of current consumption. The potential policy tools that government can employ are the wage tax rate (x) and the tax rate on capital (α).

Figures 4.2.5 and 4.2.6[7] show some evidence that higher taxation on capital returns, ceteris paribus (keeping wage tax rates constant), destabilize the economy (see Figure 4.2.5), while on the contrary, higher tax rates on wages, ceteris paribus (keeping tax rates on capital constant), stabilize the economy (see Figure 4.2.6).

The stabilization effect of wage tax rates appears to be equal in magnitude with the destabilization effect of an equal tax rate on capital. This fact leads to the peculiar result that when the two rates are equal, the case that government applies a simple lump sum tax,

[7]Points that are colored red (blue) have a positive (negative) Lyapunov exponent and thus the corresponding parameter values give rise to chaos (stability).

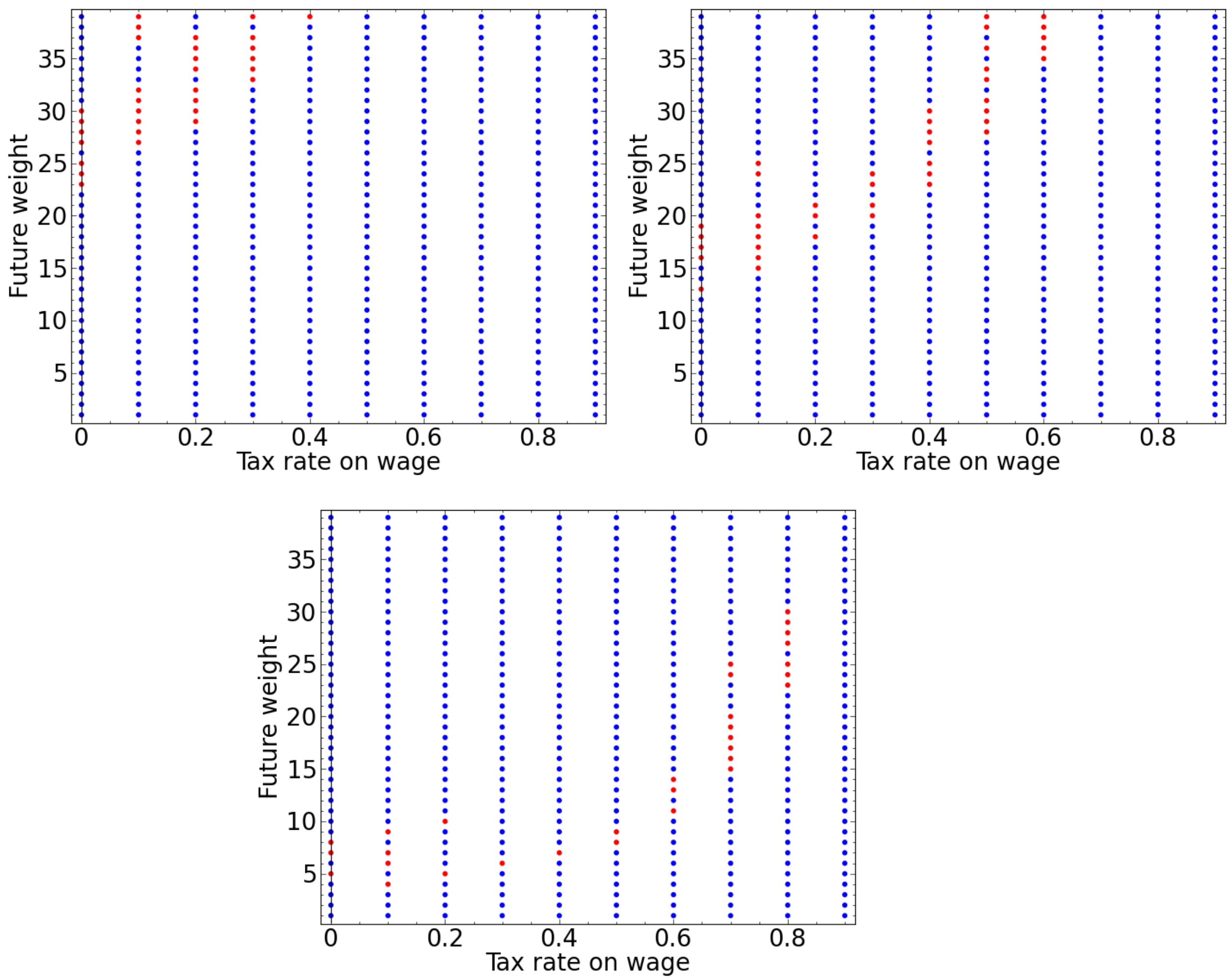

Figure 4.2.6. The Lyaponov Exponent with respect to the tax rate on wage: upper left graph ($\alpha = 0$), upper right graph ($\alpha = 0.4$), lower graph ($\alpha = 0.8$)

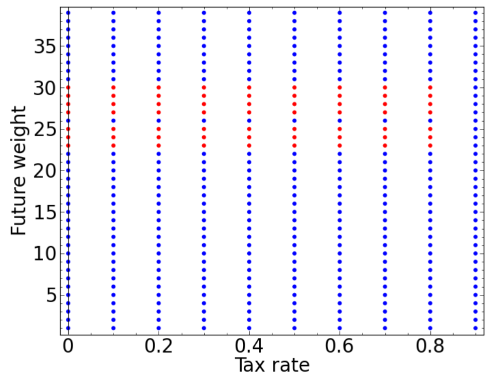

Figure 4.2.7. Lyaponov Exponent – Lump sum Tax ($\tau = \alpha = x$)

there is no effect on the stability of the economy (see Figure 4.2.7). Based on the equal magnitude of stabilization/destabilization effect that the tax rate on wage and capital have, respectively, we make the following conjecture.

Conjecture 4.2.1. *Let $h(k)$ and $\theta(k)$ be defined as the capital accumulation in the Unproductive Government Spending framework and the Basic Model respectively. Then h is topologically conjugate with θ. In addition, if $\alpha = x$, then h the has the same qualitative behavior for the different values of γ*[8] *as θ.*

Therefore the issue of fiscal policy is not the nominal amount of tax; what matters most is the structural content of the intervention. Not all forms of government intervention are consistent with stabilizing the economy, as the evidence from the model illustrates.

Regarding growth, although taxation can smooth instability and constrain the dynamic behavior of the economy, stabilization is achieved at the expense of decreased private consumption; the availability of funds decreases proportionally with respect to the tax rate. The data that describe the trade off between stability and consumption are shown in

[8] A conjugacy is a change of variables that transforms one map into another. The transformation should be a homeomorphism, so that all topological features are presented. This implies that all the qualitative characteristics of the bifurcation diagram will remain the same. The additional argument states that if $\alpha = x$ then the first bifurcation in the Unproductive Government Spending framework will occur for $\gamma \approx 16$, the second for $\gamma \approx 22$. However, the actual values of k for each γ will be lower; the graph will shift upward, since the increased taxation constrains the availability of investment funds. More evidence for the conjecture is provided in the Appendix A.2.1.

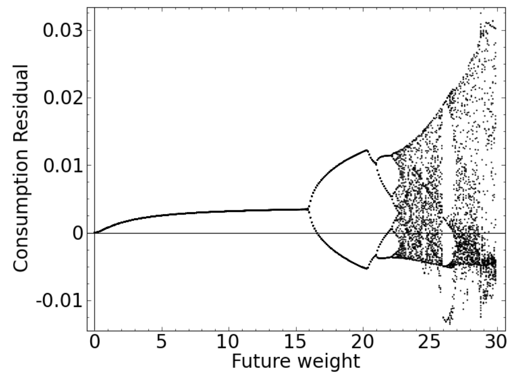

Figure 4.2.8. Private Consumption Residual between Basic Model and Unproductive Government Spending framework: Consumption($\alpha = 0$, $x = 0$)$-$Consumption($\alpha = .0$, $x = .3$)

Figure 4.2.8, which plots the residual of private consumption between no taxes and after taxes. A positive residual indicates that the steady state level of private consumption is higher in a laissez fair capitalist economy, while a negative value indicates the opposite. Over the full range of plausible values of future weight, the residual of private consumption[9] is mostly positive, validating our assumption that the price of stability is some perceived growth. The few points that are below the line are due to the more clustered cycles that arise in the big government framework.

Despite the decreased level of private consumption though, the unproductive nature of government spending increases total consumption and thus the net effect can be relatively small, depending on the magnitude of tax and the time preference of the agents. Figure 4.2.9 plots the residual of total consumption (Private + Government) and provides evidence that total consumption remains relatively constant (residuals are close to zero), at least as long the economy in both frameworks is stable for the corresponding parameter values. Consequently, an increase in tax rates can have a negligible effect on economic per-

[9]Residual of private consumption = Consumption(B, β, n, δ, γ, i, $\alpha = 0$, $x = 0$) $-$ Consumption(B, β, n, δ, γ, i, $\alpha \geq 0$, $x \geq 0$)

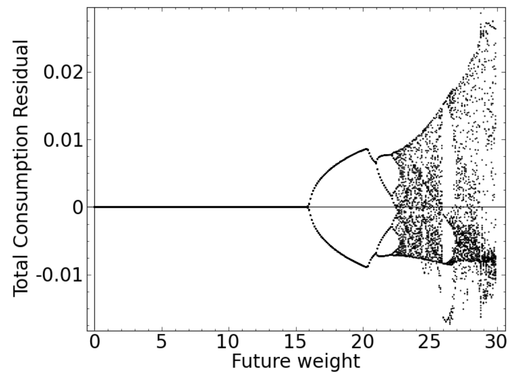

Figure 4.2.9. Total Consumption Residual: Consumption($\alpha = 0$, $x = 0$) − (Private Consumption($\alpha = .0$, $x = .3$) + Unproductive Government Spending($\alpha = .0$, $x = .3$))

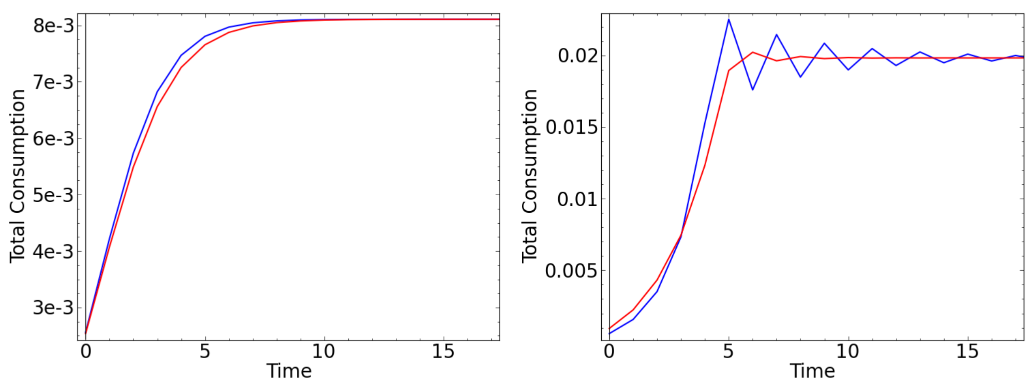

Figure 4.2.10. Transitional Dynamics and taxation. The blue line corresponds to the Basic Model ($x = \alpha = 0$), red line to the Unproductive Government case ($x = .2$, $\alpha = 0$). In the left graph $\gamma = 5$, right graph $\gamma = 14$

formance, assuming no frictions (i.e. payments to service debt or current account deficits) that would alter the provision of the total amount on domestic consumption.

Of course, this result cannot adequately reflect the full trade off between stability and welfare. The analysis of transitional dynamics[10] in Figure 4.2.10 shows that total consumption and the other macroeconomic variables (output and capital accumulation), follow different trajectories in the two frameworks, before they converge at the same steady state. If

[10] By transitional dynamics we mean the path (trajectory) that a variable follows with respect to time. It differs from bifurcation diagram, which shows only the steady states and ignores the process of convergence, in the sense that it provides full information about the evolution of the variable, including the steady state.

the agents have a high time preference (small value of γ), the level of consumption is high (savings are low) and thus the tax rate decreases even further the availability of investment funds. As a result, the economy follows a dumped trajectory comparing to a more 'liberal' environment. However, as agents decrease their time preferences (γ increases) and increase the level of savings, total consumption increases initially by the unproductive spending of government. In general, the higher the tax rate on wages the smaller the perturbation of the economy before it settles to the steady state. Thus, under the realistic assumption that an economy never settles down to its steady state, government intervention can constrain the dynamic behavior of the economy at a cost of some welfare.

The effect of unproductive government spending on consumption, as well as the welfare of the society depends on the amount of aggregate tax rate (τ), which in turn depends not only on the magnitude of tax rates, but also the magnitude of productivity rates. Recall from equation 3.2.8 that

$$\tau = (1-\beta)x + \beta\alpha.$$

Thus, given a high rate of output elasticity of capital (β), an increase of capital tax rate increases government revenue, and therefore unproductive spending[11] and total consumption more than an equal increase of wage tax rate would. However, the increased taxation on capital slows down the expansion of the economy, decreasing the steady state levels of both, output and consumption. Therefore, an increase in the tax rate on capital, assuming a high output elasticity, increases total consumption at the transitional path, but results in a reduced steady state value (see Figure 4.2.11).

Finally, the results deduced from the Basic Model, are still valid for the Unproductive Government Spending framework. The stability of the economy is still positively related with the value of output elasticity (β), while it is negatively related with the value of

[11] Conditional always on the value of future weight, because instability can arise for smaller values of γ.

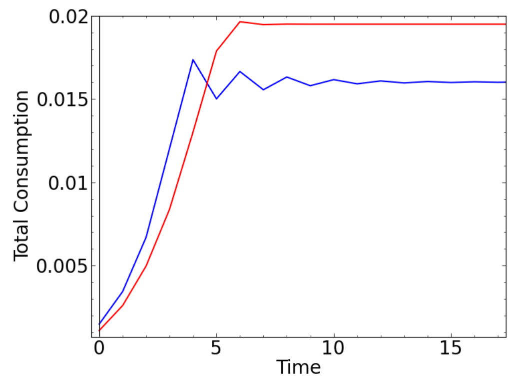

Figure 4.2.11. Transitional Dynamics under different taxation systems: blue line ($\alpha = .2$, $x = 0$), red line ($\alpha = 0$, $x = .2$)

future weight (γ) (positively related with the time preference), as well as the population and technological growth rates (n), and the rate of depreciation(δ).

If any conclusion can be made, then it is that there is no universal doctrine for the maximization of society's welfare, if the stability of the economy is taken into consideration. After all, as it has been mentioned in Chapter 1, a mathematical model is a simplification of a real process. For example, in real-world economies, government spending programs could affect the time preference of agents. The fact that any model is at best an idealization of how real-world economies operate, restrains us from focusing on the derivation of formal analytical results that justify the existence of an optimal tax rate that maximizes the welfare. Instead, we should follow a dialectic view as Minsky would do. In general, the selection of any policy framework should be based on the value of output elasticity, and to some extent on the population growth and rate of depreciation, and adjust mainly with respect to the time preferences of the agents and the state of the economy (whether it is assumed to be at a transitional path or at its steady state). If policy formers have strong evidence that the economy is not close to unstable regions, they can focus on the maximization of society's welfare and growth. Otherwise, they should be wise and sacrifice some potential growth, which could threaten the stability of the economy.

4.2.3 Productive Government Spending

The existence of a number of variables complicates the analysis of the Productive Government Spending framework, and therefore, the implications of the model are divided in two categories. First, we explore the stability of the system and then the performance of the economy.

Regarding the stability, the results derived from the Basic Model and the Unproductive Government Spending framework are still valid and can be applied within this framework, after a few adjustments. Recall from Section 3.3 that the productive function is given by

$$y_t = \overline{B} k_t^{\overline{\beta}}, \tag{4.2.2}$$

where $\overline{B} \equiv B^{\frac{1}{1-\zeta}} (i\tau)^{\overline{\zeta}}$, $\overline{\beta} \equiv \dfrac{\beta}{1-\zeta}$ and $\overline{\zeta} \equiv \dfrac{\zeta}{1-\zeta}$.

Although the stability of the economy is still positively related with the share of capital on output ($\overline{\beta}$), $\overline{\beta}$ is not solely determined by the output elasticity of the private sector, but it is also affected by the output elasticity of public capital. Thus, from equation 4.2.3 we get that

$$\overline{\beta}(\beta, \zeta) = \frac{\beta}{1-\zeta} \tag{4.2.3}$$

Despite the differences in the determination of the values between the three frameworks, the similar form of the capital accumulation equations makes us conclude the following conjecture.

Conjecture 4.2.2. *Let $g(k)$ be the capital accumulation of the Productive Government Spending framework and $\theta(k)$ be the capital accumulation of the Unproductive Government Spending framework. Then g is topologically conjugate with θ.*[12] *In addition, if $\overline{\beta} = \beta$, then g has the same qualitative behavior for the different values of γ as θ.*

[12]The conjecture follows from the analytical results presented in Appendix A.1.1

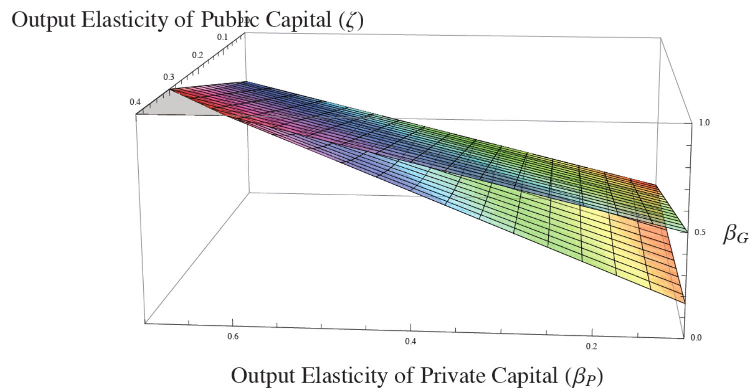

Figure 4.2.12. Relationship between output elasticity of capital in Basic Model and Productive Government Spending framework. The upper curve graphs β_B, while the lower curve graphs $\beta_G = \dfrac{\beta_P}{1-\zeta}$, assuming $\beta_P + \zeta = \beta_B$

According to the Conjecture 4.2.2, increased productivity[13] (B) cannot prevent a crisis. It is the economic environment (values of n, δ and γ) that ultimately determines the probability that a cycle will occur. With some abstraction, the conjecture can be interpreted as evidence that instability is an inherent characteristic of a capitalist economy.

In order to avoid any confusion in our analysis, let β_B denote the output elasticity of capital in the Basic Model, β_P denote the output elasticity of private sector in the Productive Government Spending framework and $\beta_G \equiv \overline{\beta} = \dfrac{\beta_P}{1-\zeta}$ denote the output elasticity of total capital in the Productive Government Spending framework. Figure 4.2.12 plots the relationship between β_B and β_G. It is straightforward to show mathematically that if $\beta_P + \zeta = \beta_B$, then $\beta_G < \beta_B$. This implies that if the output elasticity of capital (β_B) in the Basic Model is decomposed between the output elasticity of private sector

[13]Even if $\overline{B}$ in the Productive Government Spending framework has not the same value as B in the Unproductive Government Spending framework, and thus in extension with the Basic Model, the bifurcation diagrams in all three models still have the same qualitative results,i.e. instability will rise for the same value of future weight, assuming no taxes are imposed since taxes alter the qualitative behavior of the system as we saw in the previous section. For example, if $\zeta = .2$ and $\beta = .536$, then $\overline{\beta} = .67$ and therefore if no taxes are imposed bifurcation and two period cycles occur for $\gamma = 16$, which is the same as in the Basic Model and the Unproductive Government Spending framework. Therefore, technology (B) can only affect the growth of the economy. An increase (decrease) in B increases shifts the bifurcation diagram upwards (downwards).

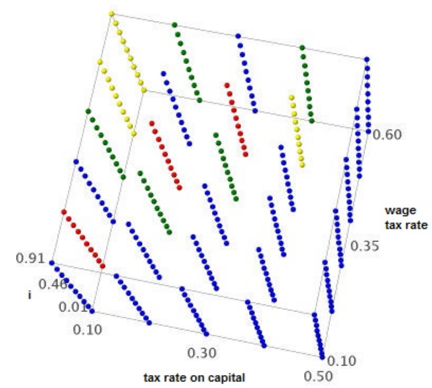

Figure 4.2.13. Lyaponov Exponent – Productive Government Spending

capital (β_P) and public sector capital (ζ), then the economy is destabilized since $\beta_G < \beta_B$; bifurcation and chaotic regions rise for smaller values of future weight.

The policy implementation which follows is that the government should not undertake projects that could have been performed by the private sector with the same efficiency. Instead, the government should engage only in projects that change the structure of production process and increase the productivity of private capital (increase β_P).

We now turn the focus of the analysis on growth. Figure 4.2.11 is a three dimensional representation of a four dimensional graph. The axes of the graphs indicate the value of the parameters at each point, while the color[14] indicates the value of the average Lyapunov exponent for the corresponding parameter values. Figure 4.2.11 provides some evidence that the composition of government spending does not affect the stability[15] of the economy since the value of the Lyapunov exponent remains relatively constant for different spending programs. As a result, the distribution of government resources can focus solely on raising

[14] blue: $L \leq -.4$, yellow: $-.4 \leq L \leq -.2$, green: $-.2 \leq L \leq 0$ and red: $L > 0$

[15] Conjecture 4.2.2, and stability analysis of the Basic Model showed that B (and thus $\overline{B}$) does not influence the stability of the system. Also, in Section 3.3 we saw that i affects only the value of $\overline{B}$. Therefore the allocation of government budget does not affect the stability of the economy.

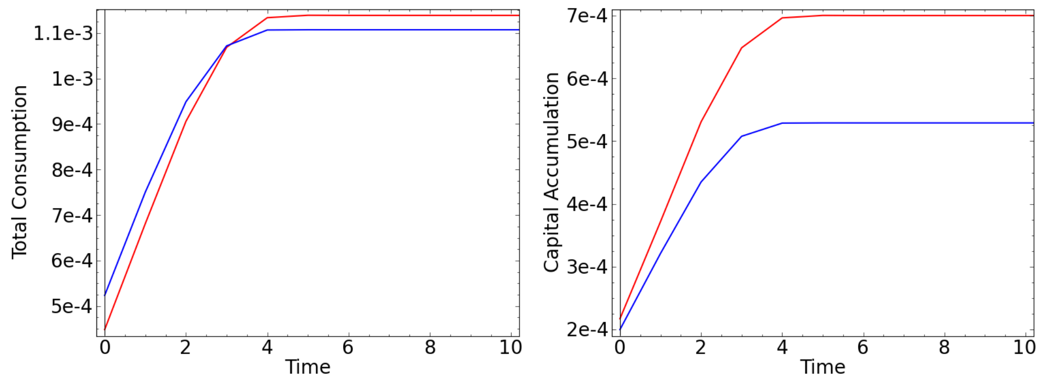

Figure 4.2.14. Productive–Unproductive Government Spending low time preference: blue line($x = .2$,$\alpha = .2$, $i = .5$,$\gamma = 7$),red line($x = .2$,$\alpha = .2$, $i = .7$, $\gamma = 7$)

the welfare of the economy, without any concern about its impact on the stability. However, we should not forget that the way the government budget is financed does affect its stability[16].

The impact of various allocations of government budget on the growth of the economy will be investigated assuming constant output elasticities. It will therefore be considered that β_P and ζ are exogenous and that i (the proportion that government spends productively) does not affect ζ.

Under these assumptions, capital accumulation and growth are affected more severely, when the input (capital or labor) with the highest share in output is taxed. For example, if output elasticity of labor is greater than that of capital ($\beta_G < .5$), an increase in the wage tax rate lowers the level of capital accumulation more than an equal increase in the tax rate on capital.

Regarding consumption, as one would expect, an increase of the unproductive government spending has only temporary positive effects on welfare and total consumption. The divergence of resources from production constrains the potential expansion of the econ-

[16]Recall from Section 4.2.2 that an increase in wage tax rate (x) stabilizes the economy, while an increase in the tax rate on capital (α) destabilizes the economy.

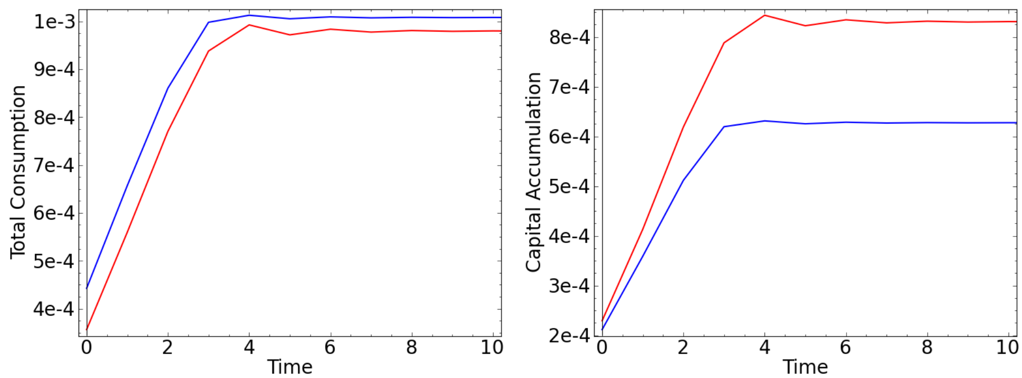

Figure 4.2.15. Productive–Unproductive Government Spending high time preference: blue line($x = .2$,$\alpha = .2$, $i = .5$, $\gamma = 14$),red line($x = .2$,$\alpha = .2$, $i = .7$, $\gamma = 14$)

omy in the long-run (see Figure 4.2.14). An increase of the unproductive spending $(1 - i)$ sacrifices future benefits in favor of current consumption. However, as time preference decreases and future weight increases, agents are more aware of future consumption and thus increase savings. In that case, unproductive government spending, although still decreases the steady state production and capital accumulation, can stimulate total consumption (see Figure 2.4.15).

In general, assuming conjecture 4.2.2, the implications of the Unproductive Government Spending framework and to some extent those of the Basic Model are valid for the Productive Government Spending framework. Therefore, if $\beta_G = \beta_B$ and $\overline{B} = B$, while all the the other parameters are also equal, then the bifurcation diagrams and the transitional dynamics of the two frameworks will be the same. Although the result might seem trivial from a mathematical perspective, the economic implications are very useful. Recall that

$$\overline{B}(B, \beta_G, \zeta, i, \alpha, x) = B^{\frac{1}{1-\zeta}} \left[i(\alpha\beta_G + (1 - \beta_G)x)\right]^{\frac{\zeta}{1-\zeta}} \tag{4.2.4}$$

while

$$\beta_G(\beta_P, \zeta) = \frac{\beta_P}{1 - \zeta} \tag{4.2.5}$$

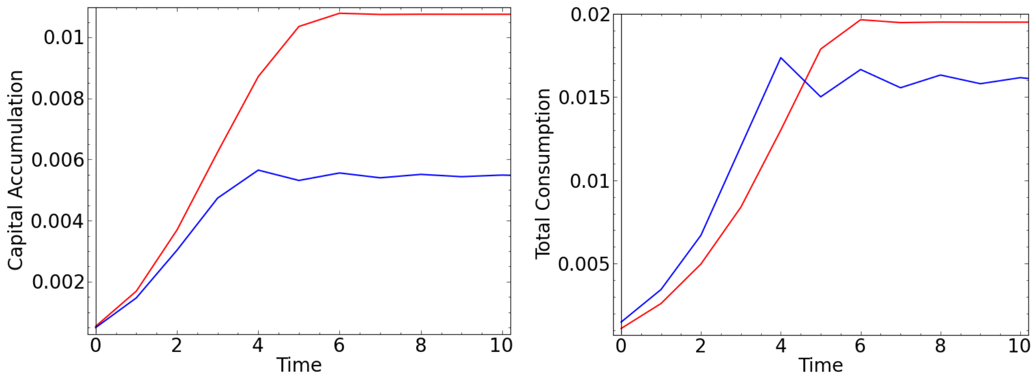

Figure 4.2.16. Productive Spending under different taxation; blue line($a = .2$,$x = .0$, β_G=.67),red line($x = .2$,$\alpha = .0$, β_G=.67)

The more productive the government sector is, the more stable[17] and expansionary[18] the economy will be.

Once again, there appears to be no universal doctrine for the maximization of society's welfare. Appropriate economic policy depends on the state of the economy. Although taxation on the input with the higher output elasticity dampens growth in the long-run more than an equal tax on the other input would, the increased taxation increases total consumption in the short-run (see Figure 4.2.16).

Despite the practical usefulness of the assumption that our variables, particularly β_P, ζ and i, are exogenous and that there is no relationship among them, in real-world economies, the variables are interconnected. As a result, the model presents a valid though general framework of economic policy which lacks, however, any specific recommendations. For example, although increased productivity of public capital stabilizes the economy, ζ is considered to be exogenous. Therefore, no argument can be made about how a favorable

[17] β_G is an increasing function of ζ. Recall that the higher the value of β_G, the more stable the economy is. Thus, ζ is positively related with the stability of the system.

[18] Recall that B is positively related with the expansion of the economy. Since $\overline{B}$ is an increasing function of ζ, the productivity of the public sector is positively related with economic growth.

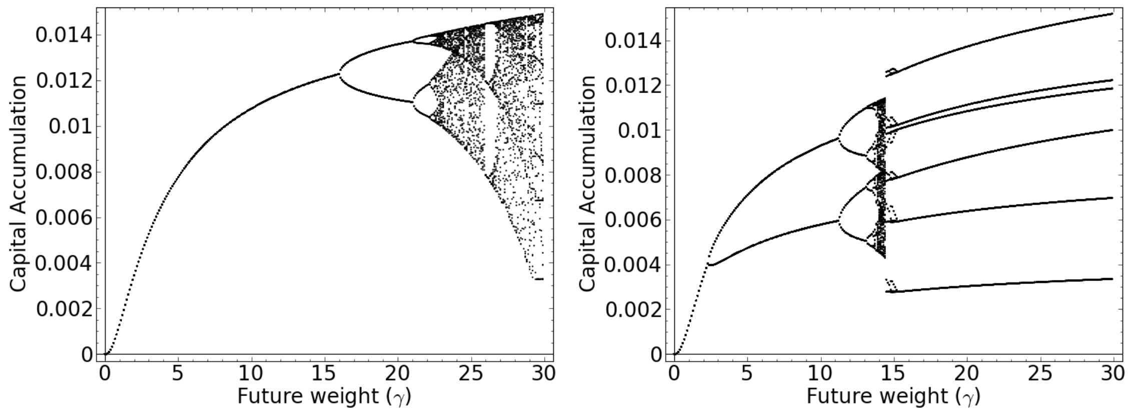

Figure 4.2.17. The impact of expectations on growth and stability: Comparison between extrapolative (left graph) and dynamic expectations (right graph)

level of it can be achieved. A more dialectic view of all the implications would thus be more appropriate.

4.2.4 Dynamic Expectations

Figure 4.2.17 plots the bifurcation diagram of the Basic Model under extrapolative and dynamic expectations (see Section 3.4) and highlights the difference in the stability between the two frameworks. As can be seen, non-stationary expectations destabilize the economy. "The fundamental instability of the capitalist economy is upward," as Minsky argued (Minsky,1984). The economy grows as long as optimistic expectations are justified. However, the optimism that prevails among agents during euphoric periods ultimately leads to instability. If the economy does not validate the prevailing optimism, either due to erroneous expectations or the natural limits in the capacity of the economy, pessimism spreads; the economy fluctuates with downward pressure until a positive growth recovers the optimism. As a result, the economy follows a non-monotonic trajectory driven by the dynamic expectations of the agents.

In such a fluid economic environment, taxation and spending programs may not be adequate to mitigate the intensity of fluctuations (see Figure 4.2.18). In order for the government to be able to stabilize the economy, policy framework should stabilize expectations and limit their dynamism through possibly the imposition of ceilings and floors. Fiscal policy though, is not an adequate instrument to smooth cyclical behavior of the economy in a highly fluid and innovative environment. A combination of well coordinated monetary and fiscal policy is therefore crucial to set favorable economic conditions and a relatively stable environment.

One of the main focuses of policy formers should be realized profits. Profits and expectation of profits define a large extent of the stability in the economy, and thus the government should use the government budget when necessary to stimulate them through the Kalecki mechanism. Otherwise, as the model predicts, the continuous adjustment of erroneous dynamic expectations will create unstable trajectories and will ultimately lead to a crisis. An interesting insight of dynamic expectations is that the trajectory of the economy is less probable to settle at an equilibrium point (steady state). Government intervention is therefore necessary to stabilize the economy and extinguish the threat of a business cycle.

4.3 Conclusions

The purpose of the study is neither to use the model for forecasting business cycles, nor to suggest explicitly optimal taxation rates, nor to identify the optimal composition of government spending. Instead the project focuses on the qualitative results that the model offers with a dialectic view, as Minsky used to do. After all, in a dynamic system with chaotic behavior errors in estimating the initial conditions or the parameters of the model would result in misleading forecasts. It is not detrimental to state once again that a model is only a simplification of the real phenomenon that it investigates. In a real-world economy, no variable can really be considered exogenous. The existence of feedback effects, no matter how small they are, do have an impact on the state of the economy.

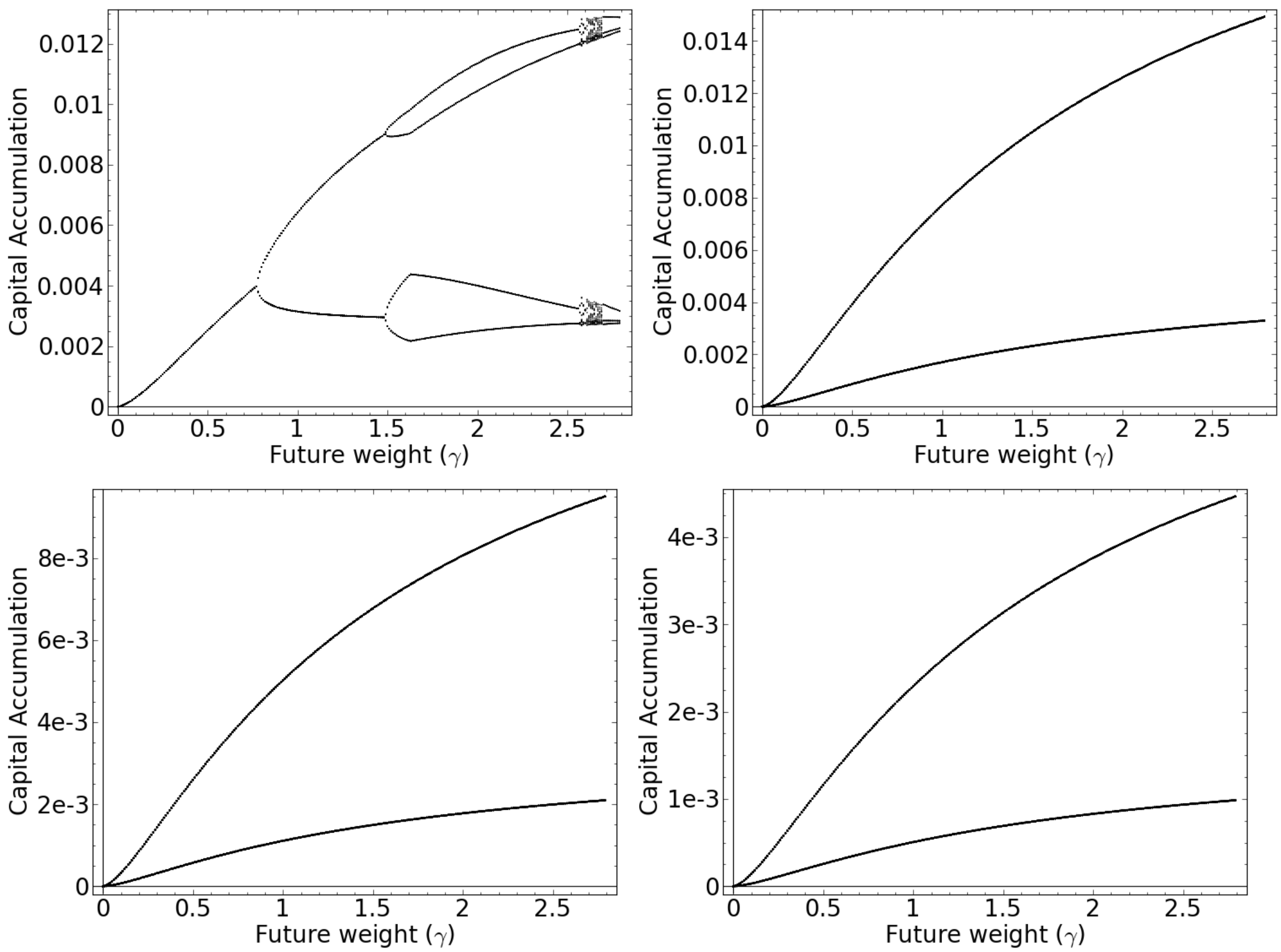

Figure 4.2.18. The impact of taxation on a highly fluid and adjustable economic environment: upper left graph(extrapolative expectations, $\beta_P = .31$, $\zeta = .15$, $\alpha = .1$, $x = .2$), upper right graph (dynamic expectations, $\beta_P = .31$, $\zeta = .15$, $\alpha = .1$, $x = .2$), lower right graph (dynamic expectations, $\beta_P = .31$, $\zeta = .15$, $\alpha = .1$, $x = .7$),lower left graph (dynamic expectations, $\beta_P = .31$, $\zeta = .15$, $\alpha = .7$, $x = .7$)

Thus the derivation of analytical results and formal proofs, would be of no usefulness. It is also common that despite their useful implications, growth models do not explicitly propose how the suggested policies can be performed. Perhaps these were the reasons that Minsky never sought to build a rigorous mathematical model to present his ideas. However, based on sound economic theory and plausible economic conditions (parameter values), the project provides evidence that verifies some of Minsky's ideas, while questions others.

First, stability is destabilizing and the driving force of instability is indeed the uncertainty under which investment decisions are made. The more dynamic the expectations are, the more unstable the economy is. The creation of ceilings and floors is a potential solution to the inherent instability. Government intervention is crucial and necessary to stabilize the economy. It is the structure though, and not the nominal amount, that is important. On a theoretical basis, any policy that can stabilize expectations is beneficiary, as long as stability is concerned. Regarding growth though, policy should be conditional first on economic environment (output elasticity of capital, technological progress and depreciation rate) and second on its current state (whether it is at the steady state or at the transitional path). A more extreme solution that could stabilize an inherently unstable economy could potentially be the socialization of investment (Minsky, 1984). The impact that such a policy would have on growth is controversial though.

Second, an important implication that was apparent in all frameworks is the existence of a trade off between stability and economic performance. Although government intervention can smooth instability and constrain the dynamic behavior of the economy, assuming non-dynamic expectations, stabilization is achieved at the expense of decreased consumption. Taxation tightens the constrains of the economy and dampens the perturbation of output fluctuation. At the same time though, it decreases the availability of funds and thus decreases private consumption. Generally speaking, economic policy should be judged by comparing its benefits to its costs on both stability and growth. However, measuring benefits and particularly costs appropriately is very difficult, specifically because the cost of the increased dynamism of an economy is considered only after a crisis has already occurred. Furthermore, the potential side effects that a policy can have complicates even more the identification of the optimal policy framework. For example, although increased tax rate on wages can stabilize the economy, a taxation burden on the labor side would most probably decrease its productivity and thus the welfare of the society. Policy makers should always keep in mind that "a perfect economic policy, good for all possible times and places, does not exist" (Minsky, 1984).

Despite all the favorable implications, the usefulness of the results should not be exaggerated, neither should the limitation and controversies of the model be overlooked. For example, a key element that is missing from the model is money. Growth models usually assume that money serves simply as a medium of exchange and thus it can be neglected without affecting the validity of the results. Such an assumption though is at least erroneous and ignores the weight that Minsky put on money as a cause of financial crisis and output fluctuations.

Another limitation of the adopted model is the fact that it presents a closed economy. Real-world economies are not closed, but open, and the flows of capital are very crucial in the determination of both stability and performance. China, for example, has experienced remarkable growth rates and a relative stability, despite its population growth and the low time preference of its agents, mainly because of its substantial export sector. The expansion of the model in order to reflect an open economy would be of particular importance and interest.

Furthermore, the assumptions and the structure of the model have their own limitations. According to phsycology, people are more prone to small losses than to small gains. Expectation dynamics might not adequately reflect the way that people actually form their expectations. In addition, different expectation dynamics would produce different results. However, the idea that dynamic expectations have a significant contribution to the stability of the system would still prevail. Beside expectation dynamics, it remains controversial why an increase in capital tax rates instead of stabilizing, destabilizes the economy. This admittedly is a peculiar result. Although, in an open economy framework, capital flight would be a plausible reason, while in a closed-economy model, this is not a feasible explanation. Furthermore, the assumption that an increase in the future weight increases savings might not be correct. People who place a high value in future consumption, instead of increasing saving, they could simply increase their productivity, since productivity determines their income.[19] In other words, a person who cares about future consumption could instead of increasing his savings, increase his work effort, because he will expect that this will increase his future wage.

Last but not least, the model questions the validity of Minsky's suggestion to move from capital intensive to labor intensive activities. According to the implications of the model, such a policy would increase the unstable regions, and fluctuations would become more probable.

The favorable implications of the model developed in the project can be strengthened further or perhaps be questioned under future research. A potential area for further research is to adjust the model in order to represent an open economy. This would be a substantial contribution and would allow to expand the focus of the project on the role of budget deficits.[20] Altervatively, someone could follow Futagami, Morita and Shibata (1993) incorporation of the public sector as an input of production and see how the treatment of public input as stock[21] rather than flow alternates the implications derived in the project. Finally, the expectation dynamics offer the opportunity to investigate the impact

[19]Recall that $w = (1-\beta)y$, where $(1-\beta)$ is the output elasticity of labor.
[20]See Appendix B for some preliminary research
[21]See Appendix B for some preliminary research

of unanticipated government policies on the trajectory that the economy follows and its overall performance.

Appendix A
Derivations

A.1 Basic Model

Capital Accumulation Identity

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s_t] = \frac{1}{1+n}[(1-\delta)k_t + y_t - c_t] \tag{A.1.1}$$

k_{t+1}: per capita capital of the $t+1^{th}$ generation

k_t: per capita capital of the t^{th} generation

y_t: per capita income

c_t: consumption

δ: rate of depreciation of existing capital

n: population growth rate.

Production Function

$$y_t = f(k_t) = BK_t^{\beta} \tag{A.1.2}$$

Income Distribution

$$y = w + rk = (1-\beta)y + \beta y \tag{A.1.3}$$

real rate of interest: $r = f'(k) = \frac{\beta y}{k} = \beta B k^{\beta-1}$

dividends: $rk = \beta k$

wage rate: $w = y - rk = (1-\beta)y$

Utility Function (Generations preference)

$$u(c, c^1) = v(c) + V(c^1) = log(c) + \gamma log(c^1)^1 \tag{A.1.4}$$

v: utility of current consumption

V: utility to the present generation from the bequest to its heirs

c: current consumption

c^1: sustainable consumption level

γ: future weight

Net rate of return

$$\rho = \frac{r - (n+\delta)}{1+n} \tag{A.1.5}$$

Sustainable level of Consumption

$$c^1 = y^1 - (n+\delta)k^1 \tag{A.1.6}$$

y^1: anticipated income for the next generation ($y^1 = w^1 + r^1 k^1$)

w^1: anticipated wage

$r^1 k^1$: anticipated dividends

r^1: anticipated rate of return

k^1: future capital stock ($k^1 = \frac{1}{1+n}[y - c + (1+\delta)k]$)

[1] Diminishing marginal utility

$$
\begin{aligned}
(6) \Rightarrow c^1 = y^1 - (n+\delta)k^1) &= w^1 + r^1k^1 - (n+\delta)k^1 \\
&= w^1 + [r^1 - (n+\delta)]k^1 \\
&\overset{(5)}{=} w^1 + [\rho^1(1+n)]k^1 \\
&\overset{(6)}{=} w^1 + [\rho^1(1+n)]\frac{1}{1+n}[y - c + (1-\delta)k] \\
&= w^1 + \rho^1[y - c + (1-\delta)k]
\end{aligned}
$$

Therefore the sustainable consumption level is

$$c^1 = w^1 + \rho^1[y - c + (1-\delta)k] \tag{A.1.7}$$

Consumption Constraint (closed economy)

$$0 \le c \le y \tag{A.1.8}$$

Laplacian and Utility maximization

$$L = v(c) + V\{w^1 + \rho^1[y - c + (1-\delta)k]\} \tag{A.1.9}$$

$$(\frac{\partial L}{\partial c}) \Rightarrow (v^{'}(c) - \rho^1 V\{w^1 + \rho^1[y - c + (1-\delta)k]\}) = \lambda, (\lambda = 0) \text{ when } c < y \tag{A.1.10}$$

$$(\frac{\partial L}{\partial c}) \Rightarrow (v^{'}(c) - \rho^1 V\{w^1 + \rho^1[y - c + (1-\delta)k]\}) = \lambda, (\lambda > 0) \text{ when } c = y \tag{A.1.11}$$

$$
\begin{aligned}
(4),(10) \Rightarrow \frac{1}{c} &= \rho^1\frac{\gamma}{w^1 + \rho^1[y - c + (1-\delta)k]} \\
c\rho^1\gamma &= w^1 + \rho^1[y - c + (1-\delta)k] \\
c\gamma &= \frac{w^1}{\rho^1} + [y - c + (1-\delta)k] \\
c\gamma + c &= \frac{w^1}{\rho^1} + [y + (1-\delta)k]
\end{aligned}
$$

$$c(1+\gamma) = \frac{w^1}{\rho^1} + [y + (1-\delta)k]$$

$$c = \frac{1}{1+\gamma}[\frac{w^1}{\rho^1} + (y + (1-\delta)k)]$$

Unconstrained Consumption Function

$$c(y, k, r^1, w^1; \gamma) = \frac{1}{1+\gamma}[\frac{w^1}{\rho^1} + (y + (1-\delta)k)] \tag{A.1.12}$$

Expectation Dynamics (stationary[2])

$$r^1 = r; \;\; \rho^1 = \rho; \;\; w^1 = w \tag{A.1.13}$$

$$(12) \Rightarrow \frac{1}{1+\gamma}[\frac{w^1}{\rho^1} + (y + (1-\delta)k)]$$

$$\overset{(13)}{=} \frac{1}{1+\gamma}[\frac{w}{\rho} + (y + (1-\delta)k)]$$

$$\overset{(5),(3)}{=} \frac{1}{1+\gamma}[\frac{\frac{(1-\beta)y}{r-(n+\delta)}}{1+n} + (y + (1-\delta)k)]$$

$$= \frac{1}{1+\gamma}[\frac{(1+n)(1-\beta)y}{r-(n+\delta)} + (y + (1-\delta)k)]$$

$$\overset{(2),(3)}{=} = \frac{1}{1+\gamma}[\frac{(1+n)(1-\beta)Bk^\beta}{\beta Bk^{\beta-1}-(n+\delta)} + (y + (1-\delta)k)]$$

Consumption-Wealth Function

$$h(k) = \frac{1}{1+\gamma}[(1-\delta)k + Bk^\beta + \frac{(1+n)(1-\beta)Bk^\beta}{\beta Bk^{\beta-1}-(n+\delta)}] \tag{A.1.14}$$

$\exists k^s$ s.t. $f(k^s) = h(k^s)$. Define

$$K^s = [0, k^s) \text{ and } K^d = [k^s, \infty) \tag{A.1.15}$$

[2]We assume that current level of income and interest are used as proxies for future values

Current Consumption

$$c(k)^{3} = \begin{cases} h(k) & k \in K^s, \\ f(k) & k \in K^d. \end{cases} \tag{A.1.16}$$

$$= \begin{cases} \dfrac{1}{1+\gamma}[(1-\delta)k + Bk^{\beta} + \dfrac{(1+n)(1-\beta)Bk^{\beta}}{\beta Bk^{\beta-1} - (n+\delta)}] & k \in K^s, \\ Bk^{\beta} & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \dfrac{1}{1+n}[(1-\delta)k + f(k) - h(k)] & k \in K^s, \\ \dfrac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{A.1.17}$$

$$\begin{cases} \dfrac{1}{1+n}\dfrac{\gamma}{1+\gamma}[(1-\delta)k + Bk^{\beta}] - \dfrac{1}{1+\gamma}\dfrac{(1-\beta)Bk^{\beta}}{\beta Bk^{\beta-1} - (n+\delta)} & k \in K^s, \\ \dfrac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

A.1.1 Existence of Stability and Instability in the Basic Model

Proposition A.1.1. *Let $\overline{k}$, be the steady state capital of the Basic Model. For each γ, n, $\delta > 0$ there exists a $\beta_b \in (0,1)$ such that $\theta'(\overline{k}) = -1$ and such that $\overline{k}$ is unstable (stable) whenever $\beta < \beta_b$ ($\beta > \beta_b$).*

Proof. To determine the stability or instability of a system, we must look at the derivative of the capital accumulation identity

$$\theta'(k) = \frac{1}{1+n}\left[\frac{1}{1+\gamma}(r+1-\delta) - \frac{1-\beta}{(1+\gamma)(1+n)\rho^2}\left((1+n)\rho + \frac{(1-\beta)r}{\beta}\right)r\right] \tag{A.1.18}$$

at the steady state.

Let $\overline{r}$ and $\overline{k}$ be the steady state rate of return and per capita capital respectively. Then in the the steady state

$$k_{t+1} = k_t = \overline{k} \tag{A.1.19}$$

$$\Rightarrow \overline{k} = \frac{1}{1+n}[(1-\delta)\overline{k} + y - c(\overline{k})]$$

[3] $k \in K^s$ if the maximization consumption level is an inferior solution of equation (2.3.8), while $k \in K^d$, if the maximization consumption level is a boundary solution

$$\Rightarrow \overline{k} = \frac{1}{1+n}\left[(1-\delta)\overline{k} + y - \frac{1}{1+\gamma}\frac{1+\overline{\rho}}{\overline{\rho}}[y-(n+\delta)\overline{k}]\right]$$

$$\Rightarrow \overline{r} = \frac{1+n}{\gamma} + (n+\delta)$$

$$\overline{r} = \frac{1+n}{\gamma} + (n+\delta) \tag{A.1.20}$$

Thus

$$\theta'(\overline{k}) = 1 - \frac{(1-\beta)\gamma^2}{(1+\gamma)(1+n)^2}\left[\frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r}\right] \tag{A.1.21}$$

For simplicity, let $\rho \equiv \theta'(\overline{k})$. If $|\theta'| < 1$, the trajectory converges to the steady state, $\overline{k}$. When $|\theta'| > 1$, $\overline{k}$ is unstable and fluctuations persist, which can be either periodic or non-periodic. Let δ, n, $\gamma > 0$. If $0 < \beta < 1$, then

$$\rho \equiv \theta'(\overline{k}) = 1 - \frac{(1-\beta)\gamma^2}{(1+\gamma)(1+n)^2}\left[\frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r}\right]$$

$$\Rightarrow \rho = 1 - \frac{(1-\beta)\gamma^2}{(1+\gamma)(1+n)^2}\left[\frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r}\right]$$

$$\Rightarrow 1 - \frac{(1-\beta)\gamma^2}{(1+\gamma)(1+n)^2}\left[\frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r}\right] - \rho = 0$$

$$\frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r} + (\rho-1)\frac{(1+n)^2(1+\gamma)}{(1-\beta)\gamma^2} = 0$$

We will refer to this equation as H_ρ. Thus,

$$H_\rho(\beta,\gamma,\delta,n) \equiv \frac{1-\beta}{\beta}\overline{r}^2 + \frac{1+n}{\gamma}\overline{r} + (\rho-1)\frac{(1+n)^2(1+\gamma)}{(1-\beta)\gamma^2} = 0 \tag{A.1.22}$$

Since $0 < \beta < 1$ and δ, n, $\gamma > 0$ it can easily be shown that $\theta' \neq 1$. Thus, to study the boundaries of instability, let $\theta' = -1$. Using equation $A.1.20$, we can derive the implicit equation

$$H_{-1}(\beta,\gamma,\delta,n) \equiv 1 + \frac{\gamma(n+\delta)}{1+n} - \frac{-1+\sqrt{1+\frac{8(1+\gamma)}{\beta}}}{2}\frac{\beta}{1-\beta} = 0 \tag{A.1.23}$$

The first term is constant for fixed γ, n, δ; the second term is continuously increasing function of β. As β goes to zero, the value of the second term goes to zero, while as β goes to one, the second term goes to infinity. Consequently, keeping γ, n and δ constant, H_{-1} changes sign as β increases from zero. By the Intermediate Value Theorem, for each γ, n and $\delta > 0$ there exists a $\beta \in (0, 1)$, such that $H_{-1}(\gamma, \beta, \delta, n) = 0$, or equivalently, such that $\theta'(\overline{k}) = -1$. $\square$

According to Proposition A.1.1, points in the parameter space below the $H_{-1} = 0$ ($\equiv \theta'(k) = 1$) curve (see Figure 4.2.1 and Figure 4.2.2) give rise to unstable steady states and points above the curve give rise to stable ones. In general, the smaller the output elasticity of capital (β) the more likely the system is to be unstable.

Regarding the comparative dynamics with respect the future weight γ, it can be shown that $H_{-1}(1/3, 0, \delta, n) = 0$ for each n, $\delta > 0$. The slope of the locus of points (γ, β), satisfying equation $H_{-1}(\beta, \gamma, \delta, n) = 0$ is

$$\frac{d\beta}{d\gamma} = -\frac{\partial H/\partial\gamma}{\partial H/\partial\beta}. \tag{A.1.24}$$

After tedious computations, we conclude that for $\gamma = 0$, the slope can be positive, zero or negative as $\dfrac{n+\delta}{1+n}$ is greater than, equal or less than 3/5. As γ increases, for any positive δ and n, the slope of the locus of points (γ, β) satisfying the equation $H_{-1}(\beta, \gamma, \delta, n) = 0$ become positive as γ increases and it approaches assymptotically $\dfrac{n+\delta}{1+n}$. Therefore,

Proposition A.1.2. *If $2n + 5\delta > 3$, then for each $\beta > 0$, there exists $\gamma_b > 0$, such that $\overline{k}$ is unstable (stable) if $\gamma > \gamma_b$ ($\gamma < \gamma_b$). If $2n + 5\delta < 3$, then there exists a $\beta_l < 1/3$ such that for all β, $\beta_l < \beta < 1/3$ there exists γ_l, and γ_u such that $\overline{k}$ is unstable for all γ such that $0 < \gamma < \gamma_l$ and $\gamma > \gamma_u$ and stable for all γ such that $\gamma_l < \gamma < \gamma_u$. Finally, if $2n + 5\delta = 3$, $\overline{k}$ is unstable (stable) for all $\beta < 1/3$ ($\beta > 1/3$).*

A.2 Unproductive Government Spending

- The government spends resources without contributing to production or capital accumulation

• Government spending is financed with proportional income taxation, at rate $\tau \in [0,1]$. Thus the government spending absorbs a fraction τ of aggregate output

$$g_t = \tau y_t. \tag{A.2.1}$$

Capital Accumulation Identity

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + y_t(1-\tau) - c_t] \tag{A.2.2}$$

The disposable income of households is now $(1-\tau)y_t$, and thus consumption is bounded between

$$0 \leq c \leq (1-\tau)y. \tag{A.2.3}$$

We assume that the rate of return on capital is taxed wih a rate α.The after-tax real rate of interest is

$$r_\tau = f'(k)(1-\alpha) = \frac{\beta y}{k}(1-\alpha), \tag{A.2.4}$$

while the wage is taxed with a rate x and the after-tax wage is give by

$$w_\tau = (1-x)(y - rk) = (1-x)(1-\beta)y. \tag{A.2.5}$$

The disposable income for a generation in terms of wages and dividends is

$$(1-\tau)y = w_\tau + r_\tau k = (1-x)(1-\beta)y + \beta y(1-\alpha). \tag{A.2.6}$$

Therefore the relationship between the tax rate on capital and wage is

$$\alpha(x) = \frac{\tau - (1-\beta)x}{\beta} \quad \text{and} \quad x(\alpha) = \frac{\tau - \beta\alpha}{1-\beta} \tag{A.2.7}$$

or respectively between income tax and wage/rate on capital tax

$$\tau = (1-\beta)x + \beta\alpha. \tag{A.2.8}$$

Since none of the tax rates can be negative, we deduce that $\tau \in [0,1]$, $\alpha \in [0, \frac{\tau}{\beta}]$ and $x \in [0, \frac{\tau}{1-\beta}]$.

Net rate of return

$$\rho_\tau = \frac{r_\tau - (n+\delta)}{1+n} \tag{A.2.9}$$

Sustainable level of consumption

$$c^1 = y^1(1-\tau) - (n+\delta)k^1 \tag{A.2.10}$$

$$(27) \Rightarrow c^1 = y^1(1-\tau) - (n+\delta)k^1 = w^1_\tau + r^1_\tau k^1 - (n+\delta)k^1$$

$$= w^1_\tau + [r^1_\tau - (n+\delta)]k^1$$

$$= w^1 + [\rho^1_\tau(1+n)]k^1$$

$$= w^1 + [\rho^1_\tau(1+n)]\frac{1}{1+n}[y(1-\tau) - c + (1-\delta)k]$$

$$= w^1_\tau + \rho^1_\tau[y(1-\tau) - c + (1-\delta)k]$$

Following the exact same derivations as in the Basic Model we derive the Unconstrained Consumption Function

$$c(y,k,r^1,w^1,\tau^1;\gamma) = \frac{1}{1+\gamma}[\frac{w^1_\tau}{\rho^1_\tau} + (y(1-\tau) + (1-\delta)k)] \tag{A.2.11}$$

Expectation Dynamics [4]

$$r^1 = r; \;\; \rho^1 = \rho; \;\; w^1 = w; \;\; \tau^1 = \tau \tag{A.2.12}$$

$$(12) \Rightarrow \frac{1}{1+\gamma}[\frac{w^1_\tau}{\rho^1_\tau} + (y(1-\tau) + (1-\delta)k)]$$

$$\overset{(29)}{=} \frac{1}{1+\gamma}[\frac{w_\tau}{\rho_\tau} + (y(1-\tau) + (1-\delta)k)]$$

$$\overset{(22),(26)}{=} \frac{1}{1+\gamma}[\frac{(1-\beta)(1-x)y}{\frac{r_\tau - (n+\delta)}{1+n}} + (y(1-\tau) + (1-\delta)k)]$$

[4] Since $r^1 = r$ and $\tau^1 = \tau$ it is deduced that $r^1_\tau = r_\tau$, etc.

$$= \frac{1}{1+\gamma}[\frac{(1+n)(1-\beta)(1-x)y}{r(1-\alpha)-(n+\delta)} + (y(1-\tau)+(1-\delta)k)]$$

$$\overset{(2),(3)}{=} \frac{1}{1+\gamma}[\frac{(1+n)(1-x)(1-\beta)Bk^{\beta}}{(1-\alpha)\beta Bk^{\beta-1}-(n+\delta)} + (y(1-\tau)+(1-\delta)k)]$$

Current Consumption

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{A.2.13}$$

$$= \begin{cases} \frac{1}{1+\gamma}[(1-\delta)k + (1-\tau)Bk^{\beta} + \frac{(1+n)(1-x)(1-\beta)Bk^{\beta}}{(1-\alpha)\beta Bk^{\beta-1}-(n+\delta)}] & k \in K^s, \\ Bk^{\beta}(1-\tau) & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \frac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \frac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{A.2.14}$$

$$\begin{cases} \frac{1}{1+n}\frac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)Bk^{\beta}] - \frac{1}{1+\gamma}\frac{(1-\beta)(1-x)Bk^{\beta}}{(1-\alpha)\beta Bk^{\beta-1}-(n+\delta)} & k \in K^s, \\ \frac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

A.2.1 Conjecture 4.2.1

The following bifurcation diagrams show evidence that when $\alpha = x$, then the qualitative behavior of the system remains the same. However, growth decreases since taxes decrease the availability of investment funds

A.3 Productive Government Spending

• Government spending is financed with proportional income taxation, at rate $\tau \in [0,1]$. Thus the government spending absorbs a fraction τ of aggregate output

$$g_t = \tau y_t. \tag{A.3.1}$$

• Let $i \in (0,1]$ be the portion of government spending that is productive. Government spending can thus be interpreted as infrastructure or other productive services. Then

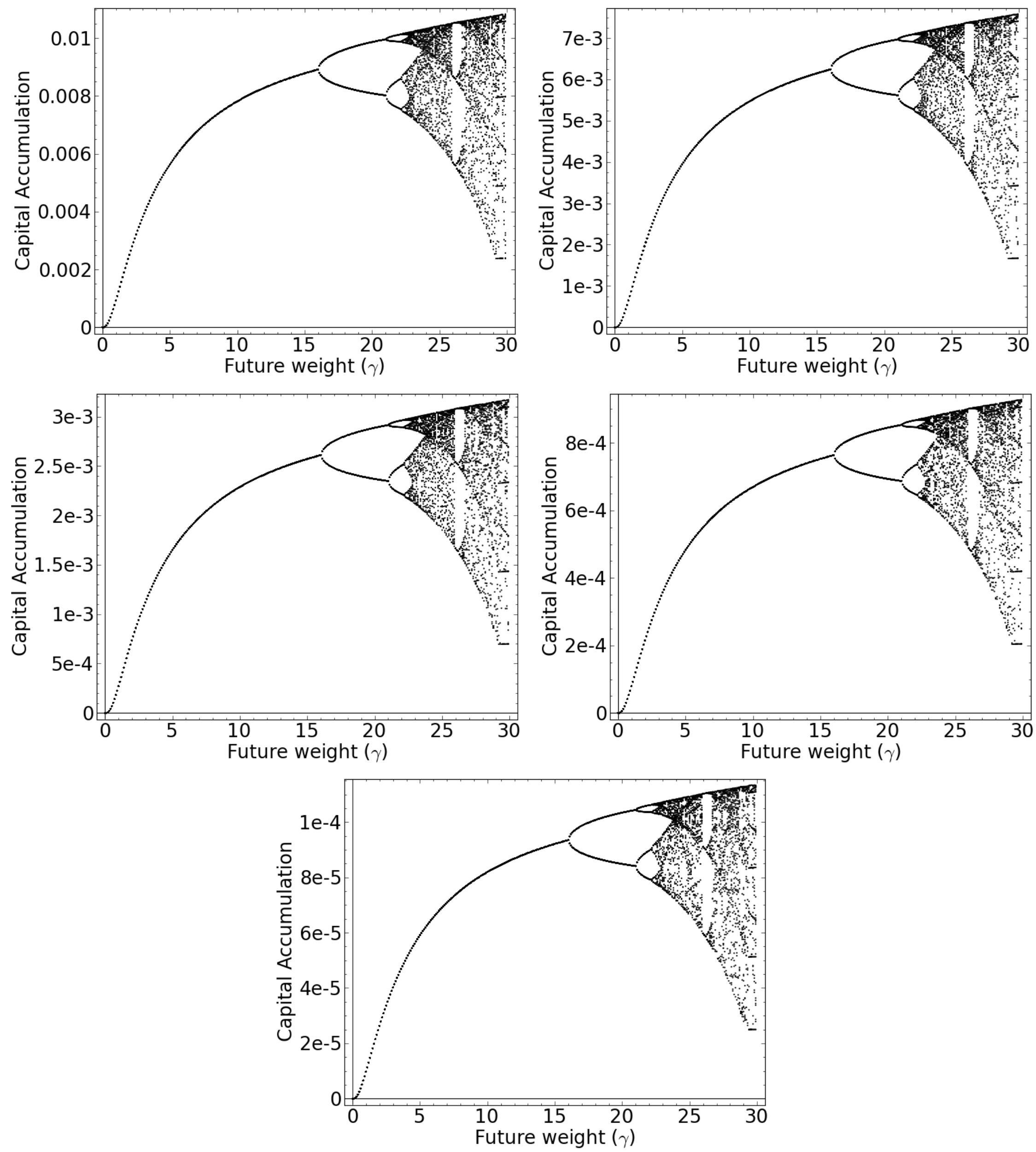

Figure A.2.1. Conjecture 4.2.1; upper left graph ($\alpha = x = .1$),upper right graph ($\alpha = x = .2$),middle left graph ($\alpha = x = .4$), middle right graph ($\alpha = x = .6$),lower graph ($\alpha = x = .8$)

$(1-i)$ is the portion that government spends without contributing to production. Therefore production is now given by

$$y_t = Bk_t^{\beta}(ig_t)^{\zeta} \tag{A.3.2}$$

Substituting $g_t = \tau y_t$ into (33) and solving for y_t, we infer

$$(33) \Rightarrow y_t^{1-\zeta} = Bk_t^{\beta}(i\tau)^{\zeta}$$

$$y_t = B^{\dfrac{1}{1-\zeta}} k_t^{\dfrac{\beta}{1-\zeta}} (i\tau)^{\dfrac{\zeta}{1-\zeta}}$$

$$y_t = B' k_t^{\beta'} (i\tau)^{\zeta'}$$

where $B' \equiv B^{\dfrac{1}{1-\zeta}}$, $\overline{\beta} \equiv \dfrac{\beta}{1-\zeta}$ and $\overline{\zeta} \equiv \dfrac{\zeta}{1-\zeta}$

Since B', i, and τ are considered as constants, let $\overline{B} \equiv B'(i\tau)^{\overline{\zeta}}$. Then the production function becomes

$$y_t = \overline{B}k_t^{\overline{\beta}} \tag{A.3.3}$$

Following the exact same process and reasoning as in Section 3.2, we deduce that

Current Consumption

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{A.3.4}$$

$$= \begin{cases} \dfrac{1}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}} + \dfrac{(1+n)(1-x)(1-\overline{\beta})\overline{B}k^{\overline{\beta}}}{(1-\alpha)\overline{\beta}\ \overline{B}k^{\overline{\beta}-1} - (n+\delta)}] & k \in K^s, \\ \overline{B}k^{\overline{\beta}}(1-\tau) & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \dfrac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \dfrac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{A.3.5}$$

$$\begin{cases} \dfrac{1}{1+n}\dfrac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}}] - \dfrac{1}{1+\gamma}\dfrac{(1-\overline{\beta})(1-x)\overline{B}k^{\overline{\beta}}}{(1-\alpha)\overline{\beta}\ \overline{B}k^{\overline{\beta}-1} - (n+\delta)} & k \in K^s, \\ \dfrac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

A.4 Adaptive Expectations

In this case we will consider non-stationary expectations. We will assume a simple and straightfoward rule of expectations; optimism feeds optimism and pessimism feeds pessimism. However, we will assume that the tax rate and the budget deficit will remain constant.

Expectation Dynamics

$$r_t^1 = (1 + \frac{r_t - r_{t-1}}{r_{t-1}})r_t \tag{A.4.1}$$

$$\rho_t^1 = (1 + \frac{\rho_t - \rho_{t-1}}{\rho_t})\rho_t \tag{A.4.2}$$

$$w_t^1 = (1 + \frac{w_t - w_{t-1}}{w_{t-1}})w_t \tag{A.4.3}$$

$$\tau_t^1 = \tau_t, \tag{A.4.4}$$

which implies that $\alpha_t^1 = \alpha_t$ and $x_t^1 = x_t$. According to the Productive Government Spending case the production function is

$$y_t = \overline{B}k_t^{\overline{\beta}} \tag{A.4.5}$$

Also the consumption function is exactly the same as in the Unproductive Government Spending case since no new assumption has been made up to that point. Thus

$$c(y, k, r^1, w^1, \tau^1; \gamma) = \frac{1}{1+\gamma}[\frac{w_\tau^1}{\rho_\tau^1} + (y(1-\tau) + (1-\delta)k)] \tag{A.4.6}$$

However, now in order to proceed and derive the consumption and capital accumulation identities we need to substitute into equation 5.1.42 the new expectation dynamics. Therefore, substituting equation 5.1.37 − 40 into equation 5.1.42, we get

$$c(y, k, r^1, w^1, \tau^1; \gamma) = \frac{1}{1+\gamma}[\frac{w_\tau^1}{\rho_\tau^1} + (y(1-\tau) + (1-\delta)k)]$$

$$= \frac{1}{1+\gamma}[\frac{1 + \frac{w - w_{-1}}{w_{-1}}w}{1 + \frac{\rho - \rho_{-1}}{\rho_{-1}}\rho} + (y(1-\tau) + (1-\delta)k)]$$

Current Consumption

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{A.4.7}$$

$$= \begin{cases} \frac{1}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}} \\ +\frac{(1+n)(1-x)(1-\overline{\beta})(\overline{B}k^{\overline{\beta}})^2}{[(1-\alpha)\overline{\beta}\;\overline{B}(k^{\overline{\beta}-1})-(n+\delta)]^2}\frac{((1-\alpha)\overline{\beta}\;\overline{B}k_{-1}^{\overline{\beta}-1}-(n+\delta))}{\overline{B}k_{-1}^{\overline{\beta}}}] & k \in K^s, \\ \overline{B}k^{\overline{\beta}}(1-\tau) & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \frac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \frac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{A.4.8}$$

$$\begin{cases} \frac{1}{1+n}\frac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\overline{\beta}}] \\ -\frac{1}{1+\gamma}\frac{(1-x)(1-\overline{\beta})(\overline{B}k^{\overline{\beta}})^2}{[(1-\alpha)\overline{\beta}\;\overline{B}(k^{\overline{\beta}-1})-(n+\delta)]^2}\frac{((1-\alpha)\overline{\beta}\;\overline{B}k_{-1}^{\overline{\beta}-1}-(n+\delta))}{\overline{B}k_{-1}^{\overline{\beta}}}] & k \in K^s, \\ \frac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

Appendix B
Future Research

B.1 Budget Deficit

• Government spending is not solely financed with proportional income taxation, but also with debt which is financed by foreigners [1]. Let the budget deficit be measured as percentage of GDP. Thus

$$BD = \pi y_t \tag{B.1.1}$$

Therefore the total government spending is given by

$$g_t = \tau y_t + \pi y_t = (\tau + \pi) y_t = \overline{\tau} y_t \tag{B.1.2}$$

• Once again, we will assume that government spends $i\%$ of its budget productively and $(1 - i)$ unproductively.

Production Function

$$y_t = B k_t^{\beta} (i g_t)^{\zeta} \tag{B.1.3}$$

[1] We exclude the portion of the debt that is financed by domestic citizens which is simply a transfer of available resources within the economy

Substituting $g_t = \overline{\tau} y_t$ into (39) and solving for y_t, we infer

$$(39) \Rightarrow y_t^{1-\zeta} = B k_t^{\beta} (i\overline{\tau})^{\zeta}$$

$$y_t = B^{\frac{1}{1-\zeta}} k_t^{\frac{\beta}{1-\zeta}} (i\overline{\tau})^{\frac{\zeta}{1-\zeta}}$$

$$y_t = B' k_t^{\beta'} (i\overline{\tau})^{\zeta'}$$

where $B' \equiv B^{\frac{1}{1-\zeta}}$, $\beta' \equiv \dfrac{1}{1-\zeta}$ and $\zeta' \equiv \dfrac{\zeta}{1-\zeta}$

Since B', i, and $\overline{\tau}$ are considered as constants, let $\overline{B} \equiv B'(i\tau)^{\zeta'}$. Then the production function becomes

$$y_t = \overline{B} k_t^{\beta'} \tag{B.1.4}$$

Following the exact same process as in the Uproductive Government Spending case, we deduce that

Current Consumption [2]

$$c(k) = \begin{cases} h(k) & k \in K^s, \\ f(k)(1-\tau) & k \in K^d. \end{cases} \tag{B.1.5}$$

$$= \begin{cases} \dfrac{1}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\beta'} + \dfrac{(1+n)(1-x)(1-\beta')\overline{B}k^{\beta'}}{(1-\alpha)\beta'\overline{B}k^{\beta'-1} - (n+\delta)}] & k \in K^s, \\ \\ \overline{B}k^{\beta'}(1-\tau) & k \in K^d. \end{cases}$$

Capital Accumulation Identity

$$k_{t+1} = \theta(k) = \begin{cases} \dfrac{1}{1+n}[(1-\delta)k + f(k)(1-\tau) - h(k)] & k \in K^s, \\ \\ \dfrac{1}{1+n}[(1-\delta)k] & k \in K^d. \end{cases} \tag{B.1.6}$$

$$\begin{cases} \dfrac{1}{1+n}\dfrac{\gamma}{1+\gamma}[(1-\delta)k + (1-\tau)\overline{B}k^{\beta'}] - \dfrac{1}{1+\gamma}\dfrac{(1-\beta')(1-x)\overline{B}k^{\beta'}}{(1-\alpha)\beta'\overline{B}k^{\beta'-1} - (n+\delta)} & k \in K^s, \\ \\ \dfrac{1-\delta}{1+n}k & k \in K^d. \end{cases}$$

[2] In the equations of consumption and capital accumulation, τ is the tax rate which is given by $\tau = (1-\beta)x + \beta\alpha$ and not $\overline{\tau}$, which incorporates the budget deficit and is compounded only in $\overline{B}$.

In a closed economy model, the incorporation of budget deficit is odd. Unless a budget constraint is considered, budget deficit appears to have only beneficial impact on the economy. On the other hand, the specification of any constraint would impose constraints on the dynamics of the model, and thus the results would be biased.

B.2 Public Capital Stock

Based on Barro's endogenous model, Futagami, Morita and Shibata (1993) developed a model that included two state variables: private and public capital stocks. The rationalization of their modeling strategy of incorporating public capital lies on the fact that many public infrastructures, such as highways, airports, and electrical and gas facilities are stock variables in nature. As they argued, in the theoretical literature on public investment, stock of public capital, instead of the flow of public services, is the productive input to private production.

In section 3.3 is was considered the current flows of government spending, rather than the services of the existing stocks, which generates additions to production. An alternative approach is to allow the government also to accumulate stocks, such as physical infrastructure, i.e. roads.

Capital Accumulation Identities

$$\text{Private Capital Stock: } k_{t+1} = \frac{1}{1+n}[(1-\delta_1)k_t + s_t] \tag{B.2.1}$$

$$\text{Public Capital Stock: } g_{t+1} = \frac{1}{1+n}[(1-\delta_2)g_t + i\tau y_t] \tag{B.2.2}$$

Production Function

$$y_t = f(k, g) = Bk_t^{\beta}(ig_t)^{\zeta} \tag{B.2.3}$$

The difficulty analyzing the implications of this model arise because the production function is a two-dimensional function that cannot be reduced to one-dimensional function since g_t is a stock variable and not a flow (we cannot substitute g_t with τy anymore). Therefore there are three inputs of production: labor, public capital and private capital. The distribution of income then should be divided to each input. However, it is not straight forward how returns on public capital should be distributed among the agents. Finally, another question that need to be addressed is what is the tax rate that government imposes on public capital. These difficulties and the time constrain restrained the analysis of this interesting case. Thus, the incorporation of budget deficit is more appropriate in an open economy model.

Appendix C
Programming

The algorithms in this section provide the basic structure of the algorithms used in the project, but do not fully cover all the variations that were applied. However, only small changes in their structure is needed to derive all the graphs presented in the project.

C.1 Mathematica

Bifurcation Diagrams for the Basic Model

```
(*Capital Accumulation=(g ((1-d) k+B k^b))/((1+n)(1+g))-((1-b) B \
k^b)/((1+g) (b B k^(-1+b)-(n+d))) Equation (23)*)
b = .67;
n = 2.4;
d = .25;
B = 1;
ValuesOfg = Range[0.1, 45, 1./10];
Orbitforg[g_] := With[
  {f = Function[k,
     If[B k^b > ( ((1 - d) k + B k^b)/(1 + g) + ((1 + n) (1 - b) B k^
          b)/((1 + g) (b*B k^(-1 + b) - (n + d)))), (
       g ((1 - d)*k + B* k^b))/((1 + n)*(1 + g)) - ((1 - b)*1*k^
        b)/((1 + g)*(b*B*k^(-1 + b) - (n + d))), ((1 - d)*
       k)/(1 + n)]]},
  NestList[f, Nest[f, 0.0001, 2000], 200]
  ]
ListPlot[Flatten[Table[{g, y}, {g, ValuesOfg}, {y, Orbitforg[g]}], 1],
  ImageSize -> 500, PlotStyle -> PointSize[Tiny],
 AxesLabel -> {\[Gamma], k},
```

```
 PlotRange -> {{.0, 45}, {0, .017}},(*GridLines-> {{16},{0}},*)
 PlotLabel -> "Capital Accumulation", BaseStyle -> {FontSize -> 20}]
(*Save capital under data to use it to find consumption, production \
and investment*)

data = Flatten[Table[{g, y}, {g, ValuesOfg}, {y, Orbitforg[g]}], 1];
(*Production=B k^b*)
Production[g_, k_] := {g, B*k^b}
ListPlot[Apply[Production, data, {1}], ImageSize -> 500,
 PlotStyle -> PointSize[Tiny], AxesLabel -> {\[Gamma], y},
 PlotRange -> {{.0, 45}, {0, .065}},(*GridLines-> {{16},{0}},*)
 PlotLabel -> "Production", BaseStyle -> {FontSize -> 20}]
(*Consumption= ((1-d) k+B k^b)/(1+g)+((1+n)(1-b) B k^b)/((1+g) (.67 1 \
k^(-1+b)-(n+d))) Equation(19)*)

Consumption[g_, k_] := {g,
  If[B k^b > ( ((1 - d) k + B k^b)/(1 + g) + ((1 + n) (1 - b) B k^
       b)/((1 + g) (b*B k^(-1 + b) - (n + d)))),  ((1 - d)*k +
      B*k^b)/(1 + g) + ((1 + n)*(1 - b)*B*k^
     b)/((1 + g)*(b*B k^(-1 + b) - (n + d))), B k^b]}
ListPlot[Apply[Consumption, data, {1}], ImageSize -> 500,
 PlotStyle -> PointSize[Tiny], AxesLabel -> {\[Gamma], c},
 PlotRange -> {{.0, 45}, {0, .062}},(*GridLines-> {{16},{0}},*)
 PlotLabel -> "Consumption", BaseStyle -> {FontSize -> 20}]
(*Investment=Saving= y-c= B k^b-( ((1-d) k+B k^b)/(1+g)+((1+n)(1-b) B \
k^b)/((1+g) (.67 1 k^(-1+b)-(n+d))))*)

Saving[g_, k_] := { g,
  If[B k^b > ( ((1 - d) k + B k^b)/(1 + g) + ((1 + n) (1 - b) B k^
       b)/((1 + g) (b*B k^(-1 + b) - (n + d)))), (B k^
      b) - ( ((1 - d)*k + B*k^b)/(1 + g) + ((1 + n)*(1 - b)*B*k^
       b)/((1 + g)*(b*B k^(-1 + b) - (n + d)))), 0]}
ListPlot[Apply[Saving, data, {1}], ImageSize -> 500,
 PlotStyle -> PointSize[Tiny], PlotRange -> {{.0, 45}, {0, .05}},
 AxesLabel -> {\[Gamma],
   S},(*PlotRange->{{.0,45},{0,.052}},*)(*GridLines-> {{16},{0}},*)
 PlotLabel -> "Investment", BaseStyle -> {FontSize -> 20}]
(*Proportion of consumption over production*)

Proportion[g_, k_] := {g,
  If[B k^b > ( ((1 - d) k + B k^b)/(1 + g) + ((1 + n) (1 - b) B k^
       b)/((1 + g) (b*B k^(-1 + b) - (n + d)))), ( ((1 - d)*k +
        B*k^b)/(1 + g) + ((1 + n)*(1 - b)*B*k^
       b)/((1 + g)*(b*B k^(-1 + b) - (n + d))))*100/(B k^b), 100]}
ListPlot[Apply[Proportion, data, {1}], ImageSize -> 500,
 PlotStyle -> PointSize[Tiny], AxesLabel -> {\[Gamma], "% of Output"},
  PlotRange -> {{.0, 45}, {0, 100}},(*GridLines-> {{16},{0}},*)
 PlotLabel -> "Consumption as % of Output",
 BaseStyle -> {FontSize -> 20}]
```

Bifurcation point between stability and instability. The code plots the curve $\theta(\bar{k}) = -1$. Points in the parameter space below a given curve give rise to unstable steady states; and points above to stable ones.

```
(*Bifurcation points between stability and instability (Points below \
the line are unstable; points above the line are stable)*)
\
(*Subscript[H, p](b,g,d,n)=(1-b)/b ((1+n)/g+n+d)^2+(1+n)/g \
((1+n)/g+n+d)+(p-1) ((1+g)(1+n)^2)/((1-b)g^2)==0 *)
n1 = 2.4;
d1 = .25;
p1 = -1;
w = .67;
sol = Solve[(1 - b1)/b1 ((1 + n1)/g1 + n1 + d1)^2 + (1 + n1)/
      g1 ((1 + n1)/g1 + n1 + d1) + (p1 -
        1) ((1 + g1) (1 + n1)^2)/((1 - b1) g1^2) == 0, {b1}];
Plot[Evaluate[Re[b1 /. sol]], {g1, 0, 45},
 PlotRange -> {{.0, 45}, {0, 1}}, AxesLabel -> {Gamma, Beta},
 GridLines -> {{16}, {w}},
 PlotLabel -> "Stable-unstable parameters zone"]
```

C.2 Sage

The following program computes the average Lyapunov exponent for the Productive Government Spending framework

```
'''Lyapunov Exponent for Capital Accumulation of the Productive Government Spending'''
from numpy import *
from pylab import *
import math

# Define Capital Accumulation Equations
def CapAccum(k,g,x,n,d,B,i,t,a,b):
    return g*((1 - d)*k + (1-t)*B*k^b)/((1 + n)*(1 + g)) - ((1 - b)*(1-x)*B*k^
   b)/((1 + g)*((1-a)*b*B*k^(-1 + b) - (n + d)))

def CapDeccum(k,g,x,n,d,B,i,t,a,b):
    return ((1 - d)*k)/(1 + n)

#Define the Derivatives of the Capital Accumulation Function
def deriv(k,g,x,n,d,B,i,t,a,b):
    return (g*(1-d+b*B*k^(-1+b)*(1-t)))/((1+g)*(1+n))+((1-a)*(1-b)*(-1+b)*b*B^2
    *k^(-2+2*b)*(1-x)) /((1+g)*(-d +(1-a)*b*B*k^(-1+b)-n)^2)-((1-b)*b*B*k^(-1+b)
    *(1-x))/((1+g)*(-d +(1-a)*b*B*k^(-1+b)-n))

def derivv(k,g,x,n,d,B,i,t,a,b):
    return (1-d)/(1+n)

#Accuracy
R = RealField(200); R
```

```
def Lyapunov(x,a,i,g,b):
    '''The function returns the Lyapunov Exponent given the the wage tax rate (x),
    the tax rate on capital (a) theportion of budget deficit that Government spends
    productively (i), the future weight (g), the output elasticity
     of Government (Government) and Private Sector (Private) '''
    t=(1-b)*x+b*a
    B=1
    n = 2.4
    d = .25
    ic = .0001
    L=0
    # Set the initial condition to the reference value
    k = ic
    # Throw away the transient iterations
    for w in[1..2000]:
        if (((1-t)*B*k^b) > (((1 - d)*k+(1-t)*B*k^b)/(1 + g)+((1 + n)*(1-x)*(1 - b)*B*k^b)
        /((1 + g)*((1-a)*b*B*k^(-1 + b) -(n + d))))):
            k=CapAccum(k,g,x,n,d,B,i,t,a,b)
        else:
            k=CapDeccum(k,g,x,n,d,B,i,t,a,b)
    # Now store the next batch of iterates
    x1 = [ ] # The iterates
    for w in [1..500]:
        if (((1-t)*B*k^b) > (((1 - d)*k+(1-t)*B*k^b)/(1 + g)+((1 + n)*(1-x)*(1 - b)*B*k^b)
        /((1 + g)*((1-a)*b*B*k^(-1 + b) -(n + d))))):
            k = CapAccum(k,g,x,n,d,B,i,t,a,b)
            x1.append( k )
        else:
            k = CapDeccum(k,g,x,n,d,B,i,t,a,b)
            x1.append( k )
    for w in [0..499]:
        k=x1[w]
        if (((1-t)*B*k^b) > (((1 - d)*k+(1-t)*B*k^b)/(1 + g)+((1 + n)*(1-x)*(1 - b)*B*k^b)
        /((1 + g)*((1-a)*b*B*k^(-1 + b) -(n + d))))):
            L=L+ln(abs(deriv(k,g,x,n,d,B,i,t,a,b)))
        else:
            L=L+ln(abs(derivv(k,g,x,n,d,B,i,t,a,b)))
    L=L/500
    return L
```

The program below, uses the *Lyapunov* function and creates a $3D$ graph, where the three axes represent the three exogenous variables, while the color of each point specifies the value of the average Lyapunov exponent.

```
x1a=[]
x1x=[]
x1i=[]
x2a=[]
x2x=[]
x2i=[]
x3a=[]
```

```
x3x=[]
x3i=[]
x4a=[]
x4x=[]
x4i=[]
#A different set of exogenous variables can be analyzed by fixing one of the three variables bel
for Government in arange(.1,.4,.05):
    for Private in arange(.3,.6,.05):
        for g in arange(.1,20,.5):
            L=Lyapunov(x,a,i,g,Government,Private)
            if (L<=-0.4):
                x1a.append(Government)
                x1x.append(Private)
                x1i.append(g)
            elif(L<=-.2):
                x4a.append(Government)
                x4x.append(Private)
                x4i.append(g)
            elif(L<=0):
                x2a.append(Government)
                x2x.append(Private)
                x2i.append(g)
            else:
                x3a.append(Government)
                x3x.append(Private)
                x3i.append(g)
l1=list_plot(zip(x1a,x1x,x1i),color=’blue’,size=10)
l2=list_plot(zip(x2a,x2x,x2i),color=’green’,size=10)
l3=list_plot(zip(x3a,x3x,x3i),color=’red’,size=10)
l4=list_plot(zip(x4a,x4x,x4i),color=’yellow’,size=10)
l1+l2+l3+l4
```

The code defines the functions of the macroeconomic variables and includes a basic algorithm that returns the trajectory of the economy, given a parameter set and an initial condition

```
from numpy import *
from pylab import *
import math


# Define Equations
def CapAccum(k,B,n,d,b,g,a,x):
    t=(1-b)*x+b*a
    return g*((1 - d)*k + (1-t)*B*k^b)/((1 + n)*(1 + g)) - ((1 - b)*(1-x)*B*k^
   b)/((1 + g)*((1-a)*b*B*k^(-1 + b) - (n + d)))

def CapDeccum(k,d,n,i,B,b):
    return ((1 - d)*k)/(1 + n)

def USpending(k,B,n,d,b,g,a,x):
    ‘‘‘
    The function returns the Unproductive Government Spending
```

```
    '''
    t=(1-b)*x+b*a
    return t*B*k^b


def Consumption(k,B,n,d,b,g,a,x):
    '''
    The function returns the Consumption
    '''
    t=(1-b)*x+b*a
    if (((1-t)*B*k^b) > (((1-d)*k+(1-t)*B*k^b+((1+n)*(1-x)*(1-b)*B*k^b)/((1-a)*b*B
    *k^(b-1)-(n+d)))/(1+g))):
        return ((1-d)*k+(1-t)*B*k^b+((1+n)*(1-x)*(1-b)*B*k^b)/((1-a)*b*B*k^(b-1)-(n+d)))/(1+g)
    else:
        return (1-t)*B*k^b

def Production(k,B,n,d,b,g,a,x):
    '''
    The function returns the level of Production (GDP)
    '''
    return B*k^b

def TotalConsumption(k,B,n,d,b,g,a,x):
    '''
    The function returns the Consumption
    '''
    return Consumption(k,B,n,d,b,g,a,x)+USpending(k,B,n,d,b,g,a,x)

def TotalConsumptionPer(k,B,n,d,b,g,a,x):
    '''
    The function returns the Total Consumption as a percentage of GDP
    '''
    return (Consumption(k,B,n,d,b,g,a,x)+USpending(k,B,n,d,b,g,a,x))
    /Production(k,B,n,d,b,g,a,x)*100


def Capital(k,B,n,d,b,g,a,x):
    t=(1-b)*x+b*a
    x1=[] #stores the values of steady state capital
    # Throw away the transient iterations
    for w in[1..2000]:
        if (((1-t)*B*k^b) > (((1-d)*k+(1-t)*B*k^b+((1+n)*(1-x)*(1-b)*B*k^b)/((1-a)*b*B
        *k^(b-1)-(n+d)))/(1+g))):
            k=CapAccum(k,B,n,d,b,g,a,x)
        else:
            k=CapDeccum(k,d,n,B,b)
        x1.append( k )
    return x1
```

The *Dynamics* returns a list plot of the transitional dynamics and the residuals, given two different set of parameters.

```
def Dynamics(k,B,n,d,b,g,a1,x1,i1,a2,x2,i2):
    CA=[]
    CA1=[]
    CA2=[]
    C1=[]
    C2=[]
    TC1=[]
    TC2=[]
    P=[]
    C=[]
    TC=[]
    gsweep=[]
    f1=Capital(k,B,n,d,b,g,a1,x1,i1)
    f2=Capital(k,B,n,d,b,g,a2,x2,i2)
    for q in [0..19]:
        CA1.append(f1[q])
        CA2.append(f2[q])
        CA.append(f1[q]-f2[q])
        C1.append(Consumption(f1[q],B,n,d,b,g,a1,x1,i1))
        C2.append(Consumption(f2[q],B,n,d,b,g,a2,x2,i2))
        TC2.append(TotalConsumption(f2[q],B,n,d,b,g,a2,x2,i2))
        TC1.append(TotalConsumption(f1[q],B,n,d,b,g,a1,x1,i1))
        #P.append(Production(f1[q],B,n,d,b,g,a1,x1,i)-Production(f2[q],B,n,d,b,g,a2,x2,i))
        #C.append(Consumption(f1[q],B,n,d,b,g,a1,x1,i)-Consumption(f2[q],B,n,d,b,g,a2,x2,i))
        #TC.append(TotalConsumption(f1[q],B,n,d,b,g,a1,x1,i)-TotalConsumption(f2[q],B,n,d,b,g,a2
        gsweep.append(q)
    w1=list_plot(zip(gsweep, CA),color='blue',fontsize=20,plotjoined=true,thickness=1.5)
    w2=list_plot(zip(gsweep, CA1),color='red',fontsize=20,plotjoined=true,thickness=1.5)
    w3=list_plot(zip(gsweep, CA2),color='blue',fontsize=20,plotjoined=true,thickness=1.5)
    w4=list_plot(zip(gsweep, C1),color='red',fontsize=20,plotjoined=true,thickness=1.5)
    w5=list_plot(zip(gsweep, C2),color='blue',fontsize=20,plotjoined=true,thickness=1.5)
    w6=list_plot(zip(gsweep, TC1),color='red',fontsize=20,plotjoined=true,thickness=1.5)
    w7=list_plot(zip(gsweep, TC2),color='blue',fontsize=20,plotjoined=true,thickness=1.5)
    w8=w2+w3
    w8.axes_labels(['Time','Capital Accumulation'])
    w8.show(xmax=10,frame=true)
    w9=w4+w5
    w9.axes_labels(['Time','Consumption'])
    w9.show(frame=true)
    w10=w6+w7
    w10.axes_labels(['Time','Total Consumption'])
    w10.show(xmax=10,frame=true)
```

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