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The Use of Stochastic Processes in Business Cycle Analysis

Hyman P. Minsky Ph.D.

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I. Introduction

First, I wish to apologize for the title of this paper: it is much too pretentious for the content. What will be considered is the use of probabalistic assumptions in the construction of Accelerator-Multiplier models: that is only a special form of business cycle models will be discussed.

Secondly, this is a paper in economics. The essential point to be made is that an examination of the generating relation for the parameters of these business cycle models leads to the conclusion that these parameters have attributes of random variables. Therefore the little this paper contributes is to economies, not to mathematics or statistics. In fact, the paper ends where the mathematics begins. The novel feature if any, consists in identifying the generating relation for the parameters of macro-economic models as naturally leading to a stochastic process.

Linear Accelerator-multiplier models have been around in business cycle analysis for some time: certainly since 1938. However, all such linear models suffer from a common defect: no matter what values are assigned to the consumption and investment parameters, the time series that the model generates is unsatisfactory. Various modifications have been suggested to circumvent this difficulty: we can mention (a) the substitution of non-linear accelerator forms for the linear forms (by Goodwin and Hicks); (b) the addition of a "synchronized" autonomous investment-impulse which is identif~~ying~~ied ~~the impulse~~ with Schumpeter's innovations (also done by Goodwin) and (2) the addition to a damped linear form of a "random" energy source (as suggested by Frisch in his 1933 Propagation and Impulse Problems paper: This was also taken up by Fisher in the A.E.R. in 1952.[1] The Fisher paper will be discussed in some detail below).

In another paper I will discuss the non-linear models: particularly that of Goodwin. Today I wish to take up the stochastic way out of the difficulty inherent in linear models. In order to do this I wish to distinguish between an "error" approach and a "random parameter" approach to such models. The above mentioned paper by G. H. Fisher is an example of the Error approach to the use of stochastic variables in business cycle theory.

II. The stochastic error approach: The Fisher Paper.

Consider a Hicks type (induced investment is a linear function of the change in income) accelerator multiplier model in which the accelerator coefficient $\beta < 1$ The time path of income is damped. If α the consumption coefficient is sufficiently small for the given β the time series generated will be damped oscillatory. This implies that the cycle will die out; therefore, unless the model is modified it is unsatisfactory for business cycle analysis. Fisher modifies this model by imposing upon this damped cycle an outside energy source, under the guise of a "random shock".

Briefly, Fisher considers a consumption function and an investment function subject to random errors: that is,

1) $C_t = \alpha Y_{t-1} + \mu_t$ and

2) $I_t = \beta(Y_{t-1} - Y_{t-2}) + \nu_t$ The μ_t and ν_t are

(1) G.H. Fisher: Some Comments on Stochastic Macro-Economic Models AER September 1952 p 515-535

λ an exogenous government investment.

random errors. We therefore have that

3) $Y_t = (\alpha + \beta) Y_{t-1} - \beta Y_{t-2} + w_t + \lambda$ where $u_t + v_t = w_t$. Fisher assumes that w_t is normally distributed with a mean of zero and a variance $\sigma^2_{w_t}$. Fisher ~~who~~ also assumes that $\alpha = .7, \beta = .5$, which generate ~~resulting in~~ a damped oscillatory time ~~paths~~ series. In order to examine how such a random shock model ~~subject to random shocks~~ behaves over time, Fisher computed Y_t for 100 time periods, obtaining his w_t by random selections from a simulated normal population of the w_t. The error model, in contrast to the mechanical model, exhibited a persistent cycle: the random shocks counteracted the damping effect due to the assumed values of α and β.

However Hicks in his volume on the Trade Cycle concluded that "the theory of damped fluctuations and erratic shocks prove unacceptable ... "for the correlation between successive cycles is quite small." This objection is true but irrelevant. The rejection is based upon the unwarranted assumption that the period generated by such a random shock model is basically the period of the cycle due to accelerator-multiplier mechanism.

The random shocks in this case are samples drawn from a universe. In such drawings you expect to have runs of various lengths of similar valued shocks. Such runs will tend to build up the amplitude of a deviation from the equilibrium level of income. These large deviations will lead to the persistence of the oscillatory movement. If the random shocks are of a significant size relative to the equilibrium level of income, the resulting time series would tend to show a small correlation between the corresponding terms of successive cycles as determined by the accelerator-multiplier mechanism. Nevertheless the time series would exhibit an almost cyclical movement with varying amplitudes ~~to~~ of the individual cycles.

Fisher's approach is vulnerable to a second comment made by Hicks: " quite moderate reduction in the investment coefficient leaves us ~~with fluctuations that are mainly random:~~ with fluctuations that are mainly random: with fluctuations, that is, that remain unexplained." Fisher assumed that the standard deviation of the random shocks σ_{w_t} is 5 billion: his equilibrium level of income is 57 billions. A positive shock equal to or greater than one standard deviation will occur about 16% of the time, on the average; in Fisher's simulated population, such shocks occur about 19-1/2% of the time. Let us assume that $Y_t = Y_{t-1} = Y_{t-2} = 57$ billion and that two successive shocks of +5 occur. $Y_{t+1} = 62$, $Y_{t+2} = 68$ in this event. In Fisher's population, the mean value of the shocks $\geq$ is 7.7 . Using this mean value, we get $Y_{t+1} = 65$ $Y_{t+2} = 74$. By inspection of the time series exhibited in Fisher's article we get that an income ≥ 68 Billion occurred 6 times, and an ≥ 74 Billion 3 times. The population would on the average yield an income $\geq$ 68 Billion by means of two successive positive shocks of 5 Billion or more 4 times out of 100. Taking into account that different lengths of runs of similar signed shocks which could yield the extreme deviations we conclude that Fisher's series is essentially the result of the random shocks. ~~Essentially~~ The effect of the accelerator-multiplier ~~model~~ set up is to make each period's income ~~and~~ equal to the weighted average of two previous incomes plus or minus a shock. This Fisher model therefore is vulnerable to the contention of Hicks that it leads to an unexplained cycle: it is therefore similar to the "tomorrows income will be todays income" school of business cycle forecasters.

It is also obvious that if the variance of the shocks were smaller, for the same α and β that Fisher assumed the result would be a damped cycle: Only by leaving a great deal of the cyclical phenomena unexplained can Fisher achieve his result.

The approach to the use of stochastic processes in economic analysis which Fisher used and which Hicks criticised assumes that random shocks are attached to a systematic generating function. The introduction of this approach into economic analysis can be imputed to Haavelmo's paper on the "Probability Approach to Econometrics". The ideology of Haavelmo's approach is given by the following quotations:

> What we want are theories that, without involving us in direct logical contradiction, state that the observations will as a rule cluster in a limited subset of the set of all conceivable observations, while it is still consistent with the theory that an observation falls outside this subset now and then...[1], and

> The question is not whether probabilities exist or not but whether -- if we proceed as if they existed we are able to make statements about real phenomena that are correct for practical purposes.[2]

The approach embodied in those two quotations can be derived from two sources: (1) the residual variations in correlation analysis after the systematic effect of the "variables" has been eliminated, and (2) errors of observation where the fallibility of humans and of the measuring instruments combine to yield observations which do not, in detail, conform to the real world values. The Haavelmo approach leads to the formulation of economic problems in the light of statistical testing techniques. This is an appropriate transformation of economic models where the problem is to apply such tests to

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1. Haavelmo, T: The Probability approach in Econometrics: Econometrica Vol 12 Supplement July 1944 p 40.

2. <u>Ibid.</u>

economic data. However, it is not the appropriate approach to the construction of a "stochastic model".[3]

III. The Random Parameter Approach

As an alternative to the Haavelmo errors of observation and unexplained residuals approach to the use of stochastic processes in economics, we can contrast a formulation of an accelerator-multiplier model in which the parameters have stochastic properties. Such a model assumes ~~postulates~~ that the behavior of the economy can be best explained in terms of ~~and its processes contain~~ elements which are in their very nature random ~~nature random~~ variables. This assumption ~~postulate~~ will be embodied in statements which assert that the values of certain attributes of the elementary economic units, firms or housholds, after allowing for the constraints of market conditions, technological, production, or utility relations, and specified behavior principles, may still take on any of a set of values. These attributes will be characterized by a probability distribution. Therefore, in any model in which such an attribute enters as a parameter, the variables of the model are not strictly determinate. ~~It may be true that the time process involved in the generation of these attributes may yield a strictly determinate asymptotic distribution of the values of the parameters and therefore of the variables.~~ "To characterize the economic

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3. The tendency in such 'testing' stochastic models to assume that the random term is distributed normally, with a zero mean is carrying the assumption made in errors of measurement analysis into their theory, and this is consistent with the origin of this approach. Certainly in economic analysis we should expect to find that the probability distribution of the random variables which are generated by economic processes may deviate from the 'normal' probability distribution. We should expect the processes of economic life would often lead to 'equally likely' alternatives (a rectangular distributinn) and to distributions in which 'almost all' events would have the same value: 'Poisson' type distributions.

process with the aid of a random process implies that certain parameters, e.g. the output of a firm, its profitability during a given period, its investment decisions are regarded as variables that with given probabilities assume given values; i.e. they are considered random variables. The probability distribution of a random variable or of a combination of such random variables at a certain moment is determined by the past of the economic process."[4]

In attempting to set up models for investment behavior economists have to rely upon expected values to achieve a meaningful statement. Expectation relations are inherently of ~~the~~ a nature ~~where~~ such that for the different economic units different expectations can co-exist; and the 'distribution of expectations' becomes an element in the aggregate investment relation. The investment relation (i.e. the amount of investment forthcoming during any period of time) is one that is not strictly determined by the observable and measurable variables of the economic system.

If we attempt to apply the random process approach, as stated above, to an accelerator-multiplier model of income determination, we have, naturally, to assume that the 'α's' and the 'β's' are the random variables. For the observable and measurable determinants of income in these models are the previous periods income (ignoring "errors of measurement"). Present income is not strictly determined by these historic variables, if we assume that the α and β parameters of the income-generating model are random variables.

4. Paraphrase of a statement of O. Lundberg, "On Random Processes and their Applicability to Sickness and Accident Insurance," Almquist and Wicksell's Boktryckeri, A. B. Uppsala, 1940, p. 3.

In what follows we will specialize by taking up ~~why one~~ only one induced investment coefficient. Obviously, similar considerations enter into the determination of the consumption coefficient.

A model of the economy which yields an accelerator coefficient that naturally is of the nature of a probability distribution can be easily constructed. Let us assume that each firm is an element in a Marshallian Industry, that it is a unit in a set of firms producing a homogenous product. The firms in the industry vary in a manner which is consistent with the doctrine of the representative firm: differences in their cost structure, production function, and in the nature (perhaps speed ~~spread)~~ of their reaction to changes. The economy consists of many such industries, and in each industry we assume that the behavior of the firms is determined by the industry parameters and not by the situation in other industries.

A change in income implies that the set of demand curves for the products of the particular industries shifts. However, firms are the investing units. What is needed for each industry is a transformation of the shift in the industry demand curve into a change in a parameter upon which the firms in an industry base their investment decisions. The impact upon a firm of a shift in the industry demand curve depends upon the market structure of the industry. In a competitive industry, a shift in demand affects firms by means of a change in the market price of the product. This change in price implies that, at the old price, a quantity different from the quantity actually taken would now be taken. Let us assume that the investment decision of firms is based upon the firms' estimate of the change in the quantity that the market will take at the price that ruled prior to the shift in demand. Each firm ~~will~~ estimates the quantity of the product which it would be profitable for it to produce by allowing its plant size to vary. The investment by a particular firm which is induced by a change in income will be the change in fixed capital necessary to alter its plant size, plus whatever change in working capital ~~that~~ is needed to produce the new optimum output. Such induced investment in a competitive industry may take place by means of a change in the number of firms in the industry rather than by means of an alteration in the size of the plants of existing firms.

We therefore have a particular firm investment relation of the form:

[illegible] on γ (see below) [illegible] γ as a function of shift and i

4. $i_{\lambda\rho}(t) = \gamma\left(Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)\right)$ where

i = investment by a particular firm

λ = industry index

ρ = firm index

γ = coefficient of induced investment for a particular firm

$Q_{\lambda\rho}(t-1)$: estimate by the ρ firm of the quantity demanded at the price of t-2 during the period t-1

$Q_{\lambda}(t-2)$: quantity actually taken at the price of t-2 during the period t-2

$Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)$: the ρth firm's estimate of the industry demand curve shift

The amount of investment that takes place in an industry will be given by:

5. $i_{\lambda}(t) = \sum_{\rho} i_{\lambda\rho}(t) = \sum \gamma_{\lambda\rho}\left(Q_{\lambda\rho}(t-1) - Q_{\lambda}(t-2)\right)$.

Heroically assuming that all firms in the λ industry have the same estimates of $Q_{\lambda\rho}(t-1)$ ~~are the same~~[5], we have

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5. That is, every firm in the λ industry has the same estimate of the elasticity of demand for the product.

6. $i_\lambda(t) = [Q_\lambda(t-1) - Q_\lambda(t-2)] \sum_\rho \gamma_{\lambda\rho}$

The amount of investment induced in the economy is the sum of the investment of the different industries:

7. $I_t = \sum i_\lambda(t) = \sum_\lambda [(Q_\lambda(t-1) - Q_\lambda(t-2)) \sum \gamma_{\lambda\rho}]$.

However, by the aggregate accelerator relation we also have that $I_t = \beta(Y_{t-1} - Y_{t-2})$. Therefore we have that

β_t

~~$I_t = \quad_t(Y_{t-1} - Y_{t-2}) = [(Q \quad (t-1) - Q \quad (t-2)) \quad]$~~

8. $\beta_t(Y_{t-1} - Y_{t-2}) = \sum_\lambda [(Q_\lambda(t-1) - Q_\lambda(t-2) \sum \gamma_{\lambda\rho}]$

9. $$\beta_t = \frac{\sum_\lambda [(Q_\lambda(t-1) - Q_\lambda(t-2)) \sum \gamma_{\lambda\rho}]}{Y_{t-1} - Y_{t-2}}$$

$\sum_\lambda (Y_t)$

Out

If the set of shifts in industry demand curves which is implied by a change in income ~~is determinate,~~ and ~~if~~ the impact of those shifts in industry demand curves upon particular firms investment ~~is~~ are determinate, then β_t, the coefficient of induced investment will be determinate, e.g. will not be a random variable. For the aggregate coefficient of induced investment to be a constant we have to assume that each $\gamma_{\lambda\rho}$ is independent of the size and direction of the shift in its particular industry demand curve, and that the shift in each demand curve $Q_\lambda(t-1) - Q_\lambda(t-2)$ is a fixed ratio to $Y_{t-1} - Y_{t-2}$. Then we would have that

e.g.? r.v.?

10. $$\beta_t = \frac{\sum [(Q_\lambda(t-1) - Q_\lambda(t-2)] \sum_\rho \gamma_{\lambda\rho}}{Y_{t-1} - Y_{t-2}}$$

$\sum_\lambda$ I don't understand the difference between 9 & 10

11. $\sum_\rho \gamma_{\lambda\rho} = \bar{\gamma}_\lambda$ a constant

$\bar{\gamma}_\lambda$ is the "industry" coefficient of induced investment.

You can change, define $\bar{\gamma}_\lambda$ as this means

12. $\dfrac{Q_\lambda(t-1) - Q_\lambda(t-2)}{Y_{t-1} - Y_{t-2}} = \xi_\lambda$ = constant: as p_λ (t-2) is used to estimate $Q_\lambda(t-1)$, ξ_λ is equivalent to the marginal propensity to consume a particular good.

13. $\beta_t = \sum \xi_\lambda \gamma_\lambda$ = constant. The above are the implicit assumptions in any constant accelerator coefficient formulation

If the set of shifts in industry demand curves which is associated with a change in income is determined by a process which can be considered as analagous to sampling, then at any time the shift in a particular industry's demand curve, which is the immediate cause of inducing investment, can be considered as a sample drawn from a universe. In addition, the amount of investment which a given shift in an industry demand curve will induce can be interpreted as depending upon the reactions of the affected firms, and the firms' reactions to particular stimuli may ~~have~~ be characterized by a probability distribution.[10] In both circumstances the β_t coefficient for the economy is a random variable.

Combining the ~~two~~ above arguments, we have that the probability distribution of β_t depends upon: a) the probability distribution of particular shifts in industry demand curves given a particular change in income $Y_{t-1} - Y_{t-2}$; b) the probability that a particular set of firms ω from the set of all firms Ω being affected by a

10. If an industry consiste of a large number of firms, and if the probability distribution of reactions by firms is the same, then aggregate investment will be a summation of the reaction of a large number of firms. By "laws of large numbers" the summation will tend to be stable. If an industry consists of a small number of firms, the summation of random variables which is the aggregate investment relation will tend to be unstable. This can be interpreted as implying that for competitive industries the stochastic elements in the determination of β is relatively unimportant, whereas for oligopolistic industries the stochastic element will tend to be more significant.

particular change in income $Y_{t-1} - Y_{t-2}$. If the above probability relations apply in the determination of β_t, we would no longer expect a fixed relation to exist between a change in income and a change in investment.

In order to contrast a true stochastic process with the error process which is ~~Haavelmo's~~ Fishers approach, we assume that β is a random variable whose value at any moment of time is drawn from a probability distribution which depends upon a) the structure of demand curve shifts which result from a given change in income; b) the set of firms for which the resultant demand curve shifts imply investment; c) the relation between output and capital stock for each firm. As β is a function of a subset of firms drawn from the set of all firms, it is a true random variable.

If we assume that the structure of demand curve shifts which result from a given change in income is independent of the level of income or of the change in income (that the marginal propensity to consume particular goods is constant), and if we assume that the magnitude of the individual firms accelerator coefficient is independent of the magnitude of the shift in the industry demand curves, then we have that the aggregate accelerator probability distribution is independent of the level or change in income. The probability distribution of β will be independent of the time path of income. Such a model of the accelerator process results in statements that given the value of α, the realized value of β will be in the interval which results in the economy being in either damped asymptotic, damped oscillatory, explosive oscillatory, or explosive ~~or~~ states a determinate percentage of time. Such a proposition is truly stochastic, as it is based upon a frequency distribution of the β's from which the observed values are drawn. For example, in an accelerator-multiplier model of the type $Y_t = (\alpha + \beta)Y_{t-1} - \beta(Y_{t-2})$ the following table gives the range of values of β which, for given α's, place the economy in each state:

Values of α	Values of β States of the economy			
	DAMPED		EXPLOSIVE	
	monotonic	oscillatory	oscillatory	monotonic
.9	0-.47	.47-1	1-1.73	1.73-
.8	0-.30	.30-1	1-2.10	2.10-
.7	0-.20	.20-1	1-2.40	2.40-
.6	0-.14	.14-1	1-2.66	2.66-
.5	0-.08	.08-1	1-2.92	2.92-

The probability of the economy being in any state depends upon the probability of β having the value appropriate to that state. For example, with $\alpha = .9$ the probability of the economy being explosive cyclical, depends upon the probability of β having a value of between 1 and 1.73. As we are using a Hicks type model, the probability of the economy being damped or being explosive depends solely upon the value of the β coefficient.

~~In an attempt~~ To illustrate how such a purely random variable

set up

would affect the operations of the accelerator and multiplier model, two test runs were made using the values of the constant, λ, of the marginal propensity to consume α, and of Y_{t-1} and Y_{t-2} that Fisher used in his 'random variable' model. In the first run β was assumed to have a rectangular distribution, with the values of $\beta = 0$, $= .25$, $= .50$, $= .75$, $= 1.0$, $= 1.5$, $= 2.0$, $= 3.0$, $= 4$ all being equally probable.

The resultant series exhibited a great amplitude of fluctuation in the first half of the series. Then because of a run of values of

coefficients which lead to a highly damped movement of income, the series exhibited a very damped cycle. Of course in such a series if $Y_{t-1} = Y_{t-2} = \frac{\lambda}{1 - \alpha}$, the cycle would die out. The damping of the series was so great that in the latter part of the sample the cycle well nigh disappeared.

A second test of β as a random variable was made using a triangular frequency distribution ~~of~~. The frequency distribution from which the sample of β values were drawn was:

	Relative Frequency
0	1
25	2
50	3
75	4
1.00	4
1.50	4
2.00	3
3.00	2
4.00	1

The time series which resulted does not exhibit the extreme fluctuations that the time series derived from a rectangular distribution of β exhibited. The fifty period time series also did not show the 'damping' of the cycle that the rectangular distribution exhibited. The reason is obvious: with the probability distribution in the second case, the chances of a 'run' of values of the β coefficient which leads to a highly explosive or a highly damped movement is much lower than in the rectangular distribution. As a result the extreme flucuations and the extreme damping associated with the rectangular distribution do not occur.

We could continue to analyse the implications of β being a probability distribution independent of the level of income, or the path of income by experimenting with additional frequency distribution of β, taking samples with replacements from these frequency distributions and observing, for specified values of α the resultant time series. However, the assumptions that were made:- that the structure of demand curve shifts which result from a given change in income is independent of the level of income or of the change in income and that the magnitude of the individual firms accelerator coefficient is independent of the magnitude of the shift in the industry demand curve (in order to derive the probability distribution of β independently of the level or the change in income) - are strong. Let us assume ~~assumptions by assuming~~ that the expected value (mean) of the frequency distribution of $\overline{\beta}_t$ depends upon the change in income and the difference between last periods income and the previous peak income.

14. $$\overline{\beta_t} = \varphi(Y_{t-1}, Y_{t-2}, Y_{t-1} - \overset{*}{Y}).$$ This model

where $\overline{\beta_t}$ is the mean of the frequency distribution of β, leads to ~~is~~ a non-linear stochastic model: let us assume that the value of the variance of the frequency distribution of β is independent of the value of $\overline{\beta}_t$, and that the mean and the variance are the only relevant moments of the frequency distribution of β_t. We can now write the income generating function as:

15. $$Y_t = \lambda + \alpha(Y_{t-1}) + (\overline{\beta_t} + \mu\sigma_t)[Y_{t-1} - Y_{t-2}] =$$
$$\lambda + \alpha(Y_{t-1}) + \overline{\beta}_t(Y_{t-1} - Y_{t-2}) + \mu\sigma_{\beta_t}(Y_{t-1} - Y_{t-2})$$

where the actual value of the accelerator coefficient at time t, $\beta_t = \overline{\beta_t} + \mu\sigma_{\beta_t}$. Depending upon the relation between σ_{β_t} and

$\overline{\beta_t}$, and upon the frequency distribution of β_t we have that we can assign a probability to β_t falling within any range $\beta_0 < \beta_t \leq \beta_1$ e.g. $P\,[\beta_t > \beta_0$ and $\beta_t \leq \beta_1]$. This can be written as:

$\beta_0 < \beta_t \leq \beta_1$

$$16. \quad P_\beta = \int_{\beta_0}^{\beta_1} \varphi_1(\overline{\beta_t}, \sigma_{\beta_t})\, d\beta_t \,. \qquad \div \int_{-\infty}^{\infty} \varphi_1(\overline{\beta}_t, \sigma_{\beta t})\, d\beta_t$$

For each value of α we can define the values of β_0 and β_1 which puts the economy in any of the four states, and the given values of Y_{t-1} and Y_{t-2} determine the mean value of β_t. Therefore we can determine the probability that the economy will be in the different states.

For example, the probability of damped asymptotic state for $\alpha = .8$ is

$$17. \quad P_1 = \int_0^1 \varphi(\overline{\beta_t}, \sigma_{\beta_t})\, d\beta_t, \text{ where}$$

$$18. \quad \overline{\beta_t} = \varphi(Y_{t-1} - Y_{t-2}, Y_{t-1} - \overset{*}{Y}) \,.$$

As the probability distribution of β_t is a function of the time path of income, we have that the probability of β_t being such as to place the economy in each of its four states is a function of the path of income. If the variance of the probability distribution is small, we have that the probability of the economy remaining in the expbsive state, where the value of $\overline{\beta_t}$ is high, is greater than the probability of the economy remaining in damped oscillation, where the value of $\overline{\beta_t}$ is small.

We have assumed that the effect of the rate of change of income is to shift the mean value of the frequency distribution of the accelerator coefficient, leaving the variance of the frequency distribution of the accelerator coefficient unchanged. We have also assumed that this variance of the frequency distribution of the accelerator coefficient is small ~~in~~ with respect to the 'explosive' values of the accelerator coefficient and relatively large with respect to the damped values of the accelerator coefficient. We therefore have a model in which the probability that random variation will lead to a change in the direction of the movement of income is high when income is changing slowly, but the random process has a small probability of affecting the value of the accelerator coefficient sufficiently to change the state of the model when income is changing rapidly.

A succession of high values of the accelerator coefficient in relation to the expected value of the accelerator coefficient may, if the economy is in a damped state, lead to an explosive movement. A succession of small values of the accelerator coefficient in relation to the expected value of the accelerator coefficient may, by decreasing the rate of growth of income, lower the expected value of the accelerator coefficient through a number of time periods so that if the economy had been in an explosive state, it enters a damped state. ~~This~~ Such an stochastic formulation of the accelerator generation process can be combined with ~~the~~ a model of the accelerator generating relation which leads to either explosive or stagnant states as stable states. ~~This~~ Such an inflation-stagnation model ~~did~~ does not contain a satisfactory mechanism which would result in a change of the economy from State A to State D and vice versa. A combination of the random element and the systematic element makes it unnecessary to posit 'shocks' or 'crises' of the magnitude of the stock market crash of 1929 or of World War II in order to have the economy shift from one of its stable states to another of its stable states. Although not a determinate relation, the allowance for the variance of the accelerator coefficient is an endogenous economic phenomena, for it is simply a statement to the effect that the investment reaction of a particular economic unit to a given economic change (a change in income) is, to some extent, indeterminate. As a result overall economic behavior which is due to the coefficients of macro-economic models, such as the accelerator coefficient, is, to some extent, indeterminate.

The stochastic parameter approach, combined with a non-linear formulation of the accelerator model, opens up many possibilities for empirical and theoretical work. It is obvious that I have not worked out the mathematics of the mixed integral- difference equation that this approach yields: I do not know if the solution is within the scope of existing mathematics. I intend to find out. Secondly, the possibilities of empirical work that is opened is in the direction of the relation between a change in income and the investment relation of individual firms. Do all firms tend to expand "proportionally", are the coefficients of induced investments for different firms the same: are the coefficients the same for the different industries? Only by knowledge of this sort can the real world frequency distribution of be estimated: and from this real world frequency distribution the adequacy of a stochastic formulation of the accelerator as a generator of business cycles be estimated.