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Exploring a Generalized Partial Borda Count Voting System

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Abstract

The main purpose of an election is to generate a "*fair*" end result in which everyone's opinion is gathered into a collective decision. This project focuses on Voting Theory, the mathematical study of voting systems. Because different voting systems yield different end results, the challenge begins with finding a voting system that will result in a "*fair*" election. Although there are many different voting systems, in this project we focus on the Partial Borda Count Voting System, which uses partially ordered sets (posets), instead of the linearly ordered ballots used in traditional elections, to rank its candidates. We introduce the Generalized Partial Borda Count Voting System, and explore which properties of Partial Borda are still satisfied in this general setting.

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Dedication

This project is dedicated to all the current math majors who have a passion for mathematics but feel intimidated by the subject. It may be difficult now, but it will all pay off in the end. You can do it!

Acknowledgments

First and foremost, I would like to thank God for giving me strength and directing my path through these last four years. To my parents, thank you for your unconditional love. Thank you for always supporting me and inspiring me to become the woman that I am today. To my advisor Lauren Rose, I could not have done any of this without you. Your will and your patience to help me learn and guide me has been amazing. Thank you for always motivating me and helping me acknowledge my accomplishments even when I didn't think anything of them. To the professors in the math department, your endless knowledge and passion for mathematics continue to inspire me. To my dear friends, with whom I've shared countless memories and laughs with, to those who pushed me to work hard, who refused to give up on me when I gave up on myself, and always made me think about the positive, thank you. To those who supported, motivated, and inspired me, thank you. To Afropulse, thank you for helping me find my voice.

1

Introduction

In a typical election, we have a set of candidates and a set of voters. Each voter reports their preference for the candidates in the form of a vote.

The study of voting systems, known as Voting Theory, is the mathematical treatment of the process by which democratic societies, or groups resolve the many and conflicting opinions of the members of the group into a general consensus. A valid voting method is a voting system that enforces rules to measure how votes are accumulated in order to generate a "$fair$" end result. While the process of voting is "$fair$" and simple when it involves two candidates, it becomes more difficult to assess which voting systems are fair when we increase the number of candidates.

In this project we explore the different ways of selecting a winning candidate in a given election. We focus on examining a voting system based on the unique properties analyzed in *"A Borda Count For Partially Ordered Ballots"*[1]. We modify the weight function used in their paper and distinguish the different outcomes obtained with our new weight function. To begin, we start with an example.

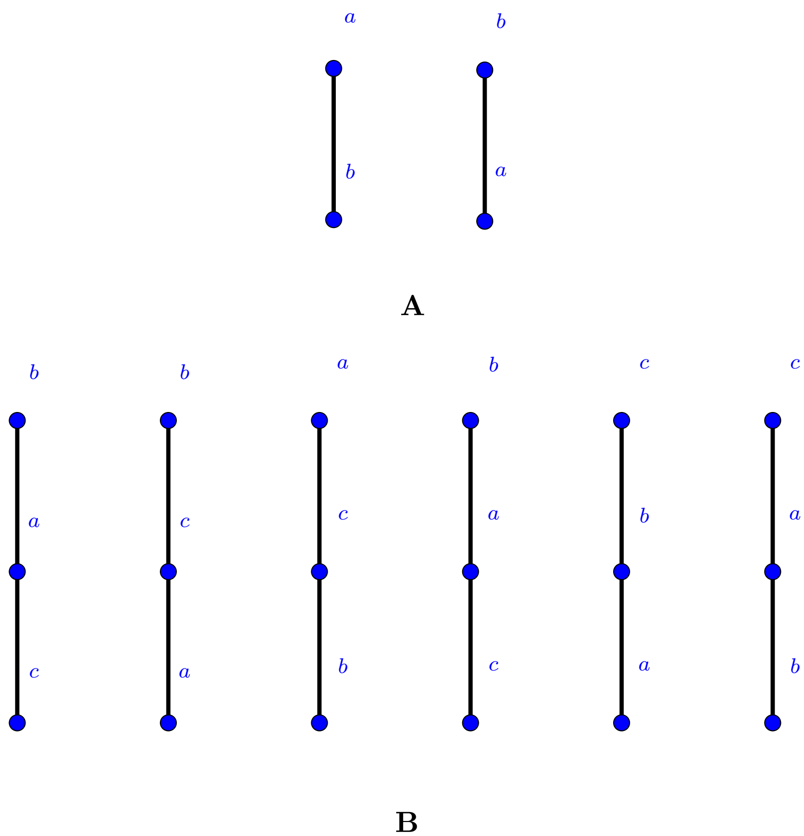


Figure 1.0.1. *2 Candidate Election(A) vs. 3 candidate Election(B)*

The following example illustrates the difference between a two candidate election and a three candidate election.

Example 1.0.1. Suppose a voter has the choice to make a decision between two candidates $\{a, b\}$. Based on his preference, the voter will always choose one candidate over the other as illustrated in Figure 1.0.1(A), *i.e.*, the voter will pick a over b, or b over a. Then the candidate the voter prefers most is the winner.

However, if the voter has to choose between three candidates, $\{a, b, c\}$, as illustrated in Figure 1.0.1(B), the notion of ranking the candidates becomes challenging. Hence, increasing the number of candidates in the election makes the decision more complicated.

$\Diamond$

Generally speaking, there are many different ways of selecting the winning candidates in an election. For illustrative purposes, we provide an informal example.

Example 1.0.2. Four students in the Bard Student Government committee are asked to pick a color that best represents the Bard student body and will serve as the new school color. The color choices are: red, yellow, blue and orange. Each student on the committee is asked to pick a color based on their preferences. Table 1.0.1 shows each students preference in descending order. We interpret this election in various ways below:

Table 1.0.1. School Colors

Student 1	**Student 2**	**Student 3**	**Student 4**
red	red	red	yellow
yellow	yellow	yellow	orange
orange	orange	blue	blue
blue	blue	orange	red

First, if we pick the winning color based on the students first choice, then red is the winner. Note that in the first row red has 3 votes, while yellow only has 1.

Second, if we pick the winning color based on the students least number of last place votes, then yellow is the winner. Note that yellow wins since it has 0 last place votes. In this case none of the students will be disappointed because none of them picked yellow as their last place vote. However, if the color red is chosen, we will have one student student disappointed, which does not result in a fair election.

$\Diamond$

Example 1.0.2 demonstrates how different voting methods yield different results. The conflict in Voting Theory lies behind conducting a "$fair$" election using different voting methods.

According to American economist Kenneth Arrow, there is no consistent method of making a fair choice among three or more candidates with preferential voting, as in the examples above. Arrow defines a preferential voting method to be a social welfare function that ranks social states as less desirable, more desirable, or indifferent for every possible pair of social states.

According to Arrow, a social welfare function must satisfy [2] :

Pareto Condition: If every voter prefers candidate x over candidate y, then the group prefers x over y.

Independence of Irrelevant Candidates: If every voters preference between x and y remains unchanged, then the group's preference between x and y will also remain unchanged (even if voter's preferences between other pairs like x and z, y and z, or z and w change).

Non-dictatorship: No single voter possesses the power to always determine the group's preference.

Theorem 1.0.3. ***(Arrow's Impossibility Theorem)*** *If C has at least three elements and the set P of individuals is finite, then it is **impossible** to find a social welfare function for C satisfying the **Pareto condition, Independence of Irrelevant Candidates,** and **non-dictatorship**.*[2]

There are many different voting systems used to conduct an election. In this project we focus on a specific method of voting called *Partial Borda Count Voting*, where we use a weight function to determine the outcome of an election. This project was motivated by the results of Cullinan, *et.al.*[1] on the properties of the Partial Borda Count Voting system. Exploring this voting method and its properties will serve as the main focus of this paper.

In Chapter 2, we explore different voting systems used in an election, mainly Preferential, Plurality, and Borda Count Voting. We then introduce a specific Fairness Criteria known as the *Majority Criterion* and show its effect on these voting systems. We also introduce partially ordered sets, and the Partial Borda Count Method.

In Chapter 3, we introduce four specific properties of voting systems and give examples of each with respect to the Partial Borda Count weighting procedure.

In Chapter 4, we introduce the generalized weight function $w = \alpha\mathbf{d} + \beta\mathbf{i}$ and explore the special case $\alpha = 3$ and $\beta = 1$, to see which properties from Chapter 3 hold. We will also see if these properties hold for any other α and β's.

Finally in Chapter 5, we introduce the future work that can be done with the properties explored in Chapters 3 and 4. We introduce tournaments, and apply these properties to simple digraphs in tournaments and explore the different properties of tournaments and their relation to posets.

2

Preliminaries

While there are many voting methods, in this section we focus on three: *Preferential Voting*, *Plurality Voting*, and *Borda Count Voting*. We show that each system yields different winners and determine whether or not each election is "$fair$".

2.1 A "$Fair$" Election

The example presented in the previous chapter, demonstrates the different effects that various voting methods have on determining the winner of an election. In this chapter we show the reader three different voting systems that determine different winners but do not necessarily seem fair. This means that some of these voting systems do not satisfy the *Fairness Criteria.*

In general, the *Fairness Criteria* determines whether or not an election is "$fair$". We give a formal definition below.

Definition 2.1.1. The **Fairness Criteria** is a group of mathematical criterions, namely, *Condorcet Criterion*, *Majority Criterion*, and *Pareto Condition*, used to determine the fairness of an election. △

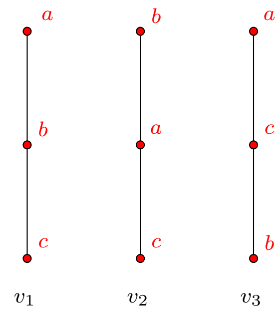


Figure 2.2.1. 3 Voter election

While we listed three criterions that determine whether an election is "$fair$", we focus on the $Majority\ Criterion$.

Definition 2.1.2. The **Majority Criterion** selects candidates as winners that have a *majority*, that is, more than half of the votes in an election. △

The next definition tells us when a voting method violates the Majority Criterion.

Definition 2.1.3. An election violates the **Majority Criterion** if some candidate has a majority of the first place votes but loses the election. △

2.2 Voting Systems

In this section we focus on the three different voting systems: *Preferential, Plurality,* and *Borda Count*. We begin with Preferential voting, which is the most commonly used method of voting. In this method of voting, voters are asked to rank the candidates from most to least preferred.

Definition 2.2.1. **Preferential Voting** occurs when a voter ranks all eligible candidates from first to last place, *i.e.*, from most to least preferred. The *winner* of the election is the candidate with the most first place choices. △

Notation: When a voter prefers a candidate a over b, then it is denoted by $a > b$.

Example 2.2.2. Figure 2.2.1 illustrates *Preferential Voting* with 3 candidates, $\{a, b, c\}$, and 3 voters, $\{v_1, v_2, v_3\}$. By the Preferential Voting method, we see that v_1 ,*i.e.*, voter 1's preference for the candidates are as follows $a > b > c$. Similarly, voter 2 prefers $b > a > c$ and voter 3 prefers $a > c > b$. Since candidate a has 2 out of the 3 spots in first place, then according to the Preferential Voting method, a is the winner.

This example illustrates that *Preferential Voting* satisfies the Majority Criterion since candidate a has over 50% of the votes. $\diamondsuit$

Remark 2.2.3. Note that since *voter* 1 prefers choice a over b and b over c, this implies a is preferred over c. This suggests transitivity within the candidates and so candidate a wins in the voter ranking. In the same manner, *voter* 2 prefers b over a and a over c. Hence, candidate b wins in the voter ranking. Similarly, *voter* 3 prefers a over c and c over b which means candidate a wins the voter ranking. $\diamondsuit$

Notation: A $Preference\ Table$ summarizes the results of all the individual preference votes for an election.

The following example shows that we can also represent a Preferential Voting System though a Preference Table.

Example 2.2.4. Recall the election from Example 1.0.2, where four students had to choose between four different colors. We can interpret this election more clearly using a preference table. *Student* 1, *Student* 2 and *Student* 3 picked red as their top preference

Table 2.2.1. Student Preference

2	**1**	**1**
red	red	yellow
yellow	yellow	orange
orange	blue	blue
blue	orange	red

and yellow as their second. Since 2 out of the 4 students in the committee prefer red over any other color, red automatically takes 50% of the vote. Additionally, since a 3rd student prefers red over any other color, red takes 75% of the votes.

Therefore by the Majority Criterion, red is the winner. ◊

We now introduce the second voting system, *Plurality Voting*. In general, Plurality Voting refers to the largest number of votes received by one candidate out of an entire group of candidates. Unlike Preferential Voting, in Plurality Voting candidates are not ranked, instead each voter votes for their top candidate.

Definition 2.2.5. Let P be a set of voters and let C be a finite set of candidates. Then **Plurality Voting** occurs when there exists $x \in C$ such that x is the candidate who polls the most votes. △

The following example illustrates the Plurality Voting System.

Example 2.2.6. Suppose we conduct a survey in a small town to predict the most favorable holiday in the area. A group of ten people are randomly chosen to take part in this election-based survey. We assume that our voters first choice is their top choice. The Plurality System ignores the second subsequent choices in the election, thus we separate the votes based on the voters top decisions. Using the Plurality method, each person selects their favorite holiday. Their choices are listed in the table below:

Table 2.2.2. Plurality Election

4	**3**	**2**	**1**
Christmas	Valentines Day	Thanksgiving	Halloween

Our results in Table 2.2.2 show that Christmas receives 4 votes, Valentines Day receives 3 votes, Thanksgiving receives 2 votes and Halloween receives 1 vote. Therefore by Plurality Voting, Christmas wins as the most preferred holiday. ◊

Notation: In Example 2.2.6, only 4 out of 10 voters picked Christmas. Since $40\% < 50\%$, then by Definition 2.1.2, the Plurality Method violates the Majority Criterion.

We now introduce the last voting system, *Borda Count Voting*.

Borda Count Voting is similar to Preferential Voting because it involves voters ranking their candidates from highest to lowest. In addition to ranking their candidates, each voter assigns a specific number of points to each candidate.

Definition 2.2.7. The **Borda Count Voting System** is a single-winner election method in which voters rank options or candidates in order of preference. △

Notation: These points are also referred to as weights, denoted w throughout the paper.

In the *Borda Count Method*, when deciding among n candidates, we give the most preferred candidate a score $n-1$. Then we give the second most preferred candidate a score of $n-2$, and the third a score of $n-3$, and so on until we reach the last candidate. The points given to a candidate by all the voters are then summed up to obtain the total score of that candidate. The candidate with the largest sum is chosen as the winner.

We now provide an example of the Borda Count voting system.

Example 2.2.8. Suppose we have an election with 4 voters who are asked to rank their preferences from highest to lowest. Candidates in this election are represented by $\{a, b, c\} \in C$. The preference ordering for each voter is as follows:

Table 2.2.3. Borda Count Election

$Voter\ 1$	$Voter\ 2$	$Voter\ 3$	$Voter\ 4$
a	b	c	b
c	a	a	a
b	c	b	c

We assign points for each candidate in the following way:

- 3 points if this is a voters first choice

- 2 points if this is a voters second choice
- 1 point if this is a voters third choice

We show our result in the table below:

Table 2.2.4. Points according to voter preference

Candidates	$Voter$ 1	$Voter$ 2	$Voter$ 3	$Voter$ 4	Sum
a	3	2	2	2	9
b	1	3	1	3	8
c	2	1	3	1	7

After distributing our points to each candidate, we notice above in Table 2.2.4 that the sum of each candidate is as follows: candidate a receives 9 points, b receives 8 points, and c receives 7 points. Even though candidate a receives the fewest number of first place votes, it is still declared the winner for this election using the Borda Count method. The Borda Count voting system therefore violates the Majority Criterion. ◊

The advantage of using Borda Count method is that it incorporates all the information from preference ballots, it takes candidates with the best average ranking and is preferable when comparing a large number of candidates. It uses the ranking information to make a formal decision instead of just using the voters best guess. The disadvantage of using Borda Count voting is that it violates the Majority Criterion.

Table 2.2.5 summarizes the three voting systems based on whether or not they violate the *Majority Criterion* and how the winner of an election is chosen.

2.3 Posets

In this section we introduce posets. In general *Completely Ordered Sets* are used to determine the linear ranking of candidates in an election. However, for this project we focus on an alternative method called *Partially Ordered Sets*.

Table 2.2.5. Compliance of Voting Systems

Voting Systems	Description	Relation to the Majority Criterion
Preferencial Voting	Determines the winner of an election by ranking each candidates from most to least preferred.	**Satisfies** the Majority Criterion
Plurality Voting	Determines the winner of an election as the candidate with the most first place votes.	**Violates** the Majority Criterion
Borda Count Voting	Determines the winner of an election by assigning a system of points to each candidate based on each voters rank.	**Violates** the Majority Criterion

Definition 2.3.1. A **Partially Ordered Set** (or *poset*, $\mathcal{P}$) is a set together with a binary relation denoted $\leq$ satisfying the following axioms [3]:

1. For all $x \in C$, $x \leq x$ (Reflexivity)
2. $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity) △

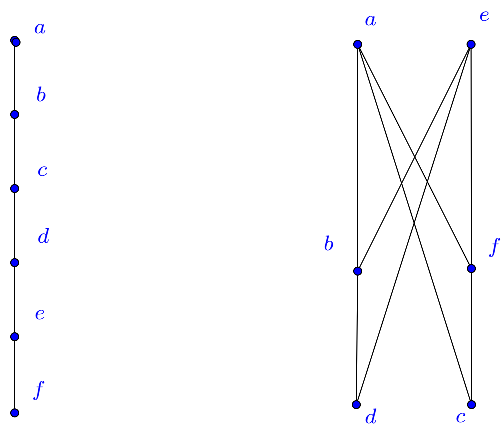
The following example gives a visual interpretation of Partially Ordered Sets and Completely Ordered Sets.

Example 2.3.2. Suppose we have an election with candidates $C = \{a, b, c, d, e, f\}$. The difference between the two sets can be visualized in Figure 2.3.1 using a *Hasse Diagram.* Note that in a *Completely Ordered Set*, the natural order of a sequence is compared by value, this is shown in Figure 2.3.1 ($\mathcal{P}_1$),*i.e.*,

$$a > b > c > d > e > f.$$

Alternatively, Figure 2.3.1 ($\mathcal{P}_2$) represents a *Partially Ordered Set* in which candidates are ranked such that:

$$a > f > c, \quad a > b > d, \quad \text{and} \quad e > b > d, \quad e > f > c.$$



$\mathcal{P}_1$=*Completely Ordered Set* $\mathcal{P}_2$=*Partially Ordered Set*

Figure 2.3.1. Complete Order and Partial Order

In this case the voter is indifferent to both a and e, and b and f, as well as b and c and d and f. $\Diamond$

Notation: We use the notation $x \geq y$ to mean $y \leq x$, we use $x < y$ to mean $x \leq y$ and $x \neq y$, and we use $x > y$ to mean $y < x$. We say two elements x and y of $\mathcal{P}$ are *comparable* if $x \leq y$ or $y \leq x$; otherwise $x \sim y$, meaning x and y are *incomparable*, [1].

Definition 2.3.3. [1] Given a partial order $\leq$, we define the **down set** and **incomparable set** of $x \in \mathcal{P}$ by

$$\text{Down}(x) = \{y \in \mathcal{P} \mid y < x\}$$

$$\text{Incomp}(x) = \{y \in \mathcal{P} \mid y \text{ is incomparable to } x\}.$$

$\triangle$

Notation: We use the notation $\mathbf{d}$ to represent the down set of a candidate in a poset such that $\mathbf{d}(a) = |Down(a)|$. We use the notation $\mathbf{i}$ to represent the incomparable set in a poset such that $\mathbf{i}(a) = |Incomp(a)|$.

We now provide an example to summarize the definitions above.

Example 2.3.4. Using the same posets $\mathcal{P}_1, \mathcal{P}_2$ in Figure 2.3.1 we compute $\mathbf{d}$(a) and $\mathbf{i}$(a) of each poset.

In $\mathcal{P}_1$, $\text{Down}(a) = \{b, c, d, e, f\}$ and $\text{Incomp}(a) = \emptyset$, so

$$\mathbf{d}(a) = |\{b, c, d, e, f\}| = 5 \text{ and } \mathbf{i}(a) = \emptyset = 0.$$

Likewise, in $\mathcal{P}_2$ $\text{Down}(a) = \{b, c, d, f\}$ and $\text{Incomp}(a) = \{e\}$, so

$$\mathbf{d}(a) = |\{b, c, d, f\}| = 4 \text{ and } \mathbf{i}(a) = |\{e\}| = 1.$$

◊

The terms and examples illustrated above will help us define *The Partial Borda Count* voting system in the next section.

2.4 Partial Borda Count

Partial Borda Count Voting is an application of the Borda Count Method that uses partially ordered posets instead of completely ordered sets to rank its candidates. Before we discuss the approach behind using Partial Borda Count Voting, we define some terms and concepts.

In a given election, we start by making decisions between candidates, where *voters* choose the ranking of each candidate. Each voter arranges the candidates in a list according to their preferences. These lists are known as *ballots*.

Definition 2.4.1. A **Ballot**, B_i is a poset on a set of candidates $C = \{a_1, \ldots, a_n\}$. △

Example 2.4.2. Let $C = \{a, b, c, d, e, f\}$ be candidates. Figure 2.3.1 gives two possible ballots, $\mathcal{P}_1$ and $\mathcal{P}_2$. ◊

Definition 2.4.3. A profile (P) is a set of ballots. △

In a Partial Borda Count Voting method, voters are allowed to submit partially ordered ballots whose rankings are determined by scores. This is known as the *scoring procedure* of a ballot. In a scoring procedure, a function takes as input a poset and outputs a weight for each candidate, based on the partial order. The basic concept involves systematically assigning weights to each candidate, then ranking them.

Notation: The points that a candidate receives is known as the *weight* of the candidate and it is determined by a mathematical system of dispensing points to each candidate known as the $weight\ function$. We give a formal definition of a weight function below.

Definition 2.4.4. Let $C = \{a_1, \ldots, a_n\}$ be a set of candidates. A **Weight Function** is a map from C to $\mathbb{R}$. △

Definition 2.4.5. A **Scoring Procedure** is a map from the set of profiles to the set of weight functions. △

The Borda Count method is the only method that assigns weights to candidates in an election. A *Social Choice Function* associates every profile in the domain with a candidate. This meaning it picks a chosen candidate(s) in P for every profile of preferences.

The following is a formal definition of the *Social Choice Function*.

Definition 2.4.6. A **Social Choice Function** is a map from a set of profiles to the set of non-empty subsets of C. △

These definitions serve as the building blocks of Borda Count Voting. We now present an example using the Partial Borda Count method.

Before we formally define the Partial Borda Weight Function used to determine the weight of each candidate in an election, we provide a short interpretation of the process with the remark below:

Remark 2.4.7. Given a partial order on C, we start by giving each $a \in C$ a weight $n-1$. Then for every pair $a, b \in C$ with $a < b$, we decrease the weight of a by 1 and increase the weight of b by 1. This means that a candidate must give away one point to any candidate that is ranked above it. In doing so a candidate receives more points for being ranked above another candidate. After distributing weights in this manner, the final weights assigned to each candidate form a weight function that we will soon define below. $\diamondsuit$

Note: Let B be a ballot on C. This means B is a *poset*. We now define the Partial Borda Weighting Procedure.

Definition 2.4.8. The **Partial Borda Weighting Procedure** is the weighting procedure that associates a function to B such that $w_B : C \to \mathbb{R}$ given by

$$w_B(a) = 2\text{Down}_B(a) + \text{Incomp}_B(a).$$

$\triangle$

Definition 2.4.9. Let P be a profile. The score of $a \in P$, denoted $s_p(a)$, is called the **Partial Borda Score** of the function and using the Partial Borda Count Weighting Function, the *Score Function* is given by,

$$s_p(a) = \sum_{B \in P} w_B(a)$$

for all Ballots B and is known as the *Partial Borda Scoring Procedure*. $\triangle$

Definition 2.4.10. We define the **Social Choice** of a profile, denoted f(P) to be

$$f(P) = \{a \in C \mid \ s(a) = \max s(b) \ \forall \ b \in C\}.$$

$\triangle$

We now formally define the Partial Borda Count.

Definition 2.4.11. Let C be a set of candidates, and let P be a profile. The **Partial Borda Count Procedure** is the weighting procedure in which:

- For each $B \in P$, we compute $w_B(x) = 2\text{Down}_B(a) + \text{Incomp}_B(a)$

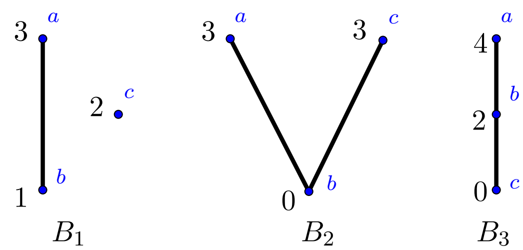


Figure 2.4.1. $w = 2\mathbf{d} + \mathbf{i}$ Profile

- For all $a \in C$, we compute $s_p(a) = \sum_{B \in P} w_B(a)$
- The winners are $f(P) = \{a \in C \mid s(a) \text{ is maximal}\}$

△

The following example illustrates the properties of a Partial Borda Count Procedure.

Example 2.4.12. Suppose in Figure 2.4.1, we conduct an election with ballots $\{B_1, B_2, B_3\} \in P$ and candidates $\{a, b, c\} \in C$. Next to each vertex, we indicate the weight of each candidate using the Partial Borda Count Weighting procedure, $w = 2\mathbf{d} + \mathbf{i}$. Note that the weights depend on the underlying poset, and not on the voter who submitted it. Let $w_{\mathbf{i}}(x) = w_{B_i}(x)$, then

- $w_1(a) = 2\mathbf{d}(a) + \mathbf{i}(a) = (2 \times 1) + (1 \times 1) = 3$
- $w_1(b) = 2\mathbf{d}(b) + \mathbf{i}(b) = (2 \times 0) + (1 \times 1) = 1$
- $w_1(c) = 2\mathbf{d}(c) + \mathbf{i}(c) = (2 \times 0) + (1 \times 2) = 2$

We look at the Hasse Diagram in Figure 2.4.1 and note the same process for each candidate in the second and third ballots. Then the score s_p of each candidate is:

$s_p(a) = 3 + 3 + 4 = 10,\ s_p(b) = 1 + 0 + 2 = 3,\ s_p(c) = 2 + 3 + 0 = 5.$

Since $s_p(a) > s_p(b)$, and $s_p(a) > s_p(c)$, we see $f(P) = \{a\}$ and therefore a is the winner. ◊

In the next section we will list some interesting observations concerning the posets of this weight function and compare it to the posets of other similar weight functions.

2.5 Total sums of the weights in a ballot; $w = 2\mathbf{d} + \mathbf{i}$ vs. $w = 3\mathbf{d} + \mathbf{i}$

In the previous section we introduced the weight function $w = 2\mathbf{d} + \mathbf{i}$, and conducted an example outputting different weights for each candidate. If we look back at Example 2.4.12, we observe that the total sum of the weights of each ballot in the profile is equal to 6 points all together.

In other words,
$\sum_{i=1}^{3} w_{B_1}(a_i) = 3 + 1 + 2 = 6$,
$\sum_{i=1}^{3} w_{B_2}(a_i) = 3 + 3 + 0 = 6$, and
$\sum_{i=1}^{3} w_{B_3}(a_i) = 4 + 2 + 0 = 6$.

In *"A Borda Count for Partially Ordered Ballots"*[1], the authors explain that the sum of a ballot can be represented by $n^2 - n$, with n representing the number of candidates in the ballot.

Given our weight function $w = 2\mathbf{d} + \mathbf{i}$, we can change $w(x) = 2\mathbf{d}(x) + \mathbf{i}(x)$ to $w(x) = 3\mathbf{d}(x) + \mathbf{i}(x)$ and short hand it to $w = 3\mathbf{d} + \mathbf{i}$. Throughout the process of modifying our weight function, we notice some interesting observations concerning the sum of the weights. It turns out that the sum of the weights of the candidates in each ballot in a profile differ depending on the weight function used. In Figure 2.5.1, we compare the the sum of the ballots in both profiles.

Example 2.5.1. Suppose we have a profile P. We notice above in Figure 2.5.1(B) that the sum of each ballot for $n = 3$ using the modified weight function is:

$$\sum_{i=1}^{3} w_{B_1}(a_i) = 4 + 4 + 0 = 8,$$

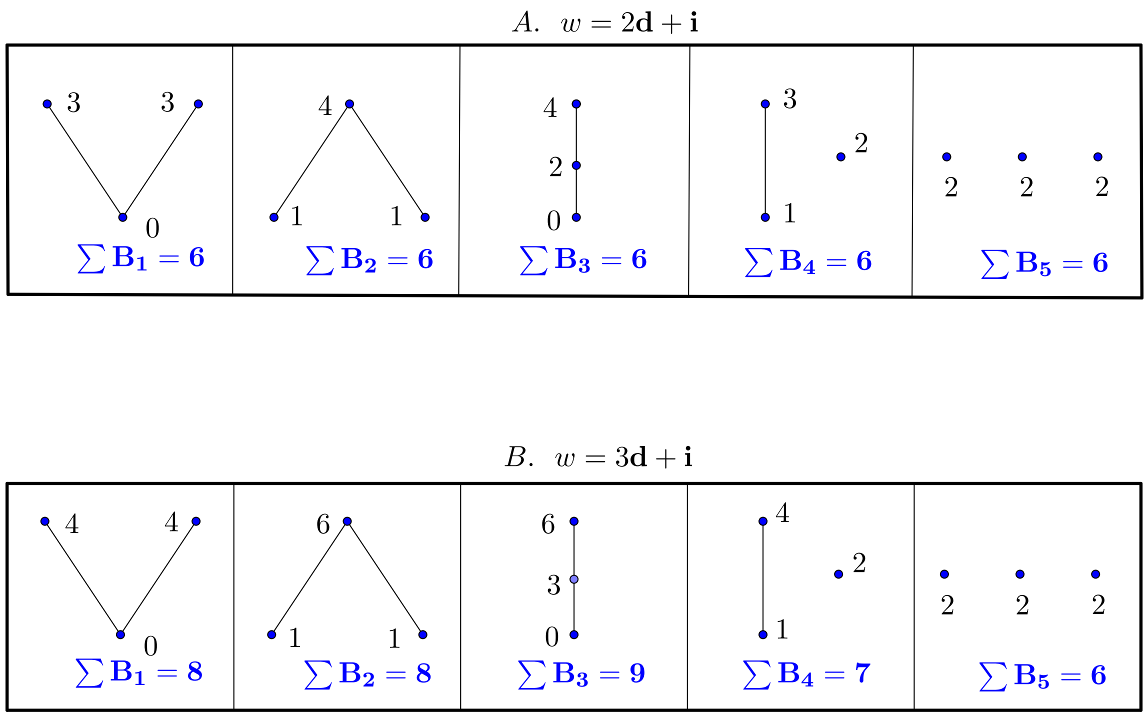


Figure 2.5.1. All possible ballots for $n = 3$

$$\sum_{i=1}^{3} w_{B_2}(a_i) = 6 + 1 + 1 = 8,$$

$$\sum_{i=1}^{3} w_{B_3}(a_i) = 6 + 3 + 0 = 9,$$

$$\sum_{i=1}^{3} w_{B_4}(a_i) = 4 + 1 + 2 = 7, \text{and}$$

$$\sum_{i=1}^{n} w_{B_5}(a_i) = 2 + 2 + 2 = 6$$

Note that we get an unequal number of total weights for each ballot in the profile. ◊

So the question now is, why do we not get consistent weights when we change our weight function?

The following theorem, proved in [1], states that the Partial Borda Count weight function is the only weight function that satisfies constant weights within a ballot.

Theorem 2.5.2. *The Partial Borda weighting procedure is the unique weighting procedure, up to affine transformation, that has constant total weights and is linear in the quantities* $down(a)$ *and* $incomp(a)$.[1]

This means that constant total weights within a ballot is only satisfied with weight function $w = (\alpha)\mathbf{d} + (\beta)\mathbf{i}$, where $\alpha = 2\beta$. This theorem can be informally proven by looking at **Remark 2.4.7** in Section 2.4 of this paper. We will omit the formal proof in this paper as it is proven in [Pg.5 [1]].

3

Properties of Voting Theory using weight functions

3.1 Unique properties of Partial Borda Count Voting

There are a considerable number of additional properties that hold for the Borda Count method. We define four of these properties that serve as key components in this paper.

Recall the following setup for the Partial Borda Count Voting System:

- P denotes a profile,*i.e.*, a set of ballots.
- C denotes a set of candidates.
- $f(P) = \{a \in C : \ s(a) \text{ is maximal}\}$.
- $w_B(a) = 2\mathbf{d}(a) + \mathbf{i}(a)$, for a particular ballot B.

Definition 3.1.1. Let $P = \{B_1 \dots B_m\}$ be a profile and let $a, b \in C$. Then

$$\pi_{ab} = |B \in P : a > b \ \ \in B|,$$

,*i.e.*, π_{ab} is the number of ballots that rank a over b in P. △

We provide an example below.

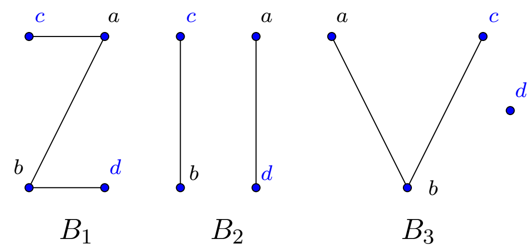


Figure 3.1.1. n=4; π_{ab}

Example 3.1.2. Suppose we have a profile $P = \{B_1, B_2, B_3\}$ with candidates $C = \{a, b, c, d\}$. If we look at Figure 3.1.1, we see that $a > b$ in B_1 and B_3, but $a \sim b$ in B_2, so $\pi_{ab} = 2$. ♢

Definition 3.1.3. A **Social Choice Function** f inputs a profile, and outputs a set of candidates ,$i.e.$, $f : \mathbb{P} \longrightarrow 2^C$, where $\mathbb{P}$ is a set of profiles, and 2^C = all subsets of C. △

We now define the four properties below:

We first define a term necessary for our first property.

Definition 3.1.4. If $P = \{B_1, \ldots, B_m\}$ and $P' = \{B'_1, \ldots, B'_m\}$ are disjoint profiles, then $P + P'$ denotes the profile $P \cup P'$. △

Definition 3.1.5. A social choice function satisfies the **Consistency Property** if for any disjoint profiles P and P', if $f(P) \cap f(P') \neq 0$, then $f(P) \cap f(P') = f(P + P')$. △

This means that within an election if the profile is split into two mutually exclusive groups and the intersection of the respective sets of winners is non-empty, then the candidates chosen by both groups are also chosen at large and vice versa.

Example 3.1.6. Consider the two disjoint profiles in Figure 3.1.2. Then using our weight function $w = 2\mathbf{d} + \mathbf{i}$, in profile P we get, $s_p(a) = 3 + 2 + 2 = 7,\ \ s_p(b) = 2 + 0 + 2 = 4,$ and $s_p(c) = 1 + 4 + 2 = 7$. In profile P' we get $s_p(a) = 2 + 3 + 1 = 6,\ s_p(b) = 4 + 0 + 1 = 5,$ and$s_p(c) = 0 + 3 + 4 = 7$. So overall $f(P) = \{a, c\},$ and $f(P') = \{c\}$. Hence we observe

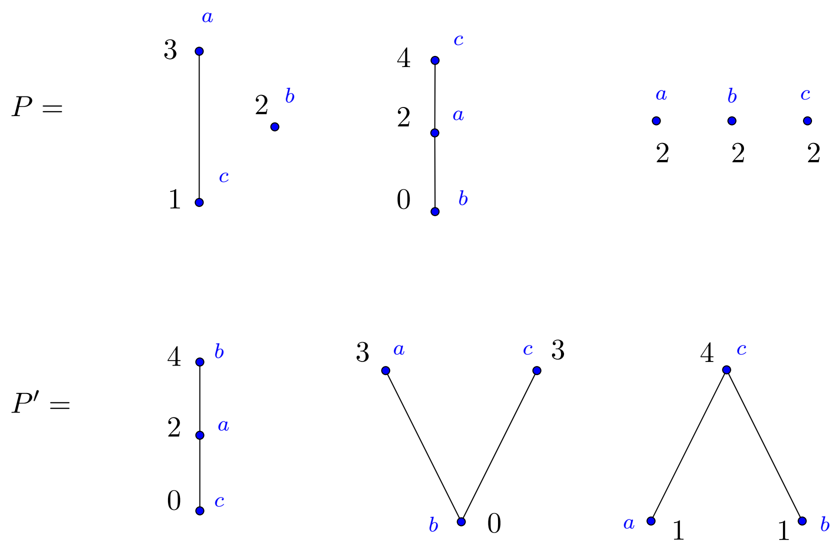


Figure 3.1.2. Consistency Property: $w = 2\mathbf{d} + \mathbf{i}$

with our results that $f(P) \cap f(P') = \{c\}$. When we combine both profiles $P + P'$, we compute $s_p(a) = 3+2+2+2+3+1 = 13$, $s_p(b) = 2+0+2+4+0+1 = 9$, and $s_p(c) = 1+4+2+0+3+4 = 14$ therefore $f(P+P') = \{c\}$. This then satisfies the *Consistency Property*. $\Diamond$

We now define the second property, the *Faithfulness Property*.

Definition 3.1.7. A social choice function satisfies the **Faithfulness Property** if for any profile P consisting of just one ballot B, if $a \in C$ and $b > a \in B$, then $a \notin f(P)$. $\triangle$

This means that in an election that consists of only one voter, the social choice preference is that of the voters, if we have b above a in a ballot, then a cannot be the winner.

Example 3.1.8. Suppose we have an election that consist of just ballot B, as in Figure 3.1.3. We have $b > a$, $b > c$, and $a \sim c$, therefore $b \in f(P)$, and $a \notin f(P)$. $\Diamond$

We now define the third property, the *Neutrality Property*.

Definition 3.1.9. A social choice function satisfies the **Neutrality Property** if for any profile P and permutation σ of C, $f(\sigma(P)) = \sigma(f(P))$. $\triangle$

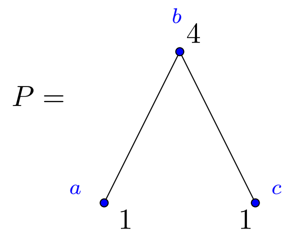


Figure 3.1.3. Faithfulness Property: $w = 2\mathbf{d} + \mathbf{i}$

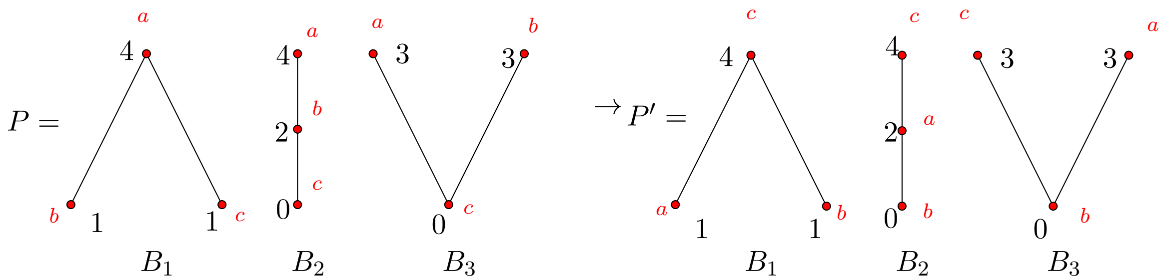


Figure 3.1.4. Neutrality Property: $w = 2\mathbf{d} + \mathbf{i}$

This means that relabeling the candidates in a profile does not negatively affect the outcome of the election.

Example 3.1.10. Consider the ballots $B_1, B_2, B_3 \in P$ in Figure 3.1.4. We observe in B_1, $a > b$, $a > c$, $b \sim c$, in B_2, $a > b > c$ and in B_3, $a > c$, $b > c$, $a \sim b$. Then after computing s_p of each candidate we get $f(P) = \{a\}$. Now let the permutation $\sigma a = c$, $\sigma b = a$, and $\sigma c = b$. Then we get $\sigma\{a, b, c\} \in P'$. Note that all the candidates are in the same order as their permutations in the new profile. Finally, we observe that $\sigma(f(P)) = \sigma(a) = \{c\}$, and $f(\sigma(P')) = \{c\}$. Therefore the property holds. ♢

The fourth property is given in the following definition.

Definition 3.1.11. A social choice function satisfies the **Cancellation Property** if for any profile P, if $\pi_{ab}(P) = \pi_{ba}(P)$ for all $a \neq b \in C$, then $f(P) = C$. △

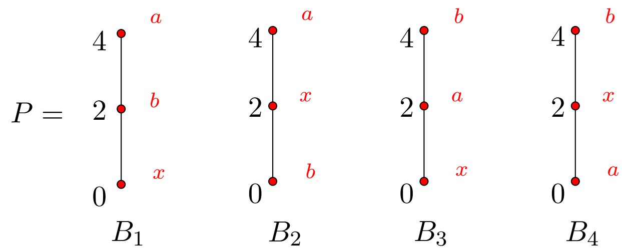


Figure 3.1.5. Cancellation Property: $w = 2\mathbf{d} + \mathbf{i}$

This means for all pairs of candidates, if the number of voters preferring C_i to C_j for all $i,\ j \in \mathbb{R}$ is equal to the number preferring the two C_j to C_i, then the two candidates should be declared winners.

Example 3.1.12. Suppose we run an election with $\{a, b, x\} \in C$. We notice in Figure 3.1.5 that in B_1, $a > b > x$, in B_2, $a > x > b$, in B_3, $b > a > x$, and in B_4, $b > x > a$. Then $s_p(a) = 4 + 4 + 2 + 0 = 10$, and $s_p(b) = 2 + 0 + 4 + 4 = 10$. In this case we have a tie, and both $a, b \in f(P)$ ◊

According to *"A Borda Count For Partially Ordered Ballots"*, all four of these properties hold for $w = 2\mathbf{d} + \mathbf{i}$. The following theorem shows that the converse is true as well.

Theorem 3.1.13. *The Partial Borda choice function is the unique social choice function that is: Consistent, Faithful, Neutral and has the Cancelation property.* [1]

Now that we know for sure that these properties have been proven to hold with $w = 2\mathbf{d} + \mathbf{i}$, we consider the following question: What would happen if we change our weight function $w(a) = 2\mathbf{d}(a) + \mathbf{i}(a)$, to an arbitrary one $w(a) = \alpha\mathbf{d}(a) + \beta\mathbf{i}(a)$ where α and $\beta \in \mathbb{R}$? Which properties would hold? This is the topic that we will explore in the next chapter.

4

Generalized Partial Borda Voting System

In this chapter we explore the generalized Partial Borda Count Voting System. We provide the reader with some examples of the Partial Borda Count Voting System using $w = 3\mathbf{d}+\mathbf{i}$ instead of $w = 2\mathbf{d}+\mathbf{i}$, as in the last chapter. We also consider a generalized weight system $w = \alpha\mathbf{d} + \beta\mathbf{i}$ and seek to see if the properties stated in the previous chapter hold for this weight function.

4.1 Partial Borda voting system properties

In Chapter 3 of this paper we introduced the Partial Borda Count Voting System with weight function $w = 2\mathbf{d} + \mathbf{i}$. This is the unique social choice function that satisfies the Consistency, Faithfulness, Neutrality and the Cancellation property. In this section we modify α and β to see if these properties still hold for weight function $w = \alpha\mathbf{d} + \beta\mathbf{i}$.

In Table 4.1.1 below, we show the results of our study for $w = \alpha\mathbf{d} + \beta\mathbf{i}$ in general and more specifically for $w = 3\mathbf{d} + \mathbf{i}$. We will prove most of these properties in this chapter.

Recall the four unique properties of the Partial Borda Count, we restate the definitions below for the readers convenience:

Table 4.1.1. Generalized Partial Borda Properties

Properties	$w = 2\mathbf{d} + \mathbf{i}$	$w = 3\mathbf{d} + \mathbf{i}$	$\alpha\mathbf{d} + \beta\mathbf{i}$
Consistency	Yes	Yes	Yes $\forall\ \alpha, \beta$
Faithfulness	Yes	Yes	Yes $\iff \alpha \geq \beta \geq 0$ and $\alpha > 0$
Neutrality	Yes	Yes	Yes $\forall\ \alpha, \beta$
Cancellation	Yes	No	Yes $\iff \alpha = 2\beta$

Definition 4.1.1. Consistency Property - For disjoint profiles P and P', if $f(P) \cap f(P') \neq 0$, then $f(P) \cap f(P') = f(P + P')$. △

Definition 4.1.2. Faithfulness Property - For any profile P consisting of just one voter, if $a \in C$ is a candidate and the voter ranks b above a for some $b \in C$, then $a \notin f(P)$. △

Definition 4.1.3. Neutrality Property - For any profile P and permutation σ of C, $f(\sigma(P)) = \sigma(f(P))$. △

Definition 4.1.4. Cancellation Property - For any profile P, if $\pi_{ab}(P) = \pi_{ba}(P)$ for all $a \neq b \in C$, then $f(P) = C$. △

4.2 *Consistency Property*

We know in Table 4.1.1 that the Consistency Property holds for $w = 2\mathbf{d} + \mathbf{i}$ from [1]. We claim that this property also satisfies $w = 3\mathbf{d} + \mathbf{i}$ as well as any number we choose to represent α and β.

Example 4.2.1. In Chapter 3, example 3.1.6, we used the weight function $w = 2\mathbf{d} + \mathbf{i}$ to illustrate the Consistency Property. Using the same profiles we now change this weight function to $w = 3\mathbf{d} + \mathbf{i}$ in Figure 4.2.1 and show that we still have consistency within the profile. We compute

$s_p(a) = w_{B_1}(a) + w_{B_2}(a) + w_{B_3}(a) = 4 + 3 + 2 = 9,$

$s_p(b) = w_{B_1}(b) + w_{B_2}(b) + w_{B_3}(b) = 2 + 0 + 2 = 4$ and

$s_p(c) = w_{B_1}(c) + w_{B_2}(c) + w_{B_3}(c) = 1 + 6 + 2 = 9,$

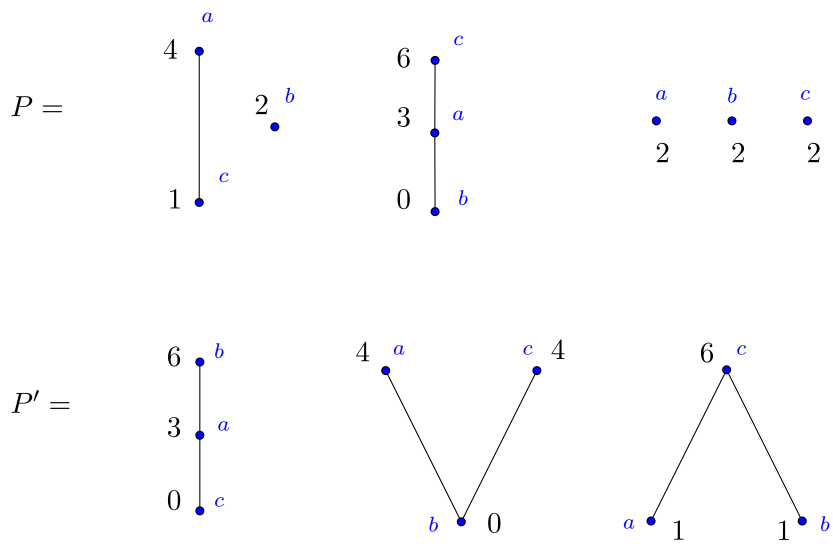


Figure 4.2.1. Consistency Property: $w = 3\mathbf{d} + \mathbf{i}$

hence $f(P) = \{a, c\}$. Likewise in P',

$s_p(a) = w_{B_1'}(a) + w_{B_2'}(a) + w_{B_3'}(a) = 3 + 4 + 1 = 8$,

$s_p(b) = w_{B_1'}(b) + w_{B_2'}(b) + w_{B_3'}(b) = 6 + 0 + 1 = 7$ and

$s_p(c) = w_{B_1'}(c) + w_{B_2'}(c) + w_{B_3'}(c) = 0 + 4 + 6 = 10$,

hence $f(P') = \{c\}$. Therefore $f(P) \cap f(P') = \{c\}$ and $f(P + P') = \{c\}$. $\Diamond$

Theorem 4.2.2. ***Consistency*** *Let $w = \alpha\boldsymbol{d} + \beta\boldsymbol{i}$ be the weight function where α, $\beta \in \mathbb{Z}$. Let $P = \{B_1, \ldots, B_k\}$ and $P' = \{B_1', \ldots, B_n'\}$ be disjoint profiles, where $f(P) \cap f(P') \neq \emptyset$. Then $f(P) \cap f(P') = f(P + P')$.*

Let $S = f(P) \cap f(P')$ and let $T = f(P + P')$.

Proof. We now prove $S \subseteq T$.

Since $S \neq \emptyset$, this means $\exists a \in S$ such that $a \in f(P)$ and $a \in f(P')$. Let w and w' denote the weight functions on P and P', and s and s' be the total score functions for each candidate associated with P and P'. Let $s(a) = \sum\limits_{B \in P} w_B(a)$ and $s'(a) = \sum\limits_{B' \in P'} w'_{B'}(a)$. Then

$s'(a) \geq s'(b)$ and $s(a) \geq s(b)$ for all $b \in C$. Let $s''(a) = \sum_{B'' \in P \cup P'} w''_{B''}(a)$. Then $s''(a) = s(a)$ + $s'(a) \geq s(b)$+ $s'(b) = s''(b)$. So $a \in f(P+P')$, and therefore $S \subseteq T$.

$T \subseteq S$

Suppose $a \in T$. Then $a \in f(P+P')$. This means that $s''(a) \geq s''(b)\ \forall\ a, b \in C$. Since $f(P) \cap (P') \neq \emptyset$, $\exists\ x \in f(P) \cap f(P')$ such that $x \in f(P)$ and $x \in f(P')$. This means that $s(x) \geq s(b)$ and $s'(x) \geq s'(b)\ \forall\ b \in C$. So in particular, $s(x) \geq s(a)$ and $s'(x) \geq s'(a)$, which then yields to $s''(x) \geq s''(a)$. But from above we stated that $s''(a) \geq s''(x)$. Hence we have $s''(a) = s''(x)$, which means that $s(x) = s(a)$ and $s'(x) = s'(a)$. Therefore we see that $a \in f(P) \cap f(P')$. So $a \in S$ and $T \subseteq S$. Hence S=T.

Therefore $f(P) \cap f(P') = f(P+P')$. □

4.3 *Faithfulness Property*

We claim in Table 4.1.1 above that the Faithfulness Property holds for every α and β as long as $\alpha \geq \beta$.

We illustrate an example below with $\alpha = 3$ and $\beta = 1$.

Example 4.3.1. Recall in Figure 3.1.3, we used weight function $w = 2\mathbf{d} + \mathbf{i}$ to find the social choice of an election. Using the same profile, we run the example with our new weight function $w = 3\mathbf{d} + \mathbf{i}$ in Figure 4.3.1. Then because we are restricted to one ballot, $s_p(b) = w(b) = 6$ and $s_p(a) = w(a) = 1$. Hence $f(P) = \{b\}$, and so $a \notin f(P)$. ◊

Before we introduce our proof, we define a some necessary terms that will aid in the construction of the proof.

Recall the Incomparable set in Definition 2.3.3 where,

$$\text{Incomp}(x) = \{y \in \mathcal{P} | y \text{ is incomparable to } x\}.$$

It is also the case that $a \sim x$ means a is *Incomparable* to x.

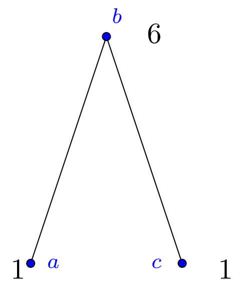


Figure 4.3.1. Faithfulness Property: $w = 3\mathbf{d} + \mathbf{i}$

Note: Since this election is restricted to a single ballot, $s_p(x) = w(x)\ \forall\, x \in C$ in this case.

The notations below are necessary for the completion of proof.

Definition 4.3.2. Let **w(a,x)** be defined as follows:

$$w(a,x) = \begin{cases} \alpha & a > x \\ 0 & a < x \\ \beta & a \sim x \\ 0 & a = x \end{cases}$$

△

Note that $w(a,x)$ keeps track of the points a gets with respect to another $x \in C$. With this definition $w(a) = \sum_{x \in C} w(a,x)$.

Example 4.3.3. The ballot in Figure 4.3.1 shows that

$w(a) = w(a,a) + w(a,b) + w(a,c) = 0 + 0 + 1 = 1.$

Similarly $w(b) = w(b,a) + w(b,b) + w(b,c) = 3 + 0 + 3 = 6$ and

$w(c) = w(c,a) + w(c,b) + w(c,c) = 1 + 0 + 0 = 1.$ ◊

We will need to break up the weight function in this way in order to prove the next theorem.

Theorem 4.3.4. ***Faithfulness***

Let $w = \alpha \boldsymbol{d} + \beta \boldsymbol{i}$ where $\alpha \geq \beta \geq 0$, and let $a, b \in C$. Let $P = \{B\}$ be a profile with a single ballot. If $b > a \in B$, then $a \notin f(P)$.

Proof. Case 1: Let $a > x$

Since $a > x$, $w(a, x) = \alpha$. We know that $b > a$ according to our theorem, so by the transitive property $b > x$, hence $w(b, x) = \alpha$. Therefore $w(b, x) \geq w(a, x)$.

Case 2: Let $a < x$

Since $a < x$, we have $w(a, x) = 0$ and $w(b, x) \geq 0$. Hence we get $w(b, x) \geq w(a, x)$.

Case 3: Let $a \sim x$

Since $a \sim x$, we have $w(a, x) = \beta$. Assume $w(b, x) = 0$. This means that $b < x$, but we already know that $a < b$, so by the transitive property, we have that $a < x$ which is a contradiction in this case because we previously stated that $a \sim x$. This then leaves us with $w(b, x) \neq 0$, therefore it must be equal to α or β. We also know that $\beta \geq \beta$, and $\alpha \geq \beta$, so we can conclude that $w(b, x) \geq w(a, x)$.

Case 4: Let $a = x$

We compute $w(a, a) = 0$ and $w(b, a) = \alpha > 0$. Then $w(a, a) < w(b, a)$.

Now, since $w(a, x) \leq w(b, x)$ for all $x \in C$ and $w(a, a) < w(b, a)$, we have $w(a) = \sum_{x \in C} w(a, x) < \sum_{x \in C} w(b, x) = w(b)$. So $w(a) < w(b)$, hence $s(a) < s(b)$. Therefore $a \notin f(P)$. ☐

We provide a counter example for $\alpha < \beta$.

Note: In order to provide a counterexample for the *Faithfulness Property* we need to choose a weight function in which $\alpha < \beta$.

Example 4.3.5. *Counterexample* :

Let $w = 2\mathbf{d} + 3\mathbf{i}$. Suppose we have a profile $P = \{B\}$ consisting of one ballot. Then $s_p(x) = w(x)$ for all $x \in C$ in this case because we only have one ballot in the profile. Then using our weight function, we see in Figure 4.3.2, $w(b) = (2 \times 4) + (3 \times 0) = 8,$ and

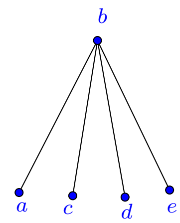


Figure 4.3.2. Faithfulness Property: $w = 2\mathbf{d} + 3\mathbf{i}$

$w(a) = w(c) = w(d) = w(e) = (2 \times 0) + (3 \times 3) = 9$. We notice that $8 < 9$, so $w(b) < w(a)$. Therefore in this case $a \in f(P)$, even though $b > a$. $\Diamond$

4.4 *Neutrality Property*

We claim in Table 4.1.1 that the *Neutrality Property* holds for all α and β. The property does not focus on the specific weight of the candidate but rather simply focuses on relabeling the candidates to see if we still receive the same winners. If the weights are not affected then the winners should not be affected.

Example 4.4.1. Recall in Figure 3.1.4, we used weight function $w = 2\mathbf{d} + \mathbf{i}$ to provide an example for the *Neutrality Property*. Using the same profile, we run the example with $w = 3\mathbf{d} + \mathbf{i}$, as in Figure 4.4.1, and we get:

$s_p(a) = w_{B_1}(a) + w_{B_2}(a) + w_{B_3}(a) = 6 + 6 + 4 = 16$,

$s_p(b) = w_{B_1}(b) + w_{B_2}(b) + w_{B_3}(b) = 1 + 3 + 4 = 8$, and

$s_p(c) = w_{B_1}(c) + w_{B_2}(c) + w_{B_3}(c) = 1 + 0 + 0 = 1$.

Now let the permutation be defined by, $\sigma(a) = c$, $\sigma(b) = a$, $\sigma(c) = b$. Then we get:

$s_p(a) = w_{B_1}(a) + w_{B_2}(a) + w_{B_3}(a) = 1 + 3 + 4 = 8$,

$s_p(b) = w_{B_1}(b) + w_{B_2}(b) + w_{B_3}(b) = 1 + 0 + 0 = 1$, and

$s_p(c) = w_{B_1}(c) + w_{B_2}(c) + w_{B_3}(c) = 6 + 6 + 4 = 16$.

Hence $\sigma(f(P)) = \sigma(a) = \{c\}$, and $f(\sigma(P')) = \{c\}$. $\Diamond$

Before we state the theorem we note a necessary term.

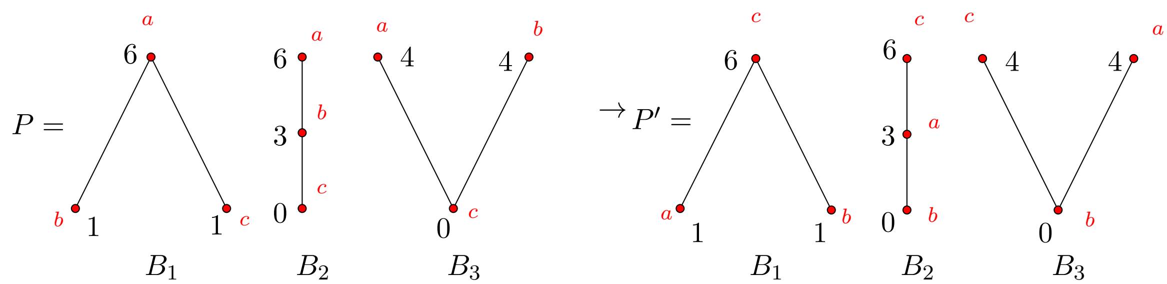


Figure 4.4.1. Neutrality Property: $w = 3\mathbf{d} + \mathbf{i}$

Notation: Let $\sigma(P)$ denote the profile in which every voter re-labels the candidates according to σ. That is, a voter prefers a over b in P if and only if that voter prefers $\sigma(a)$ to $\sigma(b)$ in $\sigma(P)$.

Theorem 4.4.2. ***Neutrality***

Let $w = \alpha \boldsymbol{d} + \beta \boldsymbol{i}$ $\forall$ $\alpha, \beta \in \mathbb{Z}$. *Let* P *be a profile and let* σ *be a permutation of* C. *Then* $f(\sigma(P) = \sigma(f(P))$.

Let $N = f(\sigma(P))$ and let $M = \sigma(f(P))$.

Proof. $M \subseteq N$

Let $a \in M$. This means that $a \in \sigma(f(P))$. This means there exists $b \in f(P)$ such that $\sigma(b) = a$. Since $b \in f(P)$, $s_p(b) \geq s_p(x)$ $\forall x \in C$. This means that $s_p(\sigma(b)) \geq s_p(\sigma(x))$, so $\sigma(b) \in f(\sigma(P))$, therefore $a \in f(\sigma(P))$ and $a \in N$. Therefore $M \subseteq N$.

$N \subseteq M$

Let $a \in N$. This means that $a \in f(\sigma(P))$, so we know that $s_p(a) \geq s_p(\sigma(x)) \, \forall x \in C$. Now there exists b such that $\sigma(b) = a$. Applying the inverse of σ to this, $s_p(\sigma^{-1}(a)) \geq s_p(\sigma^{-1}(\sigma(x)))$. This means that $s_p(b) \geq s_p(x) \forall x \in C$, which means that $b \in f(\sigma^{-1}(\sigma(P)))$. So $b \in f(P)$, then $\sigma(b) \in \sigma(f(P))$. Therefore $a \in \sigma(f(P))$ and $a \in M$, so $N \subseteq M$.

Therefore $M = N$ and $f(\sigma(P)) = \sigma(f(P))$. □

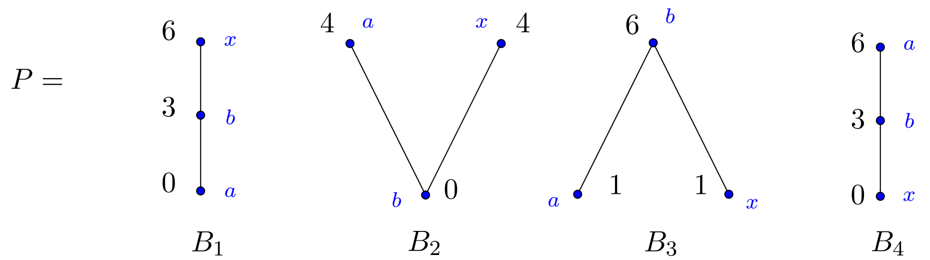


Figure 4.5.1. Cancellation Property: $w = 3\mathbf{d} + \mathbf{i}$

4.5 *Cancellation Property*

We claim in Table 4.1.1 above that the *Cancellation Property* holds for $w = 2\mathbf{d} + \mathbf{i}$ but does not hold for the weight function $w = 3\mathbf{d} + \mathbf{i}$.

As for which α and β this property holds for, we claim that the *Cancellation Property* only holds for $\alpha = 2\beta$.

Theorem 4.5.1. *The Partial Borda weighting procedure is the unique weighting procedure, up to affine transformation, that has constant total weights and is linear in the quantities down(a) and incomp(a).*

Proof. We omit the proof in this paper, as it is proved in *Theorem* 1 Pg.5 [1]. □

We provide a counterexample for $w = 3\mathbf{d} + \mathbf{i}$.

Example 4.5.2. *Counterexample* : Consider the profile in Figure 4.5.1,Using the weight function $w = 3\mathbf{d} + \mathbf{i}$. The total score of each candidate then results in :

$s_p(a) = w_{B_1}(a) + w_{B_2}(a) + w_{B_3}(a) + w_{B_4}(a) = 0 + 4 + 1 + 6 = 11$

$s_p(b) = w_{B_1}(b) + w_{B_2}(b) + w_{B_3}(b) + w_{B_4}(b) = 3 + 0 + +6 + 3 = 12$

In this case we have $\pi_{ab} = \pi_{ba}$, but we observe that $b \in f(P)$ but $a \notin f(P)$.

Therefore the weight function $w = 3\mathbf{d} + \mathbf{i}$ does not satisfy for the Cancellation Property.

◊

As you can see, most of these properties hold for our new weight function as well but, but many of them are restricted to specific conditions for both α and β.

5
Future Work

My secondary goal in this project, after generalizing the Partial Borda Weight Procedure, was to apply the function to other situations besides Voting Theory. The properties explored in the previous chapters can also be implemented in other applications such as chess games and soccer tournaments.

How can we relate the unique properties of voting systems ,*i.e.*, Consistency, Faithfulness, Neutrality, and Cancellation, to tournaments instead of posets?

In Graph Theory, a Tournament (*Tournament Graph*) is defined as a complete oriented graph in which every pair of distinct vertices is connected by a single unique directed edge. Therefore, assuming the vertices correspond to the players in a tournament, the edge between each pair of vertices is oriented from the winner to the loser.

The term “tournament” also refers to a non linear arrangement by which teams or players play against each other in order to determine the winner. We use N to denote the number of players involved in a tournament. A $N-vertex$ tournament graph corresponds to a tournament in which each member of a group of N players plays all other $N-1$ players. We will consider tournaments where ties are allowed, so essentially every tournament can

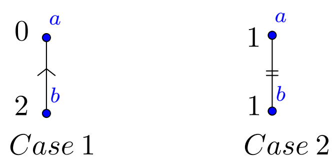


Figure 5.0.1. **2 Player Tournament**

result in a win for one player, a loss for the other player and in our case a tie. We use the Partial Borda Generalized weighting procedure, $w = \alpha\mathbf{d} + \beta\mathbf{i}$, to model different scoring procedures to determine the winner of the tournament.

We score each player with each $win = \alpha$ points, each loss counting as no points and each $tie = \beta$ points. Similar to the scoring procedure of a poset in Voting Theory, the *Down set* represents a win in each tournament, and the *Incomparable set* represents a tie in each tournament.

The example below illustrates the difference between a win and a tie.

Example 5.0.3. Assuming we have a tournament with players $\{a, b\}$, Figure 5.0.1 illustrates the different possible outcomes of winning this tournament using $w = 2\mathbf{d} + \mathbf{i}$. In the first case, player $a < b$. This means that player a is the downset of player b and player b is the winner of the tournament, so $s(b) = 2$ and $s(a) = 0$. Let f(T) represent the winner of a tournament, then $\{b\} \in f(T)$ and $a \notin f(T)$.

Likewise in the second case, the tournament can result in a tie (we use the symbol, "$=$" to represent a tie). This means that the score of player $a = 1$ and the score of player $b = 1$, so $a = b$ and $\{a, b\} \in f(T)$. ♢

Note: We define the *Score Sequence* of a tournament as the score vector of each tournament. This means that the Score Sequence of Case 1 is $(2, 0)$ and the Score Sequence of Case 2 is $(1, 1)$.

The biggest difference between a tournament and an election in a Partial Borda Count Voting System is transitivity. In voting, if $a > c$ and $b > c$, then $a > c$, but in a tournament

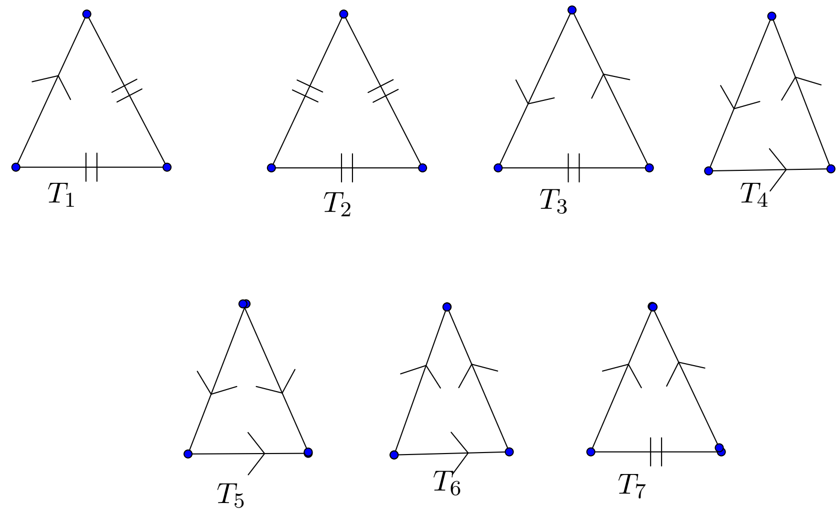


Figure 5.0.2. **3 Player Tournaments**

if player a wins over player b, and player b wins over player c, then this says nothing about the match between player a and c.

There are many different ways one can explore tournaments. We list some reasonable questions below:

1. How do the score sequences of a tournament vary as we increase the number of players?

2. Is it possible to find a formula that counts all the possible N-player tournaments? What about up to isomorphism?

3. Using the generalized weight function,$w = \alpha\mathbf{d} + \beta\mathbf{i}$, or specific weight functions such as $w = 2\mathbf{d} + \mathbf{i}$ and $w = 3\mathbf{d} + \mathbf{i}$, can the score sequences of a tournament be compared to that of a poset? Can we detect any similar patterns?

4. Figure 5.0.2 illustrates seven possible matches on a tournament graph with 3 unlabeled vertices. Each vertex represents a player. Can we illustrate all the possible combinations of a tournament with 3 players as in Figure 5.0.2 without repetition of score sequences? If so what about for N players?

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