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A Mathematical Exploration of Low-Dimensional Black Holes

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Abstract

In this paper we will be mathematically exploring low-dimensional gravitational physics and, more specifically, what it tells us about low-dimensional black holes and if there exists a Schwarzschild solution to Einstein's field equation in 2+1 dimensions. We will be starting with an existing solution in 3+1 dimensions, and then reconstructing the classical and relativistic arguments for 2+1 dimensions. Our conclusion is that in 2+1 dimensions, the Schwarzschild solution to Einstein's field equation is non-singular, and therefore it does not yield a black hole. While we still arrive at conic orbits, the relationship between Minkowski-like and Newtonian forces, energies, and geodesics in 2+1 dimensions is different than the relationship between Schwarzschild and Newtonian forces, energies, and geodesics in 3+1 dimensions.

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1
Introduction

Physics behaves in strange ways near a black hole, a consequence of the metric. This metric is a mathematical representation of the geometry of spacetime, specifically at the Schwarzschild radius. At this radius, the physics that results is very non-classical. If we tweak the model and change some of its characteristics, do the laws of physics as we know them still hold? How predictable are the results?

Einstein's field equation gives the relationship between the geometry of spacetime and the mass present in that spacetime. Karl Schwarzschild worked on a solution to the Einstein field equation that holds for a static, spherically symmetric, and asymptotically flat vacuum. Known as the Schwarzschild solution, it allows us to find the gravitational field in a vacuous spacetime outside a black hole. The Schwarzschild radius gives the boundary between the massive black hole and vacuous spacetime. We define a black hole as a massive spherically symmetric object, characterized by the Schwarzschild radius, that causes distortions in the geometry of spacetime at the boundary of the Schwarzschild radius.

We will be exploring the characteristics and behaviors of tensors, forces, energies and motion at this boundary in 3+1- and 2+1-dimensional spacetime. We say that we are working in "3+1" and "2+1" dimensions instead of "4" and "3" dimensions because we want to emphasize that one of the dimensions is time. Thus, the relativistic cases we will be exploring have three space dimensions and one time dimension, and two space dimensions and one time dimension. In the classical cases time is not considered as a dimension, so we may refer to those as "3-space" and "2-space".

There are a number of conventions of general relativity and differential geometry that we will be using throughout this paper.

i. The reader will notice that we frequently use multiple indices. Roman indices range from 1 – 2 or 1 – 3, depending on the dimension, denoting spatial spherical polar variables $r = 1$, $\theta = 2$, and $\phi = 3$. Greek indices range from 0 – 2 or 0 – 3, again depending on the dimension, and they denote time plus spatial coordinates where $t = 0$, $r = 1$, $\theta = 2$, and $\phi = 3$.

ii. When an expression has repeated indices with one as a superscript and one as a subscript, we sum over the pair of indices. So

$$x^i x_i = \sum_{i=0}^{n} x^i x_i.$$

This is referred to as Einstein summation notation.

iii. A vector or tensor with lower indices, such as A_{bc}, is referred to as a covariant vector or tensor. We also have vectors or tensors with upper indices, such as A^{bc}, that are referred to as contravariant vectors or tensors.

iv. When differentiating, the reader will occasionally observe the notation $\partial_x F$. This is simply a shorthand way of writing $\partial F/\partial x$.

v. In relativistic physics, the constant c, the speed of light in a vacuum, frequently appears in computations. We will usually change units to set $c = 1$ to simplify the expression.

We initially began this project hoping to discover how an ellipsoidal relativistic black hole behaved in 3+1 dimensions. We started off with an elliptical relativistic model in 2+1 dimensions, as lowering the dimension reduces the amount of information we have to work with, to help establish an intuition for ellipsoidal black holes. As we progressed, it became clear that we would need to work through the dynamics of 3+1- and 2+1-dimensional spherical and circular models of relativistic black holes as well as classical systems in 3-space and 2-space to contextualize and verify our solutions.

In Chapter 2, we lay a foundation for both the 2+1 classical and 3+1 relativistic cases by summarizing the argument in [5] for finding the equation of motion for a circular orbit in a plane and the energy of an orbit in 3-space with spherical polar coordinates. In Chapter 3, we develop a model of classical mechanics in 2-space to serve as a basis for comparison to our eventual 2+1-dimensional relativistic model of a black hole. In Chapter 4, we give an overview of the mathematical argument in [4] for the Schwarzschild solution in 3+1-dimensional spherical polar coordinates. This sets up the methodology we will be applying to our 2+1-dimensional model and establish the relationship between the 3-space classical model and 3+1-dimensional relativistic model that we hope to find between our 2-space classical model and 2+1-dimensional relativistic model. In Chapter 5, we build a relativistic model of a black hole in 2+1-dimensional circular polar coordinates and find a Schwarzschild-like solution to Einstein's field equation. Chapter 6 contains our conclusions, and in Chapter 7 we state possibilities for future research using what we have already found. Throughout the paper, we assume that the reader is familiar with classical mechanics, multivariable calculus, and linear algebra, and has had some exposure to general relativity and differential geometry.

2
Classical Spherical Motion in 3+1 Dimensions

We start by looking at classical motion in 3+1 dimensions to provide a context for 3+1-dimensional relativistic motion. Furthermore, 3+1-dimensional classical motion will provide a basis of comparison for 2+1-dimensional classical motion, so that we can see how the motion differs both from classical to relativistic and from 3+1 dimensions to 2+1 dimensions. The following derivation of the equations of motion for an orbit is a summary of [5, §§6.5, 6.10].

2.1 Kepler's Law of Ellipses

We start with Newton's equation for a force, using polar coordinates:

$$m\ddot{\mathbf{r}} = F(r)\mathbf{e}_r. \tag{2.1.1}$$

In 3-space, we have

$$F(r) = \frac{k}{r^2}, \tag{2.1.2}$$

where k is some constant. This yields two differential equations of motion in a circle,

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{k}{r^2} \text{ and} \tag{2.1.3}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0. \tag{2.1.4}$$

By the product rule, Equation 2.1.4 is equivalent to the conservation of angular momentum,

$$\frac{d}{dt}(r^2\dot{\theta}) = 0,$$

so

$$l = r^2\dot{\theta}, \tag{2.1.5}$$

where l is a constant. Using this constant, if we change variables to $u = \frac{1}{r}$, we have

$$\dot{r} = -l\frac{du}{d\theta} \text{ and} \tag{2.1.6}$$

$$\ddot{r} = -l^2u^2\frac{d^2u}{d\theta^2}. \tag{2.1.7}$$

Plugging (2.1.6) and (2.1.7) into Equation 2.1.3, we get

$$m\left(-l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(l^2u^4)\right) = ku^2. \tag{2.1.8}$$

Rearranging this, we have a differential equation of the orbit,

$$\frac{d^2u}{d\theta^2} + u = -\frac{k}{ml^2}. \tag{2.1.9}$$

Solutions to this differential equation are of the form

$$u = A\cos(\theta - \theta_0) + \frac{k}{ml^2}, \tag{2.1.10}$$

where A and θ_0 are constants of integration. This same solution in terms of r instead of u is

$$\boxed{r = \frac{1}{A\cos(\theta - \theta_0) + k/(ml^2)}.} \tag{2.1.11}$$

We set $\theta_0 = 0$ and rewrite Equation 2.1.11 as

$$r = \frac{ml^2/k}{1 + (Aml^2/k)\cos\theta}. \tag{2.1.12}$$

We define constants α and ϵ such that

$$\alpha = ml^2/k \quad \text{and} \quad \epsilon = A\alpha = Aml^2/k,$$

so

$$\boxed{r = \frac{\alpha}{1 + \epsilon\cos\theta}.} \tag{2.1.13}$$

For our orbit, α measures the distance of the focus from from a point on the ellipse, perpendicular to the semimajor axis, and ϵ measures the eccentricity of the orbit such that the foci are displaced from the center of the semimajor axis a by a distance ϵa.

Equation 2.1.13 describes any orbit shaped like a conic section. For a perfectly circular orbit, we expect to have $\epsilon = 0$. Values of ϵ in the range $0 < \epsilon < 1$ give elliptical orbits, $\epsilon = 1$ gives parabolic orbits, and values of $\epsilon > 1$ give hyperbolic orbits [5, p 234].

2.2 Orbital Equation of Motion

The total energy of a system is its kinetic energy plus its potential energy. This quantity is conserved for conservative forces, meaning that the total energy E is constant. Since we have a conservative force, we can say

$$E = T + V = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r). \tag{2.2.1}$$

We want to eliminate the time derivatives and change from r to $u = \frac{1}{r}$, so we take

$$\dot{r} = -l\frac{du}{d\theta} \quad \text{and} \quad \dot{\theta} = lu^2 \tag{2.2.2}$$

and plug them into Equation 2.2.1, which yields

$$E = \tfrac{1}{2}ml^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) + V(u^{-1}). \tag{2.2.3}$$

This gives a first order differential equation for $u(\theta)$.

Accounting for the potential

$$V(r) = -\frac{k}{r} = -ku, \tag{2.2.4}$$

the orbit equation is

$$E = \tfrac{1}{2}ml^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - ku. \tag{2.2.5}$$

After extensive manipulation and some integration, we have

$$u = \frac{\sqrt{b^2 - 4ac}}{-2a}\cos\left(\sqrt{-a}(\theta - \theta_0)\right) + \frac{b}{-2a}, \tag{2.2.6}$$

where $a = -1$, $b = (2k)/(ml^2)$, and $c = (2E)/(ml^2)$. Changing back from u to r and more algebraic manipulation yields

$$\boxed{r = \frac{ml^2}{k\left(1 + \sqrt{1 + 2Eml^2/k^2}\cos(\theta - \theta_0)\right)}.} \tag{2.2.7}$$

This is the orbit equation. If we set $\theta_0 = 0$ and define constants χ and ξ such that

$$\chi = ml^2/k \quad \text{and} \quad \xi = \sqrt{1 + 2Eml^2/k},$$

we get the general form of the orbit equation,

$$\boxed{r = \frac{\chi}{1 + \xi\cos(\theta)}.} \tag{2.2.8}$$

Again, we see that this orbit is a conic section.

3
Classical Circular Motion in 2+1 Dimensions

This is our own derivation. We want to find classical expressions for the gravitational force, potential, orbit equation, and vertical free-fall that we can compare to the relativistic results in Chapter 5.

3.1 Generalized Coordinate System

Changing from three spatial dimensions to two spatial dimensions does not mean that we can simply exclude the third dimension—we must rebuild the model. In addition, since we are working in two spatial dimensions, angular momentum cannot be determined using the cross product. In order to define the equations of motion in a way that is independent of the coordinate system, we turn to Hamilton's equations to define angular momentum and the energy equation in a system of i^{th}-dimensional generalized coordinates, q_i and $\dot{q}_i$.

For a kinetic energy $T(\dot{q}_i)$ and a potential energy $V(q_i)$, the Lagrangian is defined as

$$L = L(q_i, \dot{q}_i) = T(\dot{q}_i) - V(q_i). \tag{3.1.1}$$

Through a Legendre transformation, we change from coordinates $(q_i, \dot{q}_i)$ to (q_i, p_i), where p_i is the generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{3.1.2}$$

We can now define the Hamiltonian as

$$H = H(p_i, q_i) = T(p_i) + V(q_i). \tag{3.1.3}$$

From the Lagrangian and the Hamiltonian we have Hamilton's canonical equations of motion,

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i. \tag{3.1.4}$$

These equations of motion hold for any dimension and any coordinate system, which makes them valuable for our non-standard system. We are using the spatial coordinate system $q_1 = r$ and $q_2 = \theta$, with p_1 and p_2 as the momentum terms p_r and p_θ.

3.2 Forces in Two Spatial Dimensions

In a standard three-dimensional spatial coordinate system, we have a gravitational force

$$F(r) = -\frac{Cm}{r^2}$$

on a test mass for a mass m, with some constant C. However, in two spatial dimensions, we want to have a force

$$F(r) = -\frac{km}{r}, \tag{3.2.1}$$

for some constant k. A demonstration of why we are using and how we arrived at a $\frac{1}{r}$ force is given in Appendix A.

This new force changes the potential. For a $\frac{1}{r}$ force, we have a potential energy

$$V(r) = -\int F(r)dr = km\ln(r) + \beta, \tag{3.2.2}$$

where k is a constant from the force, m is the mass, and β is a constant of integration.

In 3-space the constants k and m are given experimentally by Newton and Cavendish such that

$$F(r) = G\frac{mm_0}{r} \text{ and } V(r) = -Gmm_0 \ln(r) + \beta,$$

where m_0 is a test mass and G is the gravitational constant that corresponds to our k. However, in the 2-dimensional case we have no clear idea of a ”two-dimensional experiment ” or “two-dimensional mass”, so we leave the values of k and m unspecified.

3.3 Hamilton’s Equations

For our two-dimensional force, the Lagrangian is

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - (km\ln(r) + \beta)\,. \tag{3.3.1}$$

The generalized momenta for the generalized polar coordinates r and θ are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \tag{3.3.2}$$

so

$$\dot{r} = \frac{p_r}{m} \text{ and } \dot{\theta} = \frac{p_\theta}{mr^2}. \tag{3.3.3}$$

The Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + km\ln(r) + \beta. \tag{3.3.4}$$

For our coordinate system, Hamilton’s equations are

$$\frac{\partial H}{\partial p_r} = \dot{r}, \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta}, \tag{3.3.5}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r, \qquad \frac{\partial H}{\partial \theta} = -\dot{p}_\theta. \tag{3.3.6}$$

It follows from the Hamiltonian that

$$\frac{\partial H}{\partial r} = \frac{km}{r} - \frac{p_\theta^2}{mr^3} = -\dot{p}_r \tag{3.3.7}$$

and

$$\frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta. \tag{3.3.8}$$

We now have p_θ = constant = $mr^2\dot{\theta}$. This allows us to define a two-dimensional angular momentum. For convenience, we define

$$\boxed{l = r^2\dot{\theta}} \tag{3.3.9}$$

as an 'angular momentum per unit mass' term, so $p_\theta = ml$. From Equation 3.3.3 we know

$$m\ddot{r} = \dot{p}_r$$

and from Equation 3.3.7 we have

$$\dot{p}_r = -\frac{km}{r} + \frac{p_\theta^2}{mr^3} = -\frac{km}{r} + \frac{(ml)^2}{mr^3},$$

and so

$$m\ddot{r} = -\frac{km}{r} + \frac{m(l)^2}{r^3}, \tag{3.3.10}$$

which yields the differential equation of motion. Rather than solving this equation, we pursue an energy equation to the same end.

3.4 Orbital Energies

Now that we have shown the conservation of angular momentum in two spatial dimensions, we can move onto the energy equation. We assume conservation of angular momentum and conservation of energy.

The energy equation is

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + km\ln(r) + \beta = \text{constant}. \tag{3.4.1}$$

We rewrite this with a change of variables, $u = 1/r$, such that $\dot{r} = -\dot{u}/u^2$ and $\ln(r) = -\ln(u)$. So we have

$$E = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + km\ln(r) + \beta. \tag{3.4.2}$$

Substituting $\dot{\theta} = l/r^2 = lu^2$, we get

$$E = \tfrac{1}{2}m\left(\frac{\dot{u}^2}{u^4} + l^2u^2\right) - km\ln(u) + \beta. \tag{3.4.3}$$

Rearranging this equation, we have

$$E = \tfrac{1}{2}ml^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - km\ln(u) + \beta. \tag{3.4.4}$$

Rearranging again, we get

$$\begin{aligned}\left(\frac{du}{d\theta}\right)^2 + u^2 &= (E - (-km\ln(u) + \beta))\frac{2}{ml^2},\\ \frac{du}{d\theta} &= \sqrt{\frac{2E}{ml^2} + \frac{2km\ln(u)}{ml^2} - \frac{2\beta}{ml^2} - u^2}.\end{aligned} \tag{3.4.5}$$

We approximate this differential equation by substituting in for $\ln(u)$ the first two terms of its Taylor series expansion near $u = 1$:

$$\begin{aligned}\frac{du}{d\theta} &= \sqrt{\frac{2E}{ml^2} + \frac{2km}{ml^2}\left((u-1) - \frac{(u-1)^2}{2}\right) - \frac{2\beta}{ml^2} - u^2}\\ &= \sqrt{\frac{2E}{ml^2} + \frac{2km(u-1)}{ml^2} - \frac{2km(u-1)^2}{2ml^2} - \frac{2\beta}{ml^2} - u^2}\\ &= \sqrt{\frac{2E-2\beta-km}{ml^2} + \frac{4km}{ml^2}u - \frac{km+ml^2}{ml^2}u^2}.\end{aligned} \tag{3.4.6}$$

Now, let us define constants a, b, c such that

$$a = -\frac{km+ml^2}{ml^2},\ b = \frac{4km}{ml^2},\ \text{and } c = \frac{2E-2\beta-km}{ml^2}.$$

This gives the equation

$$\frac{du}{d\theta} = \sqrt{au^2+bu+c}, \tag{3.4.7}$$

so

$$d\theta = \frac{du}{\sqrt{au^2+bu+c}}. \tag{3.4.8}$$

Integrating gives us

$$\theta - \theta_0 = \frac{1}{\sqrt{-a}}\cos^{-1}\left(-\frac{b+2au}{\sqrt{b^2-4ac}}\right).$$

Solving for u:

$$\begin{aligned}
(\theta - \theta_0)\sqrt{-a} &= \cos^{-1}\left(\frac{-b + 2au}{\sqrt{b^2 - 4ac}}\right), \\
\cos\left((\theta - \theta_0)\sqrt{-a}\right) &= \left(\frac{-b + 2au}{\sqrt{b^2 - 4ac}}\right), \\
\sqrt{b^2 - 4ac}\cos\left((\theta - \theta_0)\sqrt{-a}\right) &= -b + 2au, \\
\frac{\sqrt{b^2 - 4ac}\cos\left((\theta - \theta_0)\sqrt{-a}\right) + b}{2a} &= u.
\end{aligned} \tag{3.4.9}$$

Substituting back in for r and accounting for some small perturbation ζ (since this is an approximation), we get

$$\boxed{r = \frac{2a}{\sqrt{b^2 - 4ac}\cos\left((\theta - \theta_0)\sqrt{-a}\right) + b} + \zeta.} \tag{3.4.10}$$

Thus we have the orbital motion under conservation of energy. Note that $a < 0$, hence $\sqrt{-a}$ does not pose a problem.

If we let $\theta_0 = 0$ and define constants α, β, γ such that

$$\alpha = \frac{2a}{\sqrt{b^2 - 4ac}}, \quad \beta = \sqrt{-a}, \quad \gamma = \frac{b}{\sqrt{b^2 - 4ac}},$$

we can rewrite the orbital energy equation as

$$\boxed{r = \frac{\alpha}{\cos(\beta\theta) + \gamma} + \zeta,} \tag{3.4.11}$$

which is of a similar form as compared to the 3-dimensional case.

3.5 Classical Free-Fall

To gain some understanding of the 2-dimensional system, we decided to solve for some characteristic solutions [2]. We begin at time $t = 0$ with a particle of mass m and a speed $v = 0$ at a distance from the center $r = r_0$. Under a potential

$$V = V_0 \ln\left(\frac{r}{r_0}\right), \tag{3.5.1}$$

we want to know how long it will take for the particle to reach the center of the circle. Notice that this potential is of a slightly different form than our previous 2-space potential, but if we take Equation 3.2.2 and let $\beta = -V_0 \ln(r_0)$ then we have the same equation.

The total energy of the system is

$$E = T + V = \tfrac{1}{2}mv^2 + V_0 \ln(r/r_0), \tag{3.5.2}$$

so at time $t = 0$ we have $E = 0$. Under conservation of energy, it follows that

$$\tfrac{1}{2}mv^2 + V_0 \ln(r/r_0) = 0.$$

Solving for the velocity v and taking the negative root, since the motion is inward, gives

$$v = -\sqrt{\tfrac{2}{m}V_0 \ln(r/r_0)}. \tag{3.5.3}$$

Since velocity is defined as $v = dr/dt$, it follows that $dt = dr/v$. We integrate this and find

$$\int_0^\tau dt = \int_{r_0}^0 \frac{dr}{v}.$$

Plugging in for v and integrating yields

$$\tau = -\sqrt{m/2V_0} \int_{r_0}^0 \frac{1}{\sqrt{|\ln(r/r_0)|}}\, dr. \tag{3.5.4}$$

For ease of computation, we now change variables from r to y such that $y = r/r_0$ and $dy = dr/r_0$. This gives

$$\tau = -r_0 \sqrt{m/2V_0} \int_1^0 \frac{1}{\sqrt{-\ln(y)}}\, dy,$$

which can be solved numerically, and therefore the time for free-fall is

$$\boxed{\tau = r_0 \sqrt{\frac{m\pi}{2V_0}}.} \tag{3.5.5}$$

We compare this to the the time for classical free-fall in 3 dimensions,

$$\tau = r_0 \frac{\pi}{2} \sqrt{\frac{m}{-2V_0}}, \tag{3.5.6}$$

and we observe that they are similar results.

4
Relativistic Spherical Motion in 3+1 Dimensions

This derivation follows the one described in [4].

4.1 Measuring Spacetime

We begin by setting up the mathematical framework for spacetime

Definition 4.1.1. An m-manifold is an m-dimensional space S where each point s has a neighborhood that looks like $\mathbb{R}^m$ [6, p 225]. In our case, the manifold is spacetime.

In a standard orthogonal spherical coordinate system, the position vector $\mathbf{r}$ is

$$\mathbf{r}(t,r,\theta,\phi) = t\,\mathbf{h} + r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}. \tag{4.1.1}$$

Since we are not necessarily in Cartesian flat space, we define the natural basis vectors from the position vector. In other words, we define how to measure a quantity based on how it moves on the manifold.

Definition 4.1.2. A natural basis vector $\mathbf{e}_u$ is defined as [4, p 9]

$$\mathbf{e}_u \equiv \partial\mathbf{r}/\partial u. \tag{4.1.2}$$

The basis vectors of a coordinate system span the tangent space to the manifold. As a result, they are tangent to the curve at any point $(t_0, r_0, \theta_0, \phi_0)$.

It follows that

$$\begin{aligned}
\mathbf{e}_t &= \mathbf{h} \\
\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\
\mathbf{e}_\theta &= r\cos\theta\cos\phi\,\mathbf{i} + r\cos\theta\cos\phi\,\mathbf{j} - r\sin\theta\,\mathbf{k} \\
\mathbf{e}_\phi &= -r\sin\theta\sin\phi\,\mathbf{i} + r\sin\theta\cos\phi\,\mathbf{j}
\end{aligned}$$

A metric defines how measurements are to be taken on a manifold for quantities like lengths, distances, and angles. For black holes, it also is a way of measuring the curvature of spacetime near the singularity.

Definition 4.1.3. The covariant metric tensor $g_{\mu\nu}$ is defined as [4, p 32]

$$g_{\mu\nu} = \eta^{\mu\nu}\mathbf{e}_\mu \cdot \mathbf{e}_\nu. \tag{4.1.3}$$

The contravariant metric tensor $g^{\mu\nu}$ is defined as

$$g^{\mu\nu} = \left(g_{\mu\nu}\right)^{-1}. \tag{4.1.4}$$

The coefficient $\eta^{\mu\nu}$ corresponds to the elements of the Minkowski metric,

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.1.5}$$

A covariant metric tensor, also just known as a metric, is defined on a manifold. We will mainly be dealing with diagonal metrics, where $\mu = \nu$.

A metric allows us to measure the length of a quantity ξ on the manifold by

$$||\xi||^2 = g_{\mu\mu}(\xi^\mu)^2.$$

We will say that a "singular" metric is one that yields a physical singularity, and therefore is a metric that will produce a black hole.

The Schwarzschild solution to the Einstein equations holds for a static, spherically symmetric, and asymptotically flat vacuum spacetime. The metric for the Schwarzschild solution is of the form

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{4.1.6}$$

The functions $A(r)$ and $B(r)$ tell us how the spacetime warps as $r \to r_s$, where r_s is the Schwarzschild radius. We will solve for A and B when we solve for the Ricci tensor in the Einstein field equation.

Definition 4.1.4. A line element is the length of an infinitesimal distance on the manifold. The line element of a spatial manifold is defined as

$$ds^2 = g_{ab}dx^a dx^b, \tag{4.1.7}$$

where $a, b = 1, 2, 3$ refers to the r, θ and ϕ coordinates [4, p 45].

Similarly, in a manifold with both time and space components, we have

$$c^2 d\tau^2 \equiv g_{\mu\nu}dx^\mu dx^\nu, \tag{4.1.8}$$

where τ is the proper time, and $\mu, \nu = 0, 1, 2, 3$ correspond to t, r, θ and ϕ [4, p 84].

The Schwarzschild line element is [4, p 116]

$$\boxed{c^2 d\tau^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2.} \tag{4.1.9}$$

4.2 Geodesics

Definition 4.2.1. A geodesic is the curve that minimizes the length of a line connecting two points, and locally it looks like a straight line even though it follows the curves of the manifold. It is the differential geometry equivalent to a straight line in Euclidean space.

Definition 4.2.2. Christoffel symbols tell us how much a chosen path deviates from a geodesic on the manifold. We represent a Christoffel symbol as $\Gamma^{\mu}_{\nu\rho}$, and Christoffel symbols are symmetric, so $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$.

There are three ways we can find the Christoffel symbols.

(i) Lagrangian method We construct the Lagrangian from the metric with the equation

$$L(\dot{x}^{\rho}, x^{\rho}) \equiv \tfrac{1}{2} g_{\mu\nu}(x^{\rho})\dot{x}^{\mu}\dot{x}^{\nu}. \tag{4.2.1}$$

For a 3+1-dimensional Schwarzschild metric the Lagrangian is

$$L \equiv \tfrac{1}{2}\left(A(r)\,\dot{t}^2 - B(r)\,\dot{r}^2 - r^2\,\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2\right). \tag{4.2.2}$$

The partial derivatives of this Lagrangian are

$$\frac{\partial L}{\partial \dot{t}} = A\,\dot{t}, \qquad \frac{\partial L}{\partial t} = 0, \tag{4.2.3a}$$

$$\frac{\partial L}{\partial \dot{r}} = -B\,\dot{r}, \qquad \frac{\partial L}{\partial r} = \frac{A'}{2}\,\dot{t}^2 - \frac{B'}{2}\,\dot{r}^2 - r\,\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2, \tag{4.2.3b}$$

$$\frac{\partial L}{\partial \dot{\theta}} = -r^2\,\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -r^2\sin\theta\cos\theta\,\dot{\phi}^2, \tag{4.2.3c}$$

$$\frac{\partial L}{\partial \dot{\phi}} = -r^2\sin^2\theta\,\dot{\phi}, \qquad \frac{\partial L}{\partial \phi} = 0. \tag{4.2.3d}$$

Note that the shorthand notation A, B, A', and B' refers to $A(r), B(r), A'(r)$, and $B'(r)$.

We take the Euler-Lagrange equation,

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0, \tag{4.2.4}$$

where d/du indicates taking the total derivative, and $\mu = 0, 1, 2, 3$ refers to t, r, θ, and ϕ. The Euler-Lagrange equation gives the equations of motion for the manifold based on the Lagrangian.

Plugging 4.2.3a, 4.2.3b, 4.2.3c, and 4.2.3d into Equation 4.2.4 gives the four Euler-Lagrange equations

$$A\,\ddot{t} + A'\,\dot{t}\dot{r} = 0, \tag{4.2.5a}$$

$$-B\,\ddot{r} - \frac{A'}{2}\,\dot{t}^2 - \frac{B'}{2}\,\dot{r}^2 + r\,\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 = 0, \tag{4.2.5b}$$

$$-r^2\,\ddot{\theta} - 2r\,\dot{r}\dot{\theta} + r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0, \text{and} \tag{4.2.5c}$$

$$-r^2\sin^2\,\ddot{\phi} - 2r\sin^2\theta\,\dot{r}\dot{\phi} - 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} = 0. \tag{4.2.5d}$$

Observe that these are almost of the form of the geodesic equation,

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{4.2.6}$$

In order to get the geodesic equations from the Euler-Lagrange equations, we must solve for $\ddot{x}^\rho$. This yields the geodesic equations

$$\ddot{t} + \frac{A'}{A}\,\dot{t}\dot{r} = 0, \tag{4.2.7a}$$

$$\ddot{r} + \frac{A'}{2B}\,\dot{t}^2 + \frac{B'}{2B}\,\dot{r}^2 - \frac{r}{B}\,\dot{\theta}^2 - \frac{r\sin^2\theta}{B}\,\dot{\phi}^2 = 0, \tag{4.2.7b}$$

$$\ddot{\theta} + \frac{2}{r}\,\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \text{ and} \tag{4.2.7c}$$

$$\ddot{\phi} + \frac{2}{r}\,\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0. \tag{4.2.7d}$$

The Γ terms are the Christoffel symbols. Note that when the subscripts are unequal, so $\mu \neq \nu$, we must halve the Christoffel symbol to avoid counting it twice. Therefore, we have

$$\Gamma^t_{tr} = \frac{A}{2A}, \tag{4.2.8a}$$
$$\Gamma^r_{tt} = \frac{A'}{2B}, \tag{4.2.8b}$$
$$\Gamma^r_{rr} = \frac{B'}{2B}, \tag{4.2.8c}$$
$$\Gamma^r_{\theta\theta} = \frac{-r}{B}, \tag{4.2.8d}$$
$$\Gamma^r_{\phi\phi} = \frac{-r\sin^2\theta}{B}, \tag{4.2.8e}$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \tag{4.2.8f}$$
$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{4.2.8g}$$
$$\Gamma^\phi_{r\phi} = \frac{1}{r}, \tag{4.2.8h}$$
$$\Gamma^\phi_{\theta\phi} = \cot\theta, \tag{4.2.8i}$$

with the rest of the Christoffel symbols being zero [4, p 117].

(ii) Derivative of metric This is a straightforward yet tedious computation from the equation [4, p 70]

$$\Gamma^l_{ki} = \tfrac{1}{2} g^{lj}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}\right). \tag{4.2.9}$$

We get the same results.

(iii) Change of coordinates This method involves two sets of coordinates. An example of this method is in Appendix D.

We have shown that we can find the Christoffel symbols from the mechanics of the system, the metric directly, or how the coordinates and tensors transform. This means that we can find how curved the paths in the space are from the fundamental measurements of the manifold, from the equations governing movement across that manifold, or from the process of getting from Cartesian space to the curved manifold.

4.3 Tensors

Definition 4.3.1. A tensor of rank n is an n-dimensional block of numbers that transforms according to its indices. More precisely, a tensor $\tilde{T}$ of rank $p+q$ transforms to a tensor T of rank $p+q$ by

$$T^{i_1 \dots i_p}_{j_1 \dots j_q} = \sum_{(k),(l)} \tilde{T}^{k_1 \dots k_p}_{l_1 \dots l_q} \frac{\partial x^{i_1}}{\partial z^{k_1}} \cdots \frac{\partial x^{i_p}}{\partial z^{k_p}} \frac{\partial z^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial z^{l_q}}{\partial x^{j_q}},$$

where $x^1 \cdots x^n$ are the coordinates relative to the tensor T and $z^1 \cdots z^n$ are the coordinates relative to the tensor $\tilde{T}$ [3, p 152]. The indices of T and $\tilde{T}$ tell us the specific component of the tensor.

We mainly deal with tensors of rank 2. Think of them like a higher-order vector or an array. We find that the tensors are all symmetric, and all have a divergence of zero due to equations of continuity, equations of motion, and Maxwell's equations [4, p 101]. A symmetric tensor means that $T_{ab} = T_{ba}$ and $T^{ab} = T^{ba}$.

The first tensor we consider is the stress tensor $T^{\mu\nu}$. The stress tensor is a mathematical representation of the relationship between energy, momentum, and stress in a distribution of matter. We can think of the vacuum of space surrounding the black hole as a "perfect fluid" with no pressure and no velocity, since by definition there is no matter in a vacuum. Thus

$$T^{\mu\nu} = 0. \tag{4.3.1}$$

This gives physical conservation laws of energy and momentum. We next look at the Einstein field equation and Einstein tensor $G^{\mu\nu}$. The Einstein field equation is

$$G^{\mu\nu} = \kappa T^{\mu\nu}, \tag{4.3.2}$$

where κ is the coupling constant experimentally found to be $-8\pi G/c^4$. Qualitatively, we can think of $T^{\mu\nu}$ as the physical character and $G^{\mu\nu}$ as the geometric character in this region.

Since we set $T^{\mu\nu} = 0$, it follows that

$$G^{\mu\nu} = 0. \tag{4.3.3}$$

One key tensor is the Riemann tensor, also known as the curvature tensor. It is given by the formula

$$R^{\sigma}_{\mu\nu\rho} \equiv \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\eta}_{\mu\rho} \Gamma^{\sigma}_{\eta\nu} - \Gamma^{\eta}_{\mu\nu} \Gamma^{\sigma}_{\eta\rho}. \tag{4.3.4}$$

This tells us specifically how curved the manifold itself is, not just the paths on it.

Definition 4.3.2. A manifold is flat if $R^{\mu}_{\nu\rho\sigma} = 0$ at each point (t, u, v, w) on the manifold [4, p 103]. In other words, the Riemann tensor measures how much the manifold locally deviates from flat space.

A more specific variation of the Riemann tensor that we want to focus on is the Ricci tensor, given by

$$R_{\mu\nu} = R^{\rho}_{\mu\nu\rho}. \tag{4.3.5}$$

The process above is called contraction. If we contract again, we get the curvature scalar

$$R \equiv R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{4.3.6}$$

The Einstein tensor $G^{\mu\nu}$ is defined by the Ricci tensors and the curvature scalar. Einstein chose this formulation of $G^{\mu\nu}$ so the geometric tensor would have zero curvature when $T^{\mu\nu} = 0$. Defining the Einstein tensor only in terms of $R^{\mu\nu}$, the contravariant version of $R_{\mu\nu}$, does not give this due to a particular method of differentiation known as covariant differentiation [4, pp 76, 113], so we have

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}. \tag{4.3.7}$$

As previously stated, we have $G^{\mu\nu} = 0$, so by [4, p 113] and [3, p 400] it follows that

$$R^{\mu\nu} = 0. \tag{4.3.8}$$

Note that $R^{\mu\nu} = 0$ implies $R_{\mu\nu} = 0$ as well [7].

We now compute the Ricci tensor components. The non-zero Ricci tensor components for our four-dimensional Schwarzschild metric are

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0, \tag{4.3.9}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0, \tag{4.3.10}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0, \tag{4.3.11}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta = 0. \tag{4.3.12}$$

To find functions A and B we must decouple these differential equations in order to solve $R_{\mu\nu} = 0$. We take $R_{rr} + \frac{B}{A}R_{tt} = 0$, which simplifies to

$$A'B + AB' = 0. \tag{4.3.13}$$

So by the product rule it must be that AB = constant. As we move away from the black hole, we want our metric to look like Minkowski space, so for our boundary conditions we let $A \to c^2$ and $B \to 1$ as $r \to \infty$.

We let the constant above be c^2, so $B = c^2/A$. We plug this into $R_{\theta\theta}$ and find that

$$A(r) = c^2(1 + \tfrac{k}{r}) \quad \text{and} \quad B(r) = (1 + \tfrac{k}{r})^{-1}, \tag{4.3.14}$$

where k is a constant of integration. Based on the Newtonian mechanics and gravitational potential of the system, we find that $k = -2GM/c^2$, where G is the gravitational field constant and M is the mass of the object producing the gravitational field.

Therefore the Schwarzschild line element is

$$\boxed{c^2 d\tau^2 = c^2\left(1 - \frac{2GM}{rc^2}\right)dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2} \tag{4.3.15}$$

and the metric is

$$g_{\mu\nu} = \begin{pmatrix} c^2\left(1-\frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{4.3.16}$$

4.4 Dynamics

4.4.1 *Relativistic orbit equation*

We now want to analyze how objects move along time-like geodesics in spacetime. First, we will look at the orbit equation in the equatorial plane, which shows how the radius r changes with ϕ.

We start with Equation 4.1.9 for the line element,

$$c^2 d\tau^2 = c^2\left(1-2m/r\right)\,\dot{t}^2 - \left(1-2m/r\right)^{-1}\,\dot{r}^2 - r^2\,\dot{\phi}. \tag{4.4.1}$$

For simplicity, we can set $d\tau^2 = 1$ and assume that we are moving along the manifold at unit time. Assuming $\dot{\phi} \neq 0$, we divide through by $\dot{\phi}^2$ and get

$$\frac{c^2}{\dot{\phi}^2} = c^2\left(1-\frac{2m}{r}\right)\frac{\dot{t}^2}{\dot{\phi}^2} - \left(1-\frac{2m}{r}\right)^{-1}\frac{\dot{r}^2}{\dot{\phi}^2} - r^2.$$

Due to the spherical symmetries of t and ϕ, we can set

$$\frac{\partial L}{\partial \dot{t}} = \left(1-\frac{2m}{r}\right)\dot{t} = k \quad\text{and}\quad \frac{\partial L}{\partial \dot{\phi}} = r^2\,\dot{\phi} = h \tag{4.4.2}$$

for some constants k and h, and let $m = GM/c^2$. In addition, we let $\theta = \pi/2$, which is akin to picking the specific geodesic around the equator and computing the orbit equation for a particle moving along that geodesic. We plug these in and get

$$\frac{c^2 r^4}{h^2} = c^2\left(1-\frac{2m}{r}\right)\frac{k^2 r^4 \dot{t}^2}{h^2(1-2m/r)^2} - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\phi}\right)^2 - r^2. \tag{4.4.3}$$

Observe that instead of plugging in for $\dot{\phi}^2$ in the $\dot{r}^2$ term, we instead make use of the fact that

$$\frac{\dot{r}^2}{\dot{\phi}^2} = \left(\frac{\dot{r}}{\dot{\phi}}\right)^2 = \left(\frac{dr}{d\tau}\frac{d\tau}{d\phi}\right)^2 = \left(\frac{dr}{d\phi}\right)^2,$$

which will later allow us to solve a differential equation. We can rewrite the previous equation as

$$\left(\frac{dr}{d\phi}\right)^2 + r^2\left(1+\frac{c^2r^2}{h^2}\right)\left(1-\frac{2GM}{rc^2}\right) - \frac{c^2k^2r^4}{h^2} = 0. \tag{4.4.4}$$

We now change variables from r to u, with $u = 1/r$, to make the computation easier. This gives

$$\boxed{\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2(k^2-1)}{h^2} + \frac{2GM}{h^2}u + \frac{2GM}{c^2}u^3.} \tag{4.4.5}$$

If we let $E = c^2(k^2-1)/h^2$, we notice that this equation is in the same form as the classical Newtonian orbit equation, plus a relativistic correction $2GMu^3/c^2$.

4.4.2 *Objects falling into the black hole*

We return to Equation 4.4.1. However, since we want to find the equation of motion for a particle falling radially from its orbit into a black hole, we set $\dot{\phi} = 0$ instead of assuming it is nonzero like we did previously. With this new condition, and substituting in Equation 4.4.2 for $\dot{t}$, we have [4, p 137]

$$\dot{r}^2 - c^2k^2 + c^2(1-2m/r) = 0. \tag{4.4.6}$$

Now, we want to find a value for the constant k. We look at a particular case of this equation, when the particle is at rest, so when $r = r_0$,we have $\dot{r} = 0$. Plugging this in, we find that $k = +\sqrt{1-2m/r_0}$. Notice that we take the positive square root of k, since we follow the convention of time moving positively forward. In addition, we note that k is not a universal constant, since it changes based on the starting radius r_0.

When deriving equations of motion, we aim for something resembling a classical force equation, meaning that it has a second derivative of a position variable. Hence we differentiate both sides of (4.4.6), which gives, with rearranging and setting $c = 1$,

$$\boxed{\ddot{r} + GM/r^2 = 0.} \tag{4.4.7}$$

This equation is in the same form as the classical Newtonian equation for vertical free fall, [5, p 63]

$$\ddot{x} + GM/r^2 = 0,$$

but the relativistic r is not the vertical distance and the relativistic derivatives are with respect to proper time τ, not coordinate time t [4, p 138].

4.4.3 Motion in a circle

We start with the radial Euler-Lagrange equation,

$$-B\,\ddot{r} - \frac{A'}{2}\,\dot{t}^2 - \frac{B'}{2}\,\dot{r}^2 + r\,\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 = 0. \tag{4.4.8}$$

Plugging in the values for A and B from (4.3.14), we have

$$\left(1 - \frac{2m}{r}\right)^{-1}\ddot{r} + \frac{mc^2}{r^2}\,\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-2}\frac{m}{r^2}\,\dot{r} - r\,\dot{\phi}^2 = 0. \tag{4.4.9}$$

Assuming this motion is taking place in the equatorial plane, we let $\theta = \frac{\pi}{2}$. Additionally, since a circle has a constant radius, we let $\dot{r} = \ddot{r} = 0$. This yields

$$\frac{mc^2}{r^2}\,\dot{t}^2 - r\,\dot{\phi}^2 = 0, \tag{4.4.10}$$

which rearranges to

$$mc^2\,\dot{t}^2 = r^3\,\dot{\phi}^2. \tag{4.4.11}$$

It follows that

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{mc^2}{r^3} \tag{4.4.12}$$

since

$$\frac{\dot{\phi}^2}{\dot{t}^2} = \left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \left(\frac{d\phi}{d\tau}\frac{d\tau}{dt}\right)^2 = \left(\frac{d\phi}{dt}\right)^2.$$

Solving this differential equation and setting $\phi = 2\pi$, $\phi_0 = 0$, and $t_0 = 0$ gives

$$\boxed{t = 2\pi\sqrt{\frac{r^3}{GM}},} \tag{4.4.13}$$

the time it takes for one revolution about the equator. This is of the same form as Kepler's third law,

$$\tau = 2\pi\sqrt{\frac{a^3}{GM_{\text{sun}}}},$$

where a is the semimajor axis [5, p 239]. Classically, coordinate time is proper time, so relativistically it is OK that this is τ and not t.

5
Relativistic Circular Motion in 2+1 Dimensions

We now build a relativistic framework for a black hole in 2+1 dimensions. This is our own derivation, following similar steps as in Chapter 4.

5.1 Metric and Line Element

In two space dimensions and one time dimension, the Schwarzschild line element looks like

$$\boxed{c^2 d\tau^2 = A(r)\, dt^2 - B(r)\, dt^2 - r^2\, d\theta^2,} \tag{5.1.1}$$

where $A(r)$ and $B(r)$ are functions of r to be determined later [4, p 116].

Therefore the metric is of the form

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 \\ 0 & -B(r) & 0 \\ 0 & 0 & -r^2 \end{pmatrix}, \tag{5.1.2}$$

with $\mu, \nu = 0, 1, 2$ corresponding to t, r, and θ coordinates.

5.2 Geodesic Equation

We determine the geodesic equation using the Lagrangian method as outlined in §4.2. We set up the Lagrangian from the metric to find the equations of motion:

$$L = \frac{1}{2}\left(A\,\dot{t}^2 - B\,\dot{r}^2 - r^2\dot{\theta}^2\right), \tag{5.2.1}$$

where the Newtonian dot notation indicates taking a derivative with respect to proper time τ [4, p 60]. Taking the partial derivatives, we get

$$\frac{\partial L}{\partial \dot{t}} = A\,\dot{t}, \quad \frac{\partial L}{\partial t} = 0, \tag{5.2.2a}$$
$$\frac{\partial L}{\partial \dot{r}} = -B\,\dot{r}, \quad \frac{\partial L}{\partial r} = \frac{A'}{2}\,\dot{t}^2 - \frac{B'}{2}\,\dot{r}^2 - r\,\dot{\theta}^2, \tag{5.2.2b}$$
$$\frac{\partial L}{\partial \dot{\theta}} = -r^2\,\dot{\theta}, \frac{\partial L}{\partial \theta} = 0. \tag{5.2.2c}$$

Note that a prime indicates a derivative with respect to r.

The Euler-Lagrange equation is

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0, \tag{5.2.3}$$

where $\mu = 0, 1, 2$ such that $x^0 = t$, $x^1 = r$, and $x^2 = \theta$ [4, p 60]. We plug in the partial derivatives to get the Euler-Lagrange equations:

$$A\ddot{t} + A'\dot{t}\dot{r} = 0, \tag{5.2.4a}$$
$$-B\ddot{r} - B'\dot{r}^2 - \frac{A'}{2}\dot{t}^2 - \frac{B'}{2}\dot{r}^2 - r\dot{\theta}^2 = 0, \tag{5.2.4b}$$
$$-r^2\ddot{\theta} - 2r\dot{r}\dot{\theta} = 0. \tag{5.2.4c}$$

We now solve the Euler-Lagrange equations for $\ddot{x}^\mu$. This gives the geodesic equations

$$\ddot{t} + \frac{A'}{A}\dot{t}\dot{r} = 0, \tag{5.2.5a}$$
$$\ddot{r} + \frac{A'}{2B}\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 - \frac{r}{B}\dot{\theta}^2 = 0, \tag{5.2.5b}$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0. \tag{5.2.5c}$$

Using the geodesic equation,

$$\ddot{x}^{\rho} + \Gamma^{\rho}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0, \tag{5.2.6}$$

we are able to pluck out the nonzero Christoffel symbols

$$\Gamma^{t}_{tr} = \frac{A'}{2A}\,, \quad \Gamma^{r}_{tt} = \frac{A'}{2B}\,, \quad \Gamma^{r}_{rr} = \frac{B'}{2B}\,, \quad \Gamma^{r}_{\theta\theta} = \frac{-r}{B}\,, \quad \Gamma^{\theta}_{r\theta} = \frac{1}{r}\,. \tag{5.2.7}$$

Note that for Γ^{t}_{tr} and $\Gamma^{\theta}_{r\theta}$, since they have subscripts $\mu \neq \nu$, we must halve the Christoffel symbol so as to not count it twice.

5.3 Ricci Tensor

We now find the Ricci tensor components. The Ricci tensor component formula is

$$R_{\mu\nu} = \partial_{\nu}\Gamma^{\sigma}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}, \tag{5.3.1}$$

where the indices $\mu, \nu = 0, 1, 2$ correspond to t, r and θ, and $\rho, \sigma = 0, 1, 2$. We plug this equation into Sage to solve for the Ricci tensor components, as seen in Appendix E. We find that

$$R_{tt} = \frac{(A')^2}{4AB} - \frac{A''}{2B} + \frac{A'B'}{4B^2} - \frac{A'}{2rB}, \tag{5.3.2}$$

$$R_{rr} = -\frac{A'B'}{4AB} + \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{B'}{2rB}, \tag{5.3.3}$$

$$R_{\theta\theta} = \frac{rA'}{2AB} - \frac{rB'}{2B^2}, \tag{5.3.4}$$

and the rest are trivial. Following the logic of §4.3, we let $R_{tt} = R_{rr} = R_{\theta\theta} = 0$. We take $(R_{tt} \cdot B/A) + R_{rr} = 0$, to eliminate the second derivatives, which yields

$$-A'B = AB'. \tag{5.3.5}$$

This implies that AB = constant, assuming $A, B \neq 0$. With the boundary conditions $A \to c^2$, $B \to 1$ as $r \to \infty$, we let

$$AB = c^2, \tag{5.3.6}$$

so

$$B = \frac{c^2}{A}. \tag{5.3.7}$$

We substitute this into $R_{\theta\theta}$, which gives

$$\begin{aligned} \frac{rA'}{2AB} - \frac{rB'}{2B^2} &= 0, \\ \frac{A'}{2A} - \frac{B'}{2B} &= 0, \\ \frac{A'}{A} + \frac{c^2 A'}{A^2}\frac{A}{c^2} &= 0, \\ \frac{A'}{A} + \frac{A'}{A} &= 0. \end{aligned} \tag{5.3.8}$$

Since $A \neq 0$, it must be that $A' = 0$, so A = constant. Thus

$$\boxed{A = c^2} \quad \text{and} \quad \boxed{B = 1.} \tag{5.3.9}$$

This produces the line element

$$\boxed{c^2\, d\tau^2 = c^2\, dt^2 - dr^2 - r^2\, d\theta^2} \tag{5.3.10}$$

and a flat Minkowski-like metric

$$\boxed{g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -r^2 \end{pmatrix},} \tag{5.3.11}$$

the implications of which are explored further in §6.

5.4 Dynamics

Since our manifold is flat, we expect that the dynamics of the system will behave like they would in Minkowski space.

5.4.1 Orbit Equation

Looking back at the Lagrangian, which gives us equations of motion, we take $\partial L/\partial \dot{t}$ = constant and $\partial L/\partial \dot{\theta}$ = constant, so we have the relationship between coordinate time t

and proper time τ,

$$c^2\dot{t} = k, \tag{5.4.1}$$

and an equation analogous to the conservation of angular momentum,

$$r^2\dot{\theta} = h, \tag{5.4.2}$$

where k and h are constants of integration [4, p 137].

We start with the equation for the line element of the metric,

$$c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 = c^2, \tag{5.4.3}$$

with $d\tau = 1$, so that we are moving along the manifold at unit time.

Since we want orbital motion, we know $\dot{\theta} \neq 0$, so we can divide (5.4.3) through by $\dot{\theta}^2$, which gives

$$\frac{c^2\dot{t}^2}{\dot{\theta}^2} - \left(\frac{dr}{d\theta}\right)^2 - r^2 = \frac{c^2}{\dot{\theta}^2}. \tag{5.4.4}$$

We substitute in (5.4.1) and (5.4.2) for $\dot{\theta}$ and $\dot{t}$ and get

$$\frac{k^2r^4}{c^2h^2} - \left(\frac{dr}{d\theta}\right)^2 - r^2 = \frac{c^2r^4}{h^2}. \tag{5.4.5}$$

Rearranging yields

$$\left(\frac{dr}{d\theta}\right)^2 - \frac{k^2r^4}{c^2h^2} + r^2 + \frac{c^2r^4}{h^2} = 0. \tag{5.4.6}$$

We let $u = 1/r$, so $dr/d\theta = -du/(u^2d\theta)$. It follows that

$$\left(\frac{du}{d\theta}\right)^2 + \frac{k^2}{c^2h^2} - u^2 + \frac{c^2}{h^2} = 0. \tag{5.4.7}$$

We rewrite this as

$$\boxed{\left(\frac{du}{d\theta}\right)^2 = u^2 - \frac{k^2 + c^4}{c^2h^2}.} \tag{5.4.8}$$

This is the relativistic orbital energy equation for 2+1 dimensions. If we compare it with the approximated classical equation for 2+1 dimensions,

$$\left(\frac{du}{d\theta}\right)^2 = au^2 + bu + c,$$

we notice that the two equations are similar, but the relativistic equation is missing a linear term. In 3+1 dimensions, the relativistic equation is the classical equation plus a cubic correction term. However in this case, the relativistic equation is missing a term, or would have a correction term that is negative the linear term.

5.4.2 Vertical Free-Fall

We now want to find the radius r as a function of proper time τ to give an equation of motion for an object starting a radius r.

We return to Equation 5.4.3. For vertical free-fall, we set $\dot{\theta} = 0$, and we let $\dot{t} = k$ with $c = 1$ as per Equation 5.4.1. This gives

$$\dot{r} = \sqrt{k^2 - 1},$$

where $\dot{r} = dr/d\tau$. Solving this differential equation yields

$$r = \sqrt{k^2 - 1}\tau + \gamma, \tag{5.4.9}$$

where γ is a constant of integration. This means that the vertical free-fall is a constant linear function of the proper time τ. If we start the particle at a radius r_0 and solve for the time τ, we find

$$\boxed{\tau = r_0\sqrt{\frac{1}{k^2 - 1}} - \gamma,} \tag{5.4.10}$$

which is somewhat similar to the classical vertical free-fall time.

5.4.3 Motion in a Circle

We begin with the radial Euler-Lagrange equation,

$$-B\ddot{r} - B'\dot{r}^2 - \frac{A'}{2}\dot{t}^2 - \frac{B'}{2}\dot{r}^2 - r\dot{\theta}^2 = 0.$$

Substituting in (5.3.9) for A and B gives

$$-\ddot{r} - r^2\dot{\theta}^2 = 0. \tag{5.4.11}$$

Since a circle has constant radius, we have $\ddot{r} = 0$, so

$$-r^2\dot{\theta}^2 = 0. \tag{5.4.12}$$

This gives

$$\boxed{\frac{d\theta}{d\tau} = 0.} \tag{5.4.13}$$

This is not to be interpreted that there is no circular orbit, but that the circular orbit is not a geodesic in 2+1-dimensional flat space. The difference between this argument and the 3+1-dimensional argument is that a circle is a geodesic in the 3+1-dimensional spherical model, so the Euler-Lagrange equation gives the geodesic equation. However, a straight line is a geodesic in 2+1-dimensional flat space, which is why the Euler-Lagrange equation gives us a straight line. Here, the Euler-Lagrange equations do not necessarily yield the coordinate equations of motion. Instead, they yield the geodesic equations, and in this case a circular orbit is not a geodesic.

6
Conclusion

In 3 total dimensions, all the information about the curvature of the manifold is encoded in the Ricci tensor [3, p 309]. The act of setting the Ricci tensor equal to zero in 2+1 dimensions (which results from setting the stress tensor equal to zero) effectively forces the manifold to be flat. We are therefore guaranteed to get a Minkowski-like metric if we are working in a vacuum.

We will now show that our metric is Minkowski-like in 2+1-dimensional polar coordinates. Plugging our values for A and B into the metric, we have

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -r^2 \end{pmatrix} \tag{6.0.1}$$

with $c = 1$.

We choose to isolate the spatial part of the metric,

$$g_{ab}^{r,\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{6.0.2}$$

so that the total metric is written as

$$g_{\mu\nu}^{r,\theta} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ab}^{r,\theta} \end{pmatrix}. \tag{6.0.3}$$

We now employ a coordinate change to switch from polar coordinates to Cartesian coordinates. We have the coordinate transformation formulas

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \arctan(y/x),$$

and the transformation equation

$$g_{ab}^{x,y} = P^T g_{ab}^{r,\theta} P\,, \tag{6.0.4}$$

where $P = \begin{pmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{pmatrix}$ and P^T is its transpose. We solve this equation and find that

$$g_{ab}^{x,y} = \begin{pmatrix} x/r & -y/r^2 \\ y/r & x/r^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.0.5}$$

Following the same logic as above, we let

$$g_{\mu\nu}^{x,y} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ab}^{x,y} \end{pmatrix}, \tag{6.0.6}$$

which yields

$$g_{\mu\nu}^{x,y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.0.7}$$

This is a Minkowski-like metric in 2+1 dimensions. Since a Minkowski-like metric means that the space is flat, we therefore do not have a black hole solution.

Hence there are no non-singular Schwarzschild-like solutions in 2+1 dimensions for Einstein's equations. If we want to get a black hole-like solution, we will need to modify Einstein's equations with a different cosmological constant. We explore this possibility further in §7.1.

7
Future Research

7.1 Modifying Einstein's Field Equations

We have found that a Schwarzschild solution to Einstein's field equations does not yield a black hole in 2+1 dimensions. However, we could modify Einstein's field equation with a cosmological constant Λ so that a Schwarzschild solution would be non-singular.

Einstein's modified field equation is [4, p 201]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{7.1.1}$$

Researchers have found a solution to this for a charged Einstein-Maxwell field with a negative cosmological constant [1]. It is possible that Einstein's field equation modified by a cosmological constant might allow for a 2+1-dimensional non-singular Schwarzschild solution.

7.2 Relaxing Schwarzschild's Assumptions

Schwarzschild found his solution to Einstein's field equations based on four assumptions: a) the field was static, b) the field was spherically symmetric, c) the spacetime was empty, and d) the spacetime was asymptotically flat. Relaxing one or more of these assumptions could result in a black hole solution to Einstein's field equation in 2+1 dimensions.

a. If the field was not static, we could let the metric vary with t, which would have us solving for functions $A(t, r)$ and $B(t, r)$. The mathematics of dynamical systems may prove useful in pursuing a solution of this nature.

b. See §7.3.

c. Relaxing this condition is tantamount to a Robertson-Walker-type approach that includes radiation and mass-density in the stress tensor, which would require time-dependent modifications of the metric.

d. This condition is what gives the boundary conditions $A(r) = c^2$ and $B(r) = 1$ as $r \to \infty$. It is quite possible that there exists a Schwarzschild solution to Einstein's field equations in 2+1 dimensions that satisfies a different set of boundary conditions, one that would not necessarily have an asymptotically flat spacetime.

7.3 Relativistic Elliptical Motion in 2+1 Dimensions

Initially, we were pursuing a 2+1-dimensional Schwarzschild solution in elliptical polar coordinates that we could then rotate about the major axis to give an ellipsoidal polar solution in 3+1 dimensions.

7.3.1 *Metric and Line Element*

The equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2,$$

or

$$x = ar\cos\theta, \quad y = br\sin\theta$$

for two constants a, b. Let $a = \alpha$ and $b = 1$, where α is some nonzero stretching constant. Note that when $\alpha = 1$, the elliptical case will reduce to the circular case. So we have $x = \alpha r \cos\theta$ and $y = r\sin\theta$. This gives the position vector

$$\mathbf{r} = \alpha r \cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}. \tag{7.3.1}$$

The natural basis vectors are

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \alpha\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \tag{7.3.2}$$

and

$$\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -\alpha r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}. \tag{7.3.3}$$

We take the dot products of the natural basis vectors to find the elements of the spatial metric:

$$\mathbf{e}_r \cdot \mathbf{e}_r = \alpha^2\cos^2\theta + \sin^2\theta, \tag{7.3.4a}$$

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = (1-\alpha^2) r\sin\theta\cos\theta, \tag{7.3.4b}$$

$$\mathbf{e}_\theta \cdot \mathbf{e}_r = (1-\alpha^2) r\sin\theta\cos\theta, \tag{7.3.4c}$$

$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = r^2(\alpha^2\sin^2\theta + \cos^2\theta). \tag{7.3.4d}$$

Therefore our spatial metric is

$$g_{ij} = \begin{pmatrix} \alpha^2\cos^2\theta + \sin^2\theta & (1-\alpha^2) r\sin\theta\cos\theta \\ (1-\alpha^2) r\sin\theta\cos\theta & r^2(\alpha^2\sin^2\theta + \cos^2\theta) \end{pmatrix}, \tag{7.3.5}$$

which reduces to the circular correctly.

At $r \to \infty$, we expect that an ellipse will be circularly symmetric. Therefore, our boundary conditions will be $A(r) \to c^2$ and $B(r) \to 1$ as $r \to \infty$, the same as the circle.

Including the time component, we have

$$g_{\gamma\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(\alpha^2\cos^2\theta + \sin^2\theta) & -(1-\alpha^2) r\sin\theta\cos\theta \\ 0 & -(1-\alpha^2) r\sin\theta\cos\theta & -r^2(\alpha^2\sin^2\theta + \cos^2\theta) \end{pmatrix}, \tag{7.3.6}$$

which gives the line element

$$c^2 d\tau^2 = A(r)\,dt^2 - B(r)(\alpha^2\cos^2\theta + \sin^2\theta)\,dr^2 - 2((1-\alpha^2)r\sin\theta\cos\theta\,dr d\theta - r^2(\alpha^2\sin^2\theta + \cos^2\theta)\,d\theta^2. \tag{7.3.7}$$

We include functions A and B to account for time dilation and the radius of the event horizon.

7.3.2 Lagrangian

Following the same method as in §4.2, the Lagrangian for our line element is

$$L = \tfrac{1}{2}\left(A\,\dot{t}^2 - B(\alpha^2\cos^2\theta + \sin^2\theta)\,\dot{r}^2 - 2(1-\alpha^2)r\sin\theta\cos\theta\,\dot{r}\dot{\theta} - r^2(\alpha^2\sin^2\theta + \cos^2\theta)\,\dot{\theta}^2\right). \tag{7.3.8}$$

The partial derivatives of this Lagrangian are

$$\frac{\partial L}{\partial \dot{t}} = A\,\dot{t}, \tag{7.3.9a}$$

$$\frac{\partial L}{\partial t} = 0, \tag{7.3.9b}$$

$$\frac{\partial L}{\partial \dot{r}} = -B(\alpha^2\cos^2\theta + \sin^2\theta)\,\dot{r}, \tag{7.3.9c}$$

$$\frac{\partial L}{\partial r} = \frac{A'}{2}\,\dot{t}\dot{r} - \frac{B'}{2}(\alpha^2\cos^2\theta + \sin^2\theta)\,\dot{r}^2 - (1-\alpha^2)\sin\theta\cos\theta\,\dot{r}\dot{\theta} - r(\alpha^2\sin^2\theta + \cos^2\theta)\,\dot{\theta}^2, \tag{7.3.9d}$$

$$\frac{\partial L}{\partial \dot{\theta}} = -r(1-\alpha^2)\sin\theta\cos\theta\,\dot{r} - r^2(\alpha^2\sin^2\theta + \cos^2\theta)\,\dot{\theta}, \tag{7.3.9e}$$

$$\frac{\partial L}{\partial \theta} = -B(1-\alpha^2)\sin\theta\cos\theta\,\dot{r}^2 - r(1-\alpha^2)(\cos^2\theta - \sin^2\theta)\,\dot{r}\dot{\theta} - r^2(\alpha^2 - 1)\sin\theta\cos\theta\,\dot{\theta}^2. \tag{7.3.9f}$$

We take Equation 5.2.3, the Euler-Lagrange equation, and plug in the partial derivatives. This yields

$$A\,\ddot{t} + A'\,\dot{t}\dot{r} = 0, \tag{7.3.10}$$

$$-B(\alpha^2\cos^2\theta + \sin^2\theta)\,\ddot{r} - \frac{A'}{2}\,\dot{t}\dot{r} - \frac{B'}{2}(\alpha^2\cos^2\theta + \sin^2\theta)\,\dot{r}^2 - 2B(1-\alpha^2)\sin\theta\cos\theta\,\dot{r}\dot{\theta} + r(\alpha^2\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2 = 0, \tag{7.3.11}$$

$$-r^2(\alpha^2\sin^2\theta+\cos^2\theta)\,\ddot{\theta}-r(1-\alpha^2)\sin\theta\cos\theta\,\ddot{r}+(B-1)(1-\alpha^2)\sin\theta\cos\theta\,\dot{r}^2$$
$$-2r(\alpha^2\sin^2\theta+\cos^2\theta)\,\dot{r}\dot{\theta}-r^2(\alpha^2-1)\sin\theta\cos\theta\,\dot{\theta}^2=0. \quad (7.3.12)$$

We notice that the third Euler-Lagrange equation has two second derivatives in it. This poses a problem, because we do not know how to put it into the geodesic form so that we can find the Christoffel symbols. However, each Euler-Lagrange correctly reduces to the corresponding equation in circular polar coordinates.

7.4 Relativistic Ellipsoidal Motion in 3+1 Dimensions

We have slightly higher hopes for a 3+1 dimensional ellipsoid to produce a black hole than a 2+1 dimensional ellipse, but an ellipsoid isn't spherically symmetric. Since a Schwarzschild solution requires spherical symmetry, we might not be able to produce an ellipsoidal black hole solution.

Appendix A

We begin by explaining the derivation of Gauss' law for three spatial dimensions [7]. In three dimensions, the inverse square field allows us to write the "joint form" of Newton's law of gravity.

The flux of the field through the surface will be

$$\oiint \mathbf{G} \cdot \mathbf{n}\, da. \tag{A.1.1}$$

With the divergence theorem, we have

$$\oiint \mathbf{G} \cdot \mathbf{n}\, da = \oiiint \nabla \cdot \mathbf{G}\, d\tau, \tag{A.1.2}$$

where $d\tau$ is an interior volume element. For a point mass at the center of the sphere, we have

$$\mathbf{G} = -\frac{km}{r^2}\mathbf{e}_r$$

for some constant k, and the surface element is

$$\mathbf{n}\, da = \mathbf{e}_r r^2 \sin\theta\, d\theta d\phi.$$

Thus

$$\oiint \mathbf{G} \cdot \mathbf{e}_r \, da = \oiint \left(-\frac{km}{r^2}\right) r^2 \sin\theta \, d\theta d\phi$$
$$= -4\pi km \tag{A.1.3}$$

since the mass density is $\rho = dm/d\tau$. We can write

$$m = \oiiint \rho d\tau,$$

so the divergence theorem gives

$$-4\pi k \oiiint \rho d\tau = \oiiint \nabla \cdot \mathbf{G} \, d\tau. \tag{A.1.4}$$

It follows that

$$\nabla \cdot \mathbf{G} = -4\pi k\rho, \tag{A.1.5}$$

and so for some potential Ξ, since $\mathbf{G} = \nabla\Xi$, we have

$$\nabla^2\Xi = -4\pi k\rho, \tag{A.1.6}$$

known as Poisson's equation.

In general relativity, the coupling constant in Einstein's field equation is determined by requiring the field potential in the "curved" manifold to reduce to the Newtonian potential in flat Minkowski space via the equivalent to Poisson's equation in "curved" space, where ∇^2 is modified by the chosen metric.

We reproduce this argument in two-dimensional space, replacing m by a 2-mass m', k by an empirical gravitational constant k', mass-volume density ρ by the surface density σ, and the 3-space divergence theorem with the 2-space version of the divergence theorem,

$$\oint \mathbf{G}' \cdot \mathbf{n} \, dl = \oiint \nabla \cdot \mathbf{G}' \, d\sigma. \tag{A.1.7}$$

A brief demonstration of how we arrived at this 2-space divergence theorem is given in Appendix B. The flux through a circle surrounding a 2-mass will be

$$\oint \mathbf{G}' \cdot \mathbf{n}\, dl = \oint \mathbf{G}' \cdot \mathbf{n}(r\, d\theta) \tag{A.1.8}$$

$$= \oiint \nabla \cdot \mathbf{G}\, da. \tag{A.1.9}$$

To provide the necessary constant flux, we choose

$$\mathbf{G} = -\frac{k'm'}{r} \tag{A.1.10}$$

so that

$$\oint \mathbf{G} \cdot \mathbf{n}\, r\, d\theta = -k'm' \oint d\theta$$
$$= -2\pi k'm' = \oiint \nabla \cdot \mathbf{G}\, da. \tag{A.1.11}$$

Now writing the surface mass density as

$$\sigma = \frac{dm'}{da} \quad \text{or} \quad m' = \oiint \sigma\, da,$$

we have

$$\nabla \cdot \mathbf{G} = -2\pi k'\sigma. \tag{A.1.12}$$

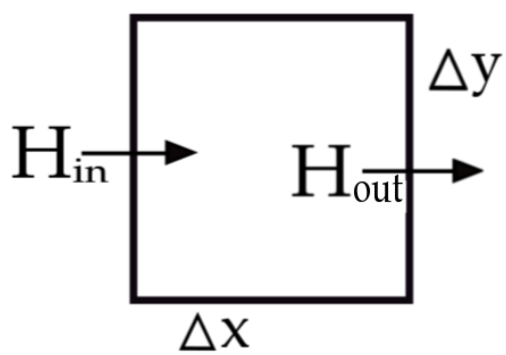
Therefore, for a potential Ξ, we have

$$\nabla^2 \Xi = -2\pi k'\sigma. \tag{A.1.13}$$

The derivation of how we obtain the potential is in Appendix C.

Appendix B

We want to define the divergence theorem in two spatial dimensions to find the flux of a field through an area. We begin with a field **H** and a differential box with side lengths Δx and Δy.



The flux in the x-direction is

$$\Phi_x = H_{\text{out}}\Delta y - H_{\text{in}}\Delta y. \tag{B.1.1}$$

We can calculate the field leaving the box,

$$H_{\text{out}} = H_{\text{in}} + \frac{\Delta \mathbf{H}}{\Delta x}\Delta x, \tag{B.1.2}$$

and plug this in Equation B.1.1, which gives

$$\Phi_x = \left(H_{\text{in}} + \frac{\Delta\mathbf{H}}{\Delta x}\Delta x\right)\Delta y - H_{\text{in}}\Delta y. \tag{B.1.3}$$

We notice that $\Delta x \Delta y$ is the area of the differential box, so we will call it dA. Therefore

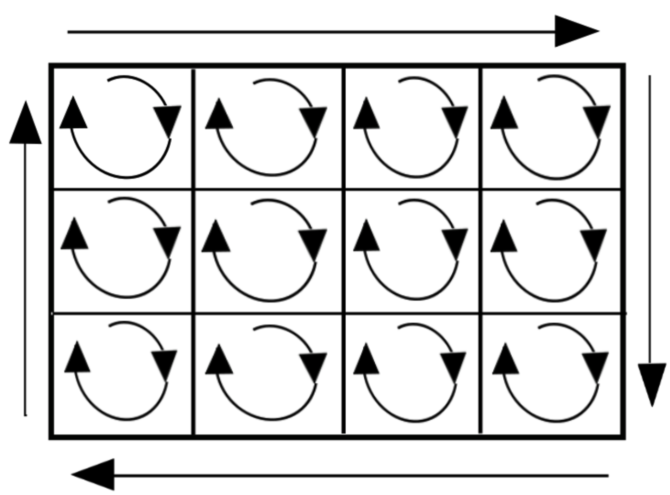
$$\Phi_x = \frac{\Delta\mathbf{H}}{\Delta x} dA = H_x \cdot \mathbf{n}\, dy, \tag{B.1.4}$$

where H_x is the x-component of the field. Finding the flux in the y-direction and the y-component of the field is done in the same way.

For a finite area composed of multiple differential boxes, we sum over all the differential boxes, which yields

$$\sum_{\text{circumference}} \mathbf{H} \cdot \mathbf{n}\, dl = \sum_{\text{interior areas}} \nabla \cdot \mathbf{H}\, dA. \tag{B.1.5}$$

The total flux is a summation of the flux through each box, each of which is a sum of the four sides. We see that the fluxes around neighboring sides effectively cancel each other out, so the total flux is an integral around the perimeter of the area.



Thus

$$\oint \mathbf{H} \cdot \mathbf{n}\, dl = \iint \nabla \cdot \mathbf{H}\, dA. \tag{B.1.6}$$

Appendix C

We begin with a 2-dimensional classical circular orbit, which has

$$\ddot{\mathbf{r}} = (\ddot{r} - r\,\dot{\theta}^2)\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\dot{\theta})\mathbf{e}_\theta. \tag{C.1.1}$$

For a force $\mathbf{F}(r) = f(r)\mathbf{e}_r$, we have

$$f(r) = \ddot{r} - r\,\dot{\theta}^2 \tag{C.1.2}$$

and an angular momentum

$$L = mr^2\,\dot{\theta}, \tag{C.1.3}$$

which can be rewritten as

$$\dot{\theta}^2 = \frac{L^2}{m^2 r^4}. \tag{C.1.4}$$

Since we are using a $\frac{1}{r}$ force

$$\mathbf{F}(r) = m\,\ddot{r}\,\mathbf{e}_r = -\frac{k}{r}\,\mathbf{e}_r, \tag{C.1.5}$$

and

$$m\left(\ddot{r} - \frac{L^2}{m^2 r^3}\right) = -\frac{k}{r},$$

then our equation for a two-dimensional force becomes

$$\mathcal{F}(r) = m\,\ddot{r} = \frac{L^2}{mr^3} - \frac{k}{r}. \tag{C.1.6}$$

To find the pseudo-potential, we can treat this as a one-dimensional problem with an attractive force $-k/r$ plus a fictional "centrifugal" repulsive force L^2/mr^3.

As a one-dimensional problem, we have a pseudo-potential

$$\begin{aligned}\mathcal{V}(r) &= -\int \mathcal{F}(r)\,dr \\ &= -\int \left(\frac{L^2}{mr^3} - \frac{k}{r}\right) dr \\ &= -\left(-\frac{L^2}{2mr^2} - k\ln(r)\right) + \beta \\ &= \frac{L^2}{2mr^2} + k\ln(r) + \beta,\end{aligned} \tag{C.1.7}$$

where β is a constant of integration.

The stable circular orbit will be where $\mathcal{V}(r)$ is a minimum, so

$$\left.\frac{d\mathcal{V}(r)}{dr}\right|_{r=r_0} = -\mathcal{F}(r)\Big|_{r=r_0} = \left.\left(\frac{L^2}{mr^3} - \frac{k}{r}\right)\right|_{r=r_0} = 0. \tag{C.1.8}$$

It follows that

$$\frac{L^2}{m(r_0)^3} = \frac{k}{r_0}, \tag{C.1.9}$$

so

$$r_0 = \frac{L}{\sqrt{km}}. \tag{C.1.10}$$

Solving for and plugging in for L with a constant radius r_0 we have

$$\dot{\theta} = \frac{1}{r_0}\sqrt{\frac{k}{m}}, \tag{C.1.11}$$

so

$$\theta = \frac{\tau}{r_0}\sqrt{\frac{k}{m}} + \gamma \tag{C.1.12}$$

where γ is a constant of integration. This gives a linear equation of motion.

Appendix D

To define the Christoffel symbols via coordinate transformation, we will be changing from Cartesian coordinates x, y to circular polar coordinates r, θ.

We begin with expressions for x and y in terms of r and θ,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{D.1.1}$$

and expressions for r and θ in terms of x and y, which are

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right). \tag{D.1.2}$$

We use the transformation rule [3, p 283]

$$\Gamma^{k'}_{p'q'} = \frac{\partial z^{k'}}{\partial z^k}\left(\Gamma^k_{pq}\frac{\partial z^p}{\partial z^{p'}}\frac{\partial z^q}{\partial z^{q'}} + \frac{\partial^2 z^k}{\partial z^{p'}\partial z^{q'}}\right), \tag{D.1.3}$$

where $k, p, q = 1, 2$ and $z^1, \cdots, z^n$ is the set of coordinates such that primed coordinates are circular polar and unprimed coordinates are Cartesian, so $z^1 = x,\ z^2 = y,\ z^{1'} = r,\ z^{2'} = \theta$. All Christoffel symbols in Cartesian coordinates are trivial, so this gives a more specific transformation rule [3, p 285]

$$\Gamma^{k'}_{p'q'} = \frac{\partial z^{k'}}{\partial z^k}\frac{\partial^2 z^k}{\partial z^{p'}\partial z^{q'}}. \tag{D.1.4}$$

Change of coordinates method of computing Christoffel symbols - Cartesian to Circular Polar

```
# VARIABLES
x = var('x')        # Euclidean coordinate x
y = var('y')        # Euclidean coordinate y
r = var('r')        # polar coordinate, radius
q = var('theta')    # polar coordinate, angle

# EQUATIONS OF MOTION
x = r * cos(q)
y = r * sin(q)

# FIRST DERIVATIVES
dxdr = x.derivative(r)
dxdq = x.derivative(q)
dydr = y.derivative(r)
dydq = y.derivative(q)

# SECOND DERIVATIVES
ddxdrdr = dxdr.derivative(r)
ddxdrdq = dxdr.derivative(q)
ddxdqdr = dxdq.derivative(r)
ddxdqdq = dxdq.derivative(q)
ddydrdr = dydr.derivative(r)
ddydrdq = dydr.derivative(q)
ddydqdr = dydq.derivative(r)
ddydqdq = dydq.derivative(q)
```

```
# VARIABLES
x = var('x')        # Euclidean coordinate x
y = var('y')        # Euclidean coordinate y
r = var('r')        # polar coordinate, radius
q = var('theta')    # polar coordinate, angle

# EQUATIONS OF MOTION
r = sqrt(x^2 + y^2)
q = arctan(y / x)

# FIRST DERIVATIVES
drdx = r.derivative(x)
drdy = r.derivative(y)
dqdx = q.derivative(x)
dqdy = q.derivative(y)
```

```
# No second derivatives needed
```

We will now plug into the transformation rule

$$\Gamma^{k'}_{p'q'} = \frac{\partial z^{k'}}{\partial z^k}\frac{\partial^2 z^k}{\partial z^{p'}\partial z^{q'}}$$

to obtain the Christoffel symbols. Note: Gmns = Γ^m_{ns}.

```
Grrr = drdx * ddxdrdr + drdy * ddydrdr
show(Grrr)
```

$$0$$

```
Grrq = drdx * ddxdrdq + drdy * ddydrdq
show(Grrq)
```

$$-\frac{x\sin(\theta)}{\sqrt{x^2+y^2}} + \frac{y\cos(\theta)}{\sqrt{x^2+y^2}}$$

```
Grqr = drdx * ddxdqdr + drdy * ddydqdr
show(Grqr)
```

$$-\frac{x\sin(\theta)}{\sqrt{x^2+y^2}} + \frac{y\cos(\theta)}{\sqrt{x^2+y^2}}$$

```
Grqq = drdx * ddxdqdq + drdy * ddydqdq
show(Grqq)
```

$$-\frac{rx\cos(\theta)}{\sqrt{x^2+y^2}} - \frac{ry\sin(\theta)}{\sqrt{x^2+y^2}}$$

```
Gqrr = dqdx * ddxdrdr + dqdy * ddydrdr
show(Gqrr)
```

$$0$$

```
Gqrq = dqdx * ddxdrdq + dqdy * ddydrdq
show(Gqrq)
```

$$\frac{\cos(\theta)}{\left(\frac{y^2}{x^2}+1\right)x} + \frac{y\sin(\theta)}{\left(\frac{y^2}{x^2}+1\right)x^2}$$

```
Gqqr = dqdx * ddxdqdr + dqdy * ddydqdr
```

```
show(Gqqr)
```

$$\frac{\cos(\theta)}{\left(\frac{y^2}{x^2}+1\right)x}+\frac{y\sin(\theta)}{\left(\frac{y^2}{x^2}+1\right)x^2}$$

```
Gqqq = dqdx * ddxdqdq + dqdy * ddydqdq
show(Gqqq)
```

$$-\frac{r\sin(\theta)}{\left(\frac{y^2}{x^2}+1\right)x}+\frac{ry\cos(\theta)}{\left(\frac{y^2}{x^2}+1\right)x^2}$$

Appendix E

To avoid algebraic errors due to long computations, we plug in the Christoffel symbols for 2+1 spatial dimensions, previously found via the Euler-Lagrange method in §5.2, into Sage to find the Ricci tensor components.

Ricci tensor components for 2+1 dimensions

```
# VARIABLES
t = var('t')            # 0, time
r = var('r')            # 1, radius
q = var('theta')        # 2, angle

# FUNCTIONS that determine the how the spacetime warps under a
Schwarzschild metric
A = function('A', r)    # the time dilation coefficient
Ap = A.derivative(r)    # A', the first derivative of A with
respect to r
B = function('B', r)    # gives the Schwarzschild radius
Bp = B.derivative(r)    # B', the first derivative of B with
respect to r

# the CHRISTOFFEL SYMBOLS - previously found via the Euler-
Lagrange method
G000 = 0
G001 = Ap / (2*A)
G010 = Ap / (2*A)
G011 = 0
G002 = 0
G020 = 0
G012 = 0
G021 = 0
G022 = 0
G100 = Ap / (2 * B)
G101 = 0
G110 = 0
G111 = Bp / (2 * B)
G102 = 0
G120 = 0
G121 = 0
G112 = 0
G122 = -r / B
G200 = 0
G201 = 0
G210 = 0
G211 = 0
G202 = 0
G220 = 0
G212 = 1 / r
G221 = 1 / r
G222 = 0
```

Note: Gabc = Γ^{a}_{bc}. We compute the Ricci tensor components by

$$R_{\mu\nu} = \partial_\nu \Gamma^{\sigma}_{\mu\sigma} - \partial_\sigma \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma},$$

where $\mu, \nu, \varrho, \sigma$ go from 0 to 2, where $t = 0, r = 1, \theta = 2$. In the following results, note that $D[0]A = dA/dr = Ap$.

```
R00 = derivative(G000,t) - derivative(G000,t) + G000 * G000 -
G000 * G000 + G100 * G010 - G100 * G010 + G200 * G020 - G200 *
G020 + derivative(G101,t) - derivative(G100,r) + G001 * G100 -
G000 * G101 + G101 * G110 - G100 * G111 + G201 * G120 - G200 *
G121 + derivative(G202,t) - derivative(G200,q) + G002 * G200 -
G000 * G202 + G102 * G210 - G100 * G212 + G202 * G220 - G200 *
G222

show(R00)
```

$$\frac{D[0](A)(r)^2}{4A(r)B(r)} - \frac{D[0,0](A)(r)}{2B(r)} + \frac{D[0](A)(r)D[0](B)(r)}{4B(r)^2} - \frac{D[0](A)(r)}{2rB(r)}$$

```
R11 = derivative(G010,r) - derivative(G011,t) + G010 * G001 -
G011 * G000 + G110 * G011 - G111 * G010 + G210 * G021 - G211 *
G020 + derivative(G111,r) - derivative(G111,r) + G011 * G101 -
G011 * G101 + G111 * G111 - G111 * G111 + G211 * G121 - G211 *
G121 + derivative(G212,r) - derivative(G211,q) + G012 * G201 -
G011 * G202 + G112 * G211 - G111 * G212 + G212 * G221 - G211 *
G222

show(R11)
```

$$-\frac{D[0](A)(r)D[0](B)(r)}{4A(r)B(r)} + \frac{D[0,0](A)(r)}{2A(r)} - \frac{D[0](A)(r)^2}{4A(r)^2} - \frac{D[0](B)(r)}{2rB(r)}$$

```
R22 = derivative(G020,q) - derivative(G022,t) + G020 * G002 -
G022 * G000 + G120 * G012 - G122 * G010 + G202 * G022 - G222 *
G020 + derivative(G121,q) - derivative(G122,r) + G021 * G102 -
G022 * G101 + G121 * G112 - G122 * G111 + G221 * G122 - G222 *
G121 + derivative(G222,q) - derivative(G222,q) + G022 * G202 -
G022 * G202 + G122 * G212 - G122 * G212 + G222 * G222 - G222 *
G222

show(R22)
```

$$\frac{rD[0](A)(r)}{2A(r)B(r)} - \frac{rD[0](B)(r)}{2B(r)^2}$$

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